A hyperbolic problem of second order with unilateral constraints: The vibrating string with a concave obstacle

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MICHELLE SCHATZMAN

CNRS et Université Pierre et Marie Curie, Laboratoire d'Analyse Numérique, Tour 55-65, 5ème étage, 4 Place Jussieu, 75230 Paris Cedex 05, France

The motion of a vibrating string constrained to remain above a material concave obstacle is studied. It is assumed that the string does not lose energy when it hits the obstacle. A set of natural inequations describes this model; an energy condition in an ad hoc form must be added to ensure uniqueness. Existence and uniqueness are proved for the Cauchy problem; the case of an infinite string and the case of a finite string with fixed ends are considered.

0. PRESENTATION OF THE PROBLEM

Let us consider an infinite vibrating string, and let us denote by $u(x, t)$ the transverse displacement of the point of abscissa x at the instant t. We suppose that this vibrating string is compelled to remain above an obstable, which is represented by the function $x \mapsto \varphi(x)$.

The initial position and velocity are given by

$$u(x, 0) = u_0(x), \tag{0.1}$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x). \tag{0.2}$$

The constraint is described by

$$u(x, t) \geqslant \varphi(x). \tag{0.3}$$

We need obviously the compatibility condition

$$u_0(x) \geqslant \varphi(x). \tag{0.4}$$

When the string docs not touch the obstacle, it satisfies the wave equation

$$\Box u = 0 \qquad \left(\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right).$$

It is therefore natural to precribe the following conditions:

$$\Box u \geqslant 0 \qquad \text{in the sense of distributions in} \qquad \mathbb{R} \times \mathbb{R}_*^+ = \mathbb{R} \times (0, \infty), \tag{0.5}$$

$$\text{the support of } \Box u \text{ is contained in the "set of contact," the set where } u(x, t) = \varphi(x). \tag{0.6}$$

In this paper, we prescribe

$$\varphi'' \geqslant 0. \tag{0.7}$$

This convexity condition on the function φ expresses the fact that the material obstacle $\{(x, z)/z \leqslant \varphi(x)\}$ is concave. Condition (0.7) ensures that there will not be "too many" reflections on the obstacle, which would cause the method described hereafter to fail.

We need information on the nature of the reflection on the obstacle and we choose to impose that energy be conserved; we shall, in fact, need a *local energy conservation* condition, which we will write as

$$\frac{\partial}{\partial x}\left(-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right) = 0$$

$$\text{in the sense of distributions in} \qquad \mathbb{R} \times \mathbb{R}_*^+ . \tag{0.8}$$

The aim of this paper is to prove that, if u_0 is in $H^1_{\text{loc}}(\mathbb{R})$ and u_1 is in $L^2_{\text{loc}}(\mathbb{R})$, then there exists a unique function u in $L^\infty_{\text{loc}}(\mathbb{R}^+; H^1_{\text{loc}}(\mathbb{R})) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\mathbb{R}))$ with $\mathbb{R}^+ = [0, +\infty)$ which satisfies conditions (0.1)–(0.6) and (0.8). We prove an analogous result in the case of a finite vibrating string, with fixed ends.

Some results of continuity with respect to the data, of regularity, and the convergence of a numerical scheme are shown in the paper of A. Bamberger, and the author [4].

Before Amerio and Prouse, who proved the first result on the vibrating string with obstacle in [3], it seems that nobody had seriously progressed toward the solution of this problem. In the case with loss of energy, Citrini has given several results in [6, 7], along the line of Amerio and Prouse; the point obstacle has been studied by Amerio [1, 2], Citrini [6], and the author [12].

The plan is the following:

I. *Introduction.* I.1. Notations and general hypotheses. Statement of the results. I.2. Justification of the mathematical model. I.3. Idea of the proof of existence. Role of the concavity hypothesis. I.4. Comparison with the results of L. Amerio and G. Prouse. I.5. Explicit computations of examples. II. *The Line of Influence and Its Properties.* II.1. Definition and first properties of the domain of influence and of the line of influence. II.2. Study of the values of the free solution and of its derivatives on the line of influence. III. *Solution of a Linear Auxiliary Problem.* III.1. Statement of the result, idea of the proof. III.2. Computation of $\Box v$ in terms of w and σ. III.3. Partial results of existence. III.4. Energy condition. End of the proof of theorem III.1. IV. *Proof of Existence: Infinite String.*

IV.1. Proof of existence in a general case. IV.2. An alternative construction when the obstacle φ is constant. V. *Proof of Uniqueness: Infinite String.* V.1. Results of trace. V.2. Proof of uniqueness. VI. *The Finite String with Fixed Ends.* VI.1. Existence. VI.2. Uniqueness. *Appendix.* A. 1. Elementary results about integration by substitution. A.2. The set $t = \sigma(x)$ in characteristic coordinates.

I. Introduction

I.1. *Notations and General Hypotheses. Statement of the Results*

In what follows, we shall denote by

$u(x, t)$	the transverse displacement from its equilibrium position of the point of abscissa x of the string at the instant t,
φ	the obstacle,
u_0	the initial position of the string,
u_1	the initial velocity of the string.

We make the following hypotheses:

$$\varphi'' \geqslant 0 \qquad \text{(in the sense of distributions)}, \tag{I.1}$$

$$u_0 \in H^1_{\text{loc}}(\mathbb{R}) = \left\{ v \in L^2_{\text{loc}}(\mathbb{R}) \Big/ \frac{dv}{dx} \in L^2_{\text{loc}}(\mathbb{R}) \right\}, \tag{I.2}$$

where $L^2_{\text{loc}}(\mathbb{R})$ is the well-known space of locally square-integrable functions;

$$u_1 \in L^2_{\text{loc}}(\mathbb{R}), \tag{I.3}$$

$$u_0(x) \geqslant \varphi(x) \qquad \forall x, \tag{I.4}$$

(this makes sense because u_0 and φ are continuous by hypotheses (I.1) and (I.2));

$$u_1(x) \geqslant 0 \qquad \text{almost everywhere on the set } \{x / u_0(x) = \varphi(x)\}. \tag{I.5}$$

In the case of a finite string with fixed ends, we replace (I.1) by

$$\varphi'' \geqslant 0 \qquad \text{and} \qquad \varphi(0) < 0, \qquad \varphi(L) < 0, \tag{I.6}$$

and (I.2), (I.3) by

$$u_0 \in H_0^1(0, L) = \left\{ v \in L^2(0, L) \Big/ \frac{dv}{dx} \in L^2(0, L),\ v(0) = v(L) = 0 \right\}, \tag{I.7}$$

$$u_1 \in L^2(0, L). \tag{I.8}$$

We shall prove the following results:

THEOREM IV.1 (Infinite String). *Under hypotheses* (I.1)–(I.5), *there exists a unique function u such that*

$$u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R})) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(\mathbb{R})), \tag{I.9}$$

$$u(x, 0) = u_0(x) \qquad \forall x, \tag{I.10}$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x) \qquad \textit{almost everywhere}, \tag{I.11}$$

$$u(x, t) \geqslant \varphi(x) \qquad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{I.12}$$

$$\Box u \geqslant 0 \qquad \textit{in the sense of distributions in } \mathbb{R} \times \mathbb{R}^+_*, \tag{I.13}$$

$$\operatorname{supp} \Box u \subset \{(x, t)/u(x, t) = \varphi(x)\}, \tag{I.14}$$

$$\frac{\partial}{\partial x}\left(-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2\right) = 0$$
$$\textit{in the sense of distributions in } \mathbb{R} \times \mathbb{R}^+_*. \tag{I.15}$$

THEOREM V.1 (Finite String with fixed Ends). *Under hypotheses* (I.4)–(I.8), *there exists a unique function u such that*

$$u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; H_0^1(0, L)) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; L^2(0, L)) \tag{I.16}$$

and u satisfies conditions (I.10)–(I.15) *with the relevant modifications on the domain.*

For simplicity, we shall denote problem (I.9)–(I.15) by the symbol P_{∞}, and problem (I.10)–(I.16) by the symbol P_{f}.

The functional space $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R})) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(\mathbb{R}))$ is defined as the set of real functions v on $\mathbb{R} \times \mathbb{R}^+$ such that, for any finite positive L and T

$$\operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \left(\int_{-L}^{L} \left[|u(x, t)|^2 + \left|\frac{\partial u}{\partial x}(x, t)\right|^2 + \left|\frac{\partial u}{\partial t}(x, t)\right|^2 \right] dt \right) < +\infty. \tag{I.17}$$

The functional space $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; H_0^1(0, L)) \cap W^{1,\infty}_{\mathrm{loc}}(R^+; L^2(0, L))$ is defined to be the set of real functions v on $[0, L] \times R^+$ such that, for all finite positive T

$$\operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \left(\int_{0}^{L} \left[|u(x, t)|^2 + \left|\frac{\partial u}{\partial x}(x, t)\right|^2 + \left|\frac{\partial u}{\partial t}(x, t)\right|^2 \right] dt \right) < +\infty \tag{I.18}$$

and

$$u(0, t) = u(L, t) = 0 \qquad \forall t \geqslant 0.$$

It will be shown in the course of the paper that condition (I.12) implies that (I.11) has a meaning.

I.2. *Justification of the Mathematical Model*

Conditions (I.10) to (I.12) are quite natural.

Condition (I.14) implies that, when the string does not touch the obstacle, it satisfies the wave equation.

If at a time t and a point x, the string touches the obstacle with a nonzero velocity, there must necessarily be a discontinuity in the time derivative. Hence, generally, $\Box u$ cannot be a function. As the string must leave the obstacle after the shock, $\Box u$ must be positive, i.e., (I.13).

Condition (I.15) expresses that the divergence of the vector field

$$S_u = (-2u_x u_t\,,\ u_x^2 + u_t^2) \tag{I.19}$$

is zero.

The first component of this vector field is an energy flux density; it is similar to the Poynting vector which is well known in electromagnetism; see, for example, Landau and Lifschitz [11, p. 98].

The second component of S_u is but a total energy density: kinetic plus potential energy.

Condition (I.15) is satisfied on any open set where $\Box u$ is null: to convince oneself it is enough to multiply $\Box u$ by $\partial u/\partial t$, and rewrite the expression in divergence form. This classical manipulation is done in Courant and Hilbert [9, p. 660].

It is not enough—for example, in the case of the finite string with fixed ends—*to impose an integral condition of the type*

$$\int_0^L \left[\left|\frac{\partial u}{\partial t}(x,t)\right|^2 + \left|\frac{\partial u}{\partial x}(x,t)\right|^2\right] dx = \int_0^L \left(\left|\frac{du_0}{dx}\right|^2 + |u_1|^2\right) dx$$

almost everywhere in t. (I.20)

Condition (I.20) is not local, and does not ensure that the velocity is reversed everywhere when a shock happens.

Condition (I.15) *expresses the conservation of energy across any curve*, and especially the discontinuity curves of $\partial u/\partial x$ and $\partial u/\partial t$. Proposition V.3 lays down precisely this property.

I.3. *Idea of the Proof of Existence*

Role of the concavity hypothesis. Let us first consider the "*free solution*" w, or solution of the problem without obstacle, with the initial data u_0, u_1 of the problem with obstacle:

$$\begin{aligned} \Box w &= 0, \\ w(x,0) &= u_0(x), \\ \frac{\partial w}{\partial t}(x,0) &= u_1(x). \end{aligned} \tag{I.21}$$

It is well known that there exists a unique w in the functional class $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R})) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(\mathbb{R}))$ which satisfies (I.21). Such a w is a continuous function.

Let us now consider the set E:

$$E = \overline{\{(x, t) \in \mathbb{R} \times \mathbb{R}^+ / w(x, t) < \varphi(x)\}}.$$

As w is continuous, E is unambiguously defined. Define the *domain of influence* to be the union, I, of all forward wave-cones with vortex in E. I is hatched in Fig. 1.

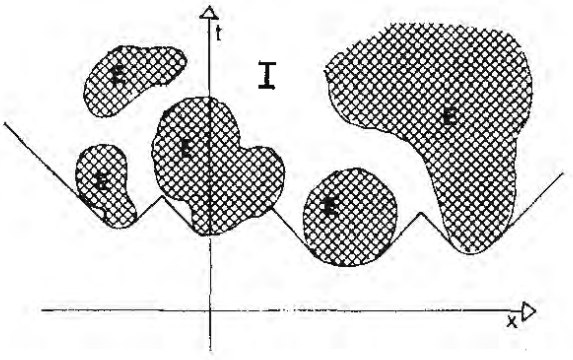

FIG. 1. Sketch of domain and line of influence.

Of course, E is included in I. By definition

$$w(x, t) \geqslant \varphi(x) \qquad \text{if} \qquad (x, t) \notin I.$$

We can easily see that if $w(x, t) > \varphi(x)$ for all (x, t) in a backward cone, then the solution u must be equal to w in that cone.

We take u to be equal to w in all the complement of I, and we must now extend u across the boundary of I.

The boundary of I, called *line of influence*, is the graph $t = \tau(x)$ of a Lipschitz-continuous function τ:

$$|\tau(x) - \tau(y)| \leqslant |x - y| \qquad \forall x, y \in \mathbb{R}.$$

Section II is devoted to studying the line of influence, and proving the following properties:

$$w(x, \tau(x)) = \varphi(x) \qquad \text{if} \qquad |\tau'(x)| < 1,$$

$$\frac{\partial w}{\partial t}(x, \tau(x)) \leqslant 0 \qquad \text{if} \qquad w(x, \tau(x)) = \varphi(x).$$

In Section III, we consider the following problem:

$$
\begin{aligned}
&v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}));\\
&\Box v|_{\{(x,t)/t\neq\sigma(x)\}} = 0,\\
&v(x,0) = v_0(x),\\
&\frac{\partial v}{\partial t}(x,0) = v_1(x),\\
&\frac{\partial^+ v}{\partial t}(x,\sigma(x)) = -\frac{\partial^- v}{\partial t}(x,\sigma(x)) \qquad \text{a.e. on } \{x/|\,\sigma'(x)| < 1 \text{ and } \sigma(x) > 0\}.
\end{aligned}
\tag{I.22}
$$

Here σ is a given nonnegative Lipschitz-continuous function, with Lipschitz constant 1, and v_0, v_1 are given, respectively, in $H^1_{\mathrm{loc}}(\mathbb{R})$ and $L^2_{\mathrm{loc}}(\mathbb{R})$.

We can prove that (I.22) possesses a unique solution v given explicitly in terms of σ and w, the free solution with data v_0 and v_1 by

$$v = w + \mathscr{E} * \mu(w),$$

where $\mathscr{E}$ is the elementary solution of the wave equation, and $\mu(w)$ is a measure given by

$$\langle \mu(w), \psi \rangle = -2 \int_{\{x/\tau(x)>0\}} (1 - \tau'(x)^2) \frac{\partial w}{\partial t}(x, \tau(x))\, \psi(x, \tau(x))\, dx,$$

To complete the proof of existence, it remains to check that conditions (I.12) and (I.13) are satisfied by u, solution of (I.22) with $\sigma = \tau$, $u_0 = v_0$ and $u_1 = v_1$. The latter condition is easily verified, but to prove the former, we rely heavily on hypothesis (I.1). We can infer indeed from (I.1) that, once the string has touched the obstacle, it does not touch it any more. From the similar hypothesis (I.6), we can deduce that once the finite string has touched the obstacle, it does not touch it again after a finite time, depending only on the geometry of the obstacle and on the initial total energy; thus we can iterate easily the above procedure and obtain a solution at a finite time after a finite number of steps.

If hypothesis (I.1) were left out, let us see what could happen: Let u_0, u_1, and φ be given by

$$
\begin{aligned}
u_0(x) &= \varphi(x) = -x^2,\\
u_1(x) &= 0.
\end{aligned}
$$

Then the free solution w is

$$w(x,t) = -x^2 - t^2$$

so that $E = I = \mathbb{R} \times \mathbb{R}^+$ and $\tau = 0$. The solution v of (I.22) satisfies

$$
\begin{aligned}
\Box v &= 0 \qquad \text{if} \qquad t > 0,\\
v(x,0) &= v_0(x) = -x^2,\\
\frac{\partial v}{\partial t}(x,0) &= v_1(x) = 0
\end{aligned}
$$

and it is readily computed:

$$v(x, t) = w(x, t) = -x^2 - t^2.$$

We see that condition (I.12) cannot be satisfied for any x and any positive t!

Of course if the initial were nowhere zero, we could build a solution of (I.22) which would be positive in a neighborhood of $\mathbb{R} \times \{0\}$, and define a new line of influence, and perhaps do the same again a certain number of times. But in general nothing guarantees that there will be only a finite number of influence lines in a finite time. The lines of influence could accumulate, and a simple argument shows that in the limit, the velocity would be zero on the limit curve, or more precisely, on the noncharacteristic parts of the limit curve where the obstacle is convex. So we need a more powerful device—which is not yet known —to prove an existence theorem without hypothesis (I.1).

Finally, the reader should note that the existence proof given here is not at all variational. Of course, conditions (I.12) to (I.14) "look variational" but it is definitely difficult to include condition (I.15) in an efficient variational formulation.

I.4. *Comparison with the Results of Amerio and Prouse*

In their first paper [3], Amerio and Prouse constructed a solution taking a rather unusual functional class of initial data. Lately, they could relax their hypothesis, in a still unpublished paper [14]. They consider a constant obstacle ($\varphi' = 0$), and they build a weak solution step by step, following the characteristics, and extending functions across the lines of influence by means of convenient formulas. They do not define a functional class in which they seek a solution of a partial differential problem, in order to have a genuine theorem of uniqueness. The fundamental idea of line of influence originated from Amerio and Prouse's paper [3], and the present work uses this idea to a large extent.

