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# BOUNDARY BEHAVIOR OF THE QUASI-HYPERBOLIC METRIC

NIKOLAI NIKOLOV AND PASCAL J. THOMAS

ABSTRACT. The precise behavior of the quasi-hyperbolic metric near a  $\mathcal{C}^{1,1}$ -smooth part of the boundary of a domain in  $\mathbb{R}^n$  is obtained.

## 1. INTRODUCTION AND RESULTS

Let  $D$  be a proper subdomain of  $\mathbb{R}^n$ . Define the *quasi-hyperbolic metric* of  $D$  by

$$h_D(a, b) = \inf_{\gamma} \int_{\gamma} \frac{\|du\|}{d_D(u)}, \quad a, b \in D$$

where  $\|\cdot\|$  is the Euclidean norm,  $d_D = \text{dist}(\cdot, \partial D)$  and the infimum is taken over all rectifiable curves  $\gamma$  in  $D$  joining  $a$  to  $b$ . By [5, Lemma 1], the infimum is attained, and any extremal curve is called *quasi-hyperbolic geodesic* (for short, *geodesic*). It turns out that the geodesics are  $\mathcal{C}^{1,1}$ -smooth (see [8, Corollary 4.8]). The quasi-hyperbolic metric arises in the theory of quasi-conformal maps.

This paper is devoted to the boundary behavior of  $h_D$ . First, we point out the following general lower bound.

**Proposition 1.** [4, Lemma 2.6] *If  $D$  is a proper subdomain of  $\mathbb{R}^n$ , then*

$$h_D(a, b) \geq 2 \log \frac{d_D(a) + d_D(b) + \|a - b\|}{2\sqrt{d_D(a)d_D(b)}}, \quad a, b \in D.$$

Observe that equality occurs if  $n = 1$  (then  $D$  is an open interval or ray).

From now, we assume that  $n \geq 2$ . Throughout the paper, we will say that  $\zeta$  is a  $\mathcal{C}^\alpha$  *smooth boundary point* of  $D$  if and only if it admits a neighborhood in which  $\partial D$  is  $\mathcal{C}^\alpha$ -smooth.

Recall that a  $\mathcal{C}^1$ -smooth boundary point  $\zeta$  of a domain  $D$  in  $\mathbb{R}^n$  is said to be *Dini-smooth* if the inner unit normal vector  $n$  to  $\partial D$  near  $\zeta$  is a

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Dini-continuous function. This means that there exists a neighborhood  $U$  of  $\zeta$  such that  $\int_0^1 \frac{\omega(t)}{t} dt < +\infty$ , where

$$\omega(t) = \omega(n, \partial D \cap U, t) := \sup\{\|n_x - n_y\| : \|x - y\| < t, x, y \in \partial D \cap U\}$$

is the respective modulus of continuity.

If  $\int_0^1 \omega(t) \frac{\log t}{t} dt > -\infty$ , then the point  $\zeta$  is called *log-Dini smooth*.

The following relations between different notions of smoothness are clear:  $\mathcal{C}^{1,\varepsilon} \Rightarrow \text{log-Dini} \Rightarrow \text{Dini} \Rightarrow \mathcal{C}^1$ .

**Theorem 2.** [9, Theorem 7] *Let  $\zeta$  be a Dini-smooth boundary point of a domain  $D$  in  $\mathbb{R}^n$ . Then for any constant  $c > 1 + \frac{\sqrt{2}}{2}$  there exists a neighborhood  $U$  of  $\zeta$  such that*

$$h_D(a, b) \leq 2 \log \left( 1 + \frac{c\|a - b\|}{\sqrt{d_D(a)d_D(b)}} \right), \quad a, b \in D \cap U.$$

Since  $h_D$  is an inner metric, we get an upper bound of  $h_D$ , similar to the lower bound from Proposition 1.

**Corollary 3.** [9, Corollary 8] *Let  $D$  be a Dini-smooth bounded domain in  $\mathbb{R}^n$ . Then there exists a constant  $c > 0$  such that*

$$h_D(a, b) \leq 2 \log \left( 1 + \frac{c\|a - b\|}{\sqrt{d_D(a)d_D(b)}} \right), \quad a, b \in D.$$

$$\begin{aligned} \text{Set now } s_D(a, b) &= 2 \sinh^{-1} \frac{\|a - b\|}{2\sqrt{d_D(a)d_D(b)}} \\ &= 2 \log \frac{\|a - b\| + \sqrt{\|a - b\|^2 + 4d_D(a)d_D(b)}}{2\sqrt{d_D(a)d_D(b)}}, \quad a, b \in D. \end{aligned}$$

Note that  $h_D = s_D$  if  $D$  is a half-space in  $\mathbb{R}^n$  (cf. [12, (2.8)]).

