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WEAK EMBEDDING PROPERTY, INNER FUNCTIONS AND ENTROPY

ALEXANDER BORICHEV, ARTUR NICOLAU AND PASCAL J. THOMAS

ABSTRACT. Following Gorkin, Mortini, and Nikolski, we say that an inner function I in $H^\infty(\mathbb{D})$ has the WEP property if its modulus at a point z is bounded from below by a function of the distance from z to the zero set of I . This is equivalent to a number of properties, and we establish some consequences of this for H^∞/IH^∞ .

The bulk of the paper is devoted to *wepable* functions, i.e. those inner functions which can be made WEP after multiplication by a suitable Blaschke product. We prove that a closed subset E of the unit circle is of finite entropy (i.e. is a Beurling–Carleson set) if and only if any singular measure supported on E gives rise to a wepable singular inner function. As a corollary, we see that singular measures which spread their mass too evenly cannot give rise to wepable singular inner functions. Furthermore, we prove that the stronger property of porosity of E is equivalent to a stronger form of wepability (*easy wepability*) for the singular inner functions with support in E . Finally, we find out the critical decay rate of masses of atomic measures (with no restrictions on support) guaranteeing that the corresponding singular inner functions are easily wepable.

To Nikolai Nikolski on occasion of his birthday

1. INTRODUCTION

1.1. **Background.** Let $H^\infty = H^\infty(\mathbb{D})$ be the algebra of bounded analytic functions on the unit disc \mathbb{D} with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. A function $I \in H^\infty$ is called *inner* if it has radial limits of modulus 1 at almost every point of the unit circle.

Any inner function I factors as $I = BS$ where B is a Blaschke product and S is a *singular* inner function, that is, an inner function without zeros in \mathbb{D} .

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A Blaschke product B is called an *interpolating* Blaschke product if its zero set $\Lambda = (z_n)_n$ forms an interpolating sequence for H^∞ , that is $H^\infty|_\Lambda = \ell^\infty|_\Lambda$. Let $\rho(z, w)$ be the pseudohyperbolic distance between the points z and w in the unit disc \mathbb{D} defined as

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|, \quad z, w \in \mathbb{D}.$$

A celebrated result of Carleson says that Λ is interpolating if and only if $\inf_{n \neq m} \rho(z_n, z_m) > 0$ and

$$(1) \quad \sup_{z \in \mathbb{D}} \sum_n \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - \bar{z}_n z|^2} < \infty.$$

It was also proved by Carleson that (??) is equivalent to the embedding $H^1 \subset L^1(d\mu)$, where H^1 is the standard Hardy space and $d\mu = \sum_n (1 - |z_n|^2)\delta_{z_n}$, δ_{z_n} being the point mass at z_n . In other words, (??) holds if and only if there exists a constant $C > 0$ such that $\sum_n (1 - |z_n|^2)|f(z_n)| \leq C\|f\|_1$, for any function f in the Hardy space H^1 consisting of the analytic functions in \mathbb{D} for which

$$\|f\|_1 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{it})| dt < \infty.$$

It is well known that a Blaschke product B is an interpolating Blaschke product if and only if there exists a constant $C = C(B) > 0$ such that

$$(2) \quad |B(z)| > C\rho(z, Z(B)),$$

where $Z(B)$ denotes the zero set of B (see the monographs [?], [?, p. 217] or [?]). This easily extends to the fact that any inner function verifying (??) is an interpolating Blaschke product.

1.2. Weak Embedding Property. In 2008, Gorkin, Mortini, and Nikolski [?] introduced the following new class of inner functions. An inner function I satisfies the *Weak Embedding Property (WEP)*, a weaker version of (??), if for any $\varepsilon > 0$ one has

$$(3) \quad \eta_I(\varepsilon) := \inf\{|I(z)| : \rho(z, Z(I)) > \varepsilon\} > 0.$$

A Blaschke product with zeros $(z_n)_n$ satisfies the WEP if and only if for every $\varepsilon > 0$,

$$\sup_{z \in \mathbb{D}, \inf_n \rho(z, z_n) > \varepsilon} \left\{ \sum \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - \bar{z}_n z|^2} \right\} < \infty,$$

which is a weakening of the Carleson embedding property (??).

Finite products of interpolating Blaschke products satisfy the WEP with $\eta_I(\varepsilon) \succeq \varepsilon^N$ and in fact, a Blaschke product B is the product of N interpolating Blaschke products if and only if there exists a constant $C = C(B) > 0$ such that $|B(z)| > C\rho(z, Z(B))^N$ for any $z \in \mathbb{D}$ [?].

However there are other inner functions that satisfy the WEP. In [?], an explicit example was presented of a Blaschke product satisfying the WEP which cannot factor into a finite product of interpolating Blaschke products. This example was extended and complemented in [?]. A different class of examples has been given in [?] showing that for every strictly increasing function $\psi : (0, 1) \rightarrow (0, 1)$ there exists a Blaschke product B satisfying the WEP such that $\eta_B(\varepsilon) = o(\psi(\varepsilon))$ as $\varepsilon \rightarrow 0$.

1.3. Operator Theory motivations. Given an inner function I consider the quotient algebra H^∞/IH^∞ . The zeros $Z(I)$ of I in \mathbb{D} are naturally embedded in the maximal ideal space \mathfrak{M} of H^∞/IH^∞ . It is proved in [?] that I satisfies the WEP if and only if H^∞/IH^∞ has no corona, that is, $Z(I)$ is dense in \mathfrak{M} .

Another condition shown to be equivalent to the WEP in [?] is the norm controlled inversion property which says that for any $\varepsilon > 0$, there exists $m(\varepsilon) > 0$ such that if $f \in H^\infty$, $\|f\|_{H^\infty} = 1$ and $\inf\{|f(z)| : z \in Z(I)\} > \varepsilon > 0$, then f is invertible in H^∞/IH^∞ and $\|1/f\|_{H^\infty/IH^\infty} \leq m(\varepsilon)$.

Consider a vector-valued version of this: for $f := (f_1, \dots, f_n) \in (H^\infty)^n$, let $\|f\|_{\infty, n}^2 := \sup_{z \in \mathbb{D}} \sum_{j=1}^n |f_j(z)|^2$ and for I inner,

$$\chi_I(f) := \inf\{\|g\|_{\infty, n} : \exists h \in H^\infty : \sum_{j=1}^n g_j f_j + hI \equiv 1\}.$$

This is like a ‘‘Corona constant’’ for the n -tuple f in the quotient space H^∞/IH^∞ .

Following Gorkin, Mortini and Nikolski, for $\delta \in (0, 1)$, $n \geq 1$, we define

$$c_n(\delta, I) := \sup \left\{ \chi_I(f) : \delta^2 \leq \inf_{\lambda \in Z(I)} \sum_{j=1}^n |f_j(\lambda)|^2, \|f\|_{\infty, n} \leq 1 \right\},$$

which is a decreasing function of δ , and

$$\delta_n(I) := \inf \{ \delta : c_n(\delta, I) < \infty \}.$$

We prove that these values do not depend on n . Let $\tilde{\delta}(I) := \inf\{\varepsilon : \eta_I(\varepsilon) > 0\}$, where $\eta_I(\varepsilon)$ is defined in (??).

An inner function I satisfies the WEP if and only if $\tilde{\delta}(I) = 0$.

The next two results are contained in Section ???. First, δ_n does not depend on n :

Proposition 1. *For any $n \geq 1$, $\delta_n(I) = \tilde{\delta}(I)$.*

Now we deal with possible rates of growth of $c_n(\delta, I)$:

Proposition 2. *For every decreasing function $\phi : (0, 1) \rightarrow (0, \infty)$ there exists a Blaschke product B such that $\delta_n(B) = 0$ and $c_n(\delta, B) \geq \phi(\delta)$, $0 < \delta < 1$, $n \geq 1$.*

Inner functions I satisfying the WEP can also be described in terms of spectral properties of the model operator acting on the Model Space $K_I = H^2/IH^2$, see [?].

1.4. Wepable functions. An inner function I is called *wepable* [?] if it can appear as a factor in a WEP inner function, i.e. if there exists J inner such that IJ satisfies the WEP. Clearly, if I is a singular inner function, thus without zeros, it cannot be WEP, but it can be wepable. It is easy to see that only the Blaschke factor in J will help make IJ a WEP function.

Let us describe some of the results in [?]. Let $dA(z)$ be area measure in the unit disc. An inner function I such that for any $\varepsilon > 0$ one has

$$(4) \quad \int_{\{z:|I(z)|<\varepsilon\}} \frac{dA(z)}{1-|z|^2} = \infty$$

is not wepable. Moreover there exist singular inner functions I satisfying (??). Hence there exist singular inner functions which are not wepable, answering a question in [?]. Condition (??) is a sort of Blaschke condition and has also appeared in [?]. It was also shown in [?] that condition (??) does not characterise (non)-wepable inner functions.

1.5. Results about the support of the singular measure. Given a measurable set $E \subset \mathbb{T} = \partial\mathbb{D}$, let $|E|$ denote its normalised length, $|\mathbb{T}| = 1$. Recall that a closed set $E \subset \mathbb{T}$ with $|E| = 0$ has finite entropy (has finite Carleson characteristic, is a Beurling–Carleson set) if

$$\mathcal{E}(E) := \sum |J_k| \log |J_k|^{-1} < \infty,$$

where $(J_k)_k$ are the connected components of $\mathbb{T} \setminus E$; more precisely, this value is the entropy of the family $(J_k)_k$. A classical result of Carleson says that a closed set $E \subset \mathbb{T}$ is the zero set of an analytic function whose derivatives of any order extend continuously to the closed unit disc if and only if E has zero length and finite entropy [?].

Given an inner function I let $\text{sing}(I)$ denote the set of points of the unit circle where I can not be extended analytically. If $I = BS$ where B is a Blaschke product with zeros $(z_n)_n$ and

$$S_\mu(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right), \quad z \in \mathbb{D},$$

where μ is a positive singular measure, then $\text{sing}(I) = (\overline{\{z_n\}} \cap \mathbb{T}) \cup \text{supp } \mu$, where $\text{supp } \mu$ denotes the (closed) support of μ .

Theorem 3. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

- (a) *Every singular inner function whose singular set is contained in E is wepable;*
- (b) *E has zero length and finite entropy.*

The sufficiency of the conditions in (b) is obtained by careful constructions of Blaschke products which are carried out in Section ??.

The necessity of the conditions in (b) is related to estimate (??) and is a consequence of the next Lemma, which may be of independent interest.