I.6. *Explicit Computation of Examples*

EXAMPLE 1. *Infinite String.* The simplest example one can devise is the following one:

$$u_0(x) = c > 0,$$

$$u_1(x) = -1,$$

$$\varphi(x) = 0.$$

Then

$$w(x, t) = c - t,$$

$$E = I = \mathbb{R}^+ \times [c, +\infty]$$

and the solution of problem P_∞ is given by

$$u(x, t) = c - t \quad \text{if} \quad t \in [0, c],$$
$$u(x, t) = t - c \quad \text{if} \quad t \geqslant c.$$

EXAMPLE 2. *Infinite String.* This example shows how a wave "coming from minus infinity" is reflected against a plane, oblique obstacle. Let us set

$$\begin{aligned} u_0(x) &= x \quad \text{if} \quad x \geqslant 0 \\ &= 0 \quad \text{if} \quad x < 0, \\ u_1(x) &= -1 \quad \text{if} \quad x \geqslant 0 \\ &= 0 \quad \text{if} \quad x < 0, \\ \varphi(x) &= ax \quad \text{with} \quad a \in (0, 1). \end{aligned}$$

Then the free solution is given by

$$w(x, t) = (x - t)^+ = \sup(0, x - t).$$

The set E is defined by

$$E = \overline{\{(x, t)/(x - t)^+ < ax\}} = \{(x, t)/x \geqslant 0 \text{ and } t \geqslant x(1 - a)\}$$

and the set I is

$$I = \{(x, t)/t \geqslant \max(-x, (1 - a)\, x)\}.$$

Then the solution of problem P_∞ is given by

$$\begin{aligned} u(x, t) &= (x - t)^+ \quad \text{if} \quad t \leqslant \max(-x, (1 - a)\, x), \\ u(x, t) &= (2a - 1)\, x + t \quad \text{if} \quad (1 - a)\, x \leqslant t \leqslant x, \\ u(x, t) &= a(x + t) \quad \text{if} \quad t \geqslant |\, x \,|\, . \end{aligned}$$

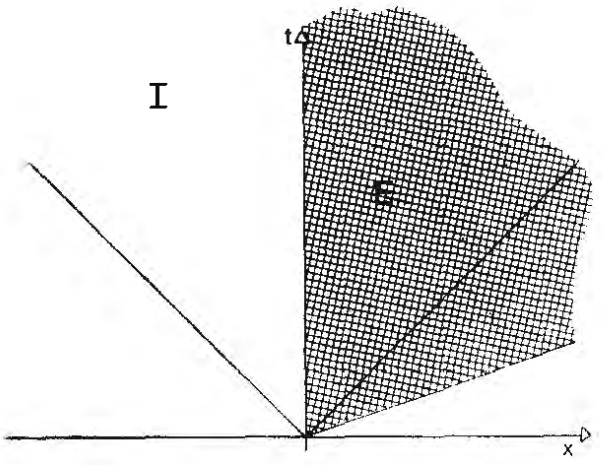

FIG. 2. The sets E and I of Example 2.

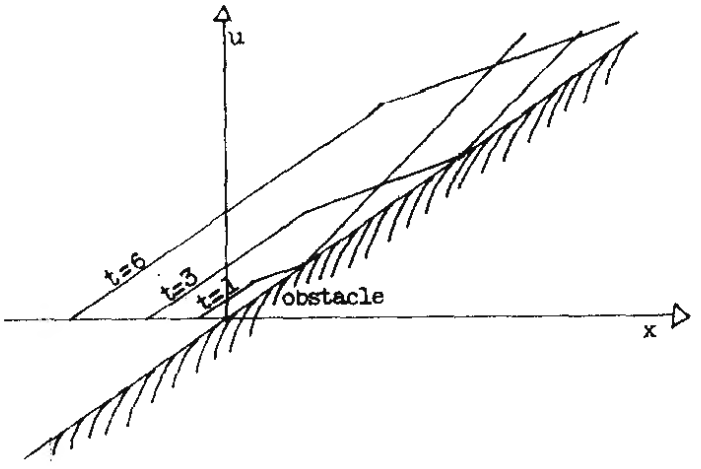

FIG. 3. Evolution of function u of Example 2.

EXAMPLE 3. *Finite String.* Given a string of unit length and data

$$u_0(x) = 0,$$
$$u_1(x) = -\pi \cos \pi x,$$
$$\varphi(x) = -\tfrac{1}{2},$$

we can compute explicitly the solution of P_f for any time t.

The initial data correspond to the free solution

$$w(x, t) = -\sin \pi t \sin \pi x,$$

so that

$$E = \{(x, t)/-\sin \pi t \sin \pi x < -\tfrac{1}{2}\}$$

and the (first) line of influence is given by

$$\tau(x) = \tfrac{1}{2} - x \qquad \text{if} \qquad 0 \leqslant x \leqslant \tfrac{1}{4},$$
$$\tau(x) = (1/\pi) \operatorname{arc\,sin}(1/(2 \sin \pi x)) \qquad \text{if} \qquad \tfrac{1}{4} \leqslant x \leqslant \tfrac{3}{4},$$
$$\tau(x) = -\tfrac{1}{2} + x \qquad \text{if} \qquad \tfrac{3}{4} \leqslant x \leqslant 1$$

(see Fig. 4).

In the set where $t \leqslant \tau(x)$, $u = w$; the velocity is reversed in those points where

$$t = \tau(x) \qquad \text{and} \qquad \tfrac{1}{4} < x < \tfrac{3}{4}.$$

Therefore, in the curvilinear triangle

$$\tau(x) \leqslant t \leqslant \tfrac{1}{2} - |\, x - \tfrac{1}{2}\,|$$

we have

$$u = -1 - w.$$

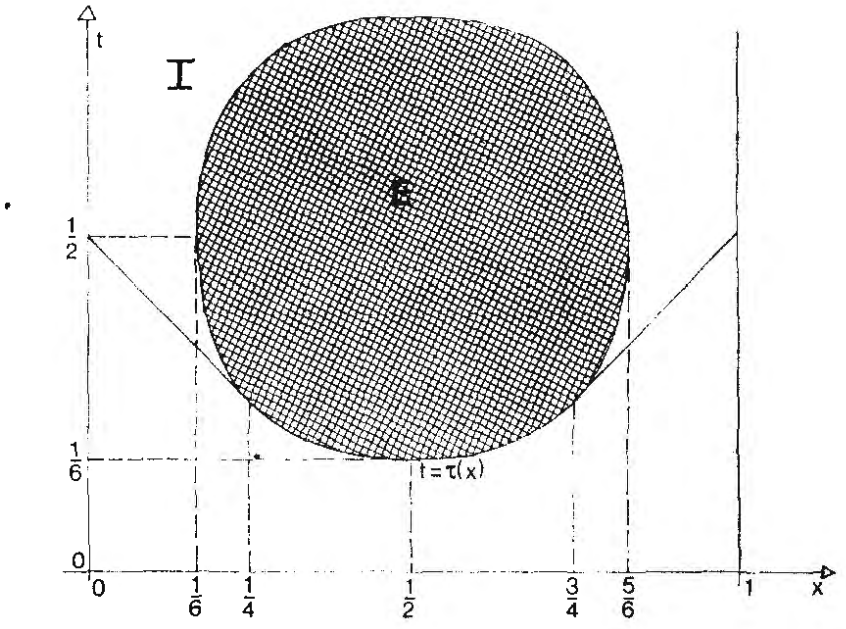

FIG. 4. The sets E and I of Example 3.

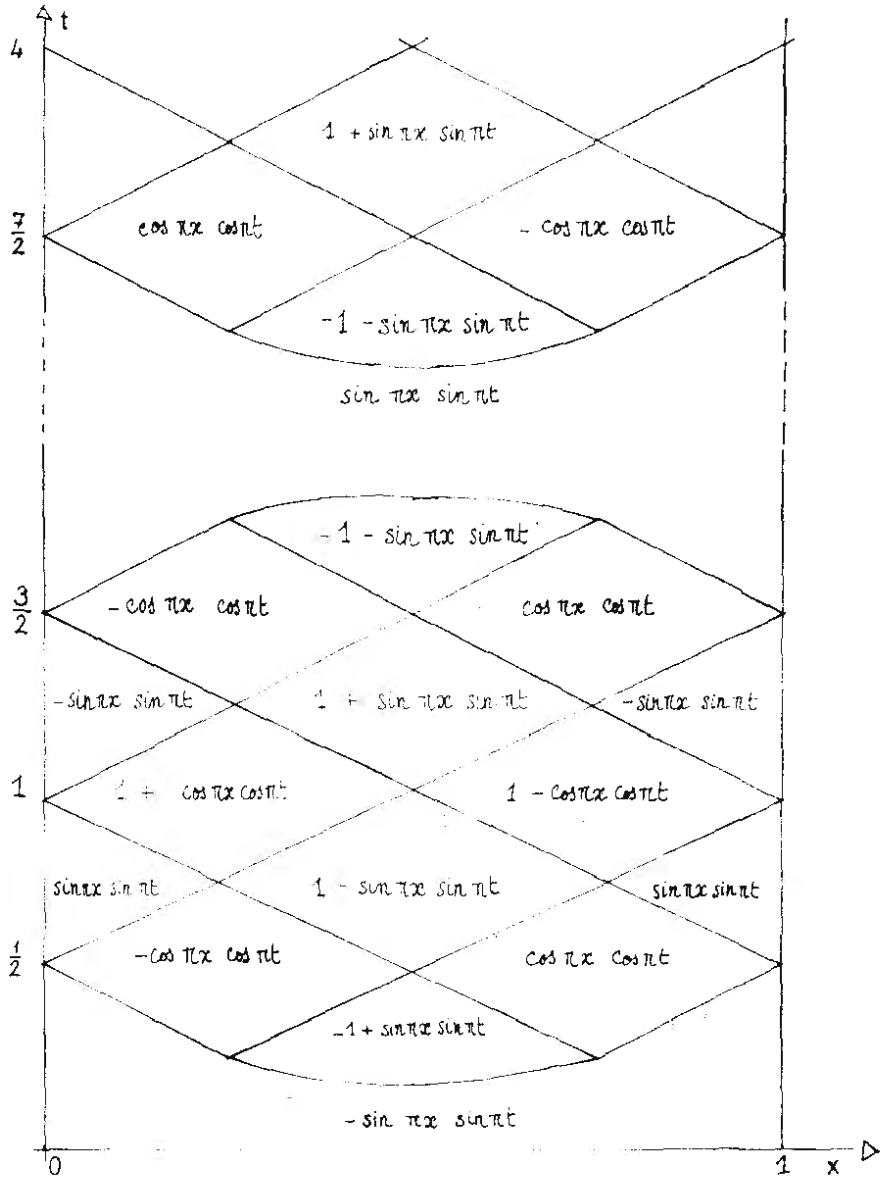

FIG. 5. Values of function u of Example 3 in the different regions of the x, t plane.

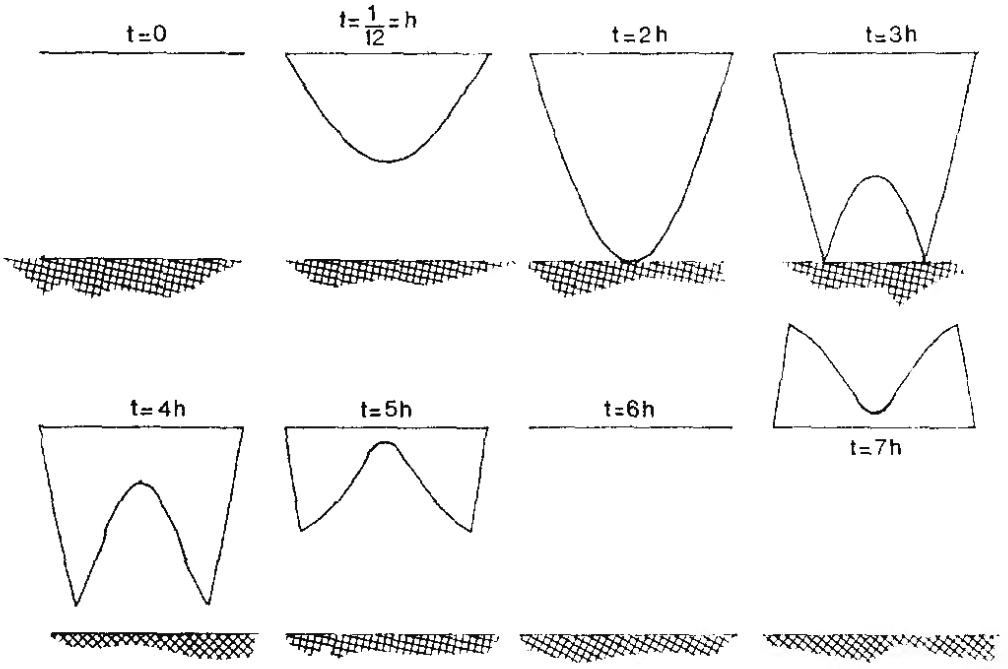

FIG. 6. Evolution of function u of Example 3.

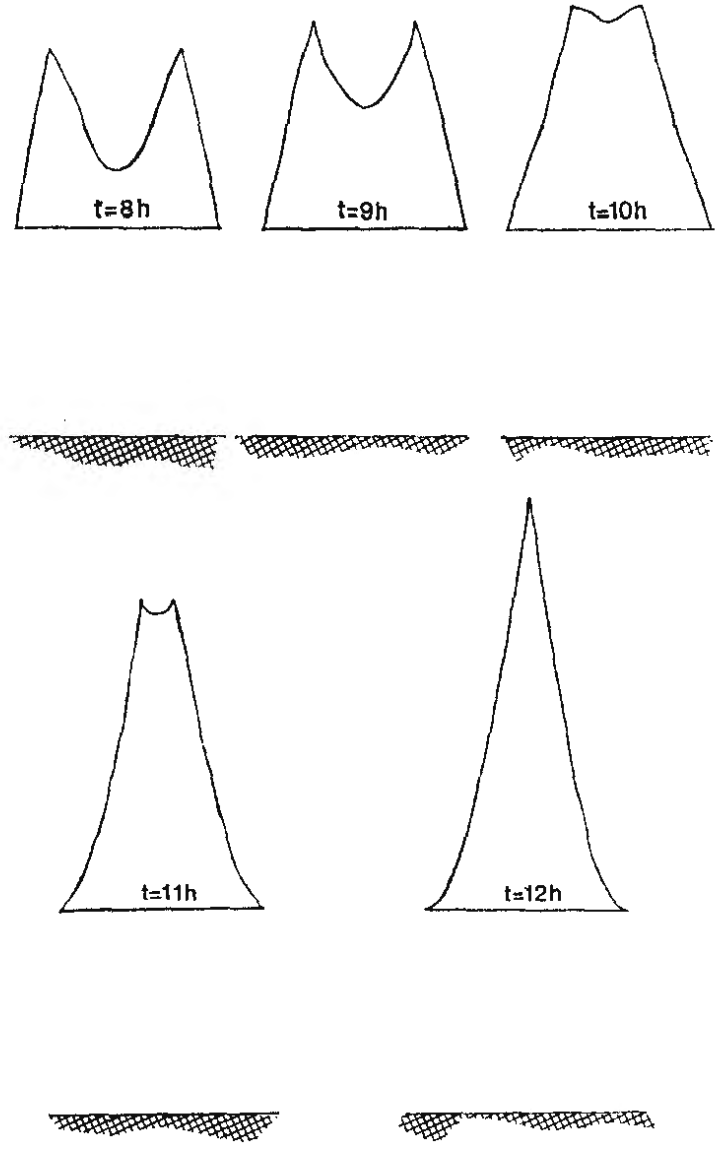

FIG. 7. Evolution of function u of Example 3 (continued).

Next, we compute u as a free solution of the wave equation, with data on the characteristic segments

$$t = \tfrac{1}{2} - |x - \tfrac{1}{2}| \qquad \text{if} \qquad |x - \tfrac{1}{2}| < \tfrac{1}{4},$$

$$t = |\tfrac{1}{2} - x| \qquad \text{if} \qquad |x - \tfrac{1}{2}| \geqslant \tfrac{1}{4},$$

as long as there is no new influence of the obstacle.

The value of u is given, according to the regions, in Fig. 5.

There is a new influence of the obstable beyond $t = \frac{7}{4}$. More precisely, we shall have the influence line:

$$\begin{aligned} \tau_1(x) &= 2 - \tau(x) \qquad \text{if} \qquad \tfrac{1}{4} \leqslant x \leqslant \tfrac{3}{4} \\ &= 2 - x \qquad \text{if} \qquad 0 \leqslant x \leqslant \tfrac{1}{4} \\ &= 1 + x \qquad \text{if} \qquad \tfrac{3}{4} \leqslant x \leqslant 1. \end{aligned}$$

We can see that here, the solution is periodic in time, with period 3, and moreover

$$u(2 - t) = u(t).$$

Figures 6 and 7 show the evolution of u for t comprised between 0 and 1, with a time step of 1/12. The computations were carried out explicitly using Fig. 5.

II. The Line of Influence and its Properties

II.1. *Definition and First Properties of the Domain of Influence and of the Line of Influence*

Let us recall that the backward cone (or cone of dependence) of a point (x, t) is the set

$$T^-_{x,t} = \{(x', t') \in \mathbb{R} \times \mathbb{R}^+ / 0 \leqslant t' \leqslant t - |x - x'|\} \tag{II.1}$$

and that the forward cone (or cone of influence) of a point (x, t) is the set

$$T^+_{x,t} = \{(x', t') \in \mathbb{R} \times \mathbb{R}^+ / t' \geqslant t + |x - x'|\}. \tag{II.2}$$

A fundamental result of the theory of hyperbolic equations is the following: Let v be a solution of the wave equation

$$\Box v = 0,$$

$$v(x, 0) = v_0(x),$$

$$\frac{\partial v}{\partial t}(x, 0) = v_1(x).$$

If v_0 and v_1 have their support in C, then the support of v is included in

$$\bigcup_{x \in C} T^+_{x,0} .$$

Let us define

$$E = \overline{\{(x, t) \in \mathbb{R} \times \mathbb{R}^+ / w(x, t) < \varphi(x)\}}. \tag{II.3}$$

One must not mistake E for the set

$$\{(x, t) \in \mathbb{R} \times \mathbb{R}^+ / w(x, t) \leqslant \varphi(x)\} \neq E.$$

We shall suppose that E is not empty; if it were, then the function w would be a solution of P_∞ and the existence problem would be solved.

We now proceed to define the domain of influence and the line of influence.

DEFINITION II.1. The domain of influence of the obstacle φ with respect to w, for t nonnegative is the set

$$I = \bigcup_{(x,t) \in E} T^+_{x,t} . \tag{II.4}$$

DEFINITION II.2. The line of influence of the obstacle φ with respect to w, for t nonnegative, or more simply, the line of influence, is the boundary of the set I.

