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A STOCHASTIC HAMILTON-JACOBI EQUATION WITH INFINITE SPEED OF PROPAGATION

PAUL GASSIAT

Abstract. We give an example of a stochastic Hamilton-Jacobi equation $du = H(Du)d\xi$ which has an infinite speed of propagation as soon as the driving signal ξ is not of bounded variation.

1. introduction

An important feature of (deterministic) Hamilton-Jacobi equations

$$\partial_t u = H(Du) \quad \text{on } (0,T)\times\mathbb{R}^N \tag{1.1}$$

is the so-called *finite speed of propagation* : assuming for instance that $H:\mathbb{R}^N\to\mathbb{R}$ is C-Lipschitz, then if u^1 and u^2 are two (viscosity) solutions of (1.1), one has

$$u^1(0,\cdot)=u^2(0,\cdot) \text{ on } B(R) \;\Rightarrow\; \forall t\geq 0,\; u^1(t,\cdot)=u^2(t,\cdot) \text{ on } B(R-Ct) \tag{1.2}$$

where by $B(R)$ we mean the ball of radius R centered at 0.

In this note, we are interested in Hamilton-Jacobi equations with rough time dependence of the form

$$\partial_t u = H(Du)\dot{\xi}(t) \quad \text{on } (0,T)\times\mathbb{R}^N, \tag{1.3}$$

where ξ is only assumed to be continuous. Of course, the above equation only makes classical (viscosity) sense for ξ in C^1, but Lions and Souganidis [2] have shown that if H is the difference of two convex functions, the solution map can be extended continuously (with respect to supremum norm) to any continuous ξ. (In typical applications, one wants to take ξ as the realization of a random process such as Brownian motion).

In fact, the Lions-Souganidis theory also gives the following result : if $H=H_1-H_2$ where H_1, H_2 are convex, C-Lipschitz, with $H_1(0)=H_2(0)=0$, then for any constant A,

$$u(0,\cdot)\equiv A \text{ on } B(R) \;\Rightarrow u(t,\cdot)\equiv A \text{ on } B(R(t))$$

where $R(t)=R-C(\max_{s\in[0,t]}\xi(s)-\min_{s\in[0,t]}\xi(s))$.

However this does not imply a finite speed of propagation for (1.3) for arbitrary initial conditions, and a natural question (as mentioned in lecture notes by Souganidis [3]) is to know whether a property analogous to (1.2) holds in that case. The purpose

of this note is to show that in general it does not : we present an example of an H such that if the total variation of ξ on $[0,T]$ is strictly greater than R, one may find initial conditions u_0^1, u_0^2 which coincide on $B(R)$, but such that for the associated solutions u^1 and u^2, one has $u^1(T,0) \neq u^2(T,0)$.

For instance, if ξ is a (realization of a) Brownian motion, then (almost surely), one may find initial conditions coinciding on balls of arbitrary large radii, but such that $u^1(t,0) \neq u^2(t,0)$ for all $t > 0$.

It should be noted that the Hamiltonian H in our example is not convex (or concave). When H is convex, some of the oscillations of the path cancel out at the PDE level[1], so that one cannot hope for simple bounds such as (2.2) below. Whether one has finite speed of propagation in this case remains an open question.

2. Main result and proof

We fix $T > 0$ and denote $\mathcal{P} = \{(t_0, \dots, t_n),\ \ 0 = t_0 \leq t_1 \leq \dots \leq t_n = T\}$ the set of partitions of $[0,T]$. Recall that the total variation of a continuous path $\xi : [0,T] \to \mathbb{R}$ is defined by

$$V_{0,T}(\xi) = \sup_{(t_0,\dots,t_n)\in\mathcal{P}} \sum_{i=0}^{n-1} |\xi(t_{i+1}) - \xi(t_i)| \,.$$

Our main result is then :

Theorem 1. *Given $\xi \in C([0,T])$, let $u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be the viscosity solution of*

$$\partial_t u = (|\partial_x u| - |\partial_y u|)\, \dot{\xi}(t) \ \text{ on } (0,T) \times \mathbb{R}^2 \tag{2.1}$$

with initial condition

$$u(0,x,y) = |x-y| + \Theta(x,y)$$

where $\Theta \geq 0$ is such that $\Theta(x,y) \geq 1$ if $\min x, y \geq R$.

One then has the estimate

$$u(T,0,0) \geq \left(\sup_{(t_0,\dots,t_n)\in\mathcal{P}} \frac{\sum_{j=0}^{n-1} |\xi(t_{j+1}) - \xi(t_j)|}{n} - \frac{R}{n} \right)_+ \wedge 1. \tag{2.2}$$

In particular, $u(T,0,0) > 0$ as soon as $V_{0,T}(\xi) > R$.

Note that since $|x-y|$ is a stationary solution of (2.1), the claims from the introduction about the speed of propagation follow.

The proof of Theorem 1 is based on the differential game associated to (2.1). Informally, the system is constituted of a pair (x,y) and the two players take turn controlling x or y depending on the sign of $\dot{\xi}$, with speed up to $|\dot{\xi}|$. The minimizing

[1]for example : for $\delta \geq 0$, $S_H(\delta) \circ S_{-H}(\delta) \circ S_H(\delta) = S_H(\delta)$, where S_H, S_{-H} are the semigroups associated to H, $-H$.

player wants x and y to be as close as possible to each other, while keeping them smaller than R. The idea is then that if the minimizing player keeps y and x stuck together, the maximizing player can lead x and y to be greater than R as long as $V_{0,T}(\xi) > R$.