We identify the interval $[0, 1)$ with the unit circle by the map $t \mapsto e^{2\pi it}$ and consider the dyadic arcs $[k2^{-n}, (k+1)2^{-n})$, $0 \leq k < 2^n$, $n \geq 0$, on the interval $[0, 1)$ (or on the unit circle). Those arcs have (normalised) length equal to 2^{-n} . Given an arc $J \subset \mathbb{T}$ of center $e^{2\pi it}$, write $z(J) = (1 - \frac{3}{4}|J|)e^{2\pi it}$.

Given a finite measure μ in the unit circle let $P[\mu]$ be its Poisson integral.

Lemma 4. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

- (a) *E has zero length and finite entropy;*
- (b) *$\sum |J| < \infty$, where the sum is taken over all dyadic arcs J such that $J \cap E \neq \emptyset$;*
- (c) *For any $C > 0$ and any positive singular measure μ of mass 1 whose support is contained in E one has $\sum |J| < \infty$, where the sum is taken over all dyadic arcs $J \subset \mathbb{T}$ such that $P[\mu](z(J)) \geq C$.*
- (d) *There exists $C > 0$ such that for any positive singular measure μ of mass 1 whose support is contained in E one has $\sum |J| < \infty$, where the sum is taken over all dyadic arcs $J \subset \mathbb{T}$ such that $P[\mu](z(J)) \geq C$.*

The condition in (c) can be understood as a discrete version of (??) with $I = S_\mu$.

The condition in (b) can be seen as a discrete version of $\int_{\Gamma(E)} \frac{dA(z)}{1-|z|^2} < \infty$, where $\Gamma(E)$ denotes the union of all the Stolz angles with vertex

on the image in the unit circle of a point of E . For a related result, see [?, Lemma A.1].

Theorem ?? and Lemma ?? will be proved in Section ?? and ??.

1.6. Results about regularity of singular measures. Positive singular measures can fairly distribute their mass. For instance, there exist singular probability measures μ on the unit circle such that

$$\sup \left\{ \left| \frac{\mu(J)}{\mu(J')} - 1 \right| + \left| \frac{\mu(J) - \mu(J')}{|J|} \right| \right\} \rightarrow 0 \quad \text{as } |J| \rightarrow 0,$$

where the supremum is taken over all pairs of adjacent arcs $J, J' \subset \mathbb{T}$ of the same length (see [?]). As a consequence of Theorem ?? we will prove that positive singular measures μ such that S_μ is wepable cannot distribute their mass as evenly (see Remark ??). Actually a Dini type condition governs the growth of the density of such measures.

Corollary 5. (a) *Let μ be a positive singular measure on the unit circle and consider $w(t) = \sup \mu(J)$, where the supremum is taken over all arcs $J \subset \mathbb{T}$ with $|J| = t$. Assume that*

$$\sum_{n \geq 1} \frac{2^{-n}}{w(2^{-n})} = \infty.$$

Then S_μ satisfies condition (??) and, hence, it is not wepable.

(b) *Let $w : [0, 1] \rightarrow [0, \infty)$ be a nondecreasing function with $w(0) = 0$ such that $w(2t) < 2w(t)$ for any $t > 0$. Assume that*

$$\sum_{n \geq 1} \frac{2^{-n}}{w(2^{-n})} < \infty.$$

Then there exists a positive singular measure μ in the unit circle satisfying $\mu(J) < w(|J|)$ for any arc $J \subset \mathbb{T}$, such that its support has zero length and finite entropy, and hence S_μ is wepable.

Next we introduce an important class of subsets of the unit circle.

Definition 6. *A closed subset E of the unit circle is called porous if there exists a constant $C > 0$ such that for any arc $J \subset \mathbb{T}$, there exists a subarc $J' \subset J \setminus E$ with $|J'| > C|J|$.*

The porosity condition (the Kotochigov condition, the (K) condition) appears naturally in the free interpolation problems for different classes of analytic functions smooth up to the boundary, see [?].

Given a porous set E , define a family of disjoint closed arcs in the following way: let $A_0 := \{J_0\}$, where $J_0 \subset \mathbb{T} \setminus E$ and $|J_0| \geq C$. For any $n \geq 0$, given A_k , $0 \leq k \leq n$, such that all the arcs in the set $\cup_{0 \leq k \leq n} A_k$

are disjoint, we have $\mathbb{T} \setminus \cup_{0 \leq k \leq n, J \in A_k} J = \cup I_j$, where $\{I_j\}$ is a finite family of disjoint open arcs. Define A_{n+1} as a set of arcs $I'_j \subset I_j \setminus E$ such that $|I'_j| \geq C|I_j|$. By induction, we see that A_{n+1} consists of 2^n arcs, $n \geq 0$, and

$$|\mathbb{T} \setminus \cup_{0 \leq k \leq n+1, J \in A_k} J| \leq (1 - C) |\mathbb{T} \setminus \cup_{0 \leq k \leq n, J \in A_k} J|.$$

Therefore, $\sum_{0 \leq k \leq n, J \in A_k} |J| \geq 1 - (1 - C)^{n+1}$, $n \geq 0$, and E is of measure 0. Furthermore, $\sum_{J \in A_n} |J| \leq |\mathbb{T} \setminus \cup_{0 \leq k \leq n-1, J \in A_k} J| \leq (1 - C)^n$, $n \geq 1$. Because of the concavity of the function $t \mapsto t \log(1/t)$, we have $\sum_{J \in A_n} |J| \log |J|^{-1} \leq C_1 n (1 - C)^n$, $n \geq 1$. Hence, E has finite entropy.

Another proof of this fact can be obtained using Lemma ?? and the next auxiliary result (Lemma ??) which is a scale invariant version of Lemma ??.

Lemma 7. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

- (a) E is porous;
- (b) There exists a constant $C > 0$ such that for any dyadic arc J one has

$$\sum_{I \in E(J)} |I| \leq C|J|,$$

where $E(J)$ is the family of the dyadic arcs $I \subset J$ such that $I \cap E \neq \emptyset$;

- (c) There exists $a \in \mathbb{N}$ such that for every $s, n \geq 0$, $J \in \mathcal{D}_n$, and for every m such that $sa \leq m < (s+1)a$, the set $J \cap E$ is covered by 2^{m-s} arcs $I \in \mathcal{D}_{n+m}$;

- (d) There exists a constant $C > 0$ such that for any finite positive measure μ with support contained in E , any number $A > 0$ and any dyadic arc $J \subset \mathbb{T}$, one has

$$\sum_{I \in E(A)} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2} \leq \frac{C}{A} P[\mu](z(J)),$$

where $E(A)$ is the family of the dyadic arcs I such that $P[\mu](z(I)) \geq A$.

Let S be a wepable inner function. Clearly, the Blaschke product by which it must be multiplied to yield a WEP function must have enough zeros in the places where $|S|$ is small, so that the smallness of the resulting product can be correlated to these zeros. However, it may happen that any Blaschke product B such that BS satisfies the WEP must have some of its zeros $(z_n)_n$ located at points where $|S|$ is close to 1; more precisely, $\limsup_{n \rightarrow \infty} |S(z_n)| = 1$. (Theorems ?? and ?? together prove the existence of such S).

Definition 8. *An inner function S will be called easily wepable if there exists a constant $m < 1$ and a Blaschke product B such that SB satisfies the WEP and $Z(B) \subset \{z \in \mathbb{D} : |S(z)| < m\}$.*

The following result will be established in Section ??.

Theorem 9. *Let E be a closed subset of the unit circle. The following conditions are equivalent:*

- (a) *Every singular inner function whose singular set is contained in E is easily wepable;*
- (b) *The set E is porous.*

In [?] it was proved that a closed set E of the unit circle is porous if and only if for any singular inner function S whose singular set is contained in E and any $a \in \mathbb{D} \setminus \{0\}$ the inner function $(S - a)/(1 - \bar{a}S)$ is a finite product of interpolating Blaschke products.

Now we describe the critical decay rate of masses of atomic measures (with no restrictions on support) guaranteeing that the corresponding singular inner functions are easily wepable.

Theorem 10. *Let $(b_s)_{s \geq 1}$ be a decreasing summable sequence of positive numbers. The following conditions are equivalent:*

- (a) *Every atomic singular inner function with point masses $(b_s)_{s \geq 1}$ is easily wepable;*
- (b)

$$(5) \quad b_s \asymp \sum_{k \geq s} b_k, \quad s \geq 1.$$

The proof will be given in Section ??.

Note that given any decreasing summable sequence of masses, this sequence can give rise to a measure μ with easily wepable singular function S_μ , simply by locating the masses at points of the form $\exp(i2^{-n})$, for instance, and applying Theorem ??. It would be interesting to have a similar statement to Theorem ?? with “wepable” instead of “easily wepable”; in particular to know whether any condition weaker than (??) can imply automatic wepability, and what rate of decrease of the (b_s) guarantees that there always exists some choice of location of the point masses which produces a non-wepable S_μ . In particular, the construction in the proof of [?, Proposition 6] shows that there exists a non-wepable atomic singular inner function as soon as the point masses decay no more rapidly than $1/(n(\log n)^2)$, $n \rightarrow \infty$.

1.7. Notation. The letter C will denote a constant whose value may change from line to line.

We denote by $\mathcal{D} = \cup_{n \geq 0} \mathcal{D}_n$ the family of the dyadic arcs, with $\mathcal{D}_n = \{J \subset \mathcal{D} : |J| = 2^{-n}\}$. Note that $\text{Card } \mathcal{D}_n = 2^n$.

Given an arc $J \subset \mathbb{T}$ of length $|J|$ and $M > 0$ let MJ be the arc of the unit circle of length $M|J|$ concentric with J . Again identifying the unit circle with the interval $[0, 1)$ we denote $Q(J) := \{re^{2\pi i\theta} : \theta \in \bar{J}, 1 - |J| \leq r < 1\}$ (the Carleson box associated to J) and $T(J) := \{re^{2\pi i\theta} : \theta \in \bar{J}, 1 - |J| \leq r \leq 1 - |J|/2\}$ (the top half of the box).

Let $z, w \in \mathbb{D}$. Later on, we use the following standard estimates:

$$(6) \quad \frac{1}{2} \cdot \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2} \leq \log \left| \frac{1 - \bar{w}z}{z - w} \right|,$$

$$(7) \quad \log \left| \frac{1 - \bar{w}z}{z - w} \right| \leq C(\delta) \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2}, \quad \rho(z, w) \geq \delta > 0.$$

1.8. Organization of the paper. In Section ??, we prove Lemmas ?? and ??, and therefore the necessity part of Theorem ?. In Section ??, we prove the remaining part of Theorem ? and Corollary ?. In Section ??, we give the proof of Theorem ?. In Section ??, we give the proof of Theorem ?, which deals with a situation where the entropy of the singular set is very well controlled. Finally, the proofs of Propositions ? and ??, which are quite independent from the rest, appear in Section ?.