The reader is referred to Fig. 8, to visualize the sets I, E, and the line of influence.

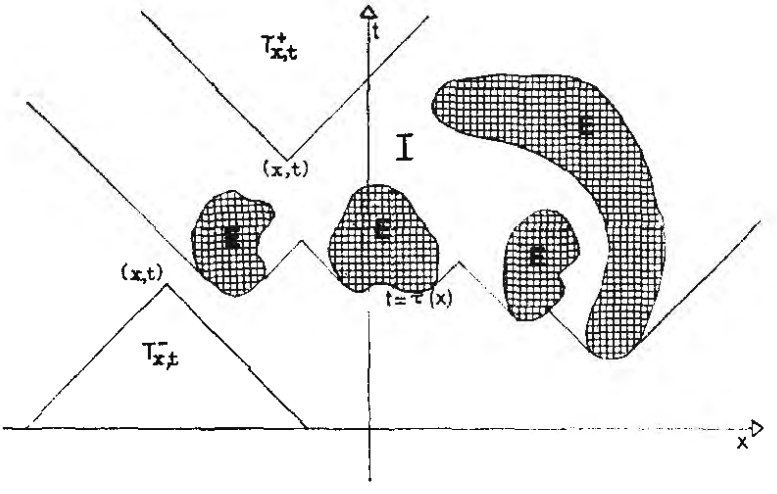

FIG. 8. The sets $T^+_{x,t}$ $T^-_{x,t}$, E, and I; the line and the domain of influence.

The names "domain in influence" and "line of influence" are derived from the fact that outside the domain of influence, the obstacle does not alter the problem, and inside the domain of influence it does. These assertions will be a result of the existence and uniqueness theorem.

The boundary of I has the following properties.

PROPOSITION II.3. *The line of influence is the graph of a Lipschitz-continuous function* τ, *with Lipschitz constant* 1,

$$I = \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ / t \geqslant \tau(x)\} \quad .$$

and therefore, I is closed.

Proof. Define on $\mathbb{R}$ a function τ by

$$\tau(x) = \inf\{t' + | x - x' | / (x', t') \in E\}. \tag{II.5}$$

Obviously if (x, t) is in I, then

$$t \geqslant \tau(x).$$

Conversely, let t be greater than or equal to $\tau(x)$. Then, for any positive ϵ, there exists $(x_\epsilon , t_\epsilon)$ in E such that

$$t \geqslant t_\epsilon + | x - x_\epsilon | - \epsilon \tag{II.6}$$

and therefore

$$| x_\epsilon | + t_\epsilon \leqslant t + | x | + \epsilon.$$

Extract from the bounded sequence $(x_\epsilon , t_\epsilon)_\epsilon$ a convergent subsequence still denoted by $(x_\epsilon , t_\epsilon)_\epsilon$. As E is closed, we shall have

$$(x_\epsilon , t_\epsilon) \to (x_0 , t_0) \in E,$$

and using (II.6)

$$t \geqslant t_0 + | x - x_0 | .$$

This shows that

$$\tau(x) = \min\{t' + | x - x' | / (x', t') \in E\} \tag{II.7}$$

and that

$$I = \{(x, t) / t \geqslant \tau(x)\}.$$

Let us show now that τ is Lipschitz-continuous, with Lipschitz constant 1. Given x, there exists (x_0 , t_0) in E such that

$$\tau(x) = t_0 + | x - x_0 | . \tag{II.8}$$

On the other hand, for any $\hat{x}$, the definition of τ implies that

$$\tau(\hat{x}) \leqslant t_0 + | \hat{x} - x_0 | . \tag{II.9}$$

We deduce from (II.8) and (II.9)

$$\tau(\hat{x}) - \tau(x) \leqslant t_0 + | \hat{x} - x_0 | - t_0 - | x - x_0 | \leqslant | x - \hat{x} | .$$

By exchanging the roles of x and $\hat{x}$, we obtain proposition II.4. ∎

II.2. *Study of the Values of the Free Solution and of Its Derivatives on the Line of Influence*

LEMMA II.5. *The following implication holds:*

$$| \tau'(x)| < 1 \Rightarrow w(x, \tau(x)) = \varphi(x). \tag{II.10}$$

Proof (by contradiction). Suppose $| \tau'(x_0)| < 1$ and

$$w(x_0 , \tau(x_0)) > \varphi(x_0); \qquad \text{i.e.,} \qquad (x_0 , \tau(x_0)) \notin E.$$

By continuity of w and φ, there exists a positive number r, such that

$$| x - x_0 |^2 + | t - \tau(x_0)|^2 \leqslant r^2 \Rightarrow (x, t) \notin E.$$

In particular, we shall have

$$| x - x_0 | \leqslant \frac{r}{2^{1/2}} \Rightarrow (x, \tau(x)) \notin E. \tag{II.11}$$

There exists (x', t') in E, by virtue of (II.7) such that

$$\tau(x_0) = t' + | x' - x_0 |$$

and, thanks to (II.11),

$$x_0 \neq x'.$$

Suppose, for instance, $x' > x_0$. Then, if $x_0 < x < x_0 + r/2^{1/2}$,

$$\begin{aligned}
\tau(x) &= (\tau(x) - \tau(x_0)) + \tau(x_0) \\
&= \tau'(x_0)\,(x - x_0) + o(x - x_0) + t' + x' - x_0 \\
&= t' + x' - x + (x - x_0)\,(1 + \tau'(x_0)) + o(x - x_0) \\
&\geqslant \tau(x') + x' - x + (x - x_0)\,(1 + \tau'(x_0)) + o(x - x_0).
\end{aligned}$$

We deduce from this last inequality that, for an x near enough to x_0,

$$\tau(x) > \tau(x') + x' - x \tag{II.12}$$

by the hypothesis $|\tau'(x_0)| < 1$.

On the other hand, the Lipschitz constant of τ is equal to 1, which contradicts (II.12). ∎

Our knowledge of the influence line is improved by the following result.

LEMMA II.6. *Let* $U = \{x/w(x, \tau(x)) > \varphi(x)\}$, *and let* (a, b) *be any connected component of* U. *Then*

$$\tau(x) = \min(\tau(a) + x - a, \tau(b) + b - x), \qquad \forall x \in [a, b]. \tag{II.13}$$

Proof. Suppose first that $(a, \tau(a))$ and (b, $\tau(b)$) both lie on one characteristic segment. For instance,

$$\tau(b) - b = \tau(a) - a. \tag{II.14}$$

By Proposition II.4, we know that

$$\tau(x) - x \leqslant \tau(a) - a, \tag{II.15}$$

$$\tau(b) - b \leqslant \tau(x) - x, \tag{II.16}$$

if $a \leqslant x \leqslant b$.

Adding inequalities (II.15) and (II.16), and comparing the result to (II.14), we obtain easily

$$\tau(x) - x = \tau(a) - a = \tau(b) - b,$$

i.e., relation (II.13).

Suppose next that $(a, \tau(a))$ and $(b, \tau(b))$ do not both lie on one characteristic segment, and set

$$\tilde{\tau}(x) = \min(\tau(a) + x - a, \tau(b) + b - x). \tag{II.17}$$

Clearly, on $[a, b]$, we must have

$$\tilde{\tau}(x) \geqslant \tau(x).$$

Suppose there exists $x_0 \in (a, b)$ such that

$$\tilde{\tau}(x_0) > \tau(x_0). \tag{II.18}$$

Therefore, by (II.7), there exists (x_1, t_1) in E such that

$$\tau(x_0) = t_1 + |x_0 - x_1|. \tag{II.19}$$

As the Lipschitz constant of τ is 1, the following inequality holds:

$$t_1 \geqslant \tau(x_1) \geqslant \tau(x_0) - | x_0 - x_1 | .$$

Hence

$$t_1 = \tau(x_1), \tag{II.20}$$

and, by definition of E

$$w(x_1 , \tau(x_1)) = \varphi(x_1). \tag{II.21}$$

The number x_1 cannot be equal to a or b. Suppose, for example, that $x_1 \leqslant x_0$. Then, write

$$\begin{aligned} \tau(a) - (x_1 - a) &\leqslant \tau(x_1) \\ &= \tau(x_0) + x_1 - x_0 \\ &< \tilde{\tau}(x_0) + x_1 - x_0 \\ &\leqslant \tau(a) + x_0 - a + x_1 - x_0 \\ &= \tau(a) + x_1 - a, \end{aligned}$$

i.e.,

$$x_1 - a > - | x_1 - a | ,$$

which implies

$$x_1 > a.$$

We could show in a similar way that

$$x_1 < b.$$

Relation (II.21) contradicts the hypothesis $w(x, \tau(x)) > \varphi(x)$, therefore relation (II.18) cannot be fulfilled. ∎

Lemma II.6 shows that on a connected component of U, the line of influence is made either of a characteristic segment (of slope $+1$ or -1) or of a characteristic segment of slope $+1$, followed by a characteristic segment of slope -1.

We introduce now the characteristic coordinates

$$\begin{aligned} \xi &= \frac{x + t}{2^{1/2}} \\ \eta &= \frac{-x + t}{2^{1/2}} \end{aligned} \tag{II.22}$$

whence

$$\begin{aligned} x &= \frac{\xi - \eta}{2^{1/2}}, \\ t &= \frac{\xi + \eta}{2^{1/2}}, \end{aligned} \tag{II.23}$$

and therefore

$$\frac{\partial}{\partial \xi} = \frac{1}{2^{1/2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right),$$
$$\frac{\partial}{\partial \eta} = \frac{1}{2^{1/2}}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right), \tag{II.24}$$

$$\frac{\partial}{\partial x} = \frac{1}{2^{1/2}}\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right),$$
$$\frac{\partial}{\partial t} = \frac{1}{2^{1/2}}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right), \tag{II.25}$$

LEMMA II.7. *Let x be such that*

$$w(x, \tau(x)) = \varphi(x).$$

If moreover $(\partial w/\partial \xi)(x, \tau(x))$ exists (respectively $(\partial w/\partial \eta)(x, \tau(x))$ exists), then the following inequality holds:

$$\frac{\partial w}{\partial \xi}(x, \tau(x)) \leqslant \frac{\varphi'(x^+)}{2^{1/2}} \tag{II.26}$$

(respectively

$$\frac{\partial w}{\partial \eta}(x, \tau(x)) \leqslant -\left(\frac{\varphi'(x^-)}{2^{1/2}}\right). \tag{II.27}$$

If $(\partial w/\partial t)(x, \tau(x))$ exists, then

$$\frac{\partial w}{\partial t}(x, \tau(x)) \leqslant 0. \tag{II.28}$$

Proof. By hypothesis

$$w(x, \tau(x)) = \varphi(x) \tag{II.29}$$

and, by definition of τ,

$$w(x + h, \tau(x) + h) \leqslant \varphi(x + h), \qquad \forall h \leqslant 0. \tag{II.30}$$

Subtracting (II.29) from (II.30) and passing to the limit, (II.26) obtains. The other two inequalities are similarly proved. ∎

As a result of Corollary A.2 note that

(II.26) holds almost everywhere on $M_\tau^+ = \{x/\tau'(x) > -1\}$,

(II.27) holds almost everywhere on $M_\tau^- = \{x/\tau'(x) < 1\}$,

and

(II.28) holds almost everywhere on $M_\tau = \{x/|\ \tau'(x)| < 1\}$.

Under the hypothesis that E is nonvoid, M_τ (and therefore M_τ^+ and M_τ^-) is not null (with respect to the Lebesgue measure); this is a result of Theorems IV.2 and III.1 below; see the end of subsection IV.1.

III. Solution of a Linear Auxiliary Problem

III.1. *Statement of the Result, Idea of the Proof*

In this section, we study a linear problem, whose solution will be a "candidate" for a solution of the nonlinear problem P_∞.

Theorem III.1. *Let σ be a nonnegative Lipschitz-continuous function with Lipschitz constant* 1, *and let v_0 be in $H^1_{\text{loc}}(\mathbb{R})$, v_1 in $L^2_{\text{loc}}(\mathbb{R})$. Then there exists a unique function u such that*

$$v \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\mathbb{R})) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; H^1_{\text{loc}}(\mathbb{R})), \tag{III.1}$$

$$\Box v\,|_{\{(x,t)/t\neq\sigma(x),\,t>0\}} = 0 \qquad (\textit{in the sense of distributions}), \tag{III.2}$$

$$v(x, 0) = v_0(x), \tag{III.3}$$

$$\frac{\partial v}{\partial t}(x, 0) = v_1(x), \tag{III.4}$$

$$\frac{\partial^+ v}{\partial t}(x, \sigma(x)) = -\frac{\partial^- v}{\partial t}(x, \sigma(x)) \quad \text{a.e. } \textit{on } \{x/\sigma(x) > 0 \textit{ and } |\sigma'(x)| < 1\}. \tag{III.5}$$

Condition (III.1) does not imply that Conditions (III.4) and (III.5) make sense. Thus, the first stage is to show (Lemma III.2) that, with the help of (III.2), Conditions (III.4) and (III.5) do make sense.

Next we shall suppose that there exists a solution v of (III.1) to (III.5), and we shall compute $\Box v$ in the open set $\mathbb{R} \times \mathbb{R}_*^+$. We shall find an explicit formula for $\mu(w) = \Box v$ in terms of σ and w, the free solution of the wave equation with initial data v_0 and v_1 (Proposition III.3).

Then, it will only remain to show that $w + \mathscr{E} * \mu(w)$ is a solution of (III.1)–(III.5), for any σ, v_0, and v_1; moreover $w + \mathscr{E} * \mu(w)$ fulfills condition (I.15) (Proposition III.6). Here $\mathscr{E}$ is the elementary solution of the wave equation.

III.2. *Computation of $\Box v$ in Terms of w and σ*

Lemma III.2. *Let v satisfy conditions* (III.1) *and* (III.2). *Then $(\partial v/\partial t)(x, 0)$ is defined almost everywhere on $\mathbb{R}$, and $(\partial^+ v/\partial t)(x, \sigma(x))$ and $(\partial^- v/\partial t)(x, \sigma(x))$ are defined almost everywhere on the set $\{x/\sigma(x) > 0$ and $|\sigma'(x)| < 1\}$.*

Proof. Let V be a connected component of $\{(x, t)/t > 0,\ t \neq \sigma(x)\}$. On any rectangle R included in V, with characteristic sides, and vertices $(x + h, t + k)$, $(x + k, x + h)$ $(x - h, t - k)$, and $(x - k, x - h)$, a solution v of the wave equation is of the form

$$v(x, t) = f(x + t) + g(x - t);$$

moreover, $f \in H^1(x + t - |h + k|, x + t + |h + k|)$ and $g \in H^1(x - t - |h - k|, x + t + |h - k|)$, by (III.1).

The functions f and g are well defined save for an additive constant.

We can join any points (x, t) and, $(\hat{x}, \hat{t})$ by a continuous arc in V, and this continuous arc can be covered by a finite number of open rectangles R_i with characteristic sides included in V. On each of these rectangles R_i we have

$$v(x, t) = f_i(x + t) + g_i(x - t)$$

with $f_i \in H^1(x_i + t_i - |h_i + k_i|, x_i + t_i + |h_i + k_i|)$ and $g_i \in H^1(x_i - t_i - |h_i - k_i|, x_i - t_i + |h_i - k_i|)$. Obviously

$$(x_i + t_i - |h_i + k_i|, x_i + t_i + |h_i + k_i|) \cap (x_{i+1} + t_{i+1} - |h_{i+1} + k_{i+1}|, x_{i+1} + t_{i+1} + |h_{i+1} + k_{i+1}|) \neq \varnothing.$$

Therefore, we can choose the additive constant in such a way that $f_i = f_{i+1}$ on the intersection of these two intervals.

Thus, because V is arcwise connected, we can define, save for an additive constant, two functions f and g such that

$$f \in H^1_{\text{loc}}(\inf(x + t/(x, t) \in V), \sup(x, t/(x, t) \in V)),$$

$$g \in H^1_{\text{loc}}(\inf(x - t/(x, t) \in V), \sup(x, t/(x, t) \in V)),$$

$$v(x, t) = f(x + t) + g(x - t), \qquad \forall (x, t) \in V.$$

More specifically, if $V = \{(x, t)/t > \sigma(x)\}$, which is, of course, connected, we know from Corollary A.3, that almost everywhere on $\{x/|\sigma'(x)| < 1\}$, $f'(x + \sigma(x))$ and $g'(x - \sigma(x))$ are defined. Then, in such a point

$$\lim_{h \downarrow 0} \frac{v(x, \sigma(x) + h) - v(x, \sigma(x))}{h}$$

$$= \lim_{h \downarrow 0} \frac{f(x + \sigma(x) + h) - f(x + \sigma(x))}{h} + \lim_{h \downarrow 0} \frac{f(x - \sigma(x) - h) - f(x - \sigma(x))}{h}$$

exists, and defines the right time-derivative of v at $(x, \sigma(x))$. We argue similarly for $(\partial^- v/\partial t)(x, \sigma(x))$ and $(\partial v/\partial t)(x, 0)$. ∎

We can now compute $\Box v$ if v is a solution of problem (III.1)–(III.5).

PROPOSITION III.3. *Let v be a solution of* (III.1)–(III.5), *then $\Box v$ is a measure $\mu(w)$ defined by*

$$\langle \mu(w), \psi \rangle = -2 \int_{\sigma(x)>0} (1 - \sigma'(x)^2)\, \psi(x, \sigma(x))\, \frac{\partial w}{\partial t}(x, \sigma(x))\, dx. \tag{III.6}$$

Proof. To avoid notational ambiguities we shall write when f is a function of x and t

$$\tilde{f}(\xi, \eta) = f\left(\frac{\xi - \eta}{2^{1/2}}, \frac{\xi + \eta}{2^{1/2}}\right), \tag{III.7}$$

whence

$$f(x, t) = \tilde{f}\left(\frac{x + t}{2^{1/2}}, \frac{-x + t}{2^{1/2}}\right). \tag{III.8}$$

Let ψ be an infinitely differentiable test function with compact support in the open half-plane. We shall suppose that this support meets only one component V of $\{(x, t)/0 < t < \sigma(x)\}$. Then

$$\langle \Box v, \psi \rangle = \langle \Box \tilde{v}, \tilde{\psi} \rangle = \langle 2\tilde{v}_{\xi\eta}, \tilde{\psi} \rangle = -2\langle \tilde{v}_{\xi}, \tilde{\psi}_{\eta} \rangle. \tag{III.9}$$

Let us denote by Y the graph of σ in characteristic coordinates

$$\eta \in Y(\xi) \Leftrightarrow \frac{\xi + \eta}{2^{1/2}} = \sigma\left(\frac{\xi - \eta}{2^{1/2}}\right). \tag{III.10}$$

By Lemma A.3, Y is a decreasing graph.