The following sharp result holds in the  $\mathcal{C}^1$ -smooth case.

**Proposition 4.** [9, Proposition 6(a)] *If  $\zeta$  is a  $\mathcal{C}^1$ -smooth boundary point of a domain  $D$  in  $\mathbb{R}^n$ , then*

$$\lim_{\substack{a, b \rightarrow \zeta \\ a \neq b}} \frac{h_D(a, b)}{s_D(a, b)} = 1.$$

Since the proof of this proposition is not long, we shall include it for completeness.

**Corollary 5.** [9, Proposition 6 (b) and p. 3] *If  $D$  is a  $C^1$ -smooth bounded domain in  $\mathbb{R}^n$ , then*

$$q_D(a, b) = \begin{cases} \frac{h_D(a, b)}{s_D(a, b)}, & a, b \in D, a \neq b \\ 1, & \text{otherwise} \end{cases}$$

*is a continuous function on  $\mathbb{R}^n \times \mathbb{R}^n$ .*

The main goal of this paper is to prove the following result related to Proposition 4.

**Theorem 6.** *If  $\zeta$  is a  $C^{1,1}$ -smooth boundary point of a domain  $D$  in  $\mathbb{R}^n$ , then*

$$\lim_{a, b \rightarrow \zeta} (h_D(a, b) - s_D(a, b)) = 0.$$

Note that Theorem 6 and Proposition 4 say the same only if  $s_D$  and  $1/s_D$  are bounded.

The assumption about regularity in Theorem 6 can be weakened in the plane.

**Proposition 7.** *If  $\zeta$  is a log-Dini smooth boundary point of a domain  $D$  in  $\mathbb{R}^2$ , then*

$$\lim_{a, b \rightarrow \zeta} (h_D(a, b) - s_D(a, b)) = 0.$$

The above results imply the following optimal version of Theorem 2.

**Corollary 8.** *Let  $\zeta$  be a  $C^{1,1}$ -smooth boundary point of a domain  $D$  in  $\mathbb{R}^n$  or  $\zeta$  be a log-Dini smooth boundary point of a domain  $D$  in  $\mathbb{R}^2$ . Then for any constant  $c > 1$  there exists a neighborhood  $U$  of  $\zeta$  such that*

$$h_D(a, b) \leq 2 \log \left( 1 + \frac{c \|a - b\|}{\sqrt{d_D(a) d_D(b)}} \right), \quad a, b \in D \cap U.$$

The rest of the paper is organized as follows: Section 2 contains the proofs of Propositions 4, 7 and Corollary 8. Section 3 contains the proof of Theorem 6. It should be mentioned that the three proofs use different flattening maps. Section 4 contains the proof of a result analogous to Corollary 8 for the Kobayashi distance.

## 2. PROOFS OF PROPOSITIONS 4, 7 AND COROLLARY 8

*Proof of Proposition 4.* After translation and rotation, we may assume that  $\zeta = 0$  and that there is a neighborhood  $U$  of 0 such that

$$D' := D \cap U = \{x \in U : r(x) := x_1 + f(x') > 0\},$$

where points of  $\mathbb{R}^n$  are denoted by  $x = (x_1, x')$ , with  $x' \in \mathbb{R}^{n-1}$ , and  $f$  is a  $\mathcal{C}^1$ -smooth function in  $\mathbb{R}^n$  with  $f(0) = 0$  and  $\nabla f(0) = 0$ .

Let  $c > 1$  and  $\theta(x) = (r(x), x')$ . We may shrink  $U$  such that

$$(1) \quad c^{-1}\|x - y\| \leq \|\theta(x) - \theta(y)\| \leq c\|x - y\|, \quad x, y \in U.$$

Choose now a neighborhood  $V \subset U$  of 0 such that  $d_{D'} = d_D$  on  $D \cap V$ . The regularity of  $D$  implies that it is a *uniform domain* near  $\zeta$  in the sense of [5]. Using, for example, [5, Corollary 2], one can find a neighborhood  $W \subset V$  of 0 such that any geodesic joining points in  $\tilde{D} = D \cap W$  is contained in  $D \cap V$ . Then  $h_D = h_{D'}$  on  $\tilde{D}^2$ .