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2. PROOFS OF LEMMAS ? AND ?

Proof of Lemma ?. (a) \Leftrightarrow (b) Let $\mathbb{T} \setminus E$ be the disjoint union of the arcs I_k , $k \geq 1$. Suppose first that $|E| = 0$. Then

$$\sum_{J \in \mathcal{D}, J \cap E \neq \emptyset} |J| = \sum_{k \geq 1} \sum_{J \in \mathcal{D}, J \cap E \neq \emptyset} |J \cap I_k| = \sum_{k \geq 1} \sum_{n \geq 0} \sum_{J \in \mathcal{D}_n, J \cap E \neq \emptyset} |J \cap I_k|.$$

Since for each k , there are at most two arcs $J \in \mathcal{D}_n$ such that $J \cap I_k \neq \emptyset$ and $J \cap E \neq \emptyset$, this last quantity is clearly

$$\lesssim \sum_{k \geq 1} \left(\sum_{0 \leq n \leq \log(1/|I_k|)} |I_k| + \sum_{n > \log(1/|I_k|)} 2^{-n} \right) \asymp \mathcal{E}(E).$$

Conversely,

$$\begin{aligned} \sum_{k \geq 1} \sum_{n \geq 0} \sum_{J \in \mathcal{D}_n, J \cap E \neq \emptyset} |J \cap I_k| &\geq \sum_{k \geq 1} \sum_{n \geq 0} \sum_{J \in \mathcal{D}_n, n \leq \log(1/|I_k|), J \cap E \neq \emptyset} |J \cap I_k| \\ &\gtrsim \sum_{k \geq 1} \sum_{0 \leq n \leq \log(1/|I_k|)} |I_k| \asymp \mathcal{E}(E). \end{aligned}$$

Next, if $|E| > 0$, then

$$\sum_{J \in \mathcal{D}, J \cap E \neq \emptyset} |J| = \infty.$$

(b) \Rightarrow (c) Arguing as above, and using the fact that $P[\mu](z) \geq C$ implies that $\text{dist}(z, \text{supp } \mu) \leq C' \sqrt{1 - |z|^2}$, where $\text{supp } \mu$ is seen as a subset of the unit circle, we obtain

$$\begin{aligned} \sum_{J \in \mathcal{D}, P[\mu](z(J)) \geq C} |J| &= \sum_{k \geq 1} \sum_{n \geq 0} \sum_{J \in \mathcal{D}_n, P[\mu](z(J)) \geq C} |J \cap I_k| \\ &\lesssim \sum_{k \geq 1} \left(\sum_{0 \leq n \leq 2 \log(1/|I_k|)} |I_k| + \sum_{n > 2 \log(1/|I_k|)} 2^{-n/2} \right) \asymp \mathcal{E}(E). \end{aligned}$$

(c) \Rightarrow (d) is evident.

(d) \Rightarrow (a) If $|E| > 0$, then we can just take $\mu_0 = K \chi_{E_1} dm$ with K to be chosen later on, $E_1 \subset E$, $|E_1| = 1/K$. By the Lebesgue density theorem, for a subset E_2 of E_1 , $|E_2| \geq |E_1|/2$ and for some $\delta > 0$ we have

$$\frac{|E_1 \cap J|}{|J|} \geq \frac{1}{2}$$

for every arc J such that $J \cap E_2 \neq \emptyset$, $|J| \leq \delta$. Hence, for $K \geq K(C)$ we obtain

$$\sum_{J \in \mathcal{D}, P[\mu_0](z(J)) \geq C} |J| \geq \sum_{J \in \mathcal{D}, |J| \leq \delta, J \cap E_1 \neq \emptyset} |J| = \infty.$$

Next, we can replace μ_0 by a Cantor type singular measure μ_1 while keeping the sum

$$\sum_{J \in \mathcal{D}, P[\mu_1](z(J)) \geq C} |J|$$

infinite.

Now, suppose that $|E| = 0$ and (b) does not hold. Let $I \subset \mathcal{D}$ be a small arc such that $I \cap E$ has infinite entropy. Let $I \setminus E$ be the disjoint

union of the arcs $I_k = (a_k, b_k)$, $k \geq 1$, and take

$$\mu = K \sum_{k \geq 1} |I_k| (\delta_{a_k} + \delta_{b_k})$$

with $K = 1/(2|I|)$, $\|\mu\| = 1$, and once again δ_x being the point mass at x . Given $J \in \mathcal{D}$, $J \subset I$, if $J \cap E \neq \emptyset$, then $\mu(J) \geq K|J|$, and for $K \geq K(C)$ we have $P[\mu](z(J)) \geq C$. Therefore,

$$\sum_{J \in \mathcal{D}, P[\mu](z(J)) \geq C} |J| \geq \sum_{J \in \mathcal{D}, J \subset I, J \cap E \neq \emptyset} |J| = \infty.$$

□

In the proof of Lemma ??, we use the following auxiliary statement.

Lemma 11. *Let u be a function positive and harmonic on the unit disc, let $A > 0$, and let \mathcal{G} be a subfamily of \mathcal{D} such that the arclength ds on $L = L(\mathcal{G}) = \cup_{I \in \mathcal{G}} \partial T(I)$ is a Carleson measure, that is (see, for instance, [?, A.5.7.2(b)])*

$$(8) \quad \sup_{z \in \mathbb{D}} \sum_{I \in \mathcal{G}} \int_{\partial T(I)} \frac{1 - |z|^2}{|1 - \bar{w}z|^2} ds(w) \leq B.$$

Assume that $u \geq A$ on L . Then for every $J \subset \mathcal{D}$ we have

$$\sum_{I \in \mathcal{G}} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2} \leq \frac{CB}{A} u(z(J))$$

for some absolute constant C .

Proof. Consider the function

$$h(z) = \sum_{I \in \mathcal{G}} \int_{\partial T(I)} \log \left| \frac{1 - \bar{w}z}{z - w} \right| \frac{ds(w)}{1 - |w|}.$$

It is harmonic on $\mathbb{D} \setminus L$ and by (??), (??), and an elementary estimate for the integral along $L \cap \{w : \rho(z, w) < 1/2\}$, we have

$$CBu(z) \geq Ah(z)$$

for $z \in \mathbb{T} \cup L$. By the maximum principle,

$$CBu(z(J)) \geq Ah(z(J)),$$

and hence, by (??),

$$C_1 Bu(z(J)) \geq A \sum_{I \in \mathcal{G}} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

□

Proof of Lemma ??. (a) \Rightarrow (c) By induction, it suffices to check the assertion for $s = 1$, $m = a$. Fix $C > 0$ such that for any arc $J \subset \mathbb{T}$, there exists a subarc $J' \subset J \setminus E$ with $|J'| > C|J|$. Let $J \in \mathcal{D}$, $|J| = 2^{-n}$. By the remarks after Definition ??, we can find a finite family of disjoint arcs $J_k \subset J \setminus E$, $1 \leq k \leq K(C)$, such that $\sum_{1 \leq k \leq K(C)} |J_k| \geq (2/3)|J|$. Given $m \geq 1$, set

$$U_m = \{I \in \mathcal{D}_{n+m} : I \cap J \cap E \neq \emptyset\}.$$

If $I \in U_m$, then $I \not\subset J_k$, $1 \leq k \leq K(C)$. Hence, $\sum_{I \in U_m} |I| \leq (|J| - \sum_{1 \leq k \leq K(C)} |J_k|) + 2K(C) \cdot 2^{-n-m} \leq 2^{-n-1}$ for $m \geq m(C)$.

(c) \Rightarrow (b) Fix $J \in \mathcal{D}$, $|J| = 2^{-n}$. Then

$$\begin{aligned} \sum_{I \in \mathcal{D}, I \subset J, I \cap E \neq \emptyset} |I| &\leq \sum_{m \geq 0} \sum_{\substack{I \in \mathcal{D}_{n+m} \\ I \subset J, I \cap E \neq \emptyset}} |I| \\ &\leq \sum_{s \geq 0} \sum_{sa \leq m < (s+1)a} 2^{m-s} 2^{-n-m} = \sum_{s \geq 0} a 2^{-n-s} = 2a|J|. \end{aligned}$$

(b) \Rightarrow (a) If E is not porous, then for every $N \geq 1$ there exist $n \geq 0$, $J \in \mathcal{D}_n$ such that if $I \in \mathcal{D}_s$, $n \leq s \leq n+N$ and $I \subset J$, then $I \cap E \neq \emptyset$. Then

$$\begin{aligned} \sum_{I \in \mathcal{D}, I \subset J, I \cap E \neq \emptyset} |I| &\geq \sum_{n \leq s \leq n+N} \sum_{I \in \mathcal{D}_s, I \subset J} |I| \\ &= \sum_{n \leq s \leq n+N} |J| = (N+1)|J|. \end{aligned}$$

(d) \Rightarrow (a) As above, if E is not porous, then for every $N \geq 1$ we can find $n \geq 0$, $J = [k2^{-n}, (k+1)2^{-n}] \in \mathcal{D}_n$ and points

$$x_\ell \in E \cap [(k2^N + \ell)2^{-n-N}, (k2^N + \ell + 1)2^{-n-N}], \quad 0 \leq \ell < 2^N.$$

Set

$$\mu = 10 \cdot 2^{-n-N} A \sum_{0 \leq \ell < 2^N} \delta_{x_\ell}.$$

We have $\text{supp } \mu \subset E$. For every arc $I \subset J$ such that $|I| \geq 2^{-n-N}$, we have $P[\mu](z(I)) \geq 2A P[\chi_I](z(I))$, so that

$$P[\mu](z(I)) \geq A, \quad I \in \mathcal{D}_m, n \leq m \leq n+N, I \subset J.$$

Therefore,

$$\sum_{I \in \mathcal{D}, P[\mu](z(I)) \geq A} \frac{|I||J|}{|1 - z(I)z(J)|^2} \gtrsim \sum_{n \leq m \leq n+N} 2^{m-n} \cdot \frac{2^{-m} 2^{-n}}{2^{-2n}} = N+1.$$

For large N this contradicts (c), because

$$P[\mu](z(J)) \leq CA.$$

(c) \Rightarrow (d) Set $u = P[\mu]$. Arguing as in the part (c) \Rightarrow (b) we obtain that

$$\sum_{I \in \mathcal{D}, I \subset J, 3I \cap E \neq \emptyset} |I| \leq C|J|, \quad J \subset \mathcal{D}.$$