$\overline{Y}$ is the function defined on

$$\{\xi / Y(\xi) \text{ is } \textit{one-valued}\}$$

with values $Y(\xi)$ on this set.

Let

$$v(x, t) = f(x + t) + g(x - t) \qquad \text{on} \qquad V, \tag{III.11}$$

$$v(x, t) = f_1(x + t) + g_1(x - t) \qquad \text{on} \qquad \{(x, t)/t > \sigma(x)\}. \tag{III.12}$$

Then using (III.10) and (III.11),

$$\langle \tilde{v}_{\xi}, \tilde{\psi}_{\eta} \rangle = \int_{\eta \leqslant \overline{Y}(\xi)} 2^{1/2} f'(\xi\, (2)^{1/2})\, \tilde{\psi}_{\eta}(\xi, \eta)\, d\xi\, d\eta$$

$$+ \int_{\eta > \overline{Y}(\xi)} 2^{1/2} f_1'(\xi\, (2)^{1/2})\, \tilde{\psi}_{\eta}(\xi, \eta)\, d\xi\, d\eta,$$

from which, after having integrated in η and substituting $\xi = (x + \sigma(x))/2^{1/2}$, we obtain

$$\langle \Box v, \psi \rangle = -2 \int [f'(x + \sigma(x)) - f_1'(x + \sigma(x))]\, \psi(x, \sigma(x))\, (1 + \sigma'(x))\, dx.$$

As u is continuous,

$$f(x + \sigma(x)) + g(x - \sigma(x)) = f_1(x + \sigma(x)) + g_1(x - \sigma(x)),$$

which we differentiate; according to Corollary A.2,

$$\begin{aligned}(1 + \sigma'(x))\, f'(x + \sigma(x)) + (1 - \sigma'(x))\, g'(x - \sigma(x)) \\ = (1 + \sigma'(x))\, f_1'(x + \sigma(x)) + (1 - \sigma'(x))\, g_1'(x - \sigma(x)) \qquad \text{a.e.}\end{aligned} \tag{III.14}$$

On the other hand, (III.5) is written:

$$\begin{aligned}f'(x + \sigma(x)) - g'(x - \sigma(x)) = -f_1'(x + \sigma(x)) + g_1'(x - \sigma(x)), \\ \text{a.e. on } \{x/|\, \sigma'(x)| < 1 \text{ and } \sigma(x) > 0\}.\end{aligned} \tag{III.15}$$

By linear combination of (III.14) and (III.15), we deduce

$$\begin{aligned}f_1'(x + \sigma(x)) = \sigma'(x)\, f'(x + \sigma(x)) + (1 - \sigma'(x))\, g'(x - \sigma(x)) \\ \text{a.e. on } \{x/|\, \sigma'(x)| < 1 \text{ and } \sigma(x) > 0\}\end{aligned} \tag{III.16}$$

and from (III.14)

$$f_1'(x + \sigma(x)) = f'(x + \sigma(x)) \qquad \text{a.e. on } \{x/\sigma'(x) = 1 \text{ and } \sigma(x) > 0\}. \tag{III.17}$$

Therefore

$$\begin{aligned}f'(x + \sigma(x)) - f_1'(x + \sigma(x)) = (f'(x + \sigma(x)) - g'(x - \sigma(x)))\, (1 - \sigma'(x)) \\ \text{a.e. on } \{x/\sigma'(x) > -1 \text{ and } \sigma(x) > 0\}.\end{aligned} \tag{III.18}$$

Carrying (III.18) into (III.13), we obtain (III.6). ∎

III.3. *Partial Results of Existence*

Let us recall that the elementary solution $\mathscr{E}$ of the wave equation in one space dimension is

$$\begin{aligned}\mathscr{E}(x, t) &= \tfrac{1}{2} \qquad \text{if} \qquad t \geqslant |\, x\, |\,, \\ \mathscr{E}(x, t) &= 0 \qquad \text{elsewhere.}\end{aligned} \tag{III.19}$$

We shall study in this section the distribution

$$v = w + \mathscr{E} * \mu(w), \tag{III.20}$$

where the distribution $\mu(w)$ is defined by (III.6).

LEMMA III.4. *Let σ be a Lipschitz-continuous function, with Lipschitz constant 1, and let v_0, v_1 be given, respectively, in $H^1_{\mathrm{loc}}(\mathbb{R})$ and $L^2_{\mathrm{loc}}(\mathbb{R})$. Then the linear form*

$$\psi \mapsto -2 \int \psi(x, \sigma(x)) \frac{\partial w}{\partial t}(x, \sigma(x)) (1 - \sigma'^2(x))\, dx$$

defined for all ψ in $\mathcal{D}(\mathbb{R} \times \mathbb{R}^+_*)$, is a measure. (Here w is the free solution of the wave equation with initial data v_0, v_1).

Proof. It is enough to show that $x \mapsto (\partial w/\partial t)(x, \sigma(x))(1 - \sigma'^2(x))$ is locally integrable. Classically

$$w(x, t) = f(x + t) + g(x - t)$$

with f and g in $H^1_{\mathrm{loc}}(\mathbb{R})$. Therefore

$$\left| \frac{\partial w}{\partial t}(x, \sigma(x)) (1 - \sigma'(x)^2) \right|$$
$$\leqslant 2 \left| f'(x + \sigma(x)) (1 + \sigma'(x)) \right| + 2 \left| g'(x - \sigma(x)) (1 - \sigma'(x)) \right|$$

and by Lemma A.1, the functions $x \mapsto (1 + \sigma'(x)) f'(x + \sigma(x))$ and $x \mapsto (1 - \sigma'(x)) g'(x - \sigma(x))$ are locally integrable. ∎

The convolution of this measure $\mu(w)$ with $\mathscr{E}$ can be defined thanks to the support condition of Schwartz [13, p. 170].

According to Lemma A.5, $\mu(w) * \mathscr{E}$ is a continuous function, because the parts of characteristics are null with respect to the measure μ.

Obviously, the function v defined by (III.20) satisfies conditions (III.2) and (III.3).

Condition (III.4) is satisfied almost everywhere on $\{x/\sigma(x) > 0\}$; it is also satisfied almost everywhere on $\{x/\sigma(x) = 0\}$ as will be proved by next lemma.

LEMMA III.5. *Let A be the set $\{x/\sigma(x) > 0\}$, and let B be the set of right and left Lebesgue points of the function*

$$x \mapsto (1 - \sigma'^2(x)) \frac{\partial w}{\partial t}(x, \sigma(x))\, 1_A(x).$$

Then, if the point x is in B, if $(\partial w/\partial t)(x, \sigma(x))$ exists, if $\sigma'(x)$ exists, and if $|\sigma'(x)| < 1$, then $(\partial^+ v/\partial t)(x, \sigma(x))$ exists.

Moreover, if $\sigma(x) > 0$, then

$$\frac{\partial^+ v}{\partial t}(x, \sigma(x)) = -\frac{\partial w}{\partial t}(x, \sigma(x)). \tag{III.21}$$

If $\sigma(x) = 0$, *then*

$$\frac{\partial^+ v}{\partial t}(x, \sigma(x)) = 0. \tag{III.22}$$

Proof. We can write formula (III.20) in the form

$$v(x, t) = w(x, t) + \tfrac{1}{2} \int_{T_{x,t}^-} d\mu(w).$$

Suppose t is greater than $\tau(x)$, and let $a(x, t)$ be the greatest number such that

$$a(x, t) - \sigma(a(x, t)) = x - t,$$

and let a' be any number such that

$$a' - \sigma(a') = x - t.$$

Then $\sigma'(s) = 1$ on $(a', a(x, t))$ and

$$\int_{a'}^{a(x,t)} 1_A(x') \frac{\partial w}{\partial t}(x', \sigma(x'))\,(1 - \sigma'(x')^2)\,dx' = 0.$$

If, similarly, $b(x, t)$ is the smallest number such that

$$b(x, t) + \sigma(b(x, t)) = x + t$$

then we may write, if t is greater than $\sigma(x)$,

$$v(x, t) = w(x, t) - \int_{a(x,t)}^{b(x,t)} 1_A(x') \frac{\partial w}{\partial t}(x', \sigma(x'))\,(1 - \sigma'(x')^2)\,dx. \tag{III.23}$$

For simplicity, set $a(x, \sigma(x) + h) = a(h)$ and $b(x, \sigma(x) + h) = b(h)$. From the hypothesis $|\sigma'(x)| < 1$, we deduce

$$\lim_{h \downarrow 0} \frac{x - a(h)}{h} = \frac{1}{1 - \sigma'(x)}, \tag{III.24}$$

$$\lim_{h \downarrow 0} \frac{b(h) - x}{h} = \frac{1}{1 + \sigma'(x)}. \tag{III.25}$$

Using (III.23), we may write

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{v(x, \sigma(x) + h) - v(x, \sigma(x))}{h} \\
&\quad = \lim_{h \downarrow 0} \frac{w(x, \sigma(x) + h) - w(x, \sigma(x))}{h} \\
&\qquad - \lim_{h \downarrow 0} \left\{ \frac{x - a(h)}{h} \cdot \frac{1}{x - a(h)} \int_{a(h)}^{x} 1_A(x') \frac{\partial u}{\partial t}(x', \sigma(x'))\,(1 - \sigma'^2(x'))\,dx' \right. \\
&\qquad \left. + \frac{b(h) - x}{h} \cdot \frac{1}{b(h) - x} \int_{x}^{b(h)} 1_A(x') \frac{\partial w}{\partial t}(x', \sigma(x'))\,(1 - \sigma'^2(x'))\,dx' \right\}
\end{aligned}$$

and by the hypotheses we made, $(\partial^+ u/\partial t)(x, \sigma(x))$ exists, and

$$\frac{\partial^+ v}{\partial t}(x, \sigma(x)) = \frac{\partial w}{\partial t}(x, \sigma(x)) - \left[\frac{1}{1 - \sigma'(x)} + \frac{1}{1 + \sigma'(x)}\right](1 - \sigma'^2(x))\, 1_A(x). \tag{III.26}$$

Relation (III.26) gives (III.21) and (III.22). ∎

Clearly, the complement of the set where $(\partial^+ v/\partial t)(x, \sigma(x))$ exists is null, so we obtain (III.4) and (III.5).

Let us compute now the derivatives of v in characteristic coordinates, in the sense of distributions.

We can write

$$\langle \mu, \psi \rangle = -2^{1/2} \int [w_\xi(x, \sigma(x)) + w_\eta(x, \sigma(x))](1 - \sigma'^2(x))\, 1_A(x)\, \psi(x, \sigma(x))\, dx. \tag{III.27}$$

By the substitutions

$$\frac{x + \sigma(x)}{2^{1/2}} = \xi \qquad \text{and} \qquad \frac{-x + \sigma(x)}{2^{1/2}} = \eta$$

we transform (III.27) into

$$\begin{aligned} \langle \mu, \psi \rangle = &-2 \int \tilde{w}_\xi(\xi, \bar{Y}(\xi))\, \tilde{\psi}(\xi, \bar{Y}(\xi)) \cdot \frac{-2\bar{Y}'(\xi)}{1 - \bar{Y}'(\xi)}\, 1_A\left(\frac{\xi - \bar{Y}(\xi)}{2^{1/2}}\right) d\xi \\ &-2 \int \tilde{w}_\eta(\bar{X}(\eta), \eta)\, \tilde{\psi}(\bar{X}(\eta), \eta) \cdot \frac{-2\bar{X}'(\eta)}{1 - \bar{X}'(\eta)}\, 1_A\left(\frac{\bar{X}(\eta) - \eta}{2^{1/2}}\right) d\eta, \end{aligned} \tag{III.28}$$

where Y is defined in (III.10), $X = Y^{-1}$, $\bar{Y}$ is the "one-valued part" of Y, and $\bar{X}$ is the "one-valued part" of X. The functions $\bar{Y}$ and $\bar{X}$ are defined everywhere except on a denumerable set.

If X is one-valued at η, and Y is one-valued at ξ, then, the value of $\tilde{v}$ is given by

$$\tilde{v}(\xi, \eta) = \tilde{w}(\xi, \eta) - \tilde{y}(\xi, \eta) - \tilde{z}(\xi, \eta), \tag{III.29}$$

where

$$\begin{aligned} \tilde{y}(\xi, \eta) &= \int_{\bar{X}(\eta)}^{\xi} \tilde{w}_\xi(\xi', \bar{Y}(\xi'))\, g(\xi')\, d\xi' \qquad \text{if} \qquad \xi \geqslant \bar{X}(\eta) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{elsewhere.} \end{aligned} \tag{III.30}$$

$$g(\xi') = \frac{-2\bar{Y}'(\xi')}{1 - \bar{Y}'(\xi')} \cdot 1_A\left(\frac{\xi' - \bar{Y}(\xi')}{2^{1/2}}\right), \tag{III.31}$$

$$\begin{aligned} \tilde{z}(\xi, \eta) &= \int_{\bar{Y}(\xi)}^{\eta} \tilde{w}_\eta(\bar{X}(\eta'), \eta')\, h(\eta')\, d\eta' \qquad \text{if} \qquad \eta \geqslant \bar{Y}(\xi) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{elsewhere.} \end{aligned} \tag{III.32}$$

$$h(\eta') = \frac{-2\bar{X}'(\eta')}{1 - \bar{X}'(\eta')} \cdot 1_A\left(\frac{\bar{X}(\eta') - \eta'}{2^{1/2}}\right). \tag{III.33}$$

Let us compute $\partial\tilde{y}/\partial\xi$ in the sense of distributions, from formula (III.30)

$$\left\langle \frac{\partial\tilde{y}}{\partial\xi}, \tilde{\psi} \right\rangle = -\left\langle \tilde{y}, \frac{\partial\tilde{\psi}}{\partial\xi} \right\rangle$$

$$= -\iint_{\xi \geqslant \bar{X}(\eta)} \frac{\partial\tilde{\psi}}{\partial\xi}(\xi, \eta) \left[\int_{\bar{X}(\eta)}^{\xi} \tilde{w}_\xi(\xi', \bar{Y}(\xi'))\, g(\xi')\, d\xi' \right] d\xi.$$

Interchange the order of integrations in ξ and ξ':

$$\left\langle \frac{\partial\tilde{y}}{\partial\xi}, \tilde{\psi} \right\rangle = \iint_{\xi' \geqslant \bar{X}(\eta)} \tilde{w}_\xi(\xi', \bar{Y}(\xi'))\, g(\xi')\, \tilde{\psi}(\xi', \eta)\, d\xi'. \tag{III.34}$$

Formula (III.34) proves that we can identify

$$\begin{aligned} \frac{\partial\tilde{y}}{\partial\xi}(\xi, \eta) &= \tilde{w}_\xi(\xi, \bar{Y}(\xi))\, g(\xi) && \text{a.e. on } \{(\xi, \eta)/\xi \geqslant \bar{X}(\eta)\}, \\ &= 0 && \text{a.e. on } \{(\xi, \eta)/\xi < \bar{X}(\eta)\}. \end{aligned} \tag{III.35}$$

Similarly

$$\begin{aligned} \frac{\partial\tilde{z}}{\partial\xi}(\xi, \eta) &= \tilde{w}_\eta(\xi, \bar{Y}(\xi))\, g(\xi) && \text{a.e. on } \{(\xi, \eta)/\xi \geqslant \bar{X}(\eta)\}, \\ &= 0 && \text{a.e. on } \{(\xi, \eta)/\xi < \bar{X}(\eta)\}. \end{aligned} \tag{III.36}$$

Now, formulas (III.35) and (III.36) allow us to compute $\partial\tilde{v}/\partial\xi$, using the fact that $\partial\tilde{w}/\partial\xi$ does not depend on η:

$$\begin{aligned} \frac{\partial\tilde{v}}{\partial\xi} &= \tilde{w}_\xi(\xi, \bar{Y}(\xi))\,(1 - g(\xi)) - \tilde{w}_\eta(\xi, \bar{Y}(\xi))\, g(\xi) && \text{a.e. on } \{(\xi, \eta)/\xi \geqslant \bar{X}(\eta)\}, \\ &= \tilde{w}_\xi(\xi, \bar{Y}(\xi)) && \text{a.e. on } \{(\xi, \eta)/\xi < \bar{X}(\eta)\}. \end{aligned} \tag{III.37}$$

Likewise

$$\begin{aligned} \frac{\partial\tilde{z}}{\partial\eta} &= \tilde{w}_\eta(\bar{X}(\eta), \eta)\, h(\eta) && \text{a.e. on } \{(\xi, \eta)/\eta \geqslant \bar{Y}(\xi)\}, \\ &= 0 && \text{a.e. on } \{(\xi, \eta)/\eta < \bar{Y}(\xi)\}; \end{aligned}$$

$$\begin{aligned} \frac{\partial\tilde{y}}{\partial\eta} &= \tilde{w}_\xi(\bar{X}(\eta), \eta)\, h(\eta) && \text{a.e. on } \{(\xi, \eta)/\eta \geqslant \bar{Y}(\xi)\}, \\ &= 0 && \text{a.e. on } \{(\xi, \eta)/\eta < \bar{Y}(\xi)\}; \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\partial\tilde{v}}{\partial\eta} &= -\tilde{w}_\xi(\bar{X}(\eta), \eta)\, h(\eta) + (1 - h(\eta))\, \tilde{w}_\eta(\bar{X}(\eta), \eta) \\ &\qquad \text{a.e. on } \{(\xi, \eta)/\eta \geqslant \bar{Y}(\xi)\}, \\ &= \tilde{w}_\eta(\bar{X}(\eta), \eta) \qquad \text{a.e. on } \{(\xi, \eta)/\eta < \bar{Y}(\xi)\}. \end{aligned} \tag{III.38}$$

III.4. *Energy Condition. End of the Proof of Theorem* III.1

The vector field S_v is given by

$$\tilde{S}_v(x, t) = (-2v_x v_t , | v_x |^2 + | v_t |^2).$$

In characteristic coordinates, S_v is transformed into

$$\tilde{S}_v(\xi, \eta) = 2^{1/2}(| \tilde{v}_\eta(\xi, \eta)|^2, | \tilde{v}_\xi(\xi, \eta)|^2). \tag{III.39}$$

The goal of this section is to prove that the divergence of $\tilde{S}_{\tilde{v}}$ (and therefore of S_v) is zero, and from that, to infer property (III.1).