Set  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_1 > 0\}$ . Using the above arguments, we may shrink  $W$  such that  $h_{\mathbb{R}_+^n} = h_{\theta(D')}$  on  $(\theta(\tilde{D}))^2$ .

On the other hand, (1) implies that (cf. [12, Exercise 3.17])

$$c^{-2}h_{D'}(z, w) \leq h_{\theta(D')}(\theta(z), \theta(w)) \leq c^2h_{D'}(z, w), \quad z, w \in D'.$$

Let  $z, w \in \tilde{D}$ . Then

$$c^{-2}h_D(z, w) \leq h_{\mathbb{R}_+^n}(\theta(z), \theta(w)) \leq c^2h_D(z, w).$$

Using (1) again, we get that

$$\begin{aligned} h_{\mathbb{R}_+^n}(\theta(z), \theta(w)) &= 2 \sinh^{-1} \frac{\|\theta(z) - \theta(w)\|}{2\sqrt{r_D(z)r_D(w)}} \\ &\leq 2 \sinh^{-1} \frac{c^2\|z - w\|}{2\sqrt{d_D(z)d_D(w)}} \leq c^2s_D(z, w). \end{aligned}$$

We obtain in the same way that

$$h_{\mathbb{R}_+^n}(\theta(z), \theta(w)) \geq c^{-2}s_D(z, w).$$

So

$$c^{-4}h_D(z, w) \leq s_D(z, w) \leq c^4h_D(z, w)$$

which implies the desired result.  $\square$

*Proof of Proposition 7.* We may find a neighborhood  $U$  of  $\zeta$  such that  $D \cap U$  is a bounded simply connected log-Dini smooth domain. Using an argument from the previous proof, we may replace  $D$  by  $D \cap U$ .

The Kellogg–Warschawski theorem (cf. [11, Theorem 3.5]) implies that there exists a conformal map  $\tilde{f}$  from the unit disc  $\mathbb{D}$  to  $D$  which extends to a  $\mathcal{C}^1$ -diffeomorphism between  $\overline{\mathbb{D}}$  to  $\overline{D}$  such that  $\tilde{f}(\zeta) = 1$  and

$$|\tilde{f}'(z) - \tilde{f}'(w)| \leq \omega^*(|z - w|), \quad z, w \in \mathbb{D},$$

where  $\tilde{\omega}^*(s) = \int_0^s \frac{\tilde{\omega}(t)}{t} dt + s \int_s^{+\infty} \frac{\tilde{\omega}(t)}{t^2} dt$  ( $s \geq 0$ ) and  $\tilde{\omega} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a bounded continuous function with  $\int_0^1 \tilde{\omega}(t) \frac{\log t}{t} dt > -\infty$ .

Then  $f(z) = \tilde{f}\left(\frac{1-z}{1+z}\right)$  maps conformally  $\mathbb{R}_+^2$  onto  $D$  and

$$|f'(z) - f'(w)| \leq \omega^*(|z - w|), \quad z, w \in G = \mathbb{R}_+^2 \cap \mathbb{D},$$

where  $\omega^*$  is defined in the same way as  $\tilde{\omega}^*$ .

The equality

$$f(w) - f(z) - f'(z)(w - z) = (w - z) \int_0^1 (f'(z + t(w - z)) - f'(z)) dt$$

implies that

$$|f(w) - f(z) - f'(z)(w - z)| \leq |w - z| \omega^*(|w - z|)$$

(since  $\omega^*$  is an increasing function). It follows that

$$(2) \quad |d_D(f(z)) - |f'(z)| d_{\mathbb{R}_+^2}(z)| \leq d_{\mathbb{R}_+^2}(z) \omega^*(d_{\mathbb{R}_+^2}(z)), \quad z \in G.$$

Since  $D$  is a uniform domain, there exists a neighborhood  $V$  of  $\zeta$  such that any geodesic  $\gamma$  joining points  $a = f(\alpha)$  and  $b = f(\beta)$  in  $D \cap V$  is contained in  $f(G)$ . It follows by (2) that one may find a constant  $C > 0$  (independent of  $a$  and  $b$ ) such that

$$h_{\mathbb{R}_+^2}(\alpha, \beta) \leq \int_{f^{-1} \circ \gamma} \frac{|du|}{d_{\mathbb{R}_+^2}(u)} \leq \int_{\gamma} \frac{|dv|}{d_D(v)} + C \int_{\gamma} \frac{\omega^*(d_D(v))}{d_D(v)} |dv|.$$

The first summand is equal to  $h_D(a, b)$ .