Hence, the arclength ds on $\cup_{I \in \mathcal{D}, 3I \cap E \neq \emptyset} \partial T(I)$ is a Carleson measure. Fix $A > 0$ and denote by \mathcal{G} the family of all $I \in \mathcal{D}$ such that $3I \cap E \neq \emptyset$ and $u(z(I)) \geq A$. An easy application of Harnack's inequality shows that $u \geq CA$ on $L(\mathcal{G})$. Fix $J \in \mathcal{D}$. Applying Lemma ??, we obtain that

$$u(z(J)) \geq CA \sum_{I \in \mathcal{G}} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

Now we need only to estimate

$$\sum_{I \in \mathcal{D}, 3I \cap E = \emptyset, u(z(I)) \geq A} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

We set

$$\mathcal{H} = \{I \in \mathcal{D} : 3I \cap E = \emptyset, u(z(I)) \geq A\}$$

and

$$\mathcal{H}_k = \{I \in \mathcal{H} : 2^{k-1}A \leq u(z(I)) < 2^k A\}, \quad k \geq 1,$$

so that $\mathcal{H} = \cup_{k \geq 1} \mathcal{H}_k$. If $I \in \mathcal{H}$, and $L \subset I$, then for any $\zeta \in \text{supp } \mu$, $|1 - \zeta \overline{z(L)}| \asymp |1 - \zeta \overline{z(I)}|$, so an easy estimate of the Poisson integral shows that

$$\frac{u(z(L))}{u(z(I))} \asymp \frac{|L|}{|I|}, \quad L \in \mathcal{D}, L \subset I.$$

Hence, every arc $I \in \mathcal{H}_k$, $k \geq 1$, contains at most C subarcs $I' \in \mathcal{H}_k$. Therefore, the arclength ds on $\cup_{I \in \mathcal{H}_k} \partial T(I)$ is a Carleson measure with Carleson constant uniformly bounded in $k \geq 1$. Lemma ?? gives now that

$$2^{-k}u(z(J)) \geq CA \sum_{I \in \mathcal{H}_k} \frac{|I||J|}{|1 - \overline{z(I)}z(J)|^2}.$$

Summing over $k \geq 1$ we complete the proof. \square

3. PROOF OF THEOREM ?? AND COROLLARY ??

The proof of the sufficiency of Theorem ?? uses the following auxiliary result.

Lemma 12. *Let E be a closed subset of the unit circle of zero length and finite entropy. Let $\mathcal{G} = \{J_n\}$ be the family of maximal dyadic subarcs $J_n \subset \mathbb{T}$ such that $2J_n \subset \mathbb{T} \setminus E$. Then*

- (a) *The interiors of J_n are pairwise disjoint and $\mathbb{T} \setminus E = \cup J_n$.*
- (b) *We have $\sum_n |J_n| \log |J_n|^{-1} < \infty$.*
- (c) *If J_n and J_m are in the same connected component of $\mathbb{T} \setminus E$ and $\overline{J_n} \cap \overline{J_m} \neq \emptyset$, then $4|J_n| \geq |J_m| \geq |J_n|/4$.*
- (d) *Let J be a connected component of $\mathbb{T} \setminus E$ and consider the subfamily $\mathcal{G}(J) = \{L_n\}$ of the arcs $L \in \mathcal{G}$ with $L \subset J$ ordered so that $|L_{n+1}| \leq |L_n|$ for any n . Then $|L_1| \geq |J|/8$, $|L_k|/4 \leq |L_{k+1}| \leq |L_k|$ and $|L_{k+4}| \leq |L_k|/2$ for any $k \geq 1$.*

Proof. The maximality gives that the interiors of J_n are pairwise disjoint. It is also clear that $\mathbb{T} \setminus E = \cup J_n$. Let us now prove (c). Assume that $|J_n| \geq |J_m|$. Since $2J_n \subset \mathbb{T} \setminus E$, every dyadic arc J adjacent to J_n of length $|J| = |J_n|/4$ satisfies $2J \subset 2J_n \subset \mathbb{T} \setminus E$. Since $\overline{J_n} \cap \overline{J_m} \neq \emptyset$, there is an arc J_n^* adjacent to J_n with $|J_n^*| = |J_n|/4$ such that $J_n^* \cap J_m \neq \emptyset$. Since $2J_n^* \subset \mathbb{T} \setminus E$, by maximality $J_n^* \subset J_m$. Hence $|J_m| \geq |J_n^*| = |J_n|/4$ and (c) is proved.

Let us now prove (d). The maximality gives that $|L_1| \geq |J|/8$. As in the argument for (c), the fact that $2L_k \subset \mathbb{T} \setminus E$ implies the existence of $L \in \mathcal{G}$ with $|L| \geq |L_k|/4$, so $|L_{k+1}| \geq |L_k|/4$. Assume that $|L_{k+4}| > |L_k|/2$. Then, since $\{L_n\}$ are ordered by length, $|L_{k+i}| = |L_k|$, $0 \leq i \leq 4$. Then, since these arcs are in the same connected component of $\mathbb{T} \setminus E$, there would exist $i \in \{0, 1, 2, 3, 4\}$ such that the dyadic arc L containing L_{k+i} of length $|L| = 2|L_{k+i}|$ satisfies $2L \subset \mathbb{T} \setminus E$. This would contradict the maximality of L_{k+i} and (d) is proved.

Finally, let us prove (b). Take a connected component J of $\mathbb{T} \setminus E$. The estimates in (d) give that $|L_k| \leq 2^{(4-k)/4}|L_1|$, $k \geq 1$. Hence,

$$\sum_{k \geq 1} |L_k| \log |L_k|^{-1} \leq C|L_1| \log |L_1|^{-1} \leq C|J| \log |J|^{-1},$$

and (b) follows because E has finite entropy. \square

Proof of Theorem ??. The implication (a) \Rightarrow (b) follows from Lemma ?? and property (??) in the following way. Assume that the set E does not satisfy the conditions in part (b) of Theorem ?. Let $\{I_j\}$ be a sequence of disjoint arcs such that the sets $E \cap I_j$ do not satisfy the conditions in part (b) of Theorem ?. By Lemma ??, there exist

positive singular measures μ_j whose support is contained in $E \cap I_j$, $\|\mu_j\| = 2^{-j}$ such that $\sum_{J \in A_j} |J| = \infty$, where the sums are taken over the collections A_j of all dyadic arcs $J \subset I_j$ satisfying $P[\mu_j](z(J)) \geq 2^j$. Set $\mu = \sum_{j \geq 1} \mu_j$ and consider the singular inner function

$$S_\mu(z) = \exp \left(- \int_0^1 \frac{e^{2\pi it} + z}{e^{2\pi it} - z} d\mu(t) \right), \quad z \in \mathbb{D}.$$

By Harnack's inequality, there exists a universal constant $K_0 > 0$ such that $P[\mu](z) \geq K_0 P[\mu](z(J))$ for any $z \in T(J)$. Since $P[\mu](z) = -\log |S_\mu(z)|$, $z \in \mathbb{D}$, for any fixed $j \geq 1$ we have

$$\cup_{J \in A_j} T(J) \subset \{z \in \mathbb{D} : |S_\mu(z)| < \exp(-K_0 2^j)\}.$$

Hence condition (??) is satisfied and the function S_μ is not wepable.

Let us now prove the implication (b) \Rightarrow (a). Let μ be a positive singular measure whose support E has zero length and finite entropy. Let \mathcal{G} be the family (defined in Lemma ??) of maximal dyadic arcs of the unit circle whose double is contained in $\mathbb{T} \setminus E$. Let \mathcal{F} be the family of dyadic arcs of \mathbb{T} which are not contained in any arc of \mathcal{G} . Then we claim

$$(9) \quad \sum_{J \in \mathcal{F}} |J| < \infty.$$

Indeed, for every $J \in \mathcal{F}$, $2J$ is the union of four dyadic arcs of equal length such that at least one of them intersects E , since otherwise $2J$ would be contained in $\mathbb{T} \setminus E$ and hence by maximality, J would be contained in an arc of the family \mathcal{G} . Since E has zero length and finite entropy, Lemma ?? gives that

$$\sum_{J \in \mathcal{F}} |J| \leq \sum_{J \in \mathcal{F}} |2J| \lesssim \sum_{J \in \mathcal{D}, J \cap E \neq \emptyset} |J| < \infty.$$

Let S be the singular inner function corresponding to the measure μ . We want to find a Blaschke product B such that SB satisfies the WEP. We will describe now the first family of zeros of B . By (??), one can pick a sequence of integers k_j increasing to infinity such that

$$\sum_{j=1}^{\infty} k_j \sum_{J \in \mathcal{F} \cap \mathcal{D}_j} |J| < \infty.$$

Next, for each $J \in \mathcal{F} \cap \mathcal{D}_j$, we choose a set of points $\Lambda(J) = \{z_n(J) : 1 \leq n \leq k_j\}$ uniformly distributed in $T(J)$. Uniform distribution means that the $z_n(J)$ are the centers of a maximal collection of disjoint

hyperbolic discs of fixed radius included in $T(J)$, so that

$$\min_{z, w \in \Lambda(J), z \neq w} |z - w| \asymp \max_{\zeta \in T(J)} \text{dist}(\zeta, \Lambda(J)) \asymp \min_{z \in \Lambda(J)} \text{dist}(z, \partial T(J)).$$

Let B_1 be the Blaschke product with zeros $Z(B_1) = \cup_{J \in \mathcal{F}} \Lambda(J)$. Since k_j tends to infinity, for any $0 < \delta < 1$ there exists $r = r(\delta) < 1$ such that

$$(10) \quad \{z \in \mathbb{D} : \rho(z, Z(B_1)) > \delta\} \subset \{z \in \mathbb{D} : |z| < r\} \bigcup \cup_{J \in \mathcal{G}} Q(J).$$

Next we will construct certain additional zeros of B which are contained in $\cup_{J \in \mathcal{G}} Q(J)$. Set $f = SB_1$. Since $-\log |S|$ is a positive harmonic function in the unit disc, Harnack's inequality gives that for any $z \in \mathbb{D}$ one has $-\log |S(z)| \leq -2 \log |S(0)| / (1 - |z|)$. Since

$$\inf_{J \in \mathcal{G}} \rho(z(J), Z(B_1)) > 0,$$

estimate (??) gives that there exists a constant $C > 0$ such that for any $J \in \mathcal{G}$ one has

$$\begin{aligned} -\log |B_1(z(J))| &\leq C \sum_{w \in Z(B_1)} \frac{1 - |z(J)|^2}{|1 - z(J)\bar{w}|^2} (1 - |w|^2) \\ &\leq \frac{C'}{1 - |z(J)|} \sum_{w \in Z(B_1)} (1 - |w|). \end{aligned}$$

Hence, there exists a constant $C > 0$ such that

$$(11) \quad \log |f(z(J))|^{-1} \leq C(1 - |z(J)|)^{-1} \asymp |J|^{-1}, \quad J \in \mathcal{G}.$$

Now, applying Lemma ?? (b), we obtain

$$\sum_{J \in \mathcal{G}} |J| \log \log |f(z(J))|^{-1} < \infty.$$

Claim. For every $J \in \mathcal{G}$ and every $z \in Q(J)$ with $1 - |z| < |J|/8$,

$$(12) \quad \log |f(z)|^{-1} \lesssim \frac{1 - |z|}{1 - |z(J)|} \log |f(z(J))|^{-1}.$$

Proof of the Claim. Let us first check that by Lemma ?? (c), there exists a constant $C > 0$ such that

$$(13) \quad \rho(z, Z(B_1)) \geq C, \quad z \in Q(J), 1 - |z| < |J|/8, J \in \mathcal{G}.$$

Indeed, fix $J \in \mathcal{G}$, $z \in Q(J)$ such that $1 - |z| < |J|/8$ and $I \in \mathcal{F}$, $w \in \Lambda(I)$. Denote by J_{\pm} two arcs in \mathcal{G} adjacent to J . Then either $J_+ \subset I$ or $J_+ \cap I = \emptyset$. Similarly, either $J_- \subset I$ or $J_- \cap I = \emptyset$. Since $|J|/4 \leq |J_{\pm}| \leq 4|J|$, we obtain that $\rho(z, w) \geq C > 0$ and (??) follows.