PROPOSITION III.6. *Let v be defined by* (III.20). *Then*

$$\nabla \cdot S_v = 0 \qquad \textit{in the sense of distributions.} \tag{III.40}$$

Proof. The divergence of $\tilde{S}_{\tilde{v}}$, in the sense of distributions is given by

$$\langle \nabla \cdot \tilde{S}_{\tilde{v}} , \tilde{\psi} \rangle = -2^{1/2} \left[\left\langle \left| \frac{\partial \tilde{v}}{\partial \eta} \right|^2, \frac{\partial \tilde{\psi}}{\partial \xi} \right\rangle + \left\langle \left| \frac{\partial \tilde{v}}{\partial \xi} \right|^2, \frac{\partial \tilde{\psi}}{\partial \eta} \right\rangle \right]. \tag{III.41}$$

Now, using formula (III.37)

$$\begin{aligned} &\int \left| \frac{\partial \tilde{v}}{\partial \xi} \right|^2 \frac{\partial \tilde{\psi}}{\partial \eta} \, d\xi \, d\eta \\ &\quad = \int_{\eta < \bar{Y}(\xi)} \left| \frac{\partial \tilde{w}}{\partial \xi} (\xi, \eta) \right|^2 \frac{\partial \tilde{\psi}}{\partial \eta} (\xi, \eta) \, d\xi \, d\eta \\ &\qquad + \int_{\eta \geqslant \bar{Y}(\xi)} | \tilde{w}_\xi(\xi, \bar{Y}(\xi)) (1 - g(\xi)) - \tilde{w}_\eta(\xi, \bar{Y}(\xi)) \, g(\xi)|^2 \frac{\partial \tilde{\psi}}{\partial \eta} (\xi, \eta) \, d\xi \, d\eta. \end{aligned}$$

Integrate by parts in η:

$$\begin{aligned} &\int \left| \frac{\partial \tilde{v}}{\partial \xi} \right|^2 \frac{\partial \tilde{\psi}}{\partial \eta} \, d\xi \, d\eta \\ &\quad = \int \tilde{\psi}(\xi, \bar{Y}(\xi)) \, [| \tilde{w}_\xi(\xi, \bar{Y}(\xi))|^2 \\ &\qquad - | \tilde{w}_\xi(\xi, \bar{Y}(\xi)) (1 - g(\xi)) - \tilde{w}_\eta(\xi, \bar{Y}(\xi)) \, g(\xi)|^2] \, d\xi. \end{aligned} \tag{III.42}$$

In the same fashion, from formula (III.38), we get

$$\begin{aligned} &\int \left| \frac{\partial \tilde{v}}{\partial \eta} \right|^2 \frac{\partial \tilde{\psi}}{\partial \xi} \, d\xi \, d\eta \\ &\quad = \int \tilde{\psi}(\bar{X}(\eta), \eta) \, [\tilde{w}_\eta(\bar{X}(\eta), \eta)|^2 \\ &\qquad - | -\tilde{w}_\xi(\bar{X}(\eta), \eta) \, h(\eta) + \tilde{w}_\eta(\bar{X}(\eta), \eta) \, (1 - h(\eta))|^2] \, d\eta. \end{aligned} \tag{III.43}$$

Add (III.42) and (III.43), and go back to the x variable by the substitutions

$$\xi = \frac{x + \sigma(x)}{2^{1/2}} \qquad \text{in} \qquad \text{(III.42)},$$

$$\eta = \frac{-x + \sigma(x)}{2^{1/2}} \qquad \text{in} \qquad \text{(III.43)}.$$

Then

$$2^{1/2} \int \left[\left| \frac{\partial \tilde{v}}{\partial \xi} \right|^2 \frac{\partial \tilde{\psi}}{\partial \eta} + \left| \frac{\partial \tilde{v}}{\partial \eta} \right|^2 \frac{\partial \tilde{\psi}}{\partial \xi} \right] d\xi \, d\eta$$

$$= \int_A \psi(x, \sigma(x)) \{ [| w_\xi |^2 - | w_\xi \sigma' - w_\eta (1 - \sigma')|^2] (1 + \sigma') \qquad \text{(III.44)}$$

$$+ [| w_\eta |^2 - | w_\xi (1 + \sigma') + w_\eta \sigma' |^2] (1 - \sigma') \} \, dx$$

and an elementary computation proves (III.40). ∎

End of the Proof of Theorem (III.1). It remains only to prove (III.1). For almost all t_0 in $\mathbb{R}^+$,

$$\text{mes}\{x/\sigma(x) = t_0\} = 0. \qquad \text{(III.45)}$$

Let D be the set

$$D = \{(x, t)/0 \leqslant t \leqslant \min(A + t_0 - | x | , t)\},$$

where t_0 satisfies (III.45), and A is an arbitrary positive number.

Let n be the exterior normal to ∂D, the boundary of D. Condition (III.45) implies that $\partial v/\partial t$ and $\partial v/\partial x$ are defined almost everywhere on $\mathbb{R} \times \{t_0\}$; the characteristic derivative $\partial v/\partial \xi$ is defined almost everywhere on the ξ characteristic going through $(-A - t_0 , 0)$, and likewise, the characteristic derivative $\partial v/\partial \eta$ is defined almost everywhere on the η characteristic through $(A + t_0 , 0)$.

Thanks to (III.40)

$$\int_{\partial D} S_u \cdot n = 0$$

and therefore

$$\int_{-A}^{A} \left(\left| \frac{\partial v}{\partial x}(x, t_0) \right|^2 + \left| \frac{\partial v}{\partial t}(x, t_0) \right|^2 \right) dx \leqslant \int_{-A-t_0}^{A+t_0} \left(| v_1(x)|^2 + \left| \frac{dv_0}{dx}(x) \right|^2 \right) dx,$$

which implies (III.1). ∎

IV. Proof of Existence: Infinite String

IV.1. *Proof of Existence in the General Case*

Let us recall the hypotheses and the statement we have in the Introduction

The function φ represents the obstacle, and

$$\varphi'' \geqslant 0 \qquad \text{in the sense of distributions.} \tag{IV.1}$$

The initial data are

$$u_0 \in H^1_{\text{loc}}(\mathbb{R}), \tag{IV.2}$$

$$u_1 \in L^2_{\text{loc}}(\mathbb{R}) \tag{IV.3}$$

with the compatibility conditions

$$u_0(x) \geqslant \varphi(x) \qquad \forall x \in \mathbb{R}, \tag{IV.4}$$

$$u_1(x) \geqslant 0 \qquad \text{a.e. on } \{x/u_0(x) = \varphi(x)\}. \tag{IV.5}$$

THEOREM IV.1 (Infinite String). *Under hypotheses* (IV.1)–(IV.5), *there exists a unique function* u *such that*

$$u \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1_{\text{loc}}(\mathbb{R})) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\mathbb{R})), \tag{IV.6}$$

$$u(x, 0) = u_0(x), \qquad \forall x \in \mathbb{R}, \tag{IV.7}$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x), \qquad \text{a.e. in } \mathbb{R}, \tag{IV.8}$$

$$u(x, t) \geqslant \varphi(x), \qquad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{IV.9}$$

$$\Box u \geqslant 0 \qquad \textit{in the sense of distributions in } \mathbb{R} \times \mathbb{R}^+_*, \tag{IV.10}$$

$$\text{supp } \Box u \subset \{(x, t)/u(x, t) = \varphi(x)\}, \tag{IV.11}$$

$$\frac{\partial}{\partial x}\left(-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2\right) = 0$$
$$\textit{in the sense of distributions in } \mathbb{R} \times \mathbb{R}^+_*. \tag{IV.12}$$

Moreover, the solution u is explicitly known:

THEOREM IV.2. *Let* w *be the free solution of the wave equation with initial data* u_0 *and* u_1

$$\Box w = 0,$$
$$w(x, 0) = u_0(x), \tag{IV.13}$$
$$\frac{\partial w}{\partial t}(x, 0) = u_1(x).$$

Define a measure $\mu(w)$ *on* $\mathbb{R} \times]0, +\infty[$ *by*

$$\langle \mu(w), \psi \rangle = -2 \int_{\tau(x)>0} \frac{\partial w}{\partial t}(x, \tau(x))\,(1 - \tau'^2(x))\,\psi(x, \tau(x))\,dx, \tag{IV.14}$$

where $x \mapsto \tau(x)$ is the line of influence of the obstacle φ with respect to w, following Definition II.2.

Then the function

$$u = w + \mathscr{E} * \mu(w), \tag{IV.15}$$

with $\mathscr{E}$ the elementary solution of the wave equation solves problem (IV.6)–(IV.12).

Proof of Theorem IV.2. By Theorem III.1, we know that u given by (IV.15) satisfies conditions (IV.6)–(IV.8) and (IV.11). Condition (IV.10) is seen to be satisfied by Lemma (II.7) and the explicit expression (IV.14) and condition (IV.12) results from Proposition III.6.

So it remains to check condition (IV.9), for $t \geqslant \tau(x)$. Set

$$Z = u - \varphi \tag{IV.16}$$

and note that

$$\Box Z = \Box u - \Box\varphi = \Box u + \varphi'' \geqslant 0, \tag{IV.17}$$

by conditions (IV.1) and (IV.10).

On the other hand

$$\tilde{Z}(\xi, \eta) = \tilde{Z}(\bar{X}(\eta), \eta) + \int_{\bar{X}(\eta)}^{\xi} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \eta)\, d\xi' \tag{IV.18}$$

if X is one-valued in η (same notations as in subsection III.2).

With notations (III.11) and (III.12), and thanks to Lemma II.5, we have

$$\begin{gathered} f(x + \tau(x)) + g(x - \tau(x)) = f_1(x + \tau(x)) + g_1(x + \tau(x)) = \varphi(x) \\ \text{a.e. on } M_\tau = \{x / |\tau'(x)| < 1 \text{ and } \tau(x) > 0\}. \end{gathered} \tag{IV.19}$$

Let us differentiate (IV.19); then by a linear combination with (III.15) which we write again as

$$f'(x + \tau(x)) - g'(x - \tau(x)) = -f_1'(x + \sigma(x)) + g_1'(x - \sigma(x)) \quad \text{a.e. on } M_\tau \tag{IV.20}$$

we obtain

$$\begin{aligned} f'(x + \tau(x)) + f_1'(x + \tau(x)) &= \varphi'(x) \quad \text{a.e. on } M_\tau, \\ g'(x - \tau(x)) + g_1'(x - \tau(x)) &= \varphi'(x) \quad \text{a.e. on } M_\tau. \end{aligned} \tag{IV.21}$$

Therefore, Lemma III.7 implies

$$f_1'(x + \tau(x)) \geqslant \frac{\varphi'(x)}{2} \quad \text{a.e. on } M_\tau. \tag{IV.22}$$

On the set $\{x/\tau(x) = 0\}$, the relation

$$f_1(x + \tau(x)) + g_1(x - \tau(x)) = \varphi(x)$$

holds (still Lemma II.5!), and condition (IV.5) implies

$$f_1'(x + \tau(x)) \geqslant \frac{\varphi'(x)}{2} \qquad \text{a.e. on} \quad \{x/\tau(x) = 0\}. \tag{IV.23}$$

Relation (III.17) implies

$$\begin{gathered}\frac{\partial}{\partial \xi}\left[f_1(x + t) + g_1(x - t)\right]|_{t=\tau(x)} = 2^{1/2} f_1'(x + \tau(x)) = \frac{\varphi'(x)}{2^{1/2}} \\ \text{a.e. on} \quad \{x/\tau'(x) = 1, \tau(x) > 0, w(x, \tau(x)) = \varphi(x)\}.\end{gathered} \tag{IV.24}$$

We can deduce from relations (IV.22)–(IV.24) that

$$\frac{\partial Z}{\partial \xi}(x, \tau(x)) \geqslant 0 \qquad \text{a.e. on} \quad \{x/\tau'(x) > -1, w(x, \tau(x)) = \varphi(x)\}, \tag{IV.25}$$

we know from (IV.15) that $\Box Z \geqslant 0$, and we can now infer from (IV.25) that

$$\frac{\partial \tilde{Z}}{\partial \xi}(\xi, \eta) \geqslant 0 \qquad \text{a.e. on} \quad \{(\xi, \eta)/\overline{Y}(\xi) \leqslant \eta \text{ and } \xi \in C\}, \tag{IV.26}$$

where C is defined as

$$C = \{\xi/\tilde{Z}(\xi, \overline{Y}(\xi)) = 0\} \tag{IV.27}$$

and thus,

$$\int_{C \cap [\overline{X}(\eta), \xi]} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \eta)\, d\xi' \geqslant 0. \tag{IV.28}$$

Recall the definition of the set U,

$$U = \{x/w(x, \tau(x)) > \varphi(x)\}.$$

Save for a null set, there is the obvious relation between U and C

$$C^c = \frac{(I + \tau)}{2^{1/2}} U,$$

and besides, on U, the function τ is known very precisely.

Let U be the (at most) denumerable union of disjoint open intervals (a_i, b_i) $(i \in I)$, define c_i by

$$c_i = \frac{\tau(b_i) + b_i - \tau(a_i) + a_i}{2},$$

and define a subset J of I by

$$J = \{i \in I / c_i > a_i\}.$$

Denote

$$\alpha_i = \frac{a_i + \tau(a_i)}{2^{1/2}}, \qquad \gamma_i = \frac{c_i + \tau(c_i)}{2^{1/2}}.$$

Then, by Lemma II.6, save for a null set, the following relation holds:

$$C^c = \bigcup_{i \in J} (\alpha_i, \gamma_i). \tag{IV.29}$$

Moreover

$$\bar{Y}(\xi) = \bar{Y}(\alpha_i) \qquad \text{if} \qquad \alpha_i \leqslant \xi < \gamma_i, \quad i \in J. \tag{IV.30}$$

On the other hand, by relation (III.37),

$$\begin{aligned} \frac{\partial \tilde{Z}}{\partial \xi}(\xi, \eta) &= \frac{\partial \tilde{w}}{\partial \xi}(\xi, \bar{Y}(\xi)) - \frac{1}{2^{1/2}} \varphi'\left(\frac{\xi - \bar{Y}(\xi)}{2^{1/2}}\right) \\ &\text{if} \qquad \alpha_i \leqslant \xi < \gamma_i, \quad \eta \geqslant \bar{Y}(\xi), \end{aligned} \tag{IV.31}$$

we can now estimate

$$\tilde{Z}(\bar{X}(\eta), \eta) + \int_{[\bar{X}(\eta), \xi] \cap C^c} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \eta)\, d\xi',$$

using relations (IV.29)–(IV.31).

Let ξ_0 belong to the interval $[\alpha_i, \gamma_i]$. Then

$$\begin{aligned} \int_{\alpha_i}^{\xi_0} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \eta)\, d\xi' &= \int_{\alpha_i}^{\xi_0} \left[\frac{\partial \tilde{w}}{\partial \xi}(\xi', \bar{Y}(\alpha_i)) - \frac{1}{2^{1/2}} \varphi'\left(\frac{\xi' - \bar{Y}(\alpha_i)}{2^{1/2}}\right)\right] d\xi' \\ &= \tilde{w}(\xi_0, \bar{Y}(\alpha_i)) - \varphi\left(\frac{(\xi_0 - \bar{Y}(\alpha_i)}{2^{1/2}}\right) \geqslant 0. \end{aligned} \tag{IV.32}$$

As we suppose that X is one-valued in η, we never have

$$\bar{X}(\eta) \in (\alpha_i, \gamma_i).$$

By the definition of the line of influence

$$\tilde{Z}(\bar{X}(\eta), \eta) \geqslant 0. \tag{IV.33}$$

If we put together (IV.32) and (IV.33), we get

$$\begin{aligned}\int_{[\bar{X}(\eta),\xi]\cap C^c} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \eta)\, d\xi' + \tilde{Z}(\bar{X}(\eta), \eta) \\ = \sum_{\{i\in J/(\alpha_i,\gamma_i)\subset[\bar{X}(\eta),\xi]\}} \int_{\alpha_i}^{\gamma_i} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \bar{Y}(\alpha_i))\, d\xi' \qquad \text{(IV.34)} \\ + \int_{\alpha_{i_0}}^{\xi} \frac{\partial \tilde{Z}}{\partial \xi}(\xi', \bar{Y}(\alpha_{i_0}))\, d\xi' + \tilde{Z}(\bar{X}(\eta), \eta) \geqslant 0.\end{aligned}$$

Using (IV.28) and (IV.34), we can see now that condition (IV.9) is satisfied. ∎

Theorem IV.2 proves the existence of a solution to problem P_∞, (IV.6)–(IV.12).

Uniqueness will be proved in next chapter.

We can infer from Theorems III.1 and IV.2 that if E is nonvoid, then $|\tau'(x)|$ $|\tau'(x)|$ is not almost everywhere equal to one. Assume indeed that

$$|\tau'(x)| = 1 \qquad \text{a.e. on } \{x/\tau(x) > 0\}. \tag{IV.35}$$

Then, w is clearly the unique solution of problem (III.5)–(III.9) as conditions (III.6) and (III.9) are automatically fulfilled. We have proved by Theorem IV.2 that

$$w + \mathscr{E} * \mu(w) \geqslant \varphi \qquad \text{on} \qquad \mathbb{R} \times \mathbb{R}^+.$$

Here, the measure $\mu(w)$ is equal to zero; therefore

$$w \geqslant \varphi \qquad \text{on} \qquad \mathbb{R} \times \mathbb{R}^+, \tag{IV.36}$$

and E is void. ∎

In the same fashion, if u_1 is almost everywhere nonnegative, E is void.

