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RANDOM WALK ON A PERTURBATION OF THE INFINITELY-FAST MIXING INTERCHANGE PROCESS

MICHELE SALVI AND FRANÇOIS SIMENHAUS

Abstract. We consider a random walk in dimension $d \geq 1$ in a dynamic random environment evolving as an interchange process with rate $\gamma > 0$. We only assume that the annealed drift is non–zero. We prove that, if we choose γ large enough, almost surely the empirical velocity of the walker X_t/t eventually lies in an arbitrary small ball around the annealed drift. This statement is thus a perturbation of the case $\gamma = +\infty$ where the environment is refreshed between each step of the walker. We extend three-way part of the results of [HS15], where the environment was given by the 1–dimensional exclusion process: (i) We deal with any dimension $d \geq 1$; (ii) We treat the much more general interchange process, where each particle carries a transition vector chosen according to an arbitrary law μ; (iii) We show that X_t/t is not only in the same direction of the annealed drift, but that it is also close to it.

AMS subject classification (2010 MSC): 60K37, 82C22, 60Fxx, 82D30.

Keywords: Random walk, dynamic random environment, interchange process, limit theorems, renormalisation.

1. Introduction, model and result

We are interested in a Markovian walker on $\mathbb{Z}^d$ whose transitions are given by an underlying random environment. The environment itself evolves in time according to an interchange process, also called random stirring process in the physical literature: We initially put a single particle on each site of $\mathbb{Z}^d$ and suppose the particles carry a transition-probability vector chosen in an i.i.d. way from some law μ. We attach independent Poisson clocks with parameter $\gamma > 0$ on each edge of $\mathbb{Z}^d$ and, when the clock attached to $\{x, y\}$ rings, the system is updated by exchanging the particles sitting in x and y. We consider a time-discrete walker $(X_t)_{t\in\mathbb{N}}$ on top of this system: For each jump it chooses a direction according to the transition-probability vector carried by the particle on which it sits. We assume that the expected drift of the transition-probability vectors is non-zero. If the environment is completely refreshed at each step of (X_t) (corresponding to $\gamma = +\infty$), the walker evolves in a homogenous environment and thus admits the expected drift as asymptotic velocity. We are concerned here with a perturbation of this regime. We actually prove that, choosing γ large enough, the empirical velocity of the walker X_t/t eventually lives in an arbitrary small ball around the expected drift.

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While the problem of the asymptotic velocity for random walks in random media has been widely studied in the case of static environments (see for example [BCR16] and references therein for the general i.i.d. setting and [BS13] for the reversible case), the case of media that evolve in time has gained a great attention from the mathematical and physical communities only in recent years. The typical examples are that of the tracer (or tagged) particle in lattice gases (as in [Spo90]) or that of random walks moving on top of an interacting particle system (such as a spin flip dynamics or a conservative particle systems). When the environment dynamics has good space-time mixing conditions, traps, i.e. "unfavorable" regions of the environment, tend to dissolve on a shorter time scale than the one that is interesting for the analysis of the walker, and therefore in many cases it has been possible to derive a law of large numbers (see, e.g., [DKL08] for environments given by a space–time mixing Markov chain, [AdHR11, dHdSS13], where the so-called cone-mixing and conditional cone-mixing conditions are introduced, [ABF16a, ABF16b] for perturbations of walks on a class of dynamic environments satisfying the Poincaré inequality, or [BZ06, dHdS14] for results on specific models). It appears more complicated to study the asymptotic properties of a walker moving in an environment that mixes slowly, for example when the underlying dynamics is conservative: Some results have been lately obtained for a walk on the simple symmetric exclusion process with strong drift conditions in [AdSV13], for a walk moving on a collection of independent particles performing simple symmetric random walks in [HdHdS+15, BHDST17a, BHDST17b] and for rather general ergodic Markovian environments with sufficiently fast polynomial mixing in [RV13]. In the slow-mixing setting, the authors of [HS15] consider a one-dimensional dynamic random environment evolving as the Simple Symmetric Exclusion Process (SSEP) with jump parameter γ. A nearest-neighbor walker moves on this environment at integer times, jumping to the right (left) with probability $\alpha > 0$ (resp. $1-\alpha$) if it finds itself on a site occupied by a particle, and to the right (left) with probability $\beta > 0$ (resp. $1-\beta$) if it is on an empty site. The authors show that if in the case $\gamma = +\infty$ the walker has a strictly positive limiting speed, then, for γ big enough, the walk will admit a strictly positive limiting speed, too. They also prove an analogous result (requiring only transience of the walk for $\gamma = 0$) for γ small enough.

In the present paper we consider a walker moving on the interchange process, or random stirring process. This interacting particle system often appears in studies of the quantum Heisenberg ferromagnetic model and its cycle and loop representations (see e.g. [Tó93] or [GUW11] for a review). Even if we do not prove a full law of large numbers, the present work can be thought as a generalization of the result about the perturbation of the $\gamma = +\infty$ regime in [HS15] (see Theorem 1.1 therein) in three different directions:

(i) We pass from dimension $d = 1$ to $d \geq 1$. We point out that most of the results for random walks in dynamic random environment have been achieved so far for dimension $d = 1$, especially in the slowly-mixing dynamics framework;

(ii) Instead of considering the SSEP for the environment dynamics, we deal with the interchange process. In particular, while in [HS15] only two possible transition-probability vectors are allowed, initially chosen with a Bernoulli product measure, in the present article we consider any vector coming from the d-dimensional simplex, sampled at time 0 with an arbitrary product measure $\mu^{\otimes \mathbb{Z}^d}$;

(iii) While in [HS15] it is proven that, for γ sufficiently large, X_t/t is just eventually positive in the average drift direction, we prove here that, eventually, X_t/t stays in fact close to the average drift.

We will follow the general strategy of [HS15], readapting and enhancing its techniques into our much more general setting. In particular, we will use concentration inequalities to control the environment behavior and a renormalization procedure for obtaining the asymptotics of the walker. We postpone to a future work the study of the full law of large numbers and of the CLT. The major obstacle is due to the high dimensionality of the problem, which makes rather hard to study the frequency of times when the walker is "ahead" of any particle it visited. This prevents one from constructing a proper renewal structure (see e.g. [AdSV13, HdHdS+15, HS15]), which is the main tool to derive such theorems.

1.1. **Notation, model and main result.** We call $\mathbb{N} := \{0, 1, ...\}$ the set of natural numbers. We write $\mathcal{I} := \{-1, \cdots, -d\} \cup \{1, \cdots, d\}$ and define $\mathcal{N} := (e_i)_{i \in \mathcal{I}}$, where, for each $i = 1, ..., d$, e_i denotes the i-th element of the canonical basis of $\mathbb{R}^d$, while $e_{-i} := -e_i$. We define $\mathcal{S} := \{s \in [0,1]^{\mathcal{I}} : s_1 + ... + s_d + s_{-1} + ... + s_{-d} = 1\}$ to be the simplex of possible transition-probability vectors and we associate to each $s \in \mathcal{S}$ its drift

$$D(s) := \sum_{i \in \mathcal{I}} s_i \, e_i \, .$$

We denote by $\Omega := (\mathcal{S}^{\mathbb{Z}^d})^{\mathbb{R}_+}$ the space of space-time environments. We think of an element ω as a time-indexed sequence of environments, where an environment is a collection of transition-probability vectors associated to each point of the lattice: $\omega_y(t) \in \mathcal{S}$ is the transition-probability vector in the site $y \in \mathbb{Z}^d$ at time $t \geq 0$, and $\omega_y^i(t)$ represents its i-th coordinate, $i \in \mathcal{I}$.

Given an environment $\omega \in \Omega$ we define the discrete-time random walk $(X_t)_{t \in \mathbb{N}}$ on $\mathbb{Z}^d$ with law P_ω as the process defined by the following: For all $k \in \mathbb{N}$, $y \in \mathbb{Z}^d$ and $i \in \mathcal{I}$ we have

$$\begin{aligned} &P_\omega(X_0 = 0) = 1 \, , \\ &P_\omega(X_{k+1} = y + e_i \,|\, X_k = y) = \omega_y^i(k) \, . \end{aligned}$$

Notice that, albeit ω has been defined for each positive real time t, the walk $(X_t)_{t \in \mathbb{N}}$ is influenced only by the environment at integer times.

We now construct a probability measure on Ω, corresponding to an interchange process where each particle carries a transition-probability vector chosen once and for all at time zero. We let μ be a probability measure on $\mathcal{S}$ and indicate with $\mathbb{E}_\mu[\cdot]$ the expectation w.r.t. μ. Our unique assumption on μ is that

$$\mathbb{E}_\mu [D] \neq 0 \, . \tag{1.1}$$

In particular, we point out that we do not require any kind of ellipticity on the environment. We consider the product law $\pi := \mu^{\otimes \mathbb{Z}^d}$ on $\mathcal{S}^{\mathbb{Z}^d}$. We finally construct a law $\mathbb{P}$ on Ω in the following way: We start from an $\eta \in \mathcal{S}^{\mathbb{Z}^d}$ chosen according to π, and we let it evolve according to the generator

$$\mathcal{L}f(\widetilde{\eta}) = \frac{\gamma}{2} \sum_{x,y\in\mathbb{Z}^d:\, \|x-y\|=1} \big(f(\widetilde{\eta}\circ\sigma_{xy}) - f(\widetilde{\eta})\big),$$

where $\sigma_{xy}(\widetilde{\eta})$ indicates the environment obtained from $\widetilde{\eta} \in \mathcal{S}^{\mathbb{Z}^d}$ by interchanging the particles in x and y, i.e. the values of $\widetilde{\eta}_x$ and $\widetilde{\eta}_y$. We indicate by $\|\cdot\|$ the usual L^2-norm. We indicate with $\mathbb{P}^\eta$ the law of the evolution of the environment started from a given $\eta \in \mathcal{S}^{\mathbb{Z}^d}$, $\mathbb{E}^\eta$ being the corresponding expectation. Notice that we will always use bold characters for indicating probabilities and expectations when we will deal with the sole environment process. In our particle representation, we see that $\mathcal{L}$ is nothing but the generator of the interchange process: We can think to assign to each edge in $\mathbb{Z}^d$ an exponential clock of parameter γ, independent of all the other clocks. Whenever the clock of an edge $\{x, y\}$ rings, we proceed with exchanging the positions of the particles in x and y. We define the law of the random walk in the dynamic environment starting from a given $\eta \in \mathcal{S}^{\mathbb{Z}^d}$ as

$$P^\eta := \mathbb{P}^\eta \times P_\omega\,,$$

and finally the annealed law, where we average over the initial configuration according to π, is

$$P := \mathbb{P} \times P_\omega\,.$$

Our goal is to prove the following theorem:

Theorem 1. *Under assumption (1.1), for all $\varepsilon > 0$, there exists $\gamma(\varepsilon)$ such that, for all $\gamma \geq \gamma(\varepsilon)$, P-a.s. it holds that, for t large enough,*

$$\frac{X_t}{t} \in B_\varepsilon\big(\mathbb{E}_\mu[D]\big)\,,$$

where $B_\varepsilon(\mathbb{E}_\mu[D])$ is the L^2-ball of radius ε around the averaged drift $\mathbb{E}_\mu[D]$.