To prove (??) consider the measure $d\sigma = d\mu + \sum_{z \in Z(B_1)} (1 - |z|^2) \delta_z$, where we recall that $d\mu$ is the positive singular measure associated to S . Then, by (??), there exists a constant $C_1 \geq 1$ such that for any $z \in Q(J)$ with $1 - |z| \leq |J|/8$, we have

$$\begin{aligned} \log |f(z)|^{-1} &\leq \log |S_\mu(z)|^{-1} + C_1 \sum_{w \in Z(B_1)} \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2} \\ &\leq C_1 \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{w}z|^2} d\sigma(w). \end{aligned}$$

Let $w \in \text{supp } \sigma$ and $z \in Q(J)$. If $w \in \text{supp } \mu$, then the support of μ does not intersect $2J$. Hence, $|\arg z - \arg w| \gtrsim |J|$, $|\arg z - \arg w| \asymp |\arg z(J) - \arg w|$, and

$$(14) \quad |1 - \bar{w}z| \asymp |1 - \bar{w}z(J)|.$$

Otherwise, if $w \in Z(B_1)$, then (??) gives (??) once again.

Hence,

$$\log |f(z)|^{-1} < \frac{C_1(1 - |z|^2)}{C_2^2(1 - |z(J)|^2)} \int_{\mathbb{D}} \frac{1 - |z(J)|^2}{|1 - \bar{w}z(J)|^2} d\sigma(w).$$

By (??), this integral is bounded by a fixed multiple of $\log |f(z(J))|^{-1}$, and estimate (??) follows. \square

Let $M(J) = \log_2 |J|^{-1}$. By Lemma ?? (b) we have that $\sum |J| M(J) < \infty$, where the sum is taken over all arcs $J \in \mathcal{G}$. Pick a sequence of positive integers t_k increasing to infinity and such that $t_{k+1} \leq t_k + 1$ and

$$\sum_{J \in \mathcal{G}} |J| \sum_{k=1}^{M(J)} t_k < \infty.$$

For $1 \leq k \leq M(J)$ consider the strips

$$\Omega_k = \Omega_k(J) = \{z \in Q(J) : 2^{k-1}|J|^2 < 1 - |z| \leq 2^k|J|^2\}.$$

Observe that Ω_k looks like a rectangle of side lengths $|J|$ and $2^{k-1}|J|^2 \leq |J|$. Choose sets $\Lambda_k(J) = \{z_j(\Omega_k)\}_{1 \leq j \leq s_k}$ of $s_k = t_k 2^{M(J)-k} = t_k / (2^k |J|)$ points uniformly distributed in Ω_k . Observe that if L is an arc contained in J of length $2^k |J|^2$, then $T(J)$ contains $\asymp s_k 2^k |J| = t_k$ points of $\Lambda_k(J)$. Let $\Lambda(J) = \cup_{1 \leq k \leq M(J)} \Lambda_k(J)$. Then

$$\sum_{z \in \Lambda(J)} (1 - |z|) = \sum_{k=1}^{M(J)} \sum_{j=1}^{s_k} (1 - |z_j(\Omega_k)|) \leq |J| \sum_{k=1}^{M(J)} t_k.$$

Thus, the set of points $\cup_{J \in \mathcal{G}} \Lambda(J)$ satisfies the Blaschke condition. Let B_2 be the Blaschke product with these zeros.

We will now show that SB_1B_2 satisfies the WEP. Fix $\eta > 0$ and consider $z \in \mathbb{D}$ such that $\rho(z, Z(B_1B_2)) > \eta$. Applying (??) we obtain that either $|z| \leq r(\eta) < 1$ or $z \in Q(J)$ for some $J \in \mathcal{G}$. In the first case there exists $\varepsilon(\eta) > 0$ such that $|S(z)B_1(z)B_2(z)| > \varepsilon(\eta)$, as needed. In the second case, assume that $z \in Q(J)$ for some fixed $J \in \mathcal{G}$. Since $\lim_{k \rightarrow \infty} t_k = \infty$ and $\rho(z, Z(B_2)) > \eta$, there exists $C_1 = C_1(\eta) > 0$ such that $1 - |z| < C_1|J|^2$. Increasing $r(\eta)$ we can guarantee that if $|z| \geq r(\eta)$, $z \in Q(J)$ for some $J \in \mathcal{G}$ and $\rho(z, Z(B_2)) > \eta$, then there exists $C_1 = C_1(\eta) > 0$ such that $1 - |z| < C_1|J|^2 < |J|/8$. This means that there exists a positive integer $k(\eta)$ independent of J such that $z \in \cup_{0 \leq k \leq k(\eta)} \Omega_k(J)$, where

$$\Omega_0(J) = \{z \in Q(J) : 1 - |z| \leq |J|^2\},$$

and $2^{k(\eta)}|J| < 1/8$. By (??) and (??) we deduce that there exists a constant $C_2 = C_2(\eta) > 0$ such that $|f(z)| \geq C_2$ for some $C_2 = C_2(\eta) > 0$. It remains to show that $|B_2(z)|$ is also bounded below by a positive constant depending only on η . Let J_{\pm} be two arcs of the family \mathcal{G} contiguous to J . Factor $B_2 = B_3B_4$, where B_3 is the Blaschke product with zeros $\Lambda(J_-) \cup \Lambda(J) \cup \Lambda(J_+)$. Using an argument similar to that in the proof of (??), by Lemma ?? (c) we get $\text{dist}(Z(B_4), Q(J)) \gtrsim |J|$. Hence there exists a constant $C > 0$ such that $|1 - \bar{w}z| > C|J|$ for any zero w of B_4 . Therefore, by (??), there exist constants $C_3, C_4 > 0$ such that

$$\log |B_4(z)|^{-1} \leq C_3 \sum_{w \in Z(B_4)} \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2} \leq C_4 \sum_{w \in Z(B_4)} (1 - |w|^2),$$

where, in the latter estimate, we have used that $1 - |z| < C_1|J|^2$. On the other hand, if we relabel the zeros of B_3 as $Z(B_3) = (z_s)_s$ and use that $\rho(z, Z(B_2)) > \eta$, estimate (??) gives that there exists a constant $C_5 = C_5(\eta) > 0$ such that

$$\begin{aligned} \log |B_3(z)|^{-1} &\leq C_5 \sum_s \frac{(1 - |z|^2)(1 - |z_s|^2)}{|1 - \bar{z}_s z|^2} \\ &\leq C_5 \frac{\sum_{z_s \in Q(z)} (1 - |z_s|)}{1 - |z|} + C_5 \sum_{j=1}^{\infty} \frac{1}{2^{2j}(1 - |z|)} \sum_{z_s \in 2^j Q(z) \setminus 2^{j-1} Q(z)} (1 - |z_s|), \end{aligned}$$

where $Q(z) = Q(\tilde{J})$ is the Carleson box defined by the arc \tilde{J} satisfying $z = z(\tilde{J})$, $2^j Q(z) = Q(2^j \tilde{J})$.

Next, again by Lemma ?? (c), $|\log_2 |J|^{-1} - \log_2 |J_{\pm}|^{-1}|$ is bounded by an absolute constant. Since $1 - |z| < C_1|J|^2$, there exists $k(\eta)$ such

that the densities of zeros $z_s \in Q(z)$ are bounded by $t_{k(\eta)}$. Thus

$$\frac{1}{1-|z|} \sum_{z_s \in Q(z)} (1-|z_s|) \leq \sum_{1 \leq k \leq k(\eta)} t_k \leq C_6(\eta).$$

Similarly, for $j \geq 1$, the densities of zeros $z_s \in 2^j Q(z)$ are bounded by $t_{k(\eta)+j}$. Since $t_{k+1} \leq t_k + 1$, we obtain that

$$\begin{aligned} \frac{1}{1-|z|} \sum_{z_s \in 2^j Q(z)} (1-|z_s|) &\lesssim 2^j \sum_{1 \leq k \leq k(\eta)+j} t_k \\ &\leq 2^j (C_6(\eta) + j t_{k(\eta)} + j(j+1)/2), \end{aligned}$$

Thus,

$$\sum_{j=1}^{\infty} \frac{1}{2^{2j}(1-|z|)} \sum_{z_s \in 2^j Q(z)} (1-|z_s|) \leq C_7(\eta).$$

Hence there exists a constant $C_8(\eta) > 0$ such that $-\log |B_3(z)| \leq C_8(\eta)$. This finishes the proof. \square

Proof of Corollary ??. (a) Fix $0 < \varepsilon < 1$ and consider $A(\varepsilon) = \{z \in \mathbb{D} : |S_\mu(z)| < \varepsilon\}$. Let \mathcal{L}_n be the collection of the dyadic arcs $J \in \mathcal{D}_n$ such that $\mu(J) > 10|J| \log \varepsilon^{-1}$ and let $\mathcal{L} = \cup_{n \geq 0} \mathcal{L}_n$. For $\xi \in J$ and $z \in T(J)$ we have $|\xi - z| \leq 3|J|$. Hence $(1-|z|^2)|\xi - z|^{-2} \geq 1/(10|J|)$ for any $\xi \in J$ and $z \in T(J)$. Therefore, $\log |S_\mu(z)|^{-1} > \mu(J)/(10|J|)$ for $z \in T(J)$ and we deduce that

$$\int_{A(\varepsilon)} \frac{dA(z)}{1-|z|^2} \geq C \sum_{J \in \mathcal{L}} |J| = C \sum_n 2^{-n} a_n,$$

where a_n is the number of dyadic arcs in the collection \mathcal{L}_n . Since μ is singular, it is concentrated on the set of points where its derivative with respect to Lebesgue measure is infinite. Hence there exists $n_0 = n_0(\varepsilon) > 0$ such that for $n \geq n_0$, one has

$$\sum_{J \in \mathcal{L}_n} \mu(J) \geq \frac{1}{2} \mu(\mathbb{T}).$$

We deduce that $a_n w(2^{-n}) \geq \mu(\mathbb{T})/2$. Thus,

$$\int_{A(\varepsilon)} \frac{dA(z)}{1-|z|^2} \geq C \mu(\mathbb{T}) \sum_{n \geq n_0} \frac{2^{-n-1}}{w(2^{-n})} = \infty,$$

which finishes the proof of (a).