IV.2. *An Alternate Construction when the Obstacle φ is Constant*

A. Bamberger has the idea of a simpler construction, which is especially convenient when φ is a constant, which can be taken equal to zero.

For this purpose, we can deduce from the results established between (IV.21) and (IV.24) the formulas

$$\begin{aligned}&\frac{\partial \tilde{u}}{\partial \xi}(\xi, \eta) = -\frac{\partial \tilde{w}}{\partial \xi}(\xi, \eta) \qquad \text{if} \quad \xi \geqslant \bar{X}(\eta), \quad \xi + \bar{Y}(\xi) > 0, \quad \bar{Y}'(\xi) < 0, \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{and} \quad \tilde{w}(\xi, \bar{Y}(\xi)) = 0; \qquad \text{(IV.37)} \\ &\frac{\partial \tilde{u}}{\partial \xi}(\xi, \eta) = +\frac{\partial \tilde{w}}{\partial \xi}(\xi, \eta) \qquad \text{elsewhere.}\end{aligned}$$

If we define almost everywhere a kernel K by

$$\begin{aligned} \tilde{K}(\xi, \eta) &= -1 \quad \text{if} \quad \xi \geqslant \bar{X}(\eta), \quad \xi + \bar{Y}(\xi) > 0, \quad \bar{Y}'(\xi) < 0, \\ &\quad \text{and} \quad \tilde{w}(\xi, \bar{Y}(\xi)) = 0; \\ &= +1 \quad \text{elsewhere} \end{aligned} \tag{IV.38}$$

then

$$\frac{\partial \tilde{u}}{\partial \xi}(\xi, \eta) = \tilde{K}(\xi, \eta) \frac{\partial \tilde{w}}{\partial \xi}(\xi, \eta)$$

and therefore, integrating with respect to ξ,

$$\tilde{u}(\xi, \eta) = \tilde{u}(\xi, -\xi) + \int_{-\xi}^{\eta} K(\xi', \eta)\, \tilde{w}_\xi(\xi', \eta)\, d\xi'. \tag{IV.39}$$

Formula (IV.39) can be converted into the still simpler expression, where $r^- = -\inf(r, 0)$,

$$\tilde{u}(\xi, \eta) = \tilde{w}(\xi, \eta) + 2 \sup\{\tilde{w}(\xi', \eta')^- / -\eta \leqslant \xi' \leqslant \xi, -\xi \leqslant \eta' \leqslant \eta\}. \tag{IV.40}$$

Formula (IV.40) will be proved, and generalizations and consequences will be given, in a joint work with Bamberger [4].

V. Proof of Uniqueness: Infinite String

The proof of uniqueness relies on several trace results which, I believe, do not appear elsewhere in the literature. The main difficulty is to define cleanly traces of derivatives u_x and u_t on a space-like curve, i.e., a curve $x \mapsto \tau(x)$ with $|\tau'|_\infty < 1$, when u is a solution of P_∞. Once these traces are defined, we can derive from (IV.12) how the derivatives are transformed across a discontinuity line. Then, by comparison with the solution built in Section IV, uniqueness is proved. Note that allowing the initial position u_0 to be equal to φ introduces an extra difficulty, as we have to give a meaning to (IV.8), using the other relations.

V.1. *Results of Trace*

We prove how condition (IV.10) can be used to define traces.

Lemma V.1. *Let $\tilde{u}$ be a function in $W^{1,1}([a_0, b_0] \times [c_0, d_0])$, such that*

$$\frac{\partial^2 \tilde{u}}{\partial \xi\, \partial \eta} \geqslant 0.$$

Then $\eta \mapsto (\partial \tilde{u}/\partial \xi)(\xi, \eta)$ is an increasing function from $[a_0, b_0]$ to $L^1(a_0, b_0)$ and $\xi \mapsto (\partial \tilde{u}/\partial \eta)(\xi, \eta)$ is an increasing function from $[a_0, b_0]$ to $L^1(c_0, d_0)$.

Proof. If we suppose that $\tilde{v}$ is smooth, we can write

$$\left\langle \frac{\partial^2 \tilde{u}}{\partial \xi\, \partial \eta}\,,\, 1_{[a,b]\times[c,d]} \right\rangle = \int_a^b d\xi \int_c^d d\eta\, \frac{\partial^2 \tilde{u}}{\partial \xi\, \partial \eta}$$

$$= \int_a^b \left[\frac{\partial \tilde{u}}{\partial \xi}(\xi, d) - \frac{\partial \tilde{u}}{\partial \xi}(\xi, c) \right] d\xi \geqslant 0$$

for arbitrary a, b in $[a_0\,,\, b_0]$ and c, d in $[c_0\,,\, d_0]$. Then the conclusion is obvious.

In the general case, regularize $\tilde{v}$ by convolution with a smooth nonnegative function, and let this function tend to a Dirac mass; the conclusion still holds. ∎

Let u be a function in $W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+_*)$, such that $\Box u$ is nonnegative. As a result of Lemma V.1, the functions

$$\eta \mapsto \lim_{h \downarrow 0} \frac{\partial \tilde{u}}{\partial \xi}(\xi, \eta + h) = \tilde{p}_+(\xi, \eta), \tag{V.1}$$

$$\eta \mapsto \lim_{h \downarrow 0} \frac{\partial \tilde{u}}{\partial \xi}(\xi, \eta - h) = \tilde{p}_-(\xi, \eta) \tag{V.2}$$

are defined on the set $\{(\xi, \eta)/\xi \notin N_\xi\,,\, \eta > -\xi\}$, with N_ξ a null set, and increasing in η. Similarly, the functions

$$\xi \mapsto \lim_{\eta \downarrow 0} \frac{\partial \tilde{u}}{\partial \eta}(\xi + h, \eta) = \tilde{q}_+(\xi, \eta), \tag{V.3}$$

$$\xi \mapsto \lim_{h \downarrow 0} \frac{\partial \tilde{u}}{\partial \eta}(\xi - h, \eta) = \tilde{q}_-(\xi, \eta) \tag{V.4}$$

are defined on the set $\{(\xi, \eta)/\eta \notin N_\eta$ and $\xi > -\eta\}$, with N_η a null set, and increasing in ξ.

Let $L^1_{\mathrm{loc}}(X;\, m\, dx)$ be the set of functions on a subset X of $\mathbb{R}$ which are locally integrable with respect to the measure $m(x)\, dx$; here m is a locally integrable function (with respect to the Lebesgue measure).

We have more precise information on $p_\pm$ and $q_\pm$ in the following proposition.

PROPOSITION V.2. *Let u be in $W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+_*)$ and let $\Box u$ be nonnegative. Let σ be an arbitrary Lipschitz-continuous function, with Lipschitz constant* 1, *and let σ be positive on $\mathbb{R}$. Then*

$$p_\pm(x, \sigma(x)) \in L^1_{\mathrm{loc}}(\mathbb{R};\, (1 + \sigma')\, dx), \tag{V.5}$$

$$q_\pm(x, \sigma(x)) \in L^1_{\mathrm{loc}}(\mathbb{R};\, (1 - \sigma')\, dx), \tag{V.6}$$

and, almost everywhere on $\{x/|\sigma'(x)| < 1\}$,

$$\begin{aligned}\frac{\partial^+ u}{\partial t}(x, \sigma(x)) &= \lim_{h \downarrow 0} \frac{u(x, \sigma(x) + h) - u(x, \sigma(x))}{h} \\ &= \frac{1}{2^{1/2}}(p_+(x, \sigma(x)) + q_+(x, \sigma(x))),\end{aligned} \tag{V.7}$$

$$\begin{aligned}\frac{\partial^- u}{\partial t}(x, \sigma(x)) &= \lim_{h \downarrow 0} \frac{u(x, \sigma(x) - h) - u(x, \sigma(x))}{-h} \\ &= \frac{1}{2^{1/2}}(p_-(x, \sigma(x)) + q_-(x, \sigma(x)).\end{aligned} \tag{V.8}$$

Proof. Let us show, for instance, that p_+ and p_- are in $L^1_{\mathrm{loc}}(\mathbb{R}, (1 + \sigma')\, dx)$, and for this purpose, fix two arbitrary numbers $a < b$. By definition,

$$p_+(x, t) = p_-(x, t) = \frac{\partial u}{\partial \xi}(x, t) \qquad \text{a.e. on} \quad \mathbb{R} \times \mathbb{R}^+_*$$

and, in the same way

$$q_+(x, t) = q_-(x, t) = \frac{\partial u}{\partial \eta}(x, t) \qquad \text{a.e. on} \quad \mathbb{R} \times \mathbb{R}^+_* .$$

We can find real numbers t_1 and t_2 such that

$$0 < t_1 \leqslant \min_{[a,b]} \sigma(x) \leqslant \max_{[a,b]} \sigma(x) \leqslant t_2$$

and

$$p_+(\cdot, t_1) = p_-(\cdot, t_1) = \frac{\partial u}{\partial \xi}(\cdot, t_1) \in L^1_{\mathrm{loc}}(\mathbb{R}),$$

$$p_+(\cdot, t_2) = p_-(\cdot, t_2) = \frac{\partial u}{\partial \xi}(\cdot, t_2) \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

The functions $\tilde{p}_+$ and $\tilde{p}_-$ are increasing in η, so that

$$\begin{aligned}&\max\left[\int_{(a+\sigma(a))/2^{1/2}}^{(b+\sigma(b))/2^{1/2}} |\tilde{p}_+(\xi, \bar{Y}(\xi))|\, d\xi, \int_{(a+\sigma(a))/2^{1/2}}^{(b+\sigma(b))/2^{1/2}} |\tilde{p}_-(\xi, \bar{Y}(\xi))|\, d\xi\right] \\ &\leqslant \max\left[\int_{(a+\sigma(a))/2^{1/2}}^{(b+\sigma(b))/2^{1/2}} \left|\frac{\partial \tilde{u}}{\partial \xi}(\xi, t_2\, 2^{1/2} - \xi)\right| d\xi,\right. \\ &\qquad \left.\int_{(a+\sigma(a))/2^{1/2}}^{(b+\sigma(b))/2^{1/2}} \left|\frac{\partial \tilde{u}}{\partial \xi}(\xi, t_1\, 2^{1/2} - \xi)\right| d\xi\right],\end{aligned} \tag{V.9}$$

which proves (V.5). Here Y is the graph of σ in characteristic coordinates. Relation (V.6) is proved in the same fashion.

On the other hand, the following equality holds for almost all λ in $(-\min_{\mathbb{R}} \sigma, +\infty)$:

$$\frac{\partial u}{\partial t}(x, \sigma(x) + \lambda) = \frac{1}{2^{1/2}} \left[\frac{\partial u}{\partial \xi}(x, \sigma(x) + \lambda) + \frac{\partial u}{\partial \eta}(x, \sigma(x) + \lambda)\right]. \qquad \text{(V.10)}$$

Let us show that

$$\lim_{\lambda \downarrow 0} \frac{\partial u}{\partial t}(x, \sigma(x) + \lambda) = \frac{1}{2^{1/2}} (p_+(x, \sigma(x)) + q_+(x, \sigma(x))) \qquad \text{(V.11)}$$

in the $L^1_{\text{loc}}(\mathbb{R}: (1 - \sigma'^2)\, dx)$ topology.

Let $A = (a + \sigma(a))/2^{1/2}$, $B = (b + \sigma(b))/2^{1/2}$; we shall estimate $p_+(x, \sigma(x)) - u_\xi(x, \sigma(x) + \lambda)$ by a computation in ξ, η coordinates.

$$\begin{aligned}
&\int_a^b \left| \frac{\partial u}{\partial \xi}(x, \sigma(x) + \lambda) - p_+(x, \sigma(x)) \right| \frac{1 + \sigma'(x)}{2^{1/2}}\, dx \\
&\quad = \int_A^B \left| \frac{\partial \tilde{u}}{\partial \xi}\left(\xi + \frac{\lambda}{2^{1/2}}, \overline{Y}(\xi) + \frac{\lambda}{2^{1/2}}\right) - \tilde{p}_+(\xi, \overline{Y}(\xi)) \right| d\xi \\
&\quad \leqslant \int_A^B \left| \frac{\partial \tilde{u}}{\partial \xi}\left(\xi + \frac{\lambda}{2^{1/2}}, \overline{Y}(\xi) + \frac{\lambda}{2^{1/2}}\right) - \tilde{p}_+\left(\xi + \frac{\lambda}{2^{1/2}}, \overline{Y}\left(\xi + \frac{\lambda}{2^{1/2}}\right)\right) \right| d\xi \\
&\qquad + \int_A^B \left| \tilde{p}_+\left(\xi + \frac{\lambda}{2^{1/2}}, \overline{Y}\left(\xi + \frac{\lambda}{2^{1/2}}\right)\right) - \tilde{p}_+(\xi, \overline{Y}(\xi)) \right| d\xi. \qquad \text{(V.12)}
\end{aligned}$$

Substitute in the first integral of (V.12), $\xi + \lambda/2^{1/2} = \zeta$, which supplies

$$\int_{A-\lambda/2^{1/2}}^{B-\lambda/2^{1/2}} \left| \frac{\partial \tilde{u}}{\partial \xi}\left(\zeta, \overline{Y}\left(\zeta - \frac{\lambda}{2^{1/2}}\right) + \frac{\lambda}{2^{1/2}}\right) - \tilde{p}_+(\zeta, \overline{Y}(\zeta)) \right| d\zeta. \qquad \text{(V.13)}$$

As $\overline{Y}$ is decreasing, and by definition of $\tilde{p}_+$,

$$\lim_{\lambda \downarrow 0} \frac{\partial \tilde{u}}{\partial \xi}\left(\zeta, \overline{Y}\left(\zeta - \frac{\lambda}{2^{1/2}}\right) + \frac{\lambda}{2^{1/2}}\right) = \tilde{p}_+(\zeta, \overline{Y}(\zeta)) \qquad \text{a.e.}$$

Estimate (V.9) and the Lebesgue dominated convergence theorem imply that expression (V.13) tends to zero as λ decreases to zero.

The same estimate (V.9) allows us to conclude that the second term of (V.12) converges to zero.

Therefore, we have shown that

$$\lim_{\lambda \downarrow 0} \frac{\partial u}{\partial \xi}(x, \sigma(x) + \lambda) - p_+(x, \sigma(x)) = 0 \qquad \text{(V.14)}$$

in the $L^1_{\text{loc}}(\mathbb{R}; (1 - \sigma')\, dx)$ topology; the result is still true in the $L^1_{\text{loc}}(\mathbb{R}; (1 - \sigma'^2)\, dx)$ topology, which is weaker. In the same fashion we show that

$$\lim_{\lambda \downarrow 0} \frac{\partial u}{\partial \eta}(x, \sigma(x) + \lambda) - q_+(x, \sigma(x)) = 0$$

in the $L^1_{\mathrm{loc}}(\mathbb{R}; (1 + \sigma')\,dx)$ topology, and therefore in the $L^1_{\mathrm{loc}}(\mathbb{R}; (1 - \sigma'^2)\,dx)$ topology. Thus we get (V.11).

To conclude, denote

$$k(x, \lambda) = u(x, \sigma(x) + \lambda); \tag{V.15}$$

we have just proved that

$$\frac{\partial k}{\partial \lambda} \in L^1_{\mathrm{loc}}(-\min_{\mathbb{R}} \sigma(x), +\infty; L^1_{\mathrm{loc}}(\mathbb{R}; (1 - \sigma'^2)\,dx)) \tag{V.16}$$

and that $\partial k/\partial\lambda$ has right and left limits at any λ. This implies that

$$\lim_{h \downarrow 0} \frac{k(\cdot, \lambda + h) - k(\cdot, \lambda)}{h} = \lim_{h \downarrow 0} \frac{\partial k}{\partial \lambda}(\cdot, \lambda + h),$$

in the $L^1_{\mathrm{loc}}(\mathbb{R}; (1 - \sigma'^2)\,dx)$ topology, i.e., for $\lambda = 0$,

$$\begin{aligned} &\frac{1}{2^{1/2}}(p_+(x, \sigma(x)) + q_+(x, \sigma(x))) \\ &\quad = \lim_{h \downarrow 0} \frac{u(x, \sigma(x) + h) - u(x, \sigma(x))}{h} = \frac{\partial^+ u}{\partial t}(x, \sigma(x)) \\ &\qquad \text{a.e. on } \{x \in \mathbb{R} / |\sigma'(x)| < 1\}. \end{aligned}$$

This completes the proof of Proposition V.2. ∎

The next result accounts for the fact that the derivative $(\partial/\partial x)\,u(x, \sigma(x) + \lambda)$ has no jumps across the line $t = \sigma(x)$.

PROPOSITION V.3. *Let u be in $W^{1,1}_{\mathrm{loc}}(R \times R^+_*)$ and let $\Box u$ be nonnegative. Let σ be an arbitrary Lipschitz-continuous function, with Lipschitz constant* 1, *and let σ be positive on $\mathbb{R}$. Then*

$$\begin{aligned} &p_+(x, \sigma(x))\,(1 + \sigma'(x)) - q_+(x, \sigma(x))\,(1 - \sigma'(x)) \\ &\quad = p_-(x, \sigma(x))\,(1 + \sigma'(x)) - q_-(x, \sigma(x))\,(1 - \sigma'(x)) \\ &\qquad \text{a.e. on } \{x / |\sigma'(x)| < 1\}. \end{aligned} \tag{V.17}$$

Proof. With notation (V.15), the following relation holds for almost all λ.

$$\frac{\partial k}{\partial x}(\cdot, \lambda) = (1 + \sigma'(\cdot))\frac{\partial u}{\partial \xi}(\cdot, \sigma(\cdot) + \lambda) - (1 - \sigma'(\cdot))\frac{\partial u}{\partial \eta}(\cdot, \sigma(\cdot) + \lambda). \tag{V.18}$$

From (V.15), we can infer that

$$\frac{\partial^2 k}{\partial \lambda\, \partial x} \in L^1_{\mathrm{loc}}(-\min_{\mathbb{R}} \sigma(x), +\infty; \mathscr{D}'(\mathbb{R})).$$

Therefore $(\partial k/\partial x)(\cdot, \lambda)$ is continuous to $\mathscr{D}'(\mathbb{R})$ with the weak topology.