We claim that the second summand tends to 0 as  $a, b \rightarrow \zeta$ . Indeed, denote by  $t$  the natural parameter of  $\gamma$  by arc length and by  $l = l(\gamma)$  the Euclidean length of  $\gamma$ . Since  $D$  is a uniform domain, then [5, Corollary 2] provides a constant  $c > 0$  (independent of  $a$  and  $b$ ) such that  $c \cdot l \leq |a - b|$  and  $d_D(\gamma(t)) \geq c \cdot \max\{t, l - t\}$ . Using that  $\frac{\omega^*(s)}{s}$  is a decreasing function, we get

$$\int_{\gamma} \frac{\omega^*(d_D(v))}{d_D(v)} |dv| \leq \frac{2}{c} \int_0^{cl/2} \frac{\omega^*(t)}{t} dt.$$

It is easy to check the log-Dini condition for  $\omega$  is equivalent to the fact that the last integral tends to 0 as  $l \rightarrow 0$  which implies our claim.

Hence

$$\liminf_{a, b \rightarrow \zeta} (h_D(a, b) - h_{\mathbb{R}_+^2}(\alpha, \beta)) \geq 0.$$

The opposite inequality

$$\limsup_{a,b \rightarrow \zeta} (h_D(a, b) - h_{\mathbb{R}_+^2}(\alpha, \beta)) \leq 0$$

follows in the same way by taking the geodesic joining  $\alpha$  and  $\beta$ .

Using (2), we have that

$$(3) \quad \lim_{\substack{a,b \rightarrow \zeta \\ a \neq b}} \frac{|a - b|}{2\sqrt{d_D(a)d_D(b)}} \cdot \frac{2\sqrt{d_{\mathbb{R}_+^2}(\alpha)d_{\mathbb{R}_+^2}(\beta)}}{|\alpha - \beta|} = 1.$$

Since  $h_{\mathbb{R}_+^2} = s_{\mathbb{R}_+^2}$  and  $\sinh^{-1} qt < \log q + \sinh^{-1} t$  for  $q > 1, t > 0$ , then

$$\lim_{a,b \rightarrow \zeta} (s_D(a, b) - h_{\mathbb{R}_+^2}(\alpha, \beta)) = 0$$

which completes the proof.  $\square$

*Proof of Corollary 8.* We may assume that  $c = 2c' - 1 \in (1, 3]$ . By Proposition 4, Theorem 6 and Proposition 7, one may find a neighborhood  $U$  of  $\zeta$  such that for  $a, b \in D \cap U$ ,

$$h_D(a, b) \leq c' s_D(a, b), \quad h_D(a, b) \leq s_D(a, b) + \log c'.$$

Then the result follows by the inequalities  $\sinh^{-1} \frac{t}{2} < \log(1+t)$  ( $t > 0$ ),  $(1+t)^{c'} < 1+ct$  ( $0 < t < 1$ ) and  $c'(1+t) < 1+ct$  ( $t > 1$ ).  $\square$

### 3. PROOF OF THEOREM 6

Theorem 6 will follow from Propositions 9 and 11 below.

For convenience, we assume that  $D$  is a domain in  $\mathbb{R}^{n+1}$  ( $n \geq 1$ ).

We first localize the problem. We choose local coordinates so that  $\zeta = 0$  and  $T_0 \partial D = \{0\} \times \mathbb{R}^n$ .

Denote points in  $\mathbb{R}^{n+1}$  by  $\bar{x} = (x_0, x) \in \mathbb{R} \times \mathbb{R}^n$ . We also write  $\mathbb{R}_+^{n+1} = \{\bar{x} \in \mathbb{R}^{n+1} : x_0 > 0\}$ .

There are a ball  $\mathcal{U} \subset \mathbb{R}^{n+1}$  centered at  $(0, 0)$  and a function  $f \in \mathcal{C}^{1,1}(\mathcal{U} \cap \mathbb{R}^n, \mathbb{R})$  such that  $f(0) = 0$  and  $Df(0) = 0$  and

$$(4) \quad D \cap \mathcal{U} = \{\bar{x} \in \mathcal{U} : x_0 > f(x)\}.$$

By shrinking the radius of  $\mathcal{U}$  further we may assume that the projection which to  $\bar{x} \in \mathcal{U} \cap D$  associates  $\pi(\bar{x})$ , the closest point in  $\partial D$  is well-defined, and that  $\mathcal{U} \subset \pi^{-1}(\mathcal{U} \cap D)$  (see [1, Lemma 4.11], or the proof of Lemma 10 (1) below).