(b) We may assume that $w(1) = 1$. Set $n_1 = 0$ and for $k = 2, 3, \dots$ let n_k be the smallest positive integer such that $w(2^{-n_k}) < 2^{-k+1}$. Since $2^{-k+1} \leq w(2^{-n_k+1}) < 2w(2^{-n_k})$, we have $2^{-k} < w(2^{-n_k}) < 2^{-k+1}$.

Hence for any $k = 1, 2, \dots$ and any integer n with $n_k \leq n < n_{k+1}$ we have $2^{-k} \leq w(2^{-n}) < 2^{-k+1}$. Let $\mathcal{L}_k = \mathcal{D}_{n_k}$, $k \geq 1$. The measure μ will be defined by prescribing inductively its mass $\mu(J)$ over any dyadic arc $J \in \cup_k \mathcal{L}_k$. Define $\mu(\mathbb{T}) = 1$. Assume that $\mu(J)$ has been defined for any arc $J \in \mathcal{L}_k$ and we will define the mass of μ over the dyadic arcs of \mathcal{L}_{k+1} . Fix $J \in \mathcal{L}_k$ and let $\mathcal{G}(J)$ be the family of arcs in \mathcal{L}_{k+1} contained in J . If $\mu(J) = 0$, define $\mu(L) = 0$ for any dyadic arc $L \in \mathcal{G}(J)$. If $\mu(J) > 0$, pick two (arbitrary) arcs $J_i \in \mathcal{G}(J)$, $i = 1, 2$, define $\mu(J_i) = \mu(J)/2$ for $i = 1, 2$ and $\mu(L) = 0$ for any other $L \in \mathcal{G}(J)$ with $L \neq J_i$, $i = 1, 2$. In other words, in each dyadic arc of generation n_k of positive measure μ , this measure distributes its mass among two (arbitrary) arcs of generation n_{k+1} and gives no mass to the others. Let J be a dyadic arc of the unit circle with $2^{-n_{k+1}} < |J| \leq 2^{-n_k}$ for a certain integer $k = k(J)$. By construction, $\mu(J) \leq 2^{-k} \leq w(|J|)$. Since any arc is contained in the union of two dyadic arcs of comparable length, there exists a constant $C > 0$ such that $\mu(J) < Cw(|J|)$ for any arc $J \subset \mathbb{T}$. Next we will show that the support of μ has finite entropy. For any integer $k \geq 1$ there are 2^{k-1} arcs J of \mathcal{L}_k with $\mu(J) > 0$. Similarly if $n_k < n \leq n_{k+1}$, then there are exactly 2^k dyadic arcs J of normalised length 2^{-n} with $\mu(J) > 0$. Then

$$\begin{aligned} \sum_{J \in \mathcal{D}, \mu(J) > 0} |J| &\leq 1 + \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} 2^{-n} 2^k \\ &\leq 1 + \sum_{k=1}^{\infty} 2^k 2^{-n_k} \leq 1 + 2 \sum_{k=1}^{\infty} \frac{2^{-n_k}}{w(2^{-n_k})} \leq 1 + 2 \sum_{n \geq 0} \frac{2^{-n}}{w(2^{-n})}. \end{aligned}$$

Applying Lemma ?? we deduce that the support of μ has finite entropy and zero Lebesgue measure. By Theorem ??, S_μ is wepable. \square

Remark 13. Let μ be a positive singular measure in the Zygmund class, that is, there exists a constant $C > 0$ such that for any pair of contiguous arcs $J, J' \subset \mathbb{T}$ of the same length one has

$$|\mu(J) - \mu(J')| \leq C|J|.$$

As in Corollary ??, consider $w(t) = \sup \mu(J)$, where the supremum is taken over all arcs $J \subset \mathbb{T}$ with $|J| = t$. Then there exists a constant $C > 0$ such that $w(2^{-n}) \leq Cn2^{-n}$ for any positive integer n and Corollary ?? gives that S_μ is non wepable.

4. ATOMIC MEASURES AND EASILY WEPABLE SINGULAR FUNCTIONS

We start with a modification of a construction from the proof of Proposition 6 in [?].

Lemma 14. *Given a number $A > 1$, there exists $m \gg 1$ satisfying the following property: for any $J \in \mathcal{D}_k$, denote by J_s , $0 \leq s < 2^m$, the 2^m arcs in \mathcal{D}_{k+m} such that $J = \cup_{0 \leq s < 2^m} J_s$; then for any choice of $x_s \in J_s$, $0 \leq s < 2^m$, there exists a set $Y \subset (x_s)_{0 \leq s < 2^m}$, with $\text{Card } Y \lesssim 2^m/A^2$, such that any measure μ of the form*

$$\mu = \sum_{y \in Y} c_y \delta_y$$

with $10 \cdot 2^{-k-m}A \leq c_y \leq 20 \cdot 2^{-k-m}A$ verifies:

- (a) $\|\mu\| \lesssim |J|/A$;
- (b) for any $\Lambda \subset \mathbb{D}$ large enough so that

$$\{I \in \mathcal{D} : P[\mu](z(I)) \geq A\} \subset \{I \in \mathcal{D} : \Lambda \cap T(I) \neq \emptyset\},$$

we have

$$\log |B_\Lambda(z(J))| \lesssim -A^2,$$

where B_Λ is the corresponding Blaschke product.

Proof. Let $J = [0, 2^{-k})$. Fix $q \in \mathbb{N}$, $q \geq A^2$, $n = q^2$, and $m \geq 1$ such that $2^{m-1} \leq q2^{2n} < 2^m$ and define

$$Y = \{x_s : s = jq2^n + \ell, 0 \leq j < 2^n, 0 \leq \ell < 2^n\}.$$

Then $\text{Card } Y = 2^{2n}$ and $\|\mu\| \asymp 2^{2n}2^{-k-m}A \asymp 2^{-k}A/q \lesssim |J|/A$. Let

$$\mathcal{L} = \cup_{0 \leq j < 2^n} [jq2^{n-k-m}, (jq+1)2^{n-k-m}] = \bigcup \{J_s : x_s \in Y\}.$$

Suppose that r and θ satisfy the properties $2^{-k-m} \leq 1-r \leq 2^{n-k-m}$

$$0 \leq \theta - jq2^{n-k-m} \leq 2^{n-k-m}, \text{ for some } 0 \leq j < 2^n.$$

Then at least half of the arc $[\theta - (1-r), \theta + (1-r))$ is contained in \mathcal{L} , so that

$$P[\mu](re^{2\pi i\theta}) \geq A.$$

Hence, for every $I \in \mathfrak{A} = \cup_{k+m-n \leq p \leq k+m} \mathcal{D}_p$ such that $I \subset \mathcal{L}$, we have

$$\Lambda \cap T(I) \neq \emptyset.$$

Thus, applying (??), we obtain that

$$\begin{aligned} \log |B_\Lambda(z(J))| &\lesssim -|J| \sum_{I \in \mathfrak{A}, I \subset \mathcal{L}, w \in \Lambda \cap T(I)} \frac{1 - |w|^2}{|1 - \bar{w}z(J)|^2} \\ &\lesssim -\frac{1}{|J|} \sum_{I \in \mathfrak{A}, I \subset \mathcal{L}, w \in \Lambda \cap T(I)} (1 - |w|) \lesssim -\frac{1}{|J|} \sum_{k+m-n \leq p \leq k+m} 2^{-p} 2^n 2^{n-k-m+p} \\ &\asymp -n 2^k \frac{2^m}{q} 2^{-k-m} = -q. \end{aligned}$$

□

Proof of Theorem ??. (a) \Rightarrow (b) If (??) does not hold, then we can choose a sequence of groups $(b_{s_n}, \dots, b_{s_n+n})_{n \geq 1}$ such that $b_{s_n} \leq 2b_{s_n+n}$. By Lemma ??, passing to a subsequence of (s_n) denoted also by (s_n) we construct a sequence of dyadic arcs J_n and measures

$$\mu_n = \sum_{s_n \leq v < s_n + 2^{m_n}} b_v \delta_{x_v}$$

such that

$$\text{supp } \mu_n \subset J_n, \quad \|\mu_n\| = o(|J_n|), \quad \text{dist}(J_n, J_{n'}) \gtrsim \max(|J_n|, |J_{n'}|).$$

Furthermore, if the sets Λ_n satisfy the property

$$(15) \quad P[\mu_n](z(I)) \geq n \implies \Lambda_n \cap T(I) \neq \emptyset, \quad I \in \mathcal{D},$$

then the corresponding Blaschke products B_{Λ_n} satisfy the estimate

$$|B_{\Lambda_n}(z(J_n))| \leq \exp(-n^2).$$

Finally, we take $x \in \mathbb{T} \setminus \cup_n J_n$ and set

$$\mu = \sum_{n \geq 1} \mu_n + \left(\sum_{s \geq 1} b_s - \sum_{n \geq 1} \|\mu_n\| \right) \delta_x.$$

Suppose that B_Λ is a Blaschke product with zero set Λ such that $S_\mu B_\Lambda$ has the WEP. Then for every $n \geq n_0$, the set $\Lambda_n = \Lambda \cap Q(J_n)$ satisfies the property (??), and hence,

$$|B_\Lambda(z(J_n))| \leq \exp(-n^2).$$

Therefore, B_Λ should have a zero in $T(J_n)$, $n \geq n_1$. However,

$$P[\mu](z(J_n)) \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, S_μ is not easily wepable.