Remark 1.1. Theorem 1 answers one of the issues left open in [HS15]. In fact, in the case of an environment driven by a one dimensional SSEP,our theorem implies the convergence of the asymptotic velocity $v(\gamma)$ (which was proven to exist in [HS15]) to the annealed drift $\mathbb{E}_\mu[D]$ when γ goes to $+\infty$.

1.2. **Outline of the paper.** In Section 2 we will prepare the technical machinery needed for the proof of Theorem 1. The section is devoted only to the properties of the underlying interchange process: We say that a site $x \in \mathbb{Z}^d$ is L-good for a configuration $\eta \in \mathcal{S}^{\mathbb{Z}^d}$ if the empirical distribution of the environment in any ball of size larger than L centred in x is close to μ, see Definition 2.2 for a precise statement. In Proposition 2.4 we prove that if a site is L–good at time 0, then it will be J–good ($J < L$) after a time t satisfying $\gamma t > L^3$ with high probability. To this end, we will need a concentration inequality, Lemma 2.5, and some estimates on the density of the particles carrying some type of transition-probability vector, Lemma 2.6. In Section 3 we pass to the proof of Theorem 1. This will be achieved through a renormalization procedure.

Proposition 3.2 deals with the initial step of the renormalization, giving the result up to any time T choosing γ large enough depending on T. Proposition 3.3 extends the result for any time $t > T$ by iteration. Finally, in Subsection 3.4 we finalize the proof.

2. Auxiliary results on the environment dynamics

In this section we will collect a series of results that concern only the environment dynamics. We fix here and for the whole section an integer $N \in \mathbb{N}$. Note that all the statements in this section involve constants that may or may not depend on N, but we will not comment on that since it will not affect the results that we want to prove. We consider a map $T : \mathcal{S} \to \mathcal{T}$, where we call $\mathcal{T} := \{0, ..., 2^N - 1\}^{\mathcal{I}}$ the space of *types*, with $T : s \mapsto T(s)$ such that, for each coordinate $i \in \mathcal{I}$,

$$T_i(s) := \max\left\{j \in \{0, ..., 2^N - 1\} : 2^{-N} j \le s_i\right\}.$$

In particular, if $s_i \neq 1$, then $\frac{T_i(s)}{2^N} \le s_i < \frac{T_i(s)+1}{2^N}$. We simply say that an element $s \in \mathcal{S}$ is of type k (where k is a $2d$-dimensional integer multi-index) if $T(s) = k$ and call $p_k := \mu(T(s) = k)$ the μ-probability that a configuration s is of type k. We point out that if μ gave positive weight only to a finite number of possible transition-probability vectors of $\mathcal{S}$, there would be no need to introduce types (notice for example that the SSEPtreated in [HS15] corresponds to having only two possible transition-probability vectors). On the other hand, in order to deal with much more general μ's, the reduction to a finite number of types of particles is fundamental for our technique to work.

Given $x \in \mathbb{Z}^d$, $L \in \mathbb{N}$, $k \in \mathcal{T}$ and $\eta \in \mathcal{S}^{\mathbb{Z}^d}$, we let

$$\langle \eta \rangle^k_{x,L} := \frac{1}{|B_L(x)|} \sum_{y \in B_L(x)} \mathbb{1}_{T(\eta_y)=k}$$

be the empirical density of particles of type k in a ball of radius L around x. We consider L^2 balls:

$$B_L(x) := \{y \in \mathbb{Z}^d : \|x - y\| \le L\},$$

so that the cardinality of a ball is $c_1 L^d \le |B_L(x)| \le c_2 L^d$ for some constants $c_1, c_2 > 0$ depending on the dimension d.

Remark 2.1. For simplicity, throughout the article we will denote by $c, c_1, c_2, ...$ some strictly positive constants the value of which might change from one expression to the other. These constants will not depend on the other variables involved (with the possible exception of d and N), unless otherwise specified.

For simplicity, for $L \in \mathbb{N}$, we abbreviate $\langle \eta \rangle^k_L := \langle \eta \rangle^k_{0,L}$ and $B_L := B_L(0)$. We also fix a decreasing sequence $(\epsilon_L)_{L \in \mathbb{N}}$ of real numbers in $(0, 1)$ by

$$\epsilon_L := \frac{1}{1 + \ln(L)}. \tag{2.1}$$

The numbers ϵ_L are meant to control the difference between the theoretical density p_k and the empirical density of particles of type k in a box of size L:

Definition 2.2. For $L \in \mathbb{N}$, we say that a site $x \in \mathbb{Z}^d$ is *L-good* for a configuration $\eta \in \mathcal{S}^{\mathbb{Z}^d}$ if

$$|\langle \eta \rangle^k_{x,L'} - p_k| \le \epsilon_L \qquad \forall L' \ge L,\ \forall k \in \mathcal{T}. \tag{2.2}$$

In a configuration η, we call the set of L-good sites $G(\eta, L)$.

Remark 2.3. Note that the definition of good sites depends on N.

In words, $x \in G(\eta, L)$ if in every ball of radius $L' \ge L$ the density of particles of type k is close to its theoretical mean, with an error of at most ϵ_L, for every possible type k. The typical deviation, under the equilibrium measure, of the empirical density in a d–dimensional ball of radius L is of order $1/L^{d/2}$. Our conservative choice of (ϵ_L) in (2.1) guarantees that a site has large probability to be good when L is large.

The following proposition is one of the main tools that we will use for proving Theorem 1.

Proposition 2.4. *Suppose that $0 \in G(\eta, L)$ for some $\eta \in \mathcal{S}^{\mathbb{Z}^d}$, $L \in \mathbb{N}$. Then there exist $c_1, c_2 > 0$ such that the following is true: For all $J \in \mathbb{N}$, $J < L$, such that $L(\epsilon_J - \epsilon_L)$ is large enough, and for all $t > 0$ such that $\gamma t > L^3$, it holds*

$$\mathbb{P}^\eta\big(0 \notin G(\eta(t), J)\big) \le c_1 \frac{\mathrm{e}^{-c_2 J^d (\epsilon_J - \epsilon_L)^2}}{(\epsilon_J - \epsilon_L)^2}.$$

In order to prove Propositions 2.4 (which will be done in Subsection 2.3) we will need the following two lemmas, that will be proven in Subsection 2.1 and Subsection 2.2, respectively. Lemma 2.5 is a concentration-of-measure kind of statement, claiming that $\langle \eta(t) \rangle^k_L$ stays close to its mean. Lemma 2.6 controls the expectation $\mathbb{E}^\eta[\langle \eta(t) \rangle^k_M]$ in terms of the initial configuration η.

Lemma 2.5. *There exists a constant $c > 0$ such that, given $\eta \in \mathcal{S}^{\mathbb{Z}^d}$, $L \in \mathbb{N}$, $k \in \mathcal{T}$ and $a \ge 0$,*

$$\mathbb{P}^\eta\Big(\big|\langle \eta(t) \rangle^k_L - \mathbb{E}^\eta[\langle \eta(t) \rangle^k_L]\big| \ge a\Big) \le 2\,\mathrm{e}^{-c\,a^2 L^d}.$$

Lemma 2.6. *For each $\alpha > 0$, there exists a constant $c > 0$ such that, given $\eta \in \mathcal{S}^{\mathbb{Z}^d}$, $M, L \in \mathbb{N}$, $k \in \mathcal{T}$ and $t \ge 1$, the two following inequalities hold:*

$$\mathbb{E}^\eta\big[\langle \eta(t) \rangle^k_M\big] \le \Big(1 + c\Big(\frac{L^{d+1}}{(\gamma t)^{(d+1)/2}} + \frac{1}{(\gamma t)^{1/2-\alpha}}\Big)\Big) \sup\Big\{\langle \eta \rangle^k_{L'} : L' \ge L\Big\} \tag{2.3}$$

$$\mathbb{E}^\eta\big[\langle \eta(t) \rangle^k_M\big] \ge \Big(1 - c\Big(\frac{L^{d+1}}{(\gamma t)^{(d+1)/2}} + \frac{1}{(\gamma t)^{1/2-\alpha}}\Big)\Big) \inf\Big\{\langle \eta \rangle^k_{L'} : L' \ge L\Big\}. \tag{2.4}$$

Remark 2.7. Lemma 2.5 provides a control of the average number of particles of a given type in a ball of size M at time t in terms of the initial particle configuration, for any $M \in \mathbb{N}$. It is not clear if the rates of decay in t of the r.h.s. of (2.3) and of (2.4) are optimal: While the term $(L/\sqrt{\gamma t})^{d+1}$ is due to the diffusive dissipation of traps, the term $1/(\gamma t)^{1/2-\alpha}$ is present for technical reasons and it could perhaps be improved. Nevertheless, not being interested here in sharp quantitative statements on γ, the bounds (2.3) and (2.4) will be more than sufficient for our purpose.

2.1. **Proof of Lemma 2.5.** We prove the two following inequalities, which imply the lemma:

$$\mathbb{P}^\eta\Big(\langle\eta(t)\rangle_L^k - \mathbb{E}^\eta[\langle\eta(t)\rangle_L^k] \geq a\Big) \leq \mathrm{e}^{-ca^2L^d}, \tag{2.5}$$

$$\mathbb{P}^\eta\Big(\langle\eta(t)\rangle_L^k - \mathbb{E}^\eta[\langle\eta(t)\rangle_L^k] \leq -a\Big) \leq \mathrm{e}^{-ca^2L^d}. \tag{2.6}$$

Let us start with (2.5). We will replicate the argument used in [HS15, Lemma 1], considering each k-particle of our system as a 1-particle in [HS15]. Even if there are only few minor modifications, we repeat here the whole proof in order to make the present article self-contained. For $0 < \delta < |B_L|$, exponential Markov inequality gives

$$\mathbb{P}^\eta\Big(\langle\eta(t)\rangle_L^k - \mathbb{E}^\eta[\langle\eta(t)\rangle_L^k] \geq a\Big) \leq \mathrm{e}^{-\delta a}\cdot \mathrm{e}^{-\delta\mathbb{E}^\eta[\langle\eta(t)\rangle_L^k]}\cdot \mathbb{E}^\eta\Big[\mathrm{e}^{\delta\langle\eta(t)\rangle_L^k}\Big]. \tag{2.7}$$