**Proposition 9.**  $\liminf_{a,b \rightarrow 0} (h_D(a, b) - s_D(a, b)) \geq 0$ .

We can define a map  $\varphi$  on  $\mathcal{U}$  by

$$\varphi(\bar{x}) = (f(x), x) + x_0 n_x,$$

where  $n_x$  is the inward unit normal to  $\partial D$  at the point  $(f(x), x)$ .

**Lemma 10.** (1) *There exists a ball  $\mathcal{U}_0 \subset \mathcal{U}$  centered at 0 such that  $\varphi|_{\mathcal{U}_0}$  is a bilipschitz homeomorphism and for any  $\bar{x} \in \mathcal{U}_0 \cap \mathbb{R}_+^{n+1}$ ,*

$$d_D \varphi(\bar{x}) = \|\varphi(\bar{x}) - (f(x), x)\| = x_0.$$

(2) *Furthermore, if  $f \in C^\alpha(\mathcal{U} \cap \mathbb{R}^n, \mathbb{R})$ , for some  $\alpha \geq 2$ , then  $\varphi|_{\mathcal{U}_0}$  is a  $C^{\alpha-1}$ -diffeomorphism, and there exists a ball  $\mathcal{U}_1 \subset \mathcal{U}_0$  centered at 0 and a constant  $C > 0$  such that for any  $\bar{x} \in \mathcal{U}_1 \cap \mathbb{R}_+^{n+1}$  and any vector  $v \in \mathbb{R}^{n+1}$ ,*

$$\|D\varphi(\bar{x}) \cdot v\| \geq (1 - Cx_0)\|v\|.$$

where  $D\varphi(\bar{x})$  stands for the differential of  $\varphi$  taken at the point  $\bar{x}$ .

(3) *In the general case where  $f \in C^{1,1}(\mathcal{U} \cap \mathbb{R}^n, \mathbb{R})$ , then there exists a  $C > 0$  such that for any  $C^1$  curve  $\gamma : [t_1, t_2] \rightarrow \mathcal{U}_1 \cap \mathbb{R}_+^{n+1}$ ,  $\varphi \circ \gamma$  is rectifiable and for any  $F \in C([t_1, t_2], \mathbb{R}_+)$ ,*

$$\int_{t_1}^{t_2} F(t) |d\varphi \circ \gamma(t)| \geq \int_{t_1}^{t_2} F(t) |d\gamma(t)| - C \int_{t_1}^{t_2} F(t) d_D(\gamma(t)) |d\gamma(t)|.$$

*Proof.* Part (1) of the lemma is classical (see [1, Theorem 4.8]). The main point is to prove that the domain has positive *reach*, that is to say that there exists  $\delta > 0$  such that if  $x \in D$  and  $d_D(x) < \delta$ , then this distance is attained at a single point, which will be the intersection of  $\partial D$  and the unique normal line to it containing  $x$  (see [1]). In other words, for  $x \in \mathcal{U}$  well chosen and  $x_0 < \delta$ ,  $\varphi$  is one-to-one.

We quickly recall the proof. Suppose  $\|\nabla f(x) - \nabla f(x')\| \leq L\|x - x'\|$  for  $(0, x), (0, x') \in \mathcal{U}_1$ , then, taking without loss of generality the projection to  $\partial D$  to be  $(0, 0)$ , for some  $\theta \in (0, 1)$ ,

$$\begin{aligned} \|(y_0, 0) - (f(x), x)\|^2 &= y_0^2 - 2y_0 \nabla f(\theta x) \cdot x + f(x)^2 + \|x\|^2 \\ &\geq y_0^2 + \|x\|^2 - 2y_0 L \|x\|^2 > y_0^2 \end{aligned}$$

for  $y_0 < 1/2L$  and  $x \neq 0$ .

Notice that a lemma in [6, Appendix], explained in detail in [7], shows that even though  $n_x$  can only be expected to be continuous with bounded derivatives, and in general of class  $C^{\alpha-1}$  when  $\varphi \in C^\alpha$ , the function  $\bar{x} \mapsto d_D(\bar{x})$  has the same regularity as  $\varphi$ .