(b) \Rightarrow (a) Given $z, w \in \mathbb{D}$, set

$$\beta(z, w) = \log_2 \frac{1}{1 - \rho(z, w)}.$$

Let $J_1, J_2 \in \mathcal{D}$, $2^{-n} = |J_1| \leq |J_2| = 2^{-m}$, $J \in \mathcal{D}$, $J_1 \subset J$, $|J| = |J_2|$, $J = [k2^{-m}, (k+1)2^{-m}]$, $J_2 = [k'2^{-m}, (k'+1)2^{-m}]$. Then $|J_1|/|J_2| = 2^{m-n}$, $\text{dist}(c(J), c(J_2))/|J_2| = d(k, k')$, where $c(I)$ is the centre of the arc I , $d(k, k') = \min(|k - k'|, 2^m - |k - k'|)$. Here we consider the distance between J and J_2 along the unit circle (identified with the interval $[0, 1)$). An easy estimate of the Blaschke factor shows that

$$\beta_{J_1, J_2} \stackrel{\text{def}}{=} \beta(z(J_1), z(J_2)) = n - m + 2 \log_2(d(k, k') + 1) + O(1).$$

Furthermore, if $T(J_1) \cap T(J_2) = \emptyset$, then

$$(16) \quad \min_{z_1 \in T(J_1), z_2 \in T(J_2)} \beta(z, w) = \beta_{J_1, J_2} + O(1).$$

Let ϕ be an increasing subadditive function on $(0, +\infty)$ such that $\phi(x) = x$, $0 < x \leq 1$, $\phi(x) \asymp \log x$, $x \rightarrow \infty$.

Let $\mu = \sum_{s \geq 1} b_s \delta_{x_s}$. For every $J \in \mathcal{D}$ we set $\lambda(J) = P[\mu](z(J))$. Harnack's inequality gives us a Lipschitz type estimate

$$(17) \quad |\phi(\lambda(J_1)) - \phi(\lambda(J_2))| \lesssim \beta_{J_1, J_2}.$$

Now, for every $J \in \mathcal{D}$ we denote by k_J the integer part of $\phi(\lambda(J))$, and choose k_J points $z_{J,1}, \dots, z_{J,k_J}$ uniformly distributed in $T(J)$. Let B be the Blaschke product with zeros at the points $z_{J,1}, \dots, z_{J,k_J}$, $J \in \mathcal{D}$. It is clear that for some $c > 0$ and $C < 1$ we have

$$Z(B) \subset \bigcup_{P[\mu](z(J)) \geq c} T(J) \subset \{w \in \mathbb{D} : |S_\mu(w)| \leq C\},$$

so if BS_μ has the WEP, S_μ is in fact easily wepable. To check that BS_μ has the WEP (and incidentally that B exists) we should find a strictly positive function ψ on $(0, 1)$ such that

$$(18) \quad |BS_\mu(z)| \geq \psi(\rho(z, Z(B))), \quad z \in \mathbb{D}.$$

Since $k_J \rightarrow \infty$ as $\inf_{T(J)} |S_\mu| \rightarrow 0$, and the zeros of B are uniformly distributed in each $T(J)$, if $\rho(z, Z(B)) \geq \varepsilon$, then $|S_\mu(z)|$ is bounded below. So to prove (??), it is enough to bound $|B(z)|$ from below. Again by the uniform distribution of zeros, if $\rho(z, Z(B)) \geq \varepsilon$ and $z \in T(J)$, then $k_J \leq k(\varepsilon)$, so the lower bound on $|B(z)|$ will hold if we can prove that:

$$\prod_{I \in \mathcal{D} \setminus \{J\}} \prod_{1 \leq s \leq k_I} \left| \frac{z_{I,s} - z(J)}{1 - z_{I,s} z(J)} \right| \geq \exp(-Ce^{k_J}), \quad J \in \mathcal{D},$$

or, equivalently,

$$(19) \quad \sum_{I \in \mathcal{D} \setminus \{J\}} 2^{-\beta_{J,I}} \phi(\lambda(I)) \lesssim \max(\lambda(J), 1), \quad J \in \mathcal{D}.$$

Fix $J \in \mathcal{D}$. Let $|J| = 2^{-n}$. By (??) and (??), we have

$$\begin{aligned} \sum_{I \in \mathcal{D}, |I| > |J|} 2^{-\beta_{J,I}} \phi(\lambda(I)) &\lesssim \sum_{I \in \mathcal{D}, |I| > |J|} 2^{-\beta_{J,I}} (\phi(\lambda(J)) + \beta_{J,I}) \\ &\lesssim \sum_{0 \leq k < n} \sum_{s=1}^{2^k} 2^{k-n} s^{-2} (\phi(\lambda(J)) + n - k + \log s + O(1)) \lesssim \max(\lambda(J), 1). \end{aligned}$$

Next, we set $\mu' = \chi_{10J}\mu$, $\mu'' = \mu - \mu'$, and define $\lambda'(I) = P[\mu'](z(I))$, $\lambda''(I) = P[\mu''](z(I))$, $I \in \mathcal{D}$. To prove (??), we need only to check that

$$\sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda'(I)) \lesssim \max(\lambda'(J), 1)$$

and

$$(20) \quad \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda''(I)) \lesssim \max(\lambda''(J), 1).$$

We have $\mu' = \sum_{s \in N'} b_s \delta_{x_s}$, and we set $a = \max\{b_s : s \in N'\}$. (If $\mu' = 0$, we just pass to μ'' .) By (??), $P[\mu'](z(J)) \asymp 2^n a$. By the subadditivity of ϕ , we have

$$\sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda'(I)) \lesssim \sum_{s \in N'} \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))).$$

Fix for a moment $s \in N'$. Without loss of generality, $x_s = 0$, and for $m \geq n$, $I = [t2^{-m}, (t+1)2^{-m}]$, $I \neq J$, we have

$$\beta_{J,I} \geq n - m + O(1), \quad P[\delta_{x_s}](z(I)) \lesssim 2^m (|t| + 1)^{-2}.$$

Hence,

$$\begin{aligned} \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(\lambda'(I)) &\lesssim \sum_{s \in N'} \sum_{m \geq n} \sum_{t \geq 1} 2^{n-m} \phi(b_s 2^m t^{-2}) \\ &= \sum_{s \in N'} \sum_{m \geq n} \sum_{t^2 \leq 2^m b_s} 2^{n-m} \phi(2^m b_s t^{-2}) + \sum_{s \in N'} \sum_{m \geq n} \sum_{t^2 > 2^m b_s} 2^{n-m} \phi(2^m b_s t^{-2}) \\ &\lesssim \sum_{s \in N'} \sum_{m \geq n} 2^{m/2} \sqrt{b_s} 2^{n-m} (\phi(2^m b_s) + 1) \\ &\lesssim \sum_{s \in N'} \sum_{m \geq n} 2^{n-m/2} \sqrt{b_s} \max(m + \log b_s, 1) \\ &\lesssim \sum_{s \in N'} 2^{n/2} \sqrt{b_s} \max(n + \log b_s, 1) \end{aligned}$$

$$\lesssim 2^{n/2} \sqrt{a} \max(n + \log a, 1) \lesssim \max(2^n a, 1) \lesssim \max(\lambda'(J), 1).$$

(Once again, we use here that by (??), $\sum_{s \in N'} \sqrt{b_s} \lesssim \sqrt{a}$.)

Next, we pass to $\mu'' = \sum_{s \in N''} b_s \delta_{x_s}$. To prove (??), we need to verify that

$$(21) \quad \sum_{s \in N''} \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))) \lesssim 1 + \sum_{s \in N''} P[b_s \delta_{x_s}](z(J)).$$

Fix $s \in N''$ and choose $r = r(s) \geq 2$ such that

$$2r|J| = \text{dist}(x_s, J).$$

Then

$$\begin{aligned} & \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))) \\ &= \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|, \text{dist}(I, J) \leq r|J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))) \\ &+ \sum_{I \in \mathcal{D} \setminus \{J\}, |I| \leq |J|, \text{dist}(I, J) > r|J|} 2^{-\beta_{J,I}} \phi(b_s P[\delta_{x_s}](z(I))) = A_1 + A_2. \end{aligned}$$

Next, we can assume that $J = [0, 2^{-n})$ and for $I = [t2^{-m}, (t+1)2^{-m})$, $m \geq n$, we have $\beta_{J,I} = m - n + 2 \log_2(1 + t2^{n-m}) + O(1)$. Hence,

$$\begin{aligned} A_1 &\lesssim \sum_{m \geq n} \sum_{t \geq 1} 2^{n-m} (1 + t2^{n-m})^{-2} \phi(b_s 2^{-m} r^{-2} 2^{2n}) \\ &\lesssim \sum_{m \geq n} \phi(2^{2n-m} b_s r^{-2}) \lesssim 2^n b_s r^{-2} \asymp P[b_s \delta_{x_s}](z(J)). \end{aligned}$$

Furthermore,

$$\begin{aligned} A_2 &\lesssim \sum_{m \geq n} \sum_{t \geq 1} 2^{n-m} r^{-2} \phi(b_s 2^m t^{-2}) \\ &= \sum_{m \geq n} \sum_{t^2 \leq 2^m b_s} 2^{n-m} r^{-2} \phi(2^m b_s t^{-2}) + \sum_{m \geq n} \sum_{t^2 > 2^m b_s} 2^{n-m} r^{-2} \phi(2^m b_s t^{-2}) \\ &\lesssim \sum_{m \geq n} 2^{m/2} \sqrt{b_s} 2^{n-m} r^{-2} \phi(2^m b_s) + \sum_{m \geq n} 2^{n-m} r^{-2} 2^m b_s 2^{-m/2} b_s^{-1/2} \\ &\lesssim 2^{n/2} \sqrt{b_s} r^{-2} \max(n + \log b_s, 1) + 2^{n/2} \sqrt{b_s} r^{-2}. \end{aligned}$$

If $b_s \geq 2^{-n}$, then

$$A_2 \lesssim 2^n b_s r^{-2} \asymp P[b_s \delta_{x_s}](z(J)).$$

Thus, to complete the proof, we need only to estimate

$$H = \sum_{s \in N'', b_s < 2^{-n}} 2^{n/2} \sqrt{b_s} r(s)^{-2}.$$

Again by (??),

$$\sum_{b_s < 2^{-n}} \sqrt{b_s} \lesssim 2^{-n/2}.$$

Hence,

$$H \lesssim 1,$$

and (??) follows. \square

5. POROUS SETS AND EASILY WEPABLE SINGULAR FUNCTIONS

Proof of Theorem ??. (b) \Rightarrow (a) Suppose that E is porous, $\text{supp } \mu \subset E$, and set $u = P[\mu]$. Set

$$\mathcal{G}_k := \{I \in \mathcal{D} : k^2 < u(z(I)) \leq (k+1)^2\}, \quad k \geq 1.$$

We claim that there exists a constant $C > 0$ such that

$$(22) \quad \sum_{k \geq 1} k \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} \leq C u(z(J)), \quad J \in \mathcal{D}.$$

Indeed, by Lemma ?? (d),

$$\begin{aligned} \sum_{k \geq 1} k \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} &= \sum_{k \geq 1} \sum_{1 \leq j \leq k} \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} \\ &= \sum_{j \geq 1} \sum_{k \geq j} \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} \\ &= \sum_{j \geq 1} \sum_{I \in \mathcal{D}, j^2 < u(z(I))} \frac{|I||J|}{|1 - \overline{z(J)}z(I)|^2} \leq C \sum_{j \geq 1} \frac{u(z(J))}{j^2}. \end{aligned}$$

Now for each integer $k \geq 1$ and each $I \in \mathcal{G}_k$ we consider the set $\Lambda(I)$ consisting of k points, uniformly distributed in $T(I)$. Let $\Lambda := \cup_{k \geq 1} \cup_{I \in \mathcal{G}_k} \Lambda(I)$. Taking $J = \partial \mathbb{D}$ in (??), we see that Λ is a Blaschke sequence. Let B be the Blaschke product with the zero set Λ .