Notice that we can write

$$\begin{aligned}\mathbb{E}^\eta\Big[\langle\eta(t)\rangle_L^k\Big] &= \frac{1}{|B_L|}\sum_{y\in B_L}\mathbb{E}^\eta[\mathbb{1}_{T(\eta_y(t))=k}]\\ &= \frac{1}{|B_L|}\sum_{y\in B_L}\sum_{z\in\mathbb{Z}^d} p(t,y-z)\mathbb{1}_{T(\eta_z)=k} = \sum_{z\in\mathbb{Z}^d} p_L(t,z)\mathbb{1}_{T(\eta_z)=k},\end{aligned} \tag{2.8}$$

where $p(t,x)$ is the heat kernel associated to the Laplacian $\gamma\Delta$, i.e. the probability that a simple random walk jumping at rate γ and starting at the origin will be in site x after time t, and

$$p_L(t,x) := \frac{1}{|B_L|}\sum_{y\in B_L} p(t,x+y)\,. \tag{2.9}$$

The third factor of (2.7) can be handled through Ligget's inequalities (see [Lig05, Prop. 1.7, pag. 366]): Let $\theta = (\theta(t))_{t\geq 0}$ be the collective motion of continuous-time independent simple random walks on $\mathbb{Z}^d$ with rate γ, so that $\theta_x(t)$ indicates the number of particles on site x at time t. We consider as starting configuration $\theta(0)$ the one such that $\theta_x(0) = \mathbb{1}_{T(\eta_x)=k}$, i.e. we let a simple random walk start in site x if and only if the particle at site x in η is of type k. We arbitrarily label these random walks and call them $(X_j(t))_{t\geq 0}$, with $j\in\mathbb{N}$. We indicate the empirical average of the number of particles at time t in a box of size L around the origin as $\langle\theta(t)\rangle_L$. Set $\mathbf{p}_L(t,x) := |B_L|\cdot p_L(t,x)$. Using Liggett's inequality for the first line, we can now bound

$$\begin{aligned}\mathbb{E}^\eta\Big[\mathrm{e}^{\delta\langle\eta(t)\rangle_L^k}\Big] &\leq \mathbb{E}^\eta\Big[\mathrm{e}^{\delta\langle\theta(t)\rangle_L}\Big] = \prod_{j\in\mathbb{N}}\mathbb{E}^\eta\Big[\mathrm{e}^{\frac{\delta}{|B_L|}\mathbb{1}_{X_j(t)\in B_L}}\Big]\\ &= \prod_{j\in\mathbb{N}}\Big(\mathrm{e}^{\frac{\delta}{|B_L|}}\mathbf{p}_L(t,X_j(0)) + \big(1-\mathbf{p}_L(t,X_j(0))\big)\Big)\\ &= \prod_{z\in\mathbb{Z}^d}\Big(\mathrm{e}^{\frac{\delta}{|B_L|}}\mathbf{p}_L(t,z) + (1-\mathbf{p}_L(t,z))\Big)^{\mathbb{1}_{T(\eta_z)=k}}\\ &\leq \exp\Big\{(\mathrm{e}^{\frac{\delta}{|B_L|}}-1)\sum_{z\in\mathbb{Z}^d}\mathbf{p}_L(t,z)\mathbb{1}_{T(\eta_z)=k}\Big\}\\ &= \exp\Big\{(\mathrm{e}^{\frac{\delta}{|B_L|}}-1)\,|B_L|\,\mathbb{E}^\eta[\langle\eta(t)\rangle_L^k]\Big\},\end{aligned} \tag{2.10}$$

where the last equality follows from (2.8). Since $\delta/|B_L| \le 1$, we can expand $\mathrm{e}^{\frac{\delta}{|B_L|}} - 1$ to the first order in (2.10) and find a constant $C < \infty$ such that

$$\mathbb{E}^\eta\big[\mathrm{e}^{\delta\langle\eta(t)\rangle_L^k}\big] \le \mathrm{e}^{\delta\mathbb{E}^\eta[\langle\eta(t)\rangle_L^k] + C\frac{\delta^2}{|B_L|}\mathbb{E}^\eta[\langle\eta(t)\rangle_L^k]}.$$

Plugging this back into (2.7), we get

$$\mathbb{P}^\eta\big(\langle\eta(t)\rangle_L^k - \mathbb{E}^\eta[\langle\eta(t)\rangle_L^k] \ge a\big) \le \mathrm{e}^{-a\delta + C\frac{\delta^2}{|B_L|}\mathbb{E}^\eta[\langle\eta(t)\rangle_L^k]}.$$

Minimizing this expression in $\delta \in [0, |B_L|]$ (see [HS15, Proof of Lemma 2.3] for more details) and remembering that $c_1 L^d \le |B_L| \le c_2 L^d$, we obtain the claim.

We are left to show (2.6). To this end, we just consider all the particles of type $j \ne k$ as being of the same type. We call $\langle\eta(t)\rangle_L^{\hat{k}} := 1 - \langle\eta(t)\rangle_L^k$ the empirical average of particles of type different form k in B_L and notice that

$$\mathbb{P}^\eta\big(\langle\eta(t)\rangle_L^k - \mathbb{E}^\eta[\langle\eta(t)\rangle_L^k] \le -a\big) = \mathbb{P}^\eta\big(\langle\eta(t)\rangle_L^{\hat{k}} - \mathbb{E}^\eta[\langle\eta(t)\rangle_L^{\hat{k}}] \ge a\big).$$

Then we just have to replay exactly the same game as above.

2.2. **Proof of Lemma 2.6.** Recall the definition (2.9). We divide $\mathbb{Z}^d$ in disjoint crowns centered in 0: $\mathbb{Z}^d = \bigcup_{n\in\mathbb{N}} C_n$ with $C_n := \{z \in \mathbb{Z}^d : n-1 < \|z\| \le n\}$. For $y \in \mathbb{Z}^d$ we define the crown of y as $C(y) := C_{n(y)}$ where $n(y)$ is the unique integer such that $y \in C_{n(y)}$. For each $n \in \mathbb{N}$ we choose representatives $\hat{x}_n, \check{x}_n \in C_n$ such that

$$p_M^+(t,n) := p_M(t,\hat{x}_n) = \max_{z\in C_n}\{p_M(t,z)\}, \quad p_M^-(t,n) := p_M(t,\check{x}_n) = \min_{z\in C_n}\{p_M(t,z)\}.$$

We also define, for $y \in \mathbb{Z}^d$, $p_M^+(t,y) := \max_{z\in C(y)}\{p_M(t,z)\}$ and $p_M^-(t,y) := \min_{z\in C(y)}\{p_M(t,z)\}$.

Claim 2.8. *Take $y \in \mathbb{Z}^d$ such that $\langle y, e_i\rangle > 0$ for some $i \in \mathcal{I}$. Then*

$$p_M(t,y) \ge p_M(t, y+e_i).$$

Corollary 2.9 (Corollary of the Claim)**.**

$$p_M^+(t,y) \ge p_M^+(t,y') \quad \textit{and} \quad p_M^-(t,y) \ge p_M^-(t,y') \qquad \forall y, y' \in \mathbb{Z}^d : \|y'\| \ge \|y\|.$$

We postpone the easy proofs of Claim 2.8 and Corollary 2.9 to, respectively, Subsection A.1 and Subsection A.2 in the Appendix.

With Corollary 2.9 at hand, we would like to proceed in a similar fashion as in [HS15], but the high dimensionality represents an obstacle to this end as, due to the geometry of the lattice, the particles starting in a given crown do not have exactly the same probability to reach the ball of radius M at a given time. For each positive function $u : \mathbb{Z}^d \to \mathbb{R}$, we can write

$$p_M(t,y)u(y) \le p_M^+(t,y)u(y) = \sum_{n\in\mathbb{N}} \big(p_M^+(t,n) - p_M^+(t,n+1)\big)u(y)\mathbb{1}_{\|y\|\le n}. \tag{2.11}$$

If we sum over all $y \in \mathbb{Z}^d$, take $u(y) = \mathbb{1}_{T(\eta_y)=k}$ and use (2.8), we obtain

$$\begin{aligned}\mathbb{E}^\eta[\langle\eta(t)\rangle_M^k] &\le \sum_{n\in\mathbb{N}} \big(p_M^+(t,n) - p_M^+(t,n+1)\big) \sum_{y\in\mathbb{Z}^d} \mathbb{1}_{T(\eta_y)=k}\mathbb{1}_{\|y\|\le n} \\ &= \sum_{n\in\mathbb{N}} \big(p_M^+(t,n) - p_M^+(t,n+1)\big)|B_n|\langle\eta\rangle_n^k. \end{aligned} \tag{2.12}$$

We abbreviate $R := \sup\{\langle\eta\rangle^k_{L'} : L' \geq L\}$ and split the final sum in (2.12) for $n < L$ and $n \geq L$. For $n < L$ we have that $|B_n|\langle\eta\rangle^k_n \leq |B_L|R$, since the r.h.s. is bigger than the total number of particles of type k at time 0 in the ball of radius L and this clearly dominates the number of particles in any smaller ball. Hence,

$$\sum_{n<L} \big(p^+_M(t,n) - p^+_M(t,n+1)\big)|B_n|\langle\eta\rangle^k_n \leq \big(p^+_M(t,0) - p^+_M(t,L)\big)|B_L|R \leq \frac{cL^{d+1}}{(\gamma t)^{\frac{d+1}{2}}}R. \tag{2.13}$$

To see why the last inequality holds, we expand, for any $z \in \mathbb{Z}^d$ and $t > 0$, in Fourier variables $p(t,z) = \mathrm{e}^{\gamma t\Delta}\delta_0(z)$. We call $\alpha(\xi) := \sum_{j=1}^d 2(\cos(2\pi\xi\cdot e_j) - 1)$ and bound, for each $z \in \mathbb{Z}^d$, $e \in \mathcal{N}$, $t > 0$,

$$\big|p(t,z) - p(t,z+e)\big| = \Big|\int_{\mathbb{R}^d} \frac{\mathrm{e}^{\gamma t\alpha(\xi)}}{(2\pi)^d}\mathrm{e}^{i\xi\cdot z}(1 - \mathrm{e}^{i\xi\cdot e})\,\mathrm{d}\xi\Big| \leq c\int_{\mathbb{R}^d} |\xi|\mathrm{e}^{-c_1\gamma t|\xi|^2}\mathrm{d}\xi \leq \frac{c_2}{(\gamma t)^{\frac{d+1}{2}}}\,. \tag{2.14}$$

Then we are done, since (2.14) is uniform in z and e and since we can find a path of neighbors leading from any point $y \in B_M$ to the point $\hat{x}_L + y$ in $\sqrt{d}L$ steps, which gives $p^+_M(t,0) - p^+_M(t,L) \leq cL/(\gamma t)^{(d+1)/2}$.