Proof of Theorem 1. By approximation we can consider $\xi \in C^1$, and in fact we consider the backward equation :

$$\begin{cases} -\partial_t v & = (|\partial_x v| - |\partial_y v|)\,\dot{\xi}(t), \\ v(T,x,y) & = |x-y| + \Theta(x,y). \end{cases} \tag{2.3}$$

We then need a lower bound on $v(0,0,0)$. Note that

$$(|\partial_x v| - |\partial_y v|)\,\dot{\xi}(t) = \sup_{|a|\le 1}\inf_{|b|\le 1}\left\{\dot{\xi}_+(t)\,(a\partial_x u + b\partial_y u) + \dot{\xi}_-(t)\,(a\partial_y u + b\partial_s u)\right\},$$

so that by classical results (e.g. [1]) one has the representation

$$v(0,0,0) = \sup_{\delta(\cdot)\in\Delta}\inf_{\beta\in\mathcal{U}} J(\delta(\beta),\beta), \tag{2.4}$$

where $\mathcal{U}$ is the set of controls (measurable functions from $[0,T]$ to $[-1,1]$) and Δ the set of progressive strategies (i.e. maps $\delta : \mathcal{U} \to \mathcal{U}$ such that if $\beta = \beta'$ a.e. on $[0,t]$, then $\delta(\beta)(t) = \delta(\beta')(t)$). Here for $\alpha, \beta \in \mathcal{U}$, the payoff is defined by

$$J(\alpha,\beta) = |x^{\alpha,\beta}(T) - y^{\alpha,\beta}(T)| + \Theta(x^{\alpha,\beta}(T), y^{\alpha,\beta}(T))$$

where

$$x^{\alpha,\beta}(0) = y^{\alpha,\beta}(0) = 0, \quad \dot{x}^{\alpha,\beta}(s) = \dot{\xi}_+(s)\alpha(s) + \dot{\xi}_-(s)\beta(s), \quad \dot{y}^{\alpha,\beta}(s) = \dot{\xi}_-(s)\alpha(s) + \dot{\xi}_+(s)\beta(s).$$

Assume $v(0,0,0) < 1$ (otherwise there is nothing to prove) and fix $\varepsilon \in (0,1)$ such that $v(0,0,0) < \varepsilon$. Consider the strategy δ^ε for the maximizing player defined as follow : for $\beta \in \mathcal{U}$, let

$$\tau_\varepsilon^\beta = \inf\left\{t \ge 0, \quad |x^{1,\beta}(t) - y^{1,\beta}(t)| \ge \varepsilon\right\},$$

and then

$$\delta^\varepsilon(\beta)(t) = \begin{cases} 1, & t < \tau_\varepsilon^\beta \\ \beta(t), & t \ge \tau_\varepsilon^\beta \end{cases}.$$

In other words, the maximizing player moves to the right at maximal speed, until the time when $|x-y| = \varepsilon$, at which point he moves in a way such that x and y stay at distance ε.

Now by (2.4), there exists $\beta \in \mathcal{U}$ with $J(\delta^\varepsilon(\beta),\beta) < \varepsilon$. Clearly for the corresponding trajectories $x(\cdot), y(\cdot)$, this means that $|x(T) - y(T)| < \varepsilon$, and by definition of α^ε

this implies $|x(t)-y(t)|\le\varepsilon$ for $t\in[0,T]$. We now fix $(t_0,\dots,t_n)\in\mathcal{P}$ and prove by induction that for $i=0,\dots,n$,

$$\min\{x(t_i),y(t_i)\}\ge\sum_{j=0}^{i-1}|\xi(t_{j+1})-\xi(t_j)|-i\varepsilon.$$

Indeed, if it is true for some index i, then assuming that for instance $\xi(t_{i+1})-\xi(t_i)\ge 0$, one has

$$\begin{aligned} x(t_{i+1}) &= x(t_i)+\int_{t_i}^{t_{i+1}}\dot\xi_+(s)ds-\int_{t_i}^{t_{i+1}}\beta(s)\dot\xi_-(s)ds \\ &\ge x(t_i)+\xi(t_{i+1})-\xi(t_i)\ \ge\ \sum_{j=0}^{i}|\xi(t_{j+1})-\xi(t_j)|-i\varepsilon \end{aligned}$$

and since $y(t_{i+1})\ge x(t_{i+1})-\varepsilon$, one also has $y(t_{i+1})\ge\sum_{j=0}^{i}|\xi(t_{j+1})-\xi(t_j)|-(i+1)\varepsilon$. The case when $\xi(t_{i+1})-\xi(t_i)\le 0$ is similar.

Since $J(\alpha^\varepsilon(\beta),\beta)\le 1$, one must necessarily have $\min\{x(T),y(T)\}\le R$, so that

$$\varepsilon\ge\frac{1}{n}\left(\sum_{j=0}^{n}|\xi(t_{j+1})-\xi(t_j)|-R\right).$$

Letting $\varepsilon\to v(0,0,0)$ and taking the supremum over $\mathcal{P}$ on the r.h.s. we obtain (2.2). □

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