On the other hand, by (V.14)

$$\lim_{\lambda \downarrow 0} \frac{\partial u}{\partial \xi}(\cdot, \sigma(\cdot) + \lambda) - p_+(\cdot, \sigma(\cdot)) = 0 \qquad \text{in} \qquad L^1_{\text{loc}}(\mathbb{R}, (1 + \sigma')\, dx)$$

and analogously

$$\lim_{\lambda \downarrow 0} \frac{\partial u}{\partial \eta}(\cdot, \sigma(\cdot) + \lambda) - q_+(\cdot, \sigma(\cdot)) = 0 \qquad \text{in} \qquad L^1_{\text{loc}}(\mathbb{R}, (1 - \sigma')\, dx).$$

Similar relations hold when the limit is taken as λ increases to zero. The (very weak) continuity of $\partial k/\partial x$ with respect to λ completes the proof. ∎

We need the weaker assumption that σ is only nonnegative; to obtain conclusions analogous to those of Propositions V.2, and V.3, we shall make a stronger assumption on u.

COROLLARY V.4. *Suppose that σ is nonnegative and*

$$\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R})); \tag{V.19}$$

let the other assumptions of Proposition V.2 *remain.*

Then, the functional inclusion of Proposition V.2 *on p_+ and q_+ as well as* (V.7) *still hold. On the other side,*

$$\begin{aligned} &p_-(x, \sigma(x)) \in L^1_{\text{loc}}(\{x/\sigma(x) > 0\}, (1 + \sigma')\, dx), \\ &q_-(x, \sigma(x)) \in L^1_{\text{loc}}(\{x/\sigma(x) > 0\}, (1 - \sigma')\, dx), \\ &\frac{\partial^- u}{\partial t}(x, \sigma(x)) = \frac{1}{2^{1/2}}(p_-(x, \sigma(x)) + q_-(x, \sigma(x))) \\ &\qquad \textit{a.e. on } \{x/|\sigma'(x)| < 1 \textit{ and } \sigma(x) > 0\}, \end{aligned} \tag{V.20}$$

and the conclusion of Proposition V.3 *is replaced by*

$$\begin{aligned} &p_+(x, \sigma(x))(1 + \sigma'(x)) - q_+(x, \sigma(x))(1 - \sigma'(x)) \\ &\quad = p_-(x, \sigma(x))(1 + \sigma'(x)) - q_-(x, \sigma(x))(1 - \sigma'(x)) \\ &\qquad\qquad \textit{a.e.} \qquad \textit{on} \qquad \{x/|\sigma'(x)| < 1 \textit{ and } \sigma(x) > 0\}. \end{aligned}$$

Proof. Hypothesis (V.19) implies that the function

$$t \mapsto p_+(\cdot - t, t)\, 1_{[a-t, b-t]}$$

which takes values in $L^1(\mathbb{R})$ and is increasing, tends to a certain function $p_+{}^0 \cdot 1_{[a,b]}$ when t decreases to zero.

This simple remark allows us to relieve ourselves of the condition $t_1 > 0$ for estimate (V.9). Then (V.11) is still true.

Concerning p_- and q_-, one must consider a compact interval $[a, b]$ included in $\{x/\sigma(x) > 0\}$, and argue as before. ∎

V.2. *Proof of Uniqueness*

Note first that conditions (IV.10) and (IV.6) allow us to apply Corollary V.4, and to take a trace of $\partial u/\partial t$ on the curve $\sigma(x) = 0$. Thus, condition (IV.8) makes sense.

With the help of subsection V.1, it is now possible to state how the time-derivative of a function satisfying the local energy conservation condition is transformed across a space-like curve. This is the goal of next proposition.

PROPOSITION V.5. *Let u be in the space $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+_*; L^2_{\mathrm{loc}}(\mathbb{R}^+)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+_*; H^1_{\mathrm{loc}}(\mathbb{R}))$, such that $\Box u$ is nonnegative, and let σ be a Lipschitz-continuous, nonnegative function, with Lipschitz constant 1. If u satisfies*

$$\frac{\partial}{\partial x}\left(-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right) = 0 \quad \textit{in the sense of distributions} \tag{V.21}$$

then it also satisfies

$$\left|\frac{\partial^+ u}{\partial t}(x, \sigma(x))\right| = \left|\frac{\partial^- u}{\partial t}(x, \sigma(x))\right| \quad \textit{a.e. on } \{x/|\sigma'(x)| < 1 \textit{ and } \sigma(x) > 0\}. \tag{V.22}$$

Proof. By Lemma V.1, the functions $\xi \mapsto |\tilde{u}_\eta(\xi, \eta)|^2$ and $\eta \mapsto |\tilde{u}_\xi(\xi, \eta)|^2$ are of bounded variation, respectively, for almost every η and almost every ξ. Therefore the distribution

$$\rho = \frac{\partial}{\partial x}\left(-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right),$$

which is written, in characteristic coordinates,

$$\tilde{\rho} = 2^{1/2}\left(\frac{\partial}{\partial \xi}\left|\frac{\partial \tilde{u}}{\partial \eta}\right|^2 + \frac{\partial}{\partial \eta}\left|\frac{\partial \tilde{u}}{\partial \xi}\right|^2\right),$$

is in fact the sum of two measures.

Let us compute the measure by ρ of the set

$$\Gamma = \{(x, t) \mid a \leqslant x \leqslant b \text{ and } t = \sigma(x)\},$$

in terms of the functions $p_\pm$ and $q_\pm$ defined in (V.1)–(V.4). We assume that $\sigma(x)$ is positive on (a, b).

$$\rho(\Gamma) = \tilde{\rho}(\tilde{\Gamma}) = 2^{1/2} \int_{(a+\sigma(a))/2^{1/2}}^{(b+\sigma(b))/2^{1/2}} [\tilde{p}_+(\xi, \overline{Y}(\xi))^2 - \tilde{p}_-(\xi, \overline{Y}(\xi))^2]\, d\xi$$

$$+ \int_{(-b+\sigma(b))/2^{1/2}}^{(-a+\sigma(a))/2^{1/2}} [\tilde{q}_+(\overline{X}(\eta), \eta)^2 - \tilde{q}_-(\overline{X}(\eta), \eta)^2]\, d\eta.$$

Substitute $\xi = (x + \sigma(x))/2^{1/2}$ in the first integral and $\eta = (-x + \sigma(x))/2^{1/2}$ in the second one. Then

$$\begin{aligned} \rho(\Gamma) = \int_a^b \{&[p_+(x, \sigma(x))^2 - p_-(x, (x))^2]\,(1 + \sigma'(x)) \\ &+ [q_+(x, \sigma(x))^2 - q_-(x, \sigma(x))^2]\,(1 - \sigma'(x))\}\, dx. \end{aligned} \tag{V.23}$$

By Corollary V.4, the following relation holds:

$$\begin{aligned} &(p_+(x, \sigma(x))\,(1 + \sigma'(x)) - q_+(x, \sigma(x))\,(1 - \sigma'(x)))^2 \\ &\quad - (p_-(x, \sigma(x))\,(1 + \sigma'(x)) - q_-(x, \sigma(x))\,(1 - \sigma'(x)))^2 = 0 \\ &\text{a.e. on } \{x/|\,\sigma'(x)| < 1 \text{ and } \sigma(x) > 0\}. \end{aligned} \tag{V.24}$$

Thanks to the identities

$$\begin{aligned} &2p_\pm^2(1 + \sigma') + 2q_\pm^2(1 - \sigma') - (p_\pm(1 + \sigma') - q_\pm(1 - \sigma'))^2 \\ &\quad = (p_\pm + q_\pm)^2\,(1 - \sigma'^2) \end{aligned}$$

we obtain by subtracting the half of (V.24) from (V.23)

$$\begin{aligned} \rho(\Gamma) = \tfrac{1}{2} \int_a^b &[(p_+(x, \sigma(x)) + q_+(x, \sigma(x)))^2 \\ &- (p_-(x, \sigma(x)) + q_-(x, \sigma(x)))^2]\,(1 - \sigma'^2(x))\, dx \end{aligned}$$

and by relations (V.7) and (V.20),

$$\rho(\Gamma) = \int_a^b \left(\left| \frac{\partial^+ u}{\partial t}(x, \sigma(x)) \right|^2 - \left| \frac{\partial^- u}{\partial t}(x, \sigma(x)) \right|^2 \right) (1 - \sigma'^2(x))\, dx.$$

As a and b are arbitrary, we obtain (V.22), if (V.21) is fulfilled. ∎

Let u be the solution of P_∞ defined in (IV.15), with associated measure $\mu = \Box u$ and let v be another solution of P_∞, with associated measure $\nu = \Box v$. Let F be the support of ν, and define a set J, analogous to the set I defined in (II.4) by

$$J = \bigcup_{(x,t) \in F} T^+_{x,t}\,. \tag{V.25}$$

Arguing as in Section II, we can see that

$$J = \{(x, t)/t \geqslant \sigma(x)\},$$

where σ is Lipschitz continuous, with Lipschitz constant 1. Moreover, if

$$|\sigma'(x)| < 1$$

then, necessarily

$$(x, \sigma(x)) \in F = \operatorname{supp} \nu$$

and, by condition (IV.11)

$$v(x, \sigma(x)) = \varphi(x). \tag{V.26}$$

Let t be greater than $\sigma(x)$. Then, by the definition of the support of a measure

$$v(x, t) = w(x, t) + \tfrac{1}{2}\nu(T^{-}_{x,t}) > w(x, t). \tag{V.27}$$

As a first stage, we shall show, by contradiction,

LEMMA V.6.

$$\tau(x) \leqslant \sigma(x) \qquad \forall x \in \mathbb{R}.$$

Proof. We need to show that

$$v(x, t) = w(x, t) = u(x, t) \qquad \forall (x, t) \quad \text{such that} \quad t \leqslant \tau(x).$$

Let

$$V = \{(x, t)/v(x, t) > w(x, t)\}$$

and suppose that

$$V \cap I^c \neq \varnothing. \tag{V.28}$$

By (V.27), necessarily

$$V \cap I^c = \{(x, t)/\sigma(x) < t < \tau(x)\},$$

and in particular

$$\nu\,|_{V \cap I^c} = 0.$$

If $\sigma(x)$ is smaller than $\tau(x)$, and if $|\sigma'(x)|$ is smaller than 1, then relation (V.26) holds, and moreover

$$v(x, t) \geqslant w(x, t) \geqslant \varphi(x) \qquad \forall t \in [0, \tau(x)];$$

hence

$$\frac{\partial w}{\partial t}(x, \sigma(x)) = 0 \qquad \text{a.e. on} \quad \{x/0 < \sigma(x) < \tau(x) \text{ and } |\sigma'(x)| < 1\}. \tag{V.29}$$

We may apply Proposition V.5, and obtain

$$\frac{\partial^+ v}{\partial t}(x, \sigma(x)) = 0 \qquad \text{a.e. on} \quad \{x/0 < \sigma(x) < \tau(x) \text{ and } |\sigma'(x)| < 1\}. \tag{V.30}$$

On the other hand, if we argue as in Proposition III.3, we can estimate $\nu|_{I^c}$ by

$$\langle \nu|_{I^c}, \psi \rangle = \int (1 - \sigma'^2(x)) \left(\frac{\partial^+ v}{\partial t}(x, \sigma(x)) - \frac{\partial^- v}{\partial t}(x, \sigma(x)) \right) \psi(x, \sigma(x))\, dx.$$

Using (V.29) and (V.30), we conclude that $\nu|_{I^c} = 0$, and therefore $V \cap I^c$ is empty. This contradicts (V.28). ∎

Let us now apply relation (V.22) to v on the line $t = \tau(x)$.

$$\left| \frac{\partial^+ v}{\partial t}(x, \tau(x)) \right| = \left| \frac{\partial w}{\partial t}(x, \tau(x)) \right| \qquad \text{a.e. on} \quad \{x/\tau(x) > 0, |\tau'(x)| < 1\}. \tag{V.31}$$

We may infer from Lemma II.27 and condition (IV.9) that

$$\begin{aligned} \frac{\partial^+ v}{\partial t}(x, \tau(x)) &= -\frac{\partial w}{\partial t}(x, \tau(x)) \\ &= \frac{\partial^+ u}{\partial t}(x, \tau(x)) \qquad \text{a.e. on} \quad \{x/\tau(x) > 0 \mid \tau'(x)| < 1\}, \end{aligned} \tag{V.32}$$

Moreover, the function $v - u$ satisfies

$$v(x, \tau(x)) - u(x, \tau(x)) = 0 \qquad \text{on} \qquad \mathbb{R}, \tag{V.33}$$

$$\Box(v - u)|_{\{(x,t)/t > \tau(x)\}} \geqslant 0. \tag{V.34}$$

We deduce from (V.32)–(V.34) that

$$v(x, t) - u(x, t) \geqslant 0 \qquad \text{on} \qquad \{(x, t)/t > \tau(x)\}. \tag{V.35}$$

Set

$$\hat{\nu} = \nu|_{\{(x,t)/t > \tau(x)\}},$$

$$\hat{F} = \operatorname{supp} \hat{\nu},$$

and

$$\hat{J} = \bigcup_{(x,t) \in \hat{F}} T^+_{x,t} = \{(x, t)/t \geqslant \hat{\sigma}(x)\}.$$

Suppose $\hat{F}$ is nonvoid. Then if t is greater than $\hat{\sigma}(x)$,

$$v(x, t) = u(x, t) + \tfrac{1}{2}\hat{v}(T^{-}_{x,t}) > u(x, t).$$

If $|\hat{\sigma}'(x)| < 1$, we obtain as in the proof of Lemma V.6 that

$$\varphi(x) = v(x, \hat{\sigma}(x)) \geqslant u((x, \hat{\sigma}(x)) \geqslant \varphi(x);$$

hence

$$\frac{\partial^{+} v}{\partial t}(x, \hat{\sigma}(x)) = 0 \qquad \text{a.e. on} \quad \{x/\hat{\sigma}(x) > \tau(x) \text{ and } |\hat{\sigma}(x)|' < 1\},$$

and we obtain a contradiction when we compute $\hat{v}$ with a formula analogous to (V.31).

Therefore $\hat{F}$ is empty, and the proof of uniqueness is completed. ∎

VI. The Finite String with Fixed Ends

The solution of problem P_∞ will enable us to solve problem P_f, and to give existence and uniqueness theorems analogous to Theorem IV.1.

We are given a function φ such that

$$\varphi'' \geqslant 0 \qquad \text{in the sense of distributions on } (0, L), \tag{VI.1}$$

$$\varphi(0) < 0, \qquad \varphi(L) < 0, \tag{VI.2}$$

and initial data

$$u_0 \in H_0^1(0, L), \tag{VI.3}$$

$$u_1 \in L^2(0, L), \tag{VI.4}$$

with the compatibility conditions

$$u_0(x) \geqslant \varphi(x) \qquad \forall x \in [0, L], \tag{VI.5}$$

$$u_1(x) \geqslant 0 \qquad \text{a.e. on} \quad \{x \in [0, L]/u_0(x) = \varphi(x)\}. \tag{VI.6}$$

Then, the following result holds:

Theorem VI.1. *Under hypotheses* (VI.1)–(VI.6) *there existe a unique function* u *such that*

$$u \in L^\infty(\mathbb{R}^+; H_0^1(0, L)) \cap W^{1,\infty}(\mathbb{R}^+; L^2(0, L)), \tag{VI.7}$$

$$u(x, 0) = u_0(x) \qquad \forall x \in \mathbb{R}, \tag{VI.8}$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x) \qquad \forall x \in \mathbb{R}, \tag{VI.9}$$

$$u(x, t) \geqslant \varphi(x) \qquad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{VI.10}$$

$$\Box u \geqslant 0 \qquad \textit{in the sense of distributions in } \mathbb{R} \times \mathbb{R}^+_*, \tag{VI.11}$$

$$\operatorname{Supp} \Box u \subset \{(x, t)/u(x, t) = \varphi(x)\}, \tag{VI.12}$$

$$\left\{\frac{\partial}{\partial x}\left(-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2\right) = 0 \right. \tag{VI.13}$$

$$\textit{in the sense of distributions in } \mathbb{R} \times \mathbb{R}^+_* .$$

VI.1. *Existence*

Extend the function φ to all of $\mathbb{R}$ by setting

$$\hat{\varphi}(x) = \varphi(0) + x\varphi'(0^+) \qquad \text{if} \qquad x \leqslant 0,$$

$$\hat{\varphi}(x) = \varphi(L) + (x - L)\,\varphi'(L^-) \qquad \text{if} \qquad x \geqslant L.$$

Extend the functions u_0 and u_1 to all of $\mathbb{R}$ by setting

$$u_i(x) = u_i(-x),$$

$$u_i(x + 2L) = u_i(x)$$

for $i = 0, 1$.

Define initial data $\hat{u}_0$, $\hat{u}_1$ on $\mathbb{R}$ by

$$\hat{u}_0(x) = \max(u_0(x), \varphi(x)), \tag{VI.14}$$

$$\begin{aligned} \hat{u}_1(x) &= u_1(x) \qquad \text{if} \qquad u_0(x) > \varphi(x), \\ &= 0 \qquad\quad \text{if} \qquad u_0(x) \leqslant \varphi(x). \end{aligned} \tag{VI.15}$$

It is elementary to check that $\hat{u}_0$, $\hat{u}_1$, and $\hat{\varphi}$ satisfy conditions (IV.1) to (IV.5). Therefore, by Theorem IV.1, problem P_∞ possesses a unique solution $\hat{u}$. Our goal is to show that $\hat{u}$, restricted to $]0, L[\times]0, \alpha[$ solves problem P_f if α is suitably chosen.

Let w be the free solution of the wave equation in all of $\mathbb{R}$, with periodic data u_0 and u_1. This function w is classically known to be odd and $2L$-periodic.