We now prove part (2). Let  $(e_0, e_1, \dots, e_n)$  be the standard basis of  $\mathbb{R}^{n+1}$ . Let  $\tilde{e}_j = \frac{\partial f}{\partial x_j}(x) e_0 + e_j$ , for  $1 \leq j \leq n$ . They form a basis of the tangent space to  $\partial D$  at  $(x, f(x))$  and  $\langle n_x, \tilde{e}_j \rangle = 0$  for  $1 \leq j \leq n$ .



Then  $D\varphi(\bar{x}) \cdot e_0 = n_x$ , and  $D\varphi(\bar{x}) \cdot e_j = \tilde{e}_j + x_0 \frac{\partial n_x}{\partial x_j}$ , for  $1 \leq j \leq n$ .

Given  $v = \sum_0^n v_j e_j$ ,

$$D\varphi(\bar{x}) \cdot v = \left( v_0 n_x + \sum_1^n v_j \tilde{e}_j \right) + x_0 \sum_1^n v_j \frac{\partial n_x}{\partial x_j} =: V_1 + V_0.$$

Clearly,  $\|V_0\| = O(x_0)\|v\|$ . By the orthogonality of  $n_x$  to the tangent space,

$$\begin{aligned} \|V_1\|^2 &= v_0^2 + \left\| \sum_1^n v_j \tilde{e}_j \right\|^2 = v_0^2 + \left\| \sum_1^n v_j e_j + \left( \sum_1^n \frac{\partial f}{\partial x_j}(x) \right) e_0 \right\|^2 \\ &= v_0^2 + \sum_1^n v_j^2 + \left| \sum_1^n \frac{\partial f}{\partial x_j}(x) \right|^2 \geq \|v\|^2. \end{aligned}$$

In the case where  $f \in \mathcal{C}^{1,1}$ , then  $\varphi \circ \gamma$  is only a Lipschitz map. By Rademacher's theorem (see e.g. [2, Theorem 3.1.6]), it is almost everywhere differentiable and the fundamental theorem of calculus holds. We then perform the same calculation as in case (2), where the integrands are defined a.e.  $\square$

*Proof of Proposition 9.* Using Lemma 10, the proof repeats the second part of the proof of Proposition 7. Suppose that  $\zeta = 0$  and that the domain  $D$  is given by a local representation as above. We may assume that the points  $a, b \in D$  are in a small enough neighborhood of 0 so that the geodesic  $\gamma$  which joins them is entirely contained in the range of invertibility of  $\varphi$  and Lemma 10 holds; we write  $a = \varphi(\bar{\alpha})$ ,  $b = \varphi(\bar{\beta})$ ,  $\gamma = \varphi(\tilde{\gamma})$ , where  $\tilde{\gamma}$  is an arc in  $\mathbb{R}_+^{n+1}$ . Then

$$h_D(a, b) = \int_\gamma \frac{\|du\|}{d_D(u)} \geq \int_{\tilde{\gamma}} \frac{\|dv\|}{d_{\mathbb{R}_+^{n+1}}(v)} - C \cdot l(\tilde{\gamma}) \geq h_{\mathbb{R}_+^{n+1}}(\bar{\alpha}, \bar{\beta}) - C' \|\bar{\alpha} - \bar{\beta}\|,$$

where  $C' > 0$  is a constant independent of  $a$  and  $b$ . Note that  $h_{\mathbb{R}_+^{n+1}} = s_{\mathbb{R}_+^{n+1}}$ . Since the differential of  $\varphi$  at  $\bar{x}$  tends to the identity as  $x \rightarrow 0$ , it follows that

$$\lim_{a, b \rightarrow \zeta} (s_{\mathbb{R}_+^{n+1}}(\bar{\alpha}, \bar{\beta}) - s_D(a, b)) = 0$$

which completes the proof.  $\square$

**Proposition 11.**  $\limsup_{a, b \rightarrow 0} (h_D(a, b) - s_D(a, b)) \leq 0$ .

The proof is similar to that of Proposition 9, using a modification of the map  $\varphi$  which depends on  $a$  and  $b$ .