We want to show now that

$$\sum_{n\geq L} \big(p^+_M(t,n) - p^+_M(t,n+1)\big)|B_n|\langle\eta\rangle^k_n \leq \Big(1 + c\Big(\frac{L^d}{(\gamma t)^{\frac{d+1}{2}}} + \frac{1}{(\gamma t)^{\frac{1}{2}-\alpha}}\Big)\Big)R\,. \tag{2.15}$$

This together with (2.13) and (2.12) will imply the first half of the lemma, inequality (2.3). For $n \geq L$ we have that $\langle\eta\rangle^k_n \leq R$ by definition and we are left to bound $\sum_{n\geq L}(p^+_M(t,n) - p^+_M(t,n+1))|B_n|$. By summing by parts we have

$$\begin{aligned}\sum_{n\geq L} \big(p^+_M(t,n) - p^+_M(t,n+1)\big)|B_n| &= \sum_{n\geq L} p^+_M(t,n)|C_n| + p^+_M(t,L)|B_{L-1}| \\ &= \sum_{y\notin B_{L-1}} p_M(t,y) + \sum_{n\geq L}\sum_{y\in C_n} \big(p^+_M(t,n) - p_M(t,y)\big) + p^+_M(t,L)|B_{L-1}|\,. \end{aligned} \tag{2.16}$$

Since by Corollary 2.9 we have, for each $y \in C_m$ with $m < L$, that $p_M(t,y) \geq p^-_M(t,m) \geq p^-_M(t,L)$, we can bound $p^+_M(t,L)|B_{L-1}| \leq \big(p^+_M(t,L) - p^-_M(t,L)\big)|B_{L-1}| + \sum_{y\in B_{L-1}} p_M(t,y)$, which inserted into (2.16) gives

$$\sum_{n\geq L} \big(p^+_M(t,n) - p^+_M(t,n+1)\big)|B_n| \leq 1 + \sum_{n\geq L} E_M(t,n)|C_n| + E_M(t,L)|B_{L-1}|\,, \tag{2.17}$$

where we have used the fact that $\sum_{y\in\mathbb{Z}^d} p_M(t,y) = 1$ and where we have called

$$E_M(t,n) := p_M(t,\hat{x}_n) - p_M(t,\check{x}_n)$$

the maximum error we commit in the n-th crown.

Claim 2.10. *For each $\varepsilon > 0$ there exist constants c_1 and c_2 such that*

$$\sum_{n\geq L} E_M(t,n)|C_n| \leq c_1\frac{1}{(\gamma t)^{1/2-\varepsilon d}} \tag{2.18}$$

and

$$E_M(t,L)|B_{L-1}| \le c_2 \frac{L^d}{(\gamma t)^{(d+1)/2}}\,. \tag{2.19}$$

The proof of Claim 2.10 is rather technical and we postpone it to Subsection A.3 in the Appendix. We show now how to conclude the proof of Lemma 2.6. Plugging (2.18) and (2.19) into (2.17) we obtain (2.15) with $\alpha = \varepsilon d$. This implies (2.3) as mentioned before.

The lower bound (2.4) can be proven in an almost identical way as the upper bound: As in (2.11), we first note that for any positive u one has $p_M(t,y)u(y) \ge \sum_{n\in\mathbb{N}}(p_M^-(t,n)-p_M^-(t,n+1))u(y)\mathbb{1}_{n\ge\|y\|}$ and obtain a lower bound with the same flavor of (2.12). Then one splits once again the sum in n: This time the case $n<L$ can be lower-bounded directly with 0. For the case $n\ge L$ we follow the same calculation as in (2.16) and produce a similar lower bound that this time reads:

$$\sum_{n\ge L}(p_M^-(t,n)-p_M^-(t,n+1))|B_n| \ge 1 - \sum_{n\ge L} E_M(t,n)|C_n| - E_M(t,L)|B_{L-1}|\,.$$

Then we can use again the results of Claim 2.10 and proceed exactly as for the upper bound (2.3), concluding the proof of the lemma.

2.3. **Proof of Proposition 2.4.** We extend the argument in [HS15] to the case of N different kinds of particles. By union bound

$$\mathbb{P}^\eta\big(0\notin G(\eta(t),J)\big) \le \sum_{k\in\mathcal{T}}\sum_{J'\ge J}\mathbb{P}^\eta\big(|\langle\eta(t)\rangle^k_{J'}-p_k|>\epsilon_J\big)\,. \tag{2.20}$$

Notice that if $p_k=0$ for some $k\in\mathcal{T}$, the relative probability in the r.h.s. is just zero. For all the other k's we want to bound

$$\mathbb{P}^\eta\big(|\langle\eta(t)\rangle^k_{J'}-p_k|>\epsilon_J\big) = \mathbb{P}^\eta\big(\langle\eta(t)\rangle^k_{J'}>p_k+\epsilon_J\big)+\mathbb{P}^\eta\big(\langle\eta(t)\rangle^k_{J'}<p_k-\epsilon_J\big)\,. \tag{2.21}$$

For the first summand we rewrite

$$\mathbb{P}^\eta\big(\langle\eta(t)\rangle^k_{J'}>p_k+\epsilon_J\big) = \mathbb{P}^\eta\big(\langle\eta(t)\rangle^k_{J'}-\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{J'}]>p_k+\epsilon_J-\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{J'}]\big)\,. \tag{2.22}$$

Since by hypothesis $\gamma t>L^3$ and $0\in G(\eta,L)$, Lemma 2.6 with α small enough guarantees that $\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{J'}]\le(1+c/L)\sup\{\langle\eta\rangle^k_{L'},L'\ge L\}\le(1+c/L)(p_k+\epsilon_L)$ so that $p_k+\epsilon_J-\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{J'}]\ge\epsilon_J-\epsilon_L-c(p_k+\epsilon_L)/L\ge(\epsilon_J-\epsilon_L)/2$, where for the last inequality we have used the hypothesis that $L(\epsilon_J-\epsilon_L)$ is large enough. This applied to (2.22) gives

$$\mathbb{P}^\eta\big(\langle\eta(t)\rangle^k_{J'}>p_k+\epsilon_J\big)\le\mathbb{P}^\eta\left(\big|\langle\eta(t)\rangle^k_{J'}-\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{J'}]\big|\ge(\epsilon_J-\epsilon_L)/2\right)\le 2\,\mathrm{e}^{-c(\epsilon_J-\epsilon_L)^2J'^d}, \tag{2.23}$$

where the last inequality follows from Lemma 2.5.

Conversely, we bound the second summand of (2.21) by noticing that, by Lemma 2.6, $\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{J'}]\ge(1-c/L)\inf\{\langle\eta\rangle^k_{L'},L'\ge L\}\ge(1-c/L)(p_k-\epsilon_L)$, so that $p_k-\epsilon_J-\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{0,J'}]\le-(\epsilon_J-\epsilon_L)/2<0$. Hence, as before,

$$\mathbb{P}^\eta\big(\langle\eta(t)\rangle^k_{J'}<p_k-\epsilon_J\big)\le\mathbb{P}^\eta\big(\big|\langle\eta(t)\rangle^k_{J'}-\mathbb{E}^\eta[\langle\eta(t)\rangle^k_{0,J'}]\big|>(\epsilon_J-\epsilon_L)/2\big)\le 2\,\mathrm{e}^{-c(\epsilon_J-\epsilon_L)^2J'^d}. \tag{2.24}$$

Plugging (2.23) and (2.24) into (2.21) and this back into (2.20) we finally obtain

$$\begin{aligned}\mathbb{P}^{\eta}\big(0 \notin G(\eta(t), J)\big) &\le \sum_{J' \ge J} \sum_{k \in \mathcal{T}: p_k > 0} 4\,\mathrm{e}^{-cJ'^d(\epsilon_J - \epsilon_L)^2} \\ &\le c_1 \sum_{k \in \mathcal{T}: p_k > 0} \frac{\mathrm{e}^{-cJ^d(\epsilon_J - \epsilon_L)^2}}{(\epsilon_J - \epsilon_L)^2} \le c_1 \left(2^N\right)^{2d} \frac{\mathrm{e}^{-cJ^d(\epsilon_J - \epsilon_L)^2}}{(\epsilon_J - \epsilon_L)^2}.\end{aligned}$$

3. Proof of Theorem 1

Given a vector $\vec{v} \in \mathbb{R}^d$, we denote, for $t \ge 0$,

$$X_t^{\vec{v}} := \langle X_t, \vec{v} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. In order to prove Theorem 1, we will actually prove the following proposition, which is clearly equivalent to Theorem 1:

Proposition 3.1 (Equivalent of Theorem 1)**.** *Assume* (1.1) *and consider* $\vec{v} \in \mathbb{R}^d$ *with norm* 1 *such that* $v := \langle \mathbb{E}_\mu[D], \vec{v} \rangle > 0$*. For all* $\varepsilon > 0$*, there exists* $\gamma(\varepsilon)$ *such that, for all* $\gamma \ge \gamma(\varepsilon)$*,* P*-a.s. it holds that, for* t *large enough,*

$$\frac{X_t^{\vec{v}}}{t} \in (v - \varepsilon, v + \varepsilon).$$

The proof of Proposition 3.1 is postponed to Subsection 3.4.

Without being too rigorous, we call *trap* a region where an anomalous density of some type of particles occurs. If all sites of a given region are L-good for some $L > 0$, we say that there are no traps larger than L in that region. We define now a sequence of positive numbers $(\phi_L)_{L \in \mathbb{N}}$. One can think of ϕ_L as the size of "acceptable" traps in a ball of radius L. We are not interested in a sharp estimate of the typical size of the maximal trap (that should be of order $\log L$). Instead, in order to simplify our computation, we choose rather to overestimate this typical size and define, for $L \in \mathbb{N}$,

$$\phi_L := L^{1/100}. \tag{3.1}$$

3.1. Outline of the proof.

The proof relies on a renormalisation procedure adapted from [HS15]. We fix throughout the rest of the section a vector $\vec{v}$ satisfying the assumption of Proposition 3.1. Heuristically, the hypothesis we want to iterate is the following: "If the box of size t at time 0 around the walker has no trap larger than ϕ_t, then, at time t, the walker is, with large probability, at a distance $(v + o(1))\,t$ from the origin in the direction $\vec{v}$".

The first step of the procedure (see Proposition 3.2) relies on a homogenization obtained by choosing a large enough finite time T and some γ large enough chosen accordingly. To prove the iteration itself (which consists in going from scale t to scale t^2) we divide the space-time box of size t^2 into boxes of size t and use Proposition 2.4 to control the size of the traps in each of these sub-boxes. This allows us to apply the iteration hypothesis at scale t.