There exists a positive number α such that

$$|x| < \alpha \qquad \text{or} \qquad |L - x| < \alpha \Rightarrow w(x, t) - \varphi(x) > 0. \tag{VI.16}$$

Indeed

$$w(x,t)-\varphi(x)$$
$$\geqslant w(0,t)-\varphi(0)-|x|^{1/2}\left(\int_0^t\left|\frac{\partial w}{\partial x}(x,t)+\varphi'(x)\right|^2 dx\right)^{1/2}-|x|\,|\varphi'(0^+)|\,, \tag{VI.17}$$

$$w(x,t)-\varphi(x)$$
$$\geqslant w(L,t)-\varphi(L)$$
$$-|L-x|^{1/2}\left(\int_0^L\left|\frac{\partial w}{\partial x}(x,t)+\varphi'(x)\right|^2 dx\right)^{1/2}-(L-x)\,|\varphi'(L^-)|\,, \tag{VI.18}$$

which supplies a possible α,

$$\alpha^{1/2}=\min\left[\frac{-\varphi(0)}{E^{1/2}+(\int_0^L\varphi'(x)^2\,dx)^{1/2}+|\varphi'(0^+)|\,L^{1/2}}\,,\right.$$
$$\left.\frac{-\varphi(L)}{E^{1/2}+(\int_0^L\varphi'(x)^2\,dx)^{1/2}+|\varphi'(L^-)|\,L^{1/2}}\right], \tag{VI.19}$$

where

$$E=\int_0^L\left(\left|\frac{du_0}{dx}\right|^2+|u_1|^2\right)dx.$$

Let $\hat{w}$ be the free solution of the wave equation with data $\hat{u}_0$, $\hat{u}_1$. Then, by (VI.16),

$$\hat{w}=w \quad \text{in} \quad T^-_{L/2,L/2+\alpha}=\{(x,t)/0\leqslant t\leqslant L/2+\alpha-|L/2-x|\}. \tag{VI.20}$$

Besides, the influence line of the obstacle $\hat{\varphi}$, with respect to $\hat{w}$, satisfies

$$\begin{aligned}&\hat{\tau}'(x)=-1 \quad \text{if} \quad x\in[0,\alpha] \quad \text{and} \quad \hat{\tau}(x)<x+\alpha,\\ &\hat{\tau}'(x)=+1 \quad \text{if} \quad x\in[L-\alpha,L+\alpha] \quad \text{and} \quad \hat{\tau}(x)<L+\alpha-x.\end{aligned} \tag{VI.21}$$

To prove this, define the sets

$$\hat{E}=\overline{\{(x',t')/\hat{w}(x',t')<\hat{\varphi}(x')\}}$$

and

$$\hat{E}'=\{(x',t')\in\hat{E} \text{ and } x'\geqslant\alpha\}.$$

Then

$$\hat{\tau}(x)=\min\{|x-x'|+t'/(x',t')\in\hat{E}\}.$$

If (x', t') is in $\hat{E}$, then by (VI.16) and (VI.20)

$$x' \geqslant \alpha \qquad \text{or} \qquad t' \geqslant x' + \alpha.$$

Then, if (x', t') is in $\hat{E}\backslash\hat{E}'$,

$$t' + | x - x' | \geqslant t' + x - x' \geqslant x + \alpha.$$

By the assumption that $\hat{\tau}(x) < x + \alpha$, we see that

$$\begin{aligned} \hat{\tau}(x) &= \min\{| x - x' | + t'/(x', t') \in \hat{E}'\} \\ &= \min\{t' + x'/(x', t') \in \hat{E}\} - x, \end{aligned}$$

and this proves (VI.21). Relation (VI.22) is similarly proved.

Note that $\hat{u}$ satisfies the boundary conditions, for $t \leqslant \alpha$. Indeed

$$\hat{u}(x, t) = \hat{w}(x, t) \qquad \text{if} \qquad t \leqslant \hat{\tau}(x);$$

then, thanks to (VI.21) and (VI.22),

$$\hat{u}(x, t) = \hat{w}(x, t) \qquad \text{if} \qquad (x, t) \in [0, L] \times [0, \alpha].$$

Then by (VI.20), and since w is odd and $2L$-periodic,

$$\hat{u}(0, t) = \hat{u}(L, t) \qquad \text{if} \qquad 0 \leqslant t \leqslant \alpha. \tag{VI.22}$$

Being a solution of P_∞, and satisfying the boundary conditions (VI.22), the function $\hat{u}$ solves problem P_f in the time interval $[0, \alpha]$.

The procedure of constructing a solution can be resumed, if we note that

$$\begin{gathered} \hat{u}(x, \alpha) \geqslant \hat{\varphi}(x) \qquad \text{on} \qquad [0, L], \\ \hat{u}(\cdot, \alpha) \in H_0(0, L), \qquad \frac{\partial^+ \hat{u}}{\partial t}(\cdot, \alpha) \in L^2(0, L), \end{gathered}$$

and moreover, the following equality holds:

$$\begin{aligned} &\int_0^L \left(\left| \frac{\partial \hat{u}}{\partial x}(x, t) \right|^2 + \left| \frac{\partial \hat{u}}{\partial t}(x, t) \right|^2 \right) dx \\ &\qquad = \int_0^L \left(\left| \frac{du_0}{dx} \right|^2 + | u_1 |^2 \right) dx \qquad \text{a.e. on } [0, \alpha]. \end{aligned} \tag{VI.23}$$

(Integrate $\nabla \cdot S_{\hat{u}}$ on $[0, L] \times [0, t]$.)

Taking as new initial data $\hat{u}(\cdot, \alpha)$ and $\hat{u}_t(\cdot, \alpha)$, we can solve again a problem P_f in a bounded time interval $[\alpha, \alpha' + \alpha]$. But the number α' depends only on the energy of the data, and therefore can be taken to be equal to α.

By recurrence problem P_f can be solved on the whole infinite rectangle $[0, L] \times \mathbb{R}^+$; relation (VI.7) holds, because (VI.23) is true almost everywhere on $\mathbb{R}^+$. ∎

VI.2. *Uniqueness*

Let v be a solution of (VI.7)–(VI.13). We shall extend v to $\mathbb{R}^+ \times [0, \alpha/2]$ as follows.

Let $\hat{u}$ be the solution of problem P_∞ with initial data (VI.14), (VI.15), and the obstacle $\hat{\varphi}$.

Set

$$\hat{v}(x, t) = v(x, t) \qquad \text{if} \qquad \max(0, |L/2 - x| - L/2 - \alpha/2) \leqslant t \leqslant \alpha/2,$$
$$\hat{v}(x, t) = \hat{u}(x, t) \qquad \text{otherwise.}$$

Then, using (VI.16), we can check that $\hat{v}$ is a solution of P_∞ with data $\hat{u}_0$, $\hat{u}_1$, and $\hat{\varphi}$.

The theorem of uniqueness for P_∞ shows that necessarily v is equal to $\hat{u}$ in $[0, L] \times [0, \alpha/2]$. This process can be iterated any number of times to prove uniqueness in $[0, L] \times \mathbb{R}^+$. ∎

Appendix

Throughout this appendix, we will denote by σ a Lipschitz-continuous function, with Lipschitz constant 1, defined on $\mathbb{R}$.

A.1. *Elementary Results about Integration by Substitution*

Lemma A.1. *The following equivalence holds:*

(i) $f \in L^1_{\text{loc}}((I + \sigma)\,\mathbb{R}, dx)$,

(ii) $f \circ (I + \sigma) \in L^1_{\text{loc}}(\mathbb{R}, (1 + \sigma')\,dx)$.

Moreover, if one of conditions (i) *or* (ii) *is satisfied, then the identity*

$$\int_a^b f(x + \sigma(x))\,(1 + \sigma'(x))\,dx = \int_{a+\sigma(a)}^{b+\sigma(b)} f(x)\,dx \tag{A.1}$$

is true for arbitrary real numbers a and b.

Proof. If ψ is an absolutely continuous, increasing function, with domain $[a, b]$ and image $[\alpha, \beta]$, then

$$f \in L^1(\alpha, \beta) \Rightarrow f \circ \psi \cdot \psi' \in L^1(a, b),$$

and

$$\int_\alpha^\beta f(x)\, dx = \int_a^b (f \circ \psi)(x)\, \psi'(x)\, dx. \tag{A.2}$$

This classical result is proved for instance by Hewitt and Stromberg [10, 18.24, 18.25, and 20.5].

Applying this result to

$$\psi = I + \sigma, \tag{A.3}$$

$[a, b]$ an arbitrary bounded interval,

$$[\alpha, \beta] = [a + \sigma(a), b + \sigma(b)],$$

the implication (i) $\Rightarrow$ (ii) obtains, with identity (A.1).

The inverse of $I + \sigma = \psi$ can be multivalued; in this lies the difficulty of the converse.

The first point to check is that if $f \circ \psi$ is measurable with respect to the measure $\psi'\, dx$, then f is measurable with respect to dx.

For this purpose, let

$$M = \{x / (f \circ \psi)(x) \geqslant a\}.$$

Then

$$\psi(M) = \{x / f(x) \geqslant a\}.$$

Set M is measurable with respect to $\psi'\, dx$. Therefore, there exists for any positive ϵ and compact K, a compact set K' such that

$$1_M |_{K'} \qquad \text{is continuous;}$$

$$\int 1_{K \setminus K'}(x)\, \psi'(x)\, dx \leqslant \epsilon.$$

As the function ψ^{-1} defined almost everywhere is increasing, it is measurable. Let K_1 be any compact set in $\mathbb{R}$, and ϵ_1 be any positive number. Then, there exists a compact K_1' such that

$$\psi^{-1} |_{K_1'} \qquad \text{is continuous;}$$

$$\int 1_{K_1 \setminus K_1'}\, dx \leqslant \epsilon_1 .$$

Then $1_{\psi(M)} = 1_M \circ \psi^{-1}$, restricted to the compact set $K_1' \cap \psi(K')$, is continuous. Besides, the measure of $K_1 \cap \psi(K) \dot{\cup} (K_1' \cap \psi(K'))$ is majorized by $\epsilon + \epsilon_1$. Therefore, $\psi(M)$ is measurable. Now the proof of (ii) $\Rightarrow$ (i) is easy by contradiction: Suppose that f is not locally integrable. Then there exists a sequence of integrable functions f_n converging to f almost everywhere on $[a + \sigma(a), b + \sigma(b)]$ such that

$$\int_{a+\sigma(a)} |f_n(x + \sigma(x))|\, dx \to +\infty;$$

$$|f_n(x)| \leqslant |f(x)| \qquad \forall n, \quad \text{a.e. in} \quad x.$$

This gives a contradiction, using identity (A.1). ∎

COROLLARY A.2. *Let f be in $W^{1,1}_{\text{loc}}(II + \sigma)\, \mathbb{R}$), then $f \circ (I + \sigma)$ is absolutely continuous on any compact interval of $\mathbb{R}$ and*

$$\frac{d}{dx}(f \circ (I + \sigma))(x) = f'(x + \sigma(x))(1 + \sigma'(x)). \tag{A.4}$$

Proof. By Lemma A.1 we have

$$\int_{\psi(a)}^{\psi(b)} f'(x)\, dx = \int_a^b f'(\psi(x))\, \psi'(x)\, dx \tag{A.5}$$

if $\psi = I + \sigma$, and $[a, b]$ is an arbitrary compact interval.

The left-hand side of (A.5) can be expressed as

$$\int_{\psi(a)}^{\psi(b)} f'(x)\, dx = f(\psi(b)) - f(\psi(a))$$

since f is absolutely continuous. This proves Corollary A.2. ∎

A.2. *The Set $t = \sigma(x)$ in Characteristic Coordinates*

The characteristic coordinates (ξ, η) are defined as

$$\xi = \frac{x + t}{2^{1/2}}, \qquad \eta = \frac{-x + t}{2^{1/2}}. \tag{A.6}$$

The graph of σ in characteristic coordinates is the set

$$\varphi = \left\{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} \Big/ \frac{\xi + \eta}{2^{1/2}} = \sigma\left(\frac{\xi - \eta}{2^{1/2}}\right)\right\}.$$

We shall write $Y^{-1} = X$ and

$$\eta \in Y(\xi) \Leftrightarrow \frac{\xi + \eta}{2^{1/2}} = \sigma\left(\frac{\xi - \eta}{2^{1/2}}\right).$$

The set Y has the following property.

LEMMA A.3. *Y is a decreasing graph.*

Proof. Let η belong to $Y(\xi)$ and η' to $Y(\xi')$, and set

$$x = \frac{\xi - \eta}{2^{1/2}}, \qquad x' = \frac{\xi' - \eta'}{2^{1/2}}.$$

Then

$$(\xi - \xi')(\eta - \eta') = \tfrac{1}{2}[x + \sigma(x) - x' - \sigma(x')][\sigma(x) - x - \sigma(x') + x']$$

and this expression is nonpositive for all x and x'. ∎

We shall denote by $\overline{Y}$ the function defined on the set

$$\{\xi / Y(\xi) \text{ is one-valued}\}$$

by

$$\{\overline{Y}(\xi)\} = Y(\xi),$$

and likewise, we define $\overline{X}$ as the "one-valued part" of $X = Y^{-1}$.

It is possible to show that Y is maximal monotone (for a definition see Brezis [5]) in $\mathbb{R} \times \mathbb{R}$.

A.3. *Relation between the Continuity of a Function u, and the Structure of $\Box u$*

LEMMA A.4. *Let u be a function of x and t such that $\Box u$ is a measure. Then, if $u = \Box u * \mathscr{E}$, the two following properties are equivalent.*

(i) *u is continuous.*

(ii) *The parts of characteristics are null with respect to $\Box u$.*

Proof. Let $\Box u = \mu^+ - \mu^-$ be the decomposition of $\Box u = \mu$ in its positive and negative parts.

Let us prove first (i) ⇒ (ii).

Take R_n to be a sequence of open characteristic rectangles with vertices L_n, M_n, P_n, Q_n such that

$$\bigcap_n R_n = R_\infty, \qquad \text{a closed segment of characteristic} \qquad R_n \supset \overline{R}_{n+1}.$$

Suppose for instance that R_∞ is a segment of ξ-characteristic, with vertices $L_\infty = M_\infty$ and $P_\infty = Q_\infty$. Then

$$L_n \to L_\infty,$$
$$M_n \to M_\infty = L_\infty,$$
$$P_n \to P_\infty,$$
$$Q_n \to Q_\infty = P_\infty.$$

The following inequalities hold:

$$\mu^+(R_n) - \mu^-(\bar{R}_n) \leqslant u(L_n) - u(M_n) - u(Q_n) + u(P_n) \leqslant \mu^+(R_n) - \mu^-(\bar{R}_n) \tag{A.7}$$

If u is continuous, we can pass to the limit in (A.7), and

$$\mu^+(R_\infty) - \mu^-(R_\infty) \leqslant 0 \leqslant \mu^+(R_\infty) - \mu^-(R_\infty),$$

which shows that $\mu(R_\infty) = 0$.

Conversely, to prove (ii) $\Rightarrow$ (i), suppose that the parts of characteristics are null with respect to $\Box u$, and fix an arbitrary point (x_0 , t_∞) with $t_\infty > 0$.

Let t_n be a sequence decreasing to t_∞, and t'_n a sequence increasing to t_∞. Then

$$\bigcap_n T^-_{x_0,t_n} = T^-_{x_0,t_\infty}, \tag{A.8}$$

$$\bigcup_n T^-_{x_0,t'_n} = \text{int } T^-_{x_0,t_\infty}. \tag{A.9}$$

As the parts of characteristics are $\Box u$-null,

$$\Box u(\text{int } T^-_{x_0,t_\infty}) = \Box u(T^-_{x_0,t_\infty}).$$

Therefore, thanks to (A.8) and (A.9),

$$\lim_{t_n \to t_\infty} u(x_0 , t_n) = \lim_{t'_n \to t_\infty} u(x_0 , t'_n) = u(x_0 , t_\infty).$$

This shows that $(\mathscr{E} * \mu^\pm)(x, t)$ is continuous with respect to t in any point. To prove the continuity of u with respect to x and t, just note the inclusions

$$T^-_{x_0,t-|x-x_0|} \subset T^-_{x,t} \subset T^-_{x_0,t+|x-x_0|}.$$

We have thus the inequalities

$$\mu^+(T^-_{x_0,t-|x-x_0|}) - \mu^-(T^-_{x_0,t+|x-x_0|}) \leqslant (\mathscr{E} * \mu)(x, t) \leqslant \mu^+(T^-_{x_0,t+|x-x_0|}) - \mu^-(T^-_{x_0,t-(x-x_0)}),$$

and this completes the proof of Lemma A.4.

References

1. L. Amerio, Su un problema di vicoli unilaterali per l'equazione non omogenea della corda vibrante, *IAC (Ist. Appl. Calc. "Mauro Picone") Publ., Ser. D* **109** (1976), 3–11.

2. L. Amerio, On the motion of a string vibrating through a moving ring with a continuously variable diameter, *Att. Acad. Naz. Lincei Rend.* **62** (1977), 134–142.
3. L. Amerio and G. Prouse, Study of the motion of a string vibrating against an obstacle, *Rend. Mat.* **2** (1975), 563–585.
4. A. Bamberger and M. Schatzman, Étude d'un problème hyperbolique avec contrainte unilatérale, to appear.
5. H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert," North–Holland and American Elsevier, Amsterdam/London/New York, 1973.
6. C. Citrini, Sull'urto parzialmente elastico o anelastico di una corda vibrante contro un ostacolo, *Atti Accad. Naz. Lincei Rend.* **59**, 5 (1975), 368–376.
7. C. Citrini, Sull'urto parzialmente elastico o anelastico di una corda vibrante contro un ostacolo, *Atti Accad. Naz. Lincei Rend.* **59**, 6 (1975), 667–676.
8. C. Citrini, The energy theorem in the impact of a string vibrating against a point shaped obstacle, *Atti Accad. Naz. Lincei Rend.* **62** (1977), 143–149.
9. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Interscience, New York/London, 1962.
10. E. Hewitt and K. Stromberg, "Real and Abstract Analysis," Springer–Verlag, New York/Heidelberg/Berlin, 1965.
11. L. Landau and E. Lifschitz, "Théorie du champ," Éditions de la Paix, Moscow.
12. M. Schatzman, Un problème hyperbolique du 2ème ordre avec contrainte unilatérale: La corde vibrante avec obstacle ponctuel, to appear.
13. L. Schwartz, "Théorie des distributions," Hermann, Paris, 1966.
14. L. Amerio and G. Prouse, paper to appear; a lecture in the Collège de France was delivered on the subject by L. Amerio in June, 1978.