Notice that the zeros of B are restricted to the sets $T(I)$ where the modulus of S_μ is small, so if we prove that BS_μ has the WEP, we will have shown that S_μ is easily wepable.

Furthermore, for any $z \in T(I)$ such that $I \in \mathcal{G}_k$ we have $\rho(z, \Lambda) \leq ck^{-1/2}$. Therefore, if $\rho(z_0, \Lambda) > \varepsilon$, then $z_0 \notin \cup_{I \in \mathcal{G}_k, k \geq C/\varepsilon^2} T(I)$. By Harnack's inequality, $u(z_0)$ is bounded above, so that $|S_\mu(z_0)| \geq \eta = \eta(\varepsilon) > 0$. Thus, to prove that BS_μ has the WEP, we only need to show that for any $\varepsilon > 0$, one has $\inf\{|B(z)| : \rho(z, \Lambda) > \varepsilon\} > 0$. Fix $\varepsilon > 0$

and z such that $\rho(z, \Lambda) > \varepsilon$ and let $J \in \mathcal{D}$ be such that $z \in T(J)$. Since $|S_\mu(z)| \geq \eta$, by Harnack's inequality, we have

$$(23) \quad u(z(J)) \leq C \log \eta^{-1}.$$

By (??),

$$\begin{aligned} \log |B(z)|^{-1} &\leq C(\varepsilon) \sum_{\lambda \in \Lambda} \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda}z|^2} \\ &= C(\varepsilon) \sum_{k=1}^{\infty} \sum_{I \in \mathcal{G}_k} \sum_{\lambda \in \Lambda \cap T(I)} \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda}z|^2} \\ &\leq C(\varepsilon) \sum_{k=1}^{\infty} k \sum_{I \in \mathcal{G}_k} \frac{|I||J|}{|1 - z(J)z(I)|^2}. \end{aligned}$$

Applying (??) and (??), we conclude that $\log |B(z)|^{-1} \leq C(\varepsilon) \log \eta^{-1}$ which completes the proof of the implication (b) \Rightarrow (a).

(a) \Rightarrow (b) Assume now that E is not porous. We can find a sequence of arcs $J_n \in \mathcal{D}_{k_n}$, $k_n \rightarrow \infty$, and a sequence of numbers $M_n \rightarrow \infty$, $n \rightarrow \infty$, such that every $J \in \mathcal{D}_{k_n+M_n}$, $J \subset J_n$, meets E . Passing to a subsequence and using Lemma ?? we obtain a sequence of arcs $J_n \in \mathcal{D}_{k_n}$, $k_n \rightarrow \infty$, and a sequence of measures μ_n such that $\text{supp } \mu_n \subset J_n \cap E$, $\|\mu_n\| = o(|J_n|)$, and $\text{dist}(J_n, J_{n'}) \gtrsim \max(|J_n|, |J_{n'}|)$. Furthermore, Lemma ?? gives that if sets $\Lambda_n \subset \mathbb{D}$ satisfy the property

$$P[\mu_n](z(I)) \geq n \implies \Lambda_n \cap T(I) \neq \emptyset, \quad I \in \mathcal{D},$$

then the corresponding Blaschke products B_{Λ_n} satisfy the estimate

$$|B_{\Lambda_n}(z(J_n))| \leq \exp(-n^2).$$

Let

$$\mu = \sum_{n \geq 1} \mu_n.$$

To conclude that S_μ is not easily wepable we use the same argument as in the part (a) \Rightarrow (b) of the proof of Theorem ??. \square

6. CORONA TYPE CONSTANTS

First, we make an easy remark: $c_n(\delta, I) \geq c_{n-1}(\delta, I)$, thus $\delta_n(I) \geq \delta_{n-1}(I)$.

Indeed, suppose that $\gamma < c_{n-1}(\delta, I)$, then there are $(f_1, \dots, f_{n-1}) =: f$ such that $\delta^2 \leq \sum_{j=1}^{n-1} |f_j(\lambda)|^2 \leq \|f\|_{\infty, n}^2 \leq 1$, and that

$$\gamma < \inf \{ \|g\|_{\infty, n-1} : \exists h \in H^\infty : \sum_{j=1}^{n-1} g_j f_j + hI \equiv 1 \}.$$

Given $\tilde{f} := (f_1, \dots, f_{n-1}, 0)$, for every $g \in (H^\infty)^n$ we obtain that $\sum_{j=1}^n g_j f_j = \sum_{j=1}^{n-1} g_j f_j$, so that $\chi_I(\tilde{f}) \geq \gamma$. Since \tilde{f} fulfils the condition to be a candidate in the supremum, we obtain that $c_n(\delta, I) \geq \gamma$, q.e.d.

Lemma 15. *For any n , $\delta_n(I) \leq \tilde{\delta}(I)$.*

Proof. Pick any number $\varepsilon_0 > \tilde{\delta}(I)$, then choose ε_1 such that $\varepsilon_0 > \varepsilon_1 > \tilde{\delta}(I)$. Suppose that $f := (f_1, \dots, f_n) \in (H^\infty)^n$ satisfies the estimates $\varepsilon_0^2 \leq \inf_{\lambda \in Z(I)} \sum_{j=1}^n |f_j(\lambda)|^2$, $\|f\|_{\infty, n} \leq 1$. Take $z \in \mathbb{D}$ such that for some $\lambda_0 \in Z(I)$ we have $\rho(z, \lambda_0) < \varepsilon_1$. Then, applying the Schwarz-Pick Lemma to the function $\varphi := f \cdot \bar{v}$, where v is a unit vector in \mathbb{C}^n parallel to $f(\lambda_0)$, we see that

$$\left(\sum_{j=1}^n |f_j(z)|^2 \right)^{1/2} \geq |\varphi(z)| \geq \frac{\varepsilon_0 - \varepsilon_1}{1 - \varepsilon_0 \varepsilon_1} =: \varepsilon_2 > 0.$$

On the other hand, suppose that $\rho(z, Z(I)) \geq \varepsilon_1$, then $|I(z)| \geq \eta_I(\varepsilon_1) > 0$. Finally,

$$\inf_{z \in \mathbb{D}} \left(\sum_{j=1}^n |f_j(z)|^2 + |I(z)|^2 \right) \geq \min(\varepsilon_2^2, \eta_I(\varepsilon_1)^2).$$

By Carleson's Corona Theorem, we can find $g \in (H^\infty)^n$, $h \in H^\infty$ with $\|g\|_{\infty, n} \leq C(\varepsilon_2, \eta_I(\varepsilon_1))$ such that $\sum_{j=1}^n g_j f_j + hI \equiv 1$, therefore $c_n(\varepsilon_0, I) < \infty$. Since this holds for any $\varepsilon_0 > \tilde{\delta}(I)$, we are done. \square

The following will end the proof of Proposition ??.

Lemma 16. $\delta_1(I) \geq \tilde{\delta}(I)$.

Proof. Let $\varepsilon_0 < \tilde{\delta}(I)$. We want to prove that $c_1(\varepsilon_0, I) = \infty$. Pick ε_1 such that $\varepsilon_0 < \varepsilon_1 < \tilde{\delta}(I)$. Then there exists an infinite sequence $(\zeta_n)_n \subset \mathbb{D}$ such that $\rho(\zeta_n, Z(I)) \geq \varepsilon_1$ and $|I(\zeta_n)| \rightarrow 0$.

Choose a subsequence (ξ_n) of this sequence, with

$$1 - \inf_k \prod_{j:j \neq k} \rho(\xi_j, \xi_k)$$

so small that the Blaschke product B with zeros (ξ_n) satisfies the property $|B(z)| \geq \varepsilon_0$ if $\rho(z, Z(B)) \geq \varepsilon_1$ (see, for instance [?, p. 395]). Then for any $\lambda \in Z(I)$ we have $|B(\lambda)| \geq \varepsilon_0$. On the other hand, for any $g, h \in H^\infty$,

$$g(\xi_n)B(\xi_n) + h(\xi_n)I(\xi_n) = h(\xi_n)I(\xi_n) \rightarrow 0, \quad n \rightarrow \infty.$$

This proves that $gB + hI \not\equiv 1$, and hence $\chi_I(B) = \infty$ and $c_1(\varepsilon_0, I) = \infty$. \square

Proof of Proposition ??. The argument is analogous to that in the proof of Lemma ?. Take a strictly increasing function $\psi : (0, 1) \rightarrow (0, 1)$ such that $\psi \cdot (\phi + 1) \leq 1$. Using the above mentioned result from [?, p. 1199] we find a Blaschke product B satisfying the WEP and such that for every $\delta \in (0, 1)$ there exists $z_\delta \in \mathbb{D}$ satisfying

$$\rho(z_\delta, Z(B)) = \delta, \quad |B(z_\delta)| \leq \psi(\delta).$$

Denote $b_\delta(z) = (z - z_\delta)/(1 - \bar{z}_\delta z)$. We have $\min_{Z(B)} |b_\delta| = \delta$. If $g, h \in H^\infty$, $gb_\delta + hB \equiv 1$, then

$$\|h\|_\infty \geq 1 + \phi(\delta),$$

and hence,

$$c_1(\delta, B) \geq \inf_{gb_\delta + hB \equiv 1} \|g\|_\infty \geq \phi(\delta), \quad \delta \in (0, 1).$$

□

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