3.2. **Initialization of the renormalization.**

Proposition 3.2. *For all $\varepsilon > 0$, there exists N large enough, $T \in \mathbb{N}$ large enough and $\gamma > 0$ large enough (e.g., $\gamma \geq \phi_T^3$) such that the following holds: Given $\eta \in \mathcal{S}^{\mathbb{Z}^d}$ satisfying*

$$y \in G(\eta, \phi_T) \qquad \forall y \in B_T,$$

it holds that

$$P^\eta\big(|X_T^{\vec{v}} - vT| \geq \varepsilon T\big) \leq \mathrm{e}^{-\phi_T^{1/4}}.$$

Proof. We will first prove that

$$P^\eta(X_T^{\vec{v}} \leq (v-\varepsilon)T) \leq \tfrac{1}{2}\,\mathrm{e}^{-\phi_T^{1/4}}. \tag{3.2}$$

Since we have no information on the particle at the origin at time 0, we just bound the first step with the worst case:

$$P^\eta\big(X_T^{\vec{v}} \leq (v-\varepsilon)T\big) \leq P^\eta\big(X_T^{\vec{v}} - X_1^{\vec{v}} \leq (v-\varepsilon)T + 1\big).$$

For $j \in \mathcal{I}$ we let $v_j := \langle \vec{v}, e_j \rangle \in [-1, 1]$. For simplicity we suppose that $v_j \neq v_i$ for all $i \neq j$ (the proof can be easily adapted to the other cases). We also define $v_{\max} := \max_{j\in\mathcal{I}}\{v_j\}$, $v_{\min} := -v_{\max}$ and their relative indices $j_{\max} := \arg\max v_{\max}$, $j_{\min} := -j_{\max}$. By eventually enlarging the original space, we consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of i.i.d. random variables under P^η with values in $\{v_{-d}, ..., v_d\}$. The Y_k's are independent of η, of the interchange process and of the random walker, and they satisfy

$$P^\eta(Y_1 = v_j) = \begin{cases} \mathbb{E}_\mu[s_j] - \delta & \text{for } j \neq j_{\min} \\ \mathbb{E}_\mu[s_{j_{\min}}] + (2d-1)\delta & \text{for } j = j_{\min}, \end{cases}$$

where $\delta > 0$ is chosen small enough and $N \in \mathbb{N}$ is chosen large enough such that $2^{-(N-1)}d < \delta < \min\{\mathbb{E}_\mu[s_i] > 0\}$ and $2dv_{\max}\delta < \varepsilon/2$. Notice that:

(i) Y_1 is well defined, or can be made well defined with little effort. In fact, if $\mathbb{E}_\mu[s_j] > 0$ for all $j \in \mathcal{I}$, Y_1 is well defined because of the choice of δ that we made. If instead there is some $j \neq j_{\min}$ such that $\mathbb{E}_\mu[s_j] = 0$, then we can easily modify the law of Y_1 so that we define $P^\eta(Y_1 = v_j) = 0$ and adapt the definition of $P^\eta(Y_1 = v_{\min})$ consequently;

(ii) $E^\eta[Y_1] > v - \varepsilon/2$. In fact, it is straightforward to check that

$$E^\eta[Y_1] = v + \delta\Big((2d-1)v_{\min} - \sum_{j\neq j_{\min}} v_j\Big) = v - 2dv_{\max}\delta > v - \varepsilon/2,$$

where we have used the symmetry of the $v_\cdot$'s for the second equality, which implies $\sum_{j\neq j_{\min}} v_j = v_{\max}$ and $v_{\min} = -v_{\max}$.

In analogy with [HS15], for $m \geq 1$ we define the events

$$\mathcal{E}_m := \big\{y \in G(\eta(m), J),\ \forall y \in B_{m+1}\big\} \qquad \text{with } J := \lfloor \phi_T^{1/2} \rfloor.$$

and $\mathcal{E}_0 := \big\{y \in G(\eta(0), \phi_T),\ \forall y \in B_1\big\}$. Notice that, by hypothesis, $P^\eta(\mathcal{E}_0) = 1$. Loosely speaking, the event $\mathcal{E}_m^c$ indicates that at step m the random walk sees around itself an unfavorable environment. Our aim is to show that

$$P^\eta(X_T^{\vec{v}} - X_1^{\vec{v}} \leq (v-\varepsilon)T + 1) \leq P^\eta\Big(\sum_{m=1}^{T-1} Y_m \leq (v-\varepsilon)T + 1\Big) + \sum_{m=1}^{T-1} \mathbb{P}^\eta(\mathcal{E}_{m-1}^c). \tag{3.3}$$

Before showing why this is true, we explain how this implies the first half of the lemma, i.e. equation (3.2). The first term of the right-hand-side of (3.3) can be bounded with classical large deviations for sequences of i.i.d. random variables: Keeping in mind that $E^\eta[Y_1] \in (v-\varepsilon/2, v)$, we have

$$P^\eta\Big(\sum_{m=1}^{T-1} Y_m \le (v-\varepsilon)T+1\Big) \le \mathrm{e}^{-cT} \le \mathrm{e}^{-\phi_T^{1/3}}\,. \tag{3.4}$$

The second term in the right hand side of (3.3) can be bounded thanks to Proposition 2.4 and by taking T and, consequently, γ large enough as

$$\begin{aligned}\sum_{m=1}^{T-1} \mathbb{P}^\eta(\mathcal{E}_{m-1}^c) &\le \sum_{m=1}^{T-1}\sum_{y\in B_m} \mathbb{P}^\eta\big(y \notin G(\eta(m-1),J)\big) \le c\sum_{m=1}^{T-1}\sum_{y\in B_m} \frac{\mathrm{e}^{-cJ^d(\varepsilon_J-\varepsilon_{\phi_T})^2}}{(\varepsilon_J-\varepsilon_{\phi_T})^2}\\ &\le cT^{d+1}\frac{\mathrm{e}^{-cJ^d(\varepsilon_J-\varepsilon_{\phi_T})^2}}{(\varepsilon_J-\varepsilon_{\phi_T})^2} \le \mathrm{e}^{-\phi_T^{d/3}}.\end{aligned} \tag{3.5}$$

Putting back (3.4) and (3.5) into (3.3) we obtain (3.2).

Let us go back to the proof of (3.3). We show that, for any $m \ge 1$ and any $a \in \mathbb{R}$,

$$P^\eta(X_{m+1}^{\vec v} - X_1^{\vec v} \le a) \le P^\eta(X_{(m-1)+1}^{\vec v} + Y_m - X_1^{\vec v} \le a) + P^\eta(\mathcal{E}_{m-1}^c)\,. \tag{3.6}$$

Iterating this formula and using Fubini gives (3.3). We bound

$$\begin{aligned}&P^\eta(X_{m+1}^{\vec v} - X_1^{\vec v} \le a)\\ &\quad\le E^\eta\Big[P^\eta(X_{m+1}^{\vec v} - X_m^{\vec v} + X_m^{\vec v} - X_1^{\vec v} \le a \,|\, \eta(m-1), X_1, X_m)\cdot \mathbb{1}_{\mathcal{E}_{m-1}}\Big] + P^\eta(\mathcal{E}_{m-1}^c)\,.\end{aligned} \tag{3.7}$$

For any $b \in \mathbb{R}$ we call $\mathcal{J}(b) := \{j \in \mathcal{I} : v_j \le b\}$ and compute

$$\begin{aligned}&P^\eta(X_{m+1}^{\vec v} - X_m^{\vec v} \le b \,|\, \eta(m-1), X_1, X_m)\\ &\quad= \sum_{j\in J(b)}\sum_{k\in\mathcal{T}} P^\eta\big(X_{m+1}^{\vec v} - X_m^{\vec v} = v_j \,|\, \eta(m-1), X_1, X_m, T(\eta_{X_m}(m)) = k\big)\\ &\qquad\qquad P^\eta(T(\eta_{X_m}(m)) = k \,|\, \eta(m-1), X_1, X_m)\\ &\quad\le \sum_{j\in\mathcal{J}(b)}\sum_{k\in\mathcal{T}} \frac{k_j+1}{2^N} P^\eta(T(\eta_{X_m}(m)) = k \,|\, \eta(m-1), X_1, X_m)\,.\end{aligned} \tag{3.8}$$

On $\mathcal{E}_{m-1}$, using Markov property, we can apply Lemma 2.6 with $M=0$, $t=1$, $L = \lfloor \phi_T \rfloor$ if $m=1$ and $L = \lfloor \phi_T^{1/2} \rfloor$ if $m \ge 2$ to control the probability in the last display, so that if T is large enough we have, since $\gamma \ge \phi_T^3$,

$$\begin{aligned}P^\eta(T(\eta_{X_m}(m)) =& k \,|\, \eta(m-1), X_1, X_m)\\ &\le \Big(1 + c\Big(\frac{L^{d+1}}{\gamma^{\frac{d+1}{2}}} + \frac{1}{\gamma^{\frac12-\alpha}}\Big)\Big)\sup\Big\{\langle\eta(m-1)\rangle_{z,L'}^k : L' \ge L, z \sim X_m\Big\}\\ &\le \Big(1 + c\frac{1}{\phi_T}\Big)(p_k + \epsilon_{\phi_T^{1/2}}) = p_k + \delta_T,\end{aligned}$$

where the last inequality is valid for any $m \geq 1$ and where δ_T is some function that goes to 0 as T grows. Putting this back into (3.8) we obtain that on $\mathcal{E}_{m-1}$

$$P^\eta(X^{\vec{v}}_{m+1} - X^{\vec{v}}_m \leq b \,|\, \eta(m-1), X_1, X_m) \leq \sum_{j \in J(b)} \sum_{k \in \mathcal{T}} \frac{k_j + 1}{2^N} p_k + c\,\delta_T$$
$$\leq \sum_{j \in J(b)} \sum_{k \in \mathcal{T}} \frac{k_j}{2^N} p_k + \frac{|\mathcal{J}(b)|}{2^N} + c\,\delta_T, \quad (3.9)$$

where $|\mathcal{J}(b)|$ is the cardinality of $\mathcal{J}(b)$. We notice that $\forall j \neq j_{\min}$ we have $\sum_{k \in \mathcal{T}} \frac{k_j}{2^N} p_k \leq P^\eta(Y_m = v_j) + \delta$. On the other hand, $\sum_{k \in \mathcal{T}} \frac{k_{j_{\min}}}{2^N} p_k \leq P(Y_m = v_{j_{\min}}) - (2d-1)\delta$. Hence on $\mathcal{E}_{m-1}$,

$$P^\eta(X^{\vec{v}}_{m+1} - X^{\vec{v}}_m \leq b \,|\, \eta(m-1), X_1, X_m) \leq P(Y_m \leq b) - (2d - |\mathcal{J}(b)|)\delta + \frac{|\mathcal{J}(b)|}{2^N} + c\,\delta_T$$
$$\leq P(Y_m \leq b)$$

if T is big enough and $|\mathcal{J}(b)| \leq 2d-1$, since we chose $\delta > 2d/2^N$. This last inequality is trivial in the case $|J(b)| = 2d$, that is $b \geq v_{\max}$, as both terms are equal to 1. Plugging the last display back into (3.7), we easily obtain (3.6).

We prove the "other half" of the lemma in a completely specular way. We need

$$P^\eta(X^{\vec{v}}_T \geq (v+\varepsilon)T) \leq \tfrac{1}{2}\,\mathrm{e}^{-\phi_T^{1/4}}, \quad (3.10)$$

since this combined with (3.2) gives the final result. As before, we bound the first step with the worst case

$$P^\eta(X^{\vec{v}}_T \geq (v+\varepsilon)T) \leq P^\eta(X^{\vec{v}}_T - X^{\vec{v}}_1 \geq (v+\varepsilon)T - 1)$$

and define a new sequence of i.i.d. random variables $(\widetilde{Y}_k)_{k \in \mathbb{N}}$ with values in $\{v_{-d}, ..., v_d\}$, independent of the interchange process and of the walker and such that

$$P^\eta(\widetilde{Y}_1 = v_j) = \begin{cases} \mathbb{E}_\mu[s_j] - \delta & \text{for } j \neq j_{\max} \\ \mathbb{E}_\mu[s_{j_{\max}}] + (2d-1)\delta & \text{for } j = j_{\max}. \end{cases}$$

Similar comments (and eventually corrections) made for the Y_j's hold for the $\widetilde{Y}_j$'s, that is, the $\widetilde{Y}_j$'s are (or can be easily made) well defined and $E^\eta[\widetilde{Y}_1] \in (v, v+\varepsilon/2)$. We keep the previous definition of the good events $(\mathcal{E}_m)_{m \in \mathbb{N}}$ and aim to prove

$$P^\eta(X^{\vec{v}}_T - X^{\vec{v}}_1 \geq (v+\varepsilon)T - 1) \leq P^\eta\Big(\sum_{m=1}^{T-1} \widetilde{Y}_m \geq (v+\varepsilon)T - 1\Big) + \sum_{m=1}^{T-1} \mathbb{P}^\eta(\mathcal{E}^c_{m-1}). \quad (3.11)$$

As we have seen before, we can use a mild large deviations argument as in (3.4) combined with (3.5) to bound (3.11) and conclude (3.10).