*Proof.* We again assume that  $a, b \in D$ , and the geodesic connecting them, all lie in a neighborhood of  $\zeta$  small enough so that any point in it has a unique closest point on  $\partial D$ . Let  $a', b'$  be the respective closest points. We take new coordinates (and obtain a new function  $f$ ) so that  $a' = 0$  (instead of  $\zeta = 0$  as in the proof of Proposition 9) and

$$D \cap \mathcal{U} = \{\bar{x} \in \mathcal{U} : x_0 > f(x_1, \dots, x_n)\}.$$

We may also assume that  $b'_2 = \dots = b'_n = 0$ . Shrinking the radius  $r$  of  $\mathcal{U}$ , we may replace  $x_1$  by  $\sigma_1(x_1)$  such that for  $\sigma = (f(\sigma_1, 0, \dots, 0), \sigma_1, 0, \dots, 0)$  one has  $\|\sigma'\| = 1$  (in other words,  $\sigma$  is parametrized by arc length). Note that  $r$  can be chosen independently of  $a$  and  $b$ . Let  $\ell$  be the length of the curve  $\sigma$  from  $a'$  to  $b'$ , so that  $\sigma(0) = a'$ ,  $\sigma(\ell) = b'$ .

Consider the map  $\varphi$  from  $\mathbb{R}_+^2$  (near 0) to  $D$  defined by

$$\varphi(x_0, x_1) = \sigma(x_1) + x_0 n_{\sigma(x_1)},$$

where  $n_{\sigma(x_1)}$  is the inward unit normal to  $\partial D$  at the point  $\sigma(x_1)$ . Then  $d_D(\varphi(\bar{x})) = x_0$  if  $x_0$  is small enough, and if  $\alpha = (d_D(a), 0)$  and  $\beta = (d_D(b), \ell)$ , we have  $\varphi(\alpha) = a$ ,  $\varphi(\beta) = b$ .

**Lemma 12.** *There exist a neighborhood  $U$  of  $\zeta$ , a neighborhood  $V$  of 0 and a constant  $C > 0$  such that for any  $a, b \in D \cap U$  and  $\bar{x} \in \mathbb{R}_+^2 \cap V$  and any vector  $v \in \mathbb{R}^2$ , then  $\alpha, \beta \in V$  and*

$$\|D\varphi(\bar{x}) \cdot v\| \leq (1 + Cx_0)\|v\|.$$

*Proof.* As in the proof of Lemma 10 (2), in the  $\mathcal{C}^2$ -smooth case,

$$D\varphi(\bar{x}) \cdot e_0 = n_{\sigma(x_1)}, \quad D\varphi(\bar{x}) \cdot e_1 = \sigma'(x_1) + x_0 \frac{\partial n_{\sigma(x_1)}}{\partial x_1}.$$

Because  $\|\sigma'\| = 1$  and is tangent to  $\partial D$ ,  $(\sigma'(x), n_x)$  form an orthonormal system, so that  $D\psi(\bar{x})$  differs from a linear isometric embedding by a term bounded by  $\left\| \frac{\partial n_{\sigma(x_1)}}{\partial x_1} \right\| x_0$ .

Geometric considerations show that  $\left\| \frac{\partial n_{\sigma(x_1)}}{\partial x_1} \right\| \leq \frac{1}{R}$  whenever there exist two balls  $B_1, B_2$  of radius  $R$ , tangent to each side of  $\partial D$  at  $\sigma(x_1)$ . The argument in the proof of Lemma 10 (1) shows there exists  $\delta > 0$  (depending only on the neighborhood  $\mathcal{U}_0$  mentioned in that lemma) such that there exist two such balls of radius  $\delta$  at each point in  $\mathcal{U}_0 \cap \partial D$ .

As in the proof of Lemma 10 (3), the  $\mathcal{C}^{1,1}$ -smooth case follows by applying Rademacher's theorem.  $\square$

The proof of Proposition 11 can be finished similarly to that of Proposition 9. Let  $\gamma$  be the geodesic joining  $\alpha$  to  $\beta$  in  $\mathbb{R}_+^2$ . Let  $U, V$  be as in Lemma 12. Shrinking  $V$  if needed so that  $\varphi(V) \subset U$ , we have

$d_D(\varphi(u)) = d_{\mathbb{R}_+^2}(u)$  for any  $u \in \gamma$ . Since  $\varphi \circ \gamma$  is a curve joining  $a$  to  $b$  in  $D$ , using Lemma 12, we get

$$\begin{aligned} h_D(a, b) &\leq \int_0^\ell \frac{\|D\varphi(\gamma(t)) \cdot \gamma'(t)\|}{d_D(\varphi \circ \gamma(t))} dt \\ &\leq h_{\mathbb{R}_+^2}(\alpha, \beta) + Cl(\gamma) < s_{\mathbb{R}_+^2}(\alpha, \beta) + C\pi\|\alpha - \beta\| \end{aligned}$$