The proof of (3.11) is similar to the one of (3.3). We proceed in the same way as before and obtain the equivalent of (3.9) that now reads as follows: For any $b \in \mathbb{R}$, $m \geq 1$, on the event $\mathcal{E}_{m-1}$ we have that

$$P^\eta(X^{\vec{v}}_{m+1} - X^{\vec{v}}_m \geq b \mid \eta(m-1), X_1, X_m) \leq \sum_{j \notin \mathcal{J}(b)} \sum_{k \in \mathcal{T}} \frac{k_j}{2^N} p_k + \frac{2d - |\mathcal{J}(b)|}{2^N} + c\,\delta_T\ ,$$

where again δ_T is some function that goes to 0 when T grows to infinity. For all $j \neq j_{\max}$ we have $\sum_{k\in\mathcal{T}} \frac{k_j}{2^N} p_k \le P(Y_m = v_j) + \delta$, while $\sum_{k\in\mathcal{T}} \frac{k_{j_{\max}}}{2^N} p_k \le P(Y_m = v_{j_{\max}}) - (2d-1)\delta$. Hence,

$$P^\eta(X^{\vec{v}}_{m+1} - X^{\vec{v}}_m \ge b \,|\, \eta(m-1), X_1, X_m) \\ \le P(Y_m \ge b) - |\mathcal{J}(b)|\delta + (2d - |\mathcal{J}(b)|)2^{-N} + c\,\delta_T \;\le\; P(Y_m \ge b)$$

if T is big enough and $|\mathcal{J}(b)| \ge 1$ as $\delta > 2d/2^N$ (here again, if $|\mathcal{J}(b)| = 0$, that is $b < v_{\min}$ the inequality is trivial). From there we prove easily the equivalent of (3.7) with the opposite inequality and conclude the proof of (3.11). □

3.3. **Renormalization step.**

Proposition 3.3. *Take any $\varepsilon > 0$. Let N, T and γ be large enough. Given $\eta \in \mathcal{S}^{\mathbb{Z}^d}$ and $t \ge T$ such that*

$$y \in G(\eta, \phi_t) \qquad \forall y \in B_t,$$

it holds that

$$P^\eta\big(|X^{\vec{v}}_t - vt| \ge \varepsilon t\big) \le \mathrm{e}^{-\phi_t^{1/4}}.$$

Proof. We consider T so large that the conclusion of Proposition 3.2 holds for the time $T^{1/3}$ and consider $t \ge T$. We first prove that

$$P^\eta\big(X^{\vec{v}}_t < (v-\varepsilon)\,t\big) \le \tfrac{1}{2}\,\mathrm{e}^{-\phi_t^{1/4}}. \tag{3.12}$$

We define a sequence of times $(t_n)_{n\in\mathbb{N}}$ such that $t_0 \in [T^{1/3}, T]$ and, for $n \ge 0$,

$$t_{n+1} \in [t_n^2, (t_n+1)^2]\,, \tag{3.13}$$

so that for some $\widetilde{n} \ge 0$ we have $t_{\widetilde{n}} = t$ (for example define $t_{\widetilde{n}} = t$ for a suitable $\widetilde{n}$ and then define recursively $t_{k-1} = \lfloor \sqrt{t_k} \rfloor$ until reaching the interval $[T^{1/3}, T]$). We also define a sequence of positive real numbers $(c_n)_{n\in\mathbb{N}}$ such that

$$v \ge c_n := \frac{1}{2}\Big(3 - \sum_{k=0}^{n} \Big(\frac{\varepsilon}{1+\varepsilon}\Big)^k\Big)\, v \ge (1-\varepsilon/2)\, v \qquad \text{for } n \in \mathbb{N}.$$

Note that $(c_n)_{n\in\mathbb{N}}$ is a decreasing sequence and that $c_0 = v$ and $\lim_{n\to\infty} c_n = v(1-\varepsilon/2)$. We want to prove by induction that, given $n \in \mathbb{N}$ and $\eta \in \mathcal{S}^{\mathbb{Z}^d}$,

$$\textit{if } \forall y \in B_{t_n} \textit{ it holds that } y \in G(\eta, \phi_{t_n}), \textit{ then } P^\eta(X^{\vec{v}}_{t_n} < c_n t_n) \le \tfrac{1}{2}\,\mathrm{e}^{-\phi_{t_n}^{1/4}}. \tag{3.14}$$

Proposition 3.2 takes care of the initialization step $n = 1$. We assume now that (3.14) holds for some $n \ge 1$ and we show that this implies the case $n+1$. By (3.13) we can write $t_{n+1} = t_n(t_n - 1) + r$, where the rest r is a number between, say, t_n and $4t_n$. We first have to wait an initial time lapse, since we can not use the iteration assumption at time 0 as the maximal trap could be of order $\phi_{t_{n+1}}$ instead of ϕ_{t_n}:

$$P^\eta(X^{\vec{v}}_{t_{n+1}} < c_{n+1}t_{n+1}) \le P^\eta(X^{\vec{v}}_{t_{n+1}} - X^{\vec{v}}_r < c_{n+1}t_{n+1} + r)\,.$$

We define on the same probability space (enlarged if necessary) a sequence $(Z_k)_{k\in\mathbb{N}}$ of i.i.d. random variables independent from η, from the interchange process and from the walker, with distribution

$$P^\eta(Z_k = c_n t_n) = 1 - \mathrm{e}^{-\phi_{t_n}^{1/4}}, \qquad P^\eta(Z_k = -t_n) = \mathrm{e}^{-\phi_{t_n}^{1/4}}.$$

Using our inductive hypothesis we aim to show that

$$P^\eta(X^{\vec{v}}_{t_{n+1}} - X^{\vec{v}}_r < c_{n+1}t_{n+1} + r) \le P^\eta\Big(\sum_{k=1}^{t_n-1} Z_k \le c_{n+1}t_{n+1} + r\Big) + \sum_{\substack{y\in B_{t_{n+1}}\\ r\le s\le t_{n+1}}} \mathbb{P}^\eta\big(y \notin G(\eta(s),\phi_{t_n})\big). \tag{3.15}$$

To obtain this, we just show that, for all $m \ge 1$ and for any $a \in \mathbb{R}$, we have

$$P^\eta(X^{\vec{v}}_{mt_n+r} - X^{\vec{v}}_r \le a) \le P^\eta(X^{\vec{v}}_{(m-1)t_n+r} - X^{\vec{v}}_r + Z_m \le a) + \mathbb{P}^\eta(\mathcal{E}^c_{m-1}), \tag{3.16}$$

where for $m \ge 0$

$$\mathcal{E}_m := \{y \in G(\eta(mt_n + r), \phi_{t_n}),\ \forall y \in B_{(m+1)t_n+r}\}$$

is the event of having a favorable environment at step m (i.e., at time $mt_n + r$).

Let us prove (3.16):

$$P^\eta(X^{\vec{v}}_{mt_n+r} - X^{\vec{v}}_r \le a) = P^\eta(X^{\vec{v}}_{mt_n+r} - X^{\vec{v}}_{(m-1)t_n+r} + X^{\vec{v}}_{(m-1)t_n+r} - X^{\vec{v}}_r \le a)$$
$$\le \sum_{x,y\in\mathbb{Z}^d} P^\eta(X^{\vec{v}}_{mt_n+r} - X^{\vec{v}}_{(m-1)t_n+r} \le a - \langle y-x,\vec{v}\rangle \,|\, A_{x,y}) P^\eta(X_{(m-1)t_n+r} = y, X_r = x) + \mathbb{P}^\eta(\mathcal{E}^c_{m-1})$$

with $A_{x,y} := \{X_{(m-1)t_n+r} = y, X_r = x, \mathcal{E}_{m-1}\}$. Thanks to the inductive step we have that

$$P^\eta(X^{\vec{v}}_{mt_n+r} - X^{\vec{v}}_{(m-1)t_n+r} \le a - \langle y-x,\vec{v}\rangle \,|\, A_{x,y}) \le P^\eta(Z_m \le a - \langle y-x,\vec{v}\rangle),$$

and therefore

$$P^\eta(X^{\vec{v}}_{mt_n+r} - X^{\vec{v}}_r \le a) \le P^\eta(Z_m + X^{\vec{v}}_{(m-1)t_n+r} - X^{\vec{v}}_r \le a) + \mathbb{P}^\eta(\mathcal{E}^c_{m-1}).$$

This proves (3.16). Iterating (3.16) and bounding $\sum_{j=1}^{t_n-1}\mathbb{P}^\eta(\mathcal{E}^c_j)$ with the union bound of the probability that no trap bigget than ϕ_{t_n} appears after time r and until time t_{n+1} in the whole ball of radius t_{n+1} gives (3.15).

We are left to bound the r.h.s. of (3.15). The first summand is equal to

$$P^\eta\Big(\sum_{k=1}^{t_n-1}\big(Z_k - E^\eta[Z_k]\big) \le c_{n+1}t_{n+1} + r - E^\eta[Z_k](t_n-1)\Big)$$
$$\le P^\eta\Big(\sum_{k=1}^{t_n-1}\big(Z_k - E^\eta[Z_k]\big) \le (t_n-1)\big(c_{n+1}t_n - E^\eta[Z_k]\big) + 8t_n\Big)$$
$$\le P^\eta\Big(\sum_{k=1}^{t_n-1}\big(Z_k - E^\eta[Z_k]\big) \le -t_n^{3/4}(t_n-1)\Big) \le \mathrm{e}^{-ct_n^{1/2}} \le \mathrm{e}^{-\phi_{t_n}^{1/3}}, \tag{3.17}$$

where we have used the fact that $c_{n+1} < 1$ and that $r \le 4t_n$ for the first inequality. The second inequality just comes from the definitions of t_n and c_n. We point out that for this inequality there might be the need to choose a T bigger than the one considered until this point, depending on ε; this constitutes no problem, since in this case it will be sufficient to repeat the proof with a bigger T (and hence a bigger γ). The last inequality in (3.17) is just a classical concentration inequality for sums of independent bounded random variables (note that $|Z_k| \le t_n$ a.s.).