(here  $\pi$  is the Ludolphine number, not the projection). The differential of  $\varphi$  is close to a linear isometric embedding of  $\mathbb{R}^2$  in  $\mathbb{R}^{n+1}$  and hence we have the asymptotic relation (3) and

$$\lim_{a, b \rightarrow \zeta} (s_{\mathbb{R}_+^2}(\alpha, \beta) - s_D(a, b)) = 0,$$

which completes the proof.  $\square$

#### 4. AN UPPER ESTIMATE FOR THE KOBAYASHI DISTANCE

Let  $D$  be a domain in  $\mathbb{C}^n$ . The Kobayashi (pseudo) distance  $k_D$  is obtained from the Lempert function

$$l_D(a, b) = \inf\{\tanh^{-1}|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = a, \varphi(\alpha) = b\},$$

$a, b \in D.$

The Lempert function does not always satisfy the triangle inequality, but setting

$$k_D(a, b) := \inf\left\{\sum_{j=0}^{m-1} l_D(a_j, a_{j+1}) : a_j \in D, a_0 = a, a_m = b, m \geq 1\right\},$$

one does obtain a (pseudo) distance, which is the largest that is dominated by  $l_D$ .

Recall that  $k_D$  is the integrated form of the Kobayashi (pseudo) metric

$$\kappa_D(a; X) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = a, \alpha\varphi'(0) = X\},$$

$a \in D, X \in \mathbb{C}^n.$

Note that Theorem 2 and Proposition 7 (even in the Dini-smooth case) hold for  $2k_D$  instead of  $h_D$  (see [9, Theorem 7] and [10, Proposition 6]). Moreover, the following result corresponds to Proposition 4.

**Proposition 13.** [9, Proposition 5(a)] *If  $\zeta$  is a  $C^1$ -smooth boundary point of a domain  $D$  in  $\mathbb{C}^n$ , then*

$$\limsup_{\substack{a, b \rightarrow \zeta \\ a \neq b}} \frac{2k_D(a, b)}{h_D(a, b)} \leq 1.$$

It turns out that Corollary 8 also holds for  $2k_D$  instead of  $h_D$ . This gives the optimal version of [3, Proposition 2.5] in the  $\mathcal{C}^{1,1}$ -smooth case.

**Proposition 14.** *Let  $\zeta$  be a  $\mathcal{C}^{1,1}$ -smooth boundary point of a domain  $D$  in  $\mathbb{C}^n$  or  $\zeta$  be a log-Dini smooth boundary point of a domain  $D$  in  $\mathbb{C}$ . Then for any constant  $c > 1$  there exists a neighborhood  $U$  of  $\zeta$  such that*

$$k_D(a, b) \leq \log \left( 1 + \frac{c||a - b||}{\sqrt{d_D(a)d_D(b)}} \right), \quad a, b \in D \cap U.$$

*Proof.* Having in mind Corollary 8, it is enough to show that

$$\limsup_{\substack{a, b \rightarrow \zeta \\ a \neq b}} \frac{2k_D(a, b) - h_D(a, b)}{||a - b||} < +\infty.$$

Since  $k_D$  is the integrated form of  $\kappa_D$  and the lengths of the quasi-hyperbolic geodesics joining points in  $D$  near  $\zeta$  are bounded up to a multiplicative constant by the Euclidean distances between the points, the last inequality will be a consequence of the following one:

$$\limsup_{\substack{a \rightarrow \zeta \\ ||X||=1}} \left( 2\kappa_D(a; X) - \frac{1}{d_D(a)} \right) < +\infty.$$

To see this, note that there exists an  $r > 0$  such that any  $a \in D$  near  $\zeta$  is contained in a (unique) ball  $\mathbb{B}_n(\tilde{a}, r) \subset D$  with  $r - ||a - \tilde{a}|| = d_D(a)$  (the inner ball condition). It remains to use that for such an  $a$  and  $||X|| = 1$  one has that

$$\kappa_D(a; X) \leq \kappa_{\mathbb{B}_n(\tilde{a}, r)}(a; X) \leq \frac{r}{r^2 - ||a - \tilde{a}||^2} < \frac{1}{2d_D(a)} + \frac{1}{4r}. \quad \square$$

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