For the second summand in the r.h.s. of (3.15) we assume (see (3.14)) that $y \in G(\eta, \phi_{t_{n+1}})$ for any $y \in B_{t_{n+1}}$ and notice that, by the choice made in (3.1),

$\gamma r \geq \gamma t_n \geq \phi^3_{t_{n+1}}$. We can therefore use Proposition 2.4 with $L = \phi_{t_{n+1}}$ and $J = \phi_{t_n}$ (we can suppose to have taken T large enough such that the condition "$L(\epsilon_J - \epsilon_L)$ large enough" is satisfied) to bound

$$\sum_{\substack{y \in B_{t_{n+1}} \\ r \leq s \leq t_{n+1}}} P^\eta\big(y \notin G(\eta(s), \phi_{t_n})\big) \leq c_1 t_{n+1}^{d+1} \frac{\exp\{-c_2\, \phi^d_{t_n} (\epsilon_{\phi_{t_n}} - \epsilon_{\phi_{t_{n+1}}})^2\}}{(\epsilon_{\phi_{t_n}} - \epsilon_{\phi_{t_{n+1}}})^2} \leq \mathrm{e}^{-\phi^{d/3}_{t_n}}, \tag{3.18}$$

where the last bound comes from the explicit expression (2.1). Plugging (3.18) and (3.17) into (3.15) gives the conclusion of (3.14), so that (3.12) is proven.

For the converse

$$P^\eta\big(X^{\vec{v}}_t > (v+\varepsilon)t\big) \leq \tfrac{1}{2}\,\mathrm{e}^{-\phi^{1/4}_t}$$

we follow a very similar strategy. We take the same definition of t_n and a new sequence of c_n's, this time increasing:

$$c_n := \frac{1}{2}\Big(1 + \sum_{k=0}^{n} \Big(\frac{\varepsilon}{1+\varepsilon}\Big)^k\Big)\, v\,.$$

The inductive step in this case will be

$$\textit{if } \forall y \in B_{t_n} \textit{ it holds that } y \in G(\eta, \phi_{t_n}), \textit{ then } P^\eta(X^{\vec{v}}_{t_n} > c_n t_n) \leq \tfrac{1}{2}\,\mathrm{e}^{-\phi^{1/4}_{t_n}}.$$

We define new random variables (Z_k) such that $Z_k = c_n t_n$ with probability $1 - \mathrm{e}^{-\phi^{1/4}_{t_n}}$ and $Z_k = t_n$ with probability $\mathrm{e}^{-\phi^{1/4}_{t_n}}$. The events $\mathcal{E}_{m-1}$ stay the same. We can now just follow the same steps as before. The only slight difference is with estimate (3.17), which now reads

$$\begin{aligned} P^\eta\Big(\sum_{k=1}^{t_n-1} Z_k \geq c_{n+1}t_{n+1} + r\Big) &\leq P^\eta\Big(\sum_{k=1}^{t_n-1} \big(Z_k - E^\eta[Z_k]\big) \geq (t_n - 1)\big(c_{n+1}t_n - E^\eta[Z_k]\big)\Big) \\ &\leq P^\eta\Big(\sum_{k=1}^{t_n-1} \big(Z_k - E^\eta[Z_k]\big) \geq t_n^{3/4}(t_n - 1)\Big) \leq \mathrm{e}^{-c\,t_n^{1/2}}, \end{aligned}$$

which holds with the same arguments as before. □

3.4. **Proof of Proposition 3.1.** We fix $\varepsilon > 0$ and pick N, T and γ large enough so that Proposition 3.3 holds. Since under $\mathbb{P}$ the initial distribution for the environment is a product measure, by a union bound we have that

$$\begin{aligned} \mathbb{P}(\exists y \in B_t \text{ s.t. } y \notin G(\eta, \phi_t)) &\leq \sum_{y \in \mathrm{B}_t} \sum_{L \geq \phi_t} \sum_{k \in \mathcal{T}} \mathbb{P}\big(|\langle \eta(t)\rangle^k_{y,L} - p_k| > \epsilon_{\phi_t}\big) \\ &\leq \sum_{y \in \mathrm{B}_t} \sum_{L \geq \phi_t} \sum_{k \in \mathcal{T}: p_k > 0} \mathrm{e}^{-cL^d \epsilon^2_{\phi_t}} \leq |\mathcal{T}|\, t^d\, \frac{\mathrm{e}^{-c\,\phi_t \epsilon^2_{\phi_t}}}{c\,\epsilon^2_{\phi_t}} \leq c_1\, \mathrm{e}^{-t^{c_2}}. \end{aligned}$$

By the Borel-Cantelli lemma, it then follows from Proposition 3.3 that $P-a.s.$, eventually $X^{\vec{v}}_t/t \in (v-\varepsilon, v+\varepsilon)$.

Appendix A.

A.1. **Proof of Claim 2.8.** Take $f : \mathbb{Z}^d \to \mathbb{R}$ such that $f(x) - f(x+e_i) \geq 0$ for all $x \in \mathbb{Z}^d$ with $\langle x, e_i\rangle \geq 0$ and such that $f(x) = f(\widetilde{x}_i)$ for all $x \in \mathbb{Z}^d$, where $\widetilde{x}_i := (x_1, x_2, ..., -x_i, ..., x_d)$. Consider $f(t,x)$, the unique solution of

$$\begin{cases} \partial_t f(t,x) &= \gamma \Delta f(t,x) \quad \forall t \geq 0 \\ f(0,x) &= f(x)\,. \end{cases}$$

Then we have that (1) $f(t,x) = f(t,\widetilde{x}_i)$ for all $t \geq 0$, $x \in \mathbb{Z}^d$ and (2) $f(t,x) - f(t,x+e_i) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{Z}^d$ with $\langle x, e_i\rangle > 0$. The first property is true by symmetry. The second can be explained by noticing, for example, that the continuous-in-time function

$$g(t,x) := \begin{cases} f(t,x) - f(t,x+e_i) & \text{if } \langle x, e_i\rangle \geq 0 \\ f(t,x) - f(t,x-e_i) & \text{otherwise} \end{cases}$$

is clearly everywhere positive for $t = 0$, but also stays positive for all $t > 0$. In fact, take a time t such that g is positive everywhere; then, in the points x where $g(t,x) = 0$, one has $\partial_t g(t,x) \geq 0$, as it is easy to verify. To conclude the proof, just take $f(t,x) = p_M(t,x)$.

A.2. **Proof of the Corollary 2.9.** If y and y' are in the same crown there is nothing to prove. Suppose then that $y' \in C_{n'}$ and $y \in C_n$ with $n' > n$. Consider $\hat{x}_{n'}$ and an $e \in \mathcal{N}$ such that $\langle \hat{x}_{n'}, e\rangle > 0$ and $\hat{x}_{n'} - me$ belongs to C_n for some $m \in \mathbb{N}$. Applying Claim 2.8 m-times, we have that $p_M(t, \hat{x}_{n'} - me) \geq p_M(t, \hat{x}_{n'})$ and therefore $p_M^+(t,y) \geq p_M^+(t,y')$. The same argument reversed gives the $p_M^-(\cdot,\cdot)$ case.

A.3. **Proof of Claim 2.10.** We focus on (2.18) and split the sum into three parts:

$$\begin{aligned} \sum_{n \geq L} E_M(t,n)|C_n| &= \sum_{n=L}^{M_- -1} E_M(t,n)|C_n| + \sum_{n=M_-}^{M_+ -1} E_M(t,n)|C_n| + \sum_{n=M_+}^{\infty} E_M(t,n)|C_n| \\ &=: H_1 + H_2 + H_3\,, \end{aligned} \tag{A.1}$$

where we have called $M_- := \lfloor M - (\gamma t)^{1/2+\varepsilon} \rfloor \vee L$ and $M_+ := \lfloor M + (\gamma t)^{1/2+\varepsilon} \rfloor \vee L$ (note that the first two sums might be empty). Part H_3 falls into a large deviations regime, since it can be dominated by the probability that a walker starting from far away enters the ball of radius M by time t:

$$H_3 = \sum_{n=M_+}^{\infty} E_M(t,n)|C_n| \leq c \sum_{n=M_+}^{\infty} p_M(t,\hat{x}_n)\, n^{d-1} = c \sum_{n=M_+}^{\infty} \frac{n^{d-1}}{M^d} P_{\hat{x}_n}(Y_t \in B_M),$$

where $(Y_t)_{t\geq 0}$ is a simple random walk that jumps at rate γt. Notice that $P_{\hat{x}_n}(Y_t \in B_M) \leq P_0(|Y_t| \geq n - M) \leq \mathrm{e}^{-c_1 (n-M)^2/(\gamma t)}$, where for the last inequality we have applied the continuous-time version of the large deviations estimate [LL10, Eq. 2.7]. Hence, as can be easily checked,

$$H_3 \leq c \sum_{n=M_+}^{\infty} \frac{n^{d-1}}{M^d} \mathrm{e}^{-c_1 (n-M)^2/(\gamma t)} \leq c_2\, \mathrm{e}^{-c_3 (\gamma t)^{2\varepsilon}}. \tag{A.2}$$

Part H_1, if not empty, can be treated similarly as it can be dominated by the probability that a walker that starts well inside the ball of radius M will leave the ball by time t:

$$H_1 = \sum_{n=L}^{M_-} E_M(t,n)|C_n| \le c \sum_{n=L}^{M_-} \frac{n^{d-1}}{M^d}\big(1 - P_{\check{x}_n}(Y_t \in B_M)\big) \le c_2\, \mathrm{e}^{-c_3(\gamma t)^{2\varepsilon}}. \tag{A.3}$$

Part H_2, when not empty, is more delicate. For $M_- \le n < M_+$ we write

$$\begin{aligned} E_M(n,t) &\le |p_M(t,\hat{x}_n) - G_M(t,\hat{x}_n)| + |G_M(t,\hat{x}_n) - G_M(t,\check{x}_n)| + |G_M(t,\check{x}_n) - p_M(t,\check{x}_n)| \\ &=: A_1(n) + A_2(n) + A_3(n) = A_1 + A_2 + A_3, \end{aligned} \tag{A.4}$$

where $G_M(t,x) := \frac{1}{|B_M|}\sum_{y\in B_M} G(t,x-y)$ and, for $z \in \mathbb{R}^d$, $G(t,z) := \frac{\exp\{-\|z\|^2/(2\gamma t)\}}{(2\pi\gamma t)^{d/2}}$ is the Gaussian kernel.

We treat together A_1 and A_3 from (A.4), since the calculation here below is valid for both of them (as can be seen simply by substituting $\hat{x}_n$ by $\check{x}_n$). We consider just A_1 for simplicity and write

$$A_1 \le \frac{1}{|B_M|} \sum_{y\in \hat{S}_1 \cup \hat{S}_2} \big|p(t,\hat{x}_n - y) - G(t,\hat{x}_n - y)\big|, \tag{A.5}$$

where $\hat{S}_1 = \hat{S}_1(n) := \{y \in B_M : \|y - \hat{x}_n\| \le (\gamma t)^{\frac{1}{2}+\varepsilon}\}$ and $\hat{S}_2 = \hat{S}_2(n) := \{y \in B_M : \|y - \hat{x}_n\| > (\gamma t)^{\frac{1}{2}+\varepsilon}\}$. The local central limit theorem (see, e.g., [LL10, Eq. (2.5)]) says that $|p(t,z) - G(t,z)| \le c\,(\gamma t)^{-d/2-1}$ for all $z \in \mathbb{Z}^d$. Therefore

$$\sum_{y\in\hat{S}_1} \big|p(t,\hat{x}_n - y) - G(t,\hat{x}_n - y)\big| \le \frac{c}{(\gamma t)^{d/2+1}}\Big((\gamma t)^{(1/2+\varepsilon)d} \wedge |B_M|\Big).$$

The sum over $\hat{S}_2$ can be bounded by

$$\sum_{y\in\hat{S}_2} p(t,\hat{x}_n - y) + G(t,\hat{x}_n - y) \le P\big(\|Y_t\| > (\gamma t)^{\frac{1}{2}+\varepsilon}\big) + P\big(\|Z\| > (\gamma t)^{\frac{1}{2}+\varepsilon}\big) \le c_1\, \mathrm{e}^{-c_2(\gamma t)^{2\varepsilon}},$$

where $(Y_t)_{t\ge 0}$ is a simple random walk on $\mathbb{Z}^d$ starting at the origin that jumps at rate γt and Z is a d-dimensional normal random variable with zero mean and covariance $\gamma t \cdot \mathrm{Id}$. Inserting the last two displays into (A.5) shows that

$$\sum_{n=M_-}^{M_+-1} (A_1 + A_3)|C_n| \le \frac{c_1}{(\gamma t)^{\frac{d}{2}+1}}\Big(\frac{(\gamma t)^{(\frac{1}{2}+\varepsilon)d}}{M^d} \wedge 1\Big) \sum_{n=M_-}^{M_+} n^{d-1} \le c_2\,(\gamma t)^{-1+\varepsilon d}, \tag{A.6}$$

as can be checked by distinguishing the case $M > t^{1/2+\varepsilon}$ and $M \le t^{1/2+\varepsilon}$.

We estimate now A_2. Using the definition of $\hat{S}_1$ and $\hat{S}_2$ given above and defining $\check{S}_1 = \check{S}_1(n) := \{y \in B_M : \|y - \check{x}_n\| \le (\gamma t)^{\frac{1}{2}+\varepsilon}\}$ and $\check{S}_2 = \check{S}_2(n) := \{y \in B_M : \|y - \check{x}_n\| > (\gamma t)^{\frac{1}{2}+\varepsilon}\}$, we write

$$\begin{aligned} A_2 \le \frac{1}{|B_M|}\Big(\Big|\sum_{y\in\hat{S}_1} G(t,\hat{x}_n - y) - \sum_{y\in\check{S}_1} G(t,\check{x}_n - y)\Big| \\ + \sum_{y\in\hat{S}_2} G(t,\hat{x}_n - y) + \sum_{y\in\check{S}_2} G(t,\check{x}_n - y)\Big). \end{aligned} \tag{A.7}$$

The sums over $\hat{S}_2$ and $\check{S}_2$ can be bounded by

$$\sum_{y\in\hat{S}_2} G(t,\hat{x}_n-y)+\sum_{y\in\check{S}_2} G(t,\hat{x}_n-y)\le 2\,P\big(\|Z\|>(\gamma t)^{1/2+\varepsilon}\big)\le c_1\,\mathrm{e}^{-c_2(\gamma t)^{2\varepsilon}}, \quad \text{(A.8)}$$

where Z is again a d-dimensional normal random variable with zero mean and variance $\gamma t\cdot\mathrm{Id}$. We are left with the sums over $\hat{S}_1$ and $\check{S}_1$ of (A.7). We use once more the triangular inequality to obtain

$$\Big|\sum_{y\in\hat{S}_1} G(t,\hat{x}_n-y)-\sum_{y\in\check{S}_1} G(t,\check{x}_n-y)\Big|\le\sum_{y\in\hat{S}_1}|G(t,\hat{x}_n-y)-\overline{G}(t,\hat{x}_n-y)|$$
$$+\Big|\sum_{y\in\hat{S}_1}\overline{G}(t,\hat{x}_n-y)-\sum_{y\in\check{S}_1}\overline{G}(t,\check{x}_n-y)\Big|+\sum_{y\in\check{S}_1}|G(t,\check{x}_n-y)-\overline{G}(t,\check{x}_n-y)|, \quad \text{(A.9)}$$

where we have called, for $z\in\mathbb{R}^d$, $\overline{G}(t,z)=\int_{K(z)}G(t,w)\,dw$, and $K(z)$ is the cube of size one centered in z. The first and third term of (A.9) can be roughly bound thanks to the mean value theorem: For each $y\in\hat{S}_1$

$$|G(t,\hat{x}_n-y)-\overline{G}(t,\hat{x}_n-y)|\le\Big|\int_{K(\hat{x}_n-y)}G(t,\hat{x}_n-y)-G(t,w)\,dw\Big|$$
$$\le c_1\frac{\|\zeta\|}{(\gamma t)^{d/2+1}}\mathrm{e}^{-\frac{\|\zeta\|^2}{2\gamma t}}\le\frac{c_2}{(\gamma t)^{(d+1)/2}},$$

with ζ some point in $K(\hat{x}_n-y)$. Hence, distinguishing between the case $M\le(\gamma t)^{1/2+\varepsilon}$ and $M>(\gamma t)^{1/2+\varepsilon}$,

$$\sum_{y\in\hat{S}_1}|G(t,\hat{x}_n-y)-\overline{G}(t,\hat{x}_n-y)|\le c\,\frac{(M\wedge(\gamma t)^{1/2+\varepsilon})^d}{(\gamma t)^{(d+1)/2}}, \quad \text{(A.10)}$$

and the same bound holds for terms of the sum in $\check{S}_1$.

For bounding the middle term in (A.9), we call $\hat{D}=\hat{D}(n):=\bigcup_{y\in\hat{S}_1}K(\hat{x}_n-y)\subseteq\mathbb{R}^d$ and $\check{D}=\check{D}(n):=\bigcup_{y\in\check{S}_1}K(\check{x}_n-y)\subseteq\mathbb{R}^d$ and notice that $\sum_{y\in\hat{S}_1}\overline{G}(t,\hat{x}_n-y)$ is exactly the probability that a Brownian Motion (W_t) started at the origin ends up in $\hat{D}$ after time γt (the same for $\check{D}$). Call $\mathcal{R}$ the rotation that brings $\check{x}_n$ on the half-line from the origin and passing through $\hat{x}_n$ and let $\check{D}':=\mathcal{R}(\check{D})$. By rotation and translation invariance of W_t we have then that

$$\Big|\sum_{y\in\hat{S}_1}\overline{G}(t,\hat{x}_n-y)-\sum_{y\in\check{S}_1}\overline{G}(t,\check{x}_n-y)\Big|=\big|P(W_t\in\hat{D}\smallsetminus\check{D}')-P(W_t\in\check{D}'\smallsetminus\hat{D})\big| \quad \text{(A.11)}$$

and we further notice that $\hat{D}\smallsetminus\check{D}'$ and $\check{D}'\smallsetminus\hat{D}$ have volume smaller than $c\,(M\wedge(\gamma t)^{1/2+\varepsilon})^{d-1}$, i.e. the smallest between the $d-1$-dimensional surface of B_M and the $d-1$-dimensional surface of $B_{t^{1/2+\varepsilon}}$ times a constant that depends on the dimension but not on M or n.

In the case $M>(\gamma t)^{1/2+\varepsilon}$, expression (A.11) can therefore be bounded by $c_1\,(\gamma t)^{-1/2+\varepsilon(d-1)}$. Using this estimate and (A.10) for bounding (A.9), and putting this together with (A.8) back into (A.7), we obtain

$$\sum_{n=M_-}^{M_+-1}A_2(n)|C_n|\le c_1\,\frac{(\gamma t)^{-\frac{1}{2}+\varepsilon(d-1)}}{M^d}\sum_{n=M_-}^{M_+}n^{d-1}\le c_2\,\frac{(\gamma t)^{\varepsilon d}}{M}\le c_3\,(\gamma t)^{-1/2+\varepsilon d}. \quad \text{(A.12)}$$

In the case $M \le (\gamma t)^{1/2+\varepsilon}$ we further notice the following: For $n \in [L, t^{1/2+\varepsilon}]$, $\hat{D}$ and $\check{D}$ are exactly the same object, so that $\hat{D}$ and $\check{D}'$ have the same volume. It follows that also $\hat{D} \smallsetminus \check{D}'$ and $\check{D}' \smallsetminus \hat{D}$ have the same volume, and this volume is smaller than $c\,M^{d-1}$ as explained before. Hence, for these n's, we can bound (A.11) by

$$\Big| \int_{\hat{D}\smallsetminus\check{D}'} G(t,z)dz - \int_{\check{D}'\smallsetminus\hat{D}} G(t,z)dz \Big| \le c_1 \frac{M^{d-1}}{(\gamma t)^{d/2}} \Big| \mathrm{e}^{-\frac{\|z_+\|^2}{2\gamma t}} - \mathrm{e}^{-\frac{\|z_-\|^2}{2\gamma t}} \Big|$$
$$\le c_2 \frac{M^{d-1}(\|z_+\| - \|z_-\|)}{(\gamma t)^{\frac{d}{2}+1}} \xi\, \mathrm{e}^{-\frac{\xi^2}{2\gamma t}} \le c_3 \frac{M^d}{(\gamma t)^{\frac{d+1}{2}}},$$

where z_+ and z_- are, respectively, the farthest and the closest point to the origin into $\{\hat{D} \smallsetminus \check{D}'\} \cup \{\check{D}' \smallsetminus \hat{D}\}$ and $\xi \in [\|z_-\|, \|z_+\|]$. For the second inequality we have used the mean value theorem, while for the third we have used the fact that the distance between z_+ and z_- is at most a constant times M and then we have took the worst-case for ξ (corresponding to $\xi \simeq (\gamma t)^{1/2}$). For $n \in (t^{1/2+\varepsilon}, M_+ - 1]$ we just bound (A.11) with $cM^{d-1}(\gamma t)^{-d/2}$. We obtain

$$\sum_{n=M_-}^{M_+-1} A_2(n)|C_n| \le \frac{c_1}{(\gamma t)^{(d+1)/2}} \sum_{n=0}^{\lfloor t^{1/2+\varepsilon}\rfloor} n^{d-1} + \frac{c_2}{M(\gamma t)^{d/2}} \sum_{n=\lfloor t^{1/2+\varepsilon}\rfloor+1}^{M_+} n^{d-1} \le c_3(\gamma t)^{-1/2+\varepsilon d}.$$

This, together with (A.12) and (A.6), shows that $H_2 \le c\,(\gamma t)^{-1/2+\varepsilon d}$. Putting back together this bound for H_2, the bound for H_3 (A.2) and for H_1 (A.3) into (A.1) gives (2.18). Our estimates for A_1, A_2 and A_3 and (A.4) also imply (2.19).

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