Electrical resistance between pairs of vertices of a conducting cube and continuum limit for a cubic resistor network

Malik Mamode

Electrical resistance between pairs of vertices of a conducting cube and continuum limit for a cubic resistor network

Journal of Physics Communications

PAPER

Electrical resistance between pairs of vertices of a conducting cube and continuum limit for a cubic resistor network

Malik Mamode

Department of Physics, Laboratoire PIMENT, University of La Réunion, France

E-mail: malik.mamode@univ-reunion.fr

Keywords: driving-point resistance, finite cubic resistor network, Madelung constant, continuum limit

OPEN ACCESS

RECEIVED
29 May 2017

REVISED
23 August 2017

ACCEPTED FOR PUBLICATION
6 September 2017

PUBLISHED
17 October 2017

Abstract

A method is given of determining analytically the driving-point resistance between any pair of vertices of: (i) a homogeneous conducting cube, (ii) a $N \times N \times N$ cubic lattice of identical resistors. For the latter, the limit $N \to \infty$ is considered and the condition of the ohmic equivalence between these two systems is investigated.

Introduction

We intend to compute the electrical resistance between different pairs of vertices of a homogeneous ohmic cube. To the author's knowledge, such a calculation deceptively simple cannot be found anywhere except the unwieldy Fourier series-based formula for the potential at any point of the cube, obtained by Weiss *et al* in [1] from which the authors discussed the derivation of the van der Pauw formula. The van der Pauw technique is commonly used in the semiconductor industry for electrical transport measurements on solid materials [2]. It permits for instance to obtain the conductivity of infinitely thin samples of arbitrary shape from measurements of current and voltage difference considering four points along the periphery of the sample. However, the van der Pauw method is valid only for infinitely thin samples so that the effects of sample thickness on the accuracy of results may be significant requiring that the van der Pauw formula be corrected and generalized (see [1, 3]). Beyond its theoretical aspects, the calculations laid down in the paper and the exact determination of driving-point resistances between any pair of vertices of a cube may contribute to improve this experimental technique.

Apart the known resistance between two distinct points of an infinite ohmic medium (see e.g. [4], page 297), the problem for the conducting cube is usually addressed in the literature as the continuum limit of a cubic resistor network (see e.g. [5, 6]). For both cases, the equations to be solved are well-known and related to Poisson-type equations such as encountered in electrostatics. But the difficulty both for the cubic medium and the finite network is to take in account the boundary conditions for deriving the exact analytical solution as simply as possible and readily usable in applications.

The bulk of this paper is devoted to the conducting cube (section 1 and followings) while the continuum limit of the cubic resistor network will be discussed in the 4th and last section. The results reported here will show that, in a largely classical context, there exists a very deep connection between the condensed matter physics and the analytic number theory.

1. Electrostatic model

Let a homogeneous conducting cube $\mathcal{C}$ of edge a and of conductivity σ. Connecting a battery between a pair of vertices (say, O and A, see figure 1) a current I is inserted (or extracted) at the point O and similarly $-I$ at the point A. The sign of I is for now unspecified and will be suitably fixed later on. Knowing the voltage difference $V_O - V_A$, the resistance may be in principle obtained as the ratio

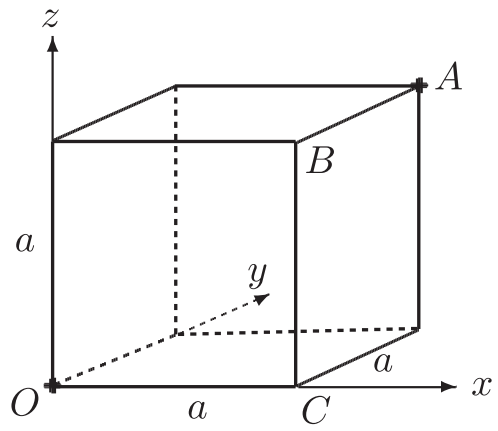

Figure 1. Geometry of the cube.

$$R_{OA} = \left| \frac{V_O - V_A}{I} \right|. \tag{1}$$

If $\mathbf{J}$ denotes the electric current density at the point $\mathbf{r} \in \mathcal{C}$, one has

$$\boldsymbol{\nabla} \cdot \mathbf{J} = I\{\delta(\mathbf{r}) - \delta(\mathbf{r} - \mathbf{r}_A)\}, \tag{2}$$

where δ is the Dirac distribution on $\mathbb{R}^3$. Hence, in virtue of the Ohm's law $\mathbf{J} = -\sigma \boldsymbol{\nabla} V$ and in the absence of other current sources or fields, the potential is solution of the following Poisson equation

$$\boldsymbol{\nabla}^2 V = -\frac{I}{\sigma}\{\delta(\mathbf{r}) - \delta(\mathbf{r} - \mathbf{r}_A)\}, \quad \mathbf{r} \in \mathcal{C} \tag{3}$$

which must be solved given Neumann boundary conditions which ensure that no current can flow through the faces of the cube, nor electric charges can accumulate on the faces. Hence, analogy with the electrostatics is immediate considering a cube of dielectric constant σ holding two opposite point charges $\pm I$.

As it is known (see e.g. [7]), the equation (3) may have an exact closed-form solution but this requires to formulate the latter boundary value problem on the 3-flat torus $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$ where Λ is the Bravais lattice $(2a\mathbb{Z})^3$. It amounts thus, by the so-called method of images [8], to periodize the problem in the three coordinate directions (period $2a \times 2a \times 2a$) and to consider a charge distribution of type body-centered cubic (bcc) for a crystal lattice, the cube $\mathcal{C}$ being thus a (quarter of) unit cell which contains two point sources $\pm 8I$ taking in account their coordination number.

So doing, we have thus led to solve on $\mathbb{T}^3$,

$$\boldsymbol{\nabla}^2 V = -\frac{8I}{\sigma}\{\delta(\mathbf{r}) - \delta(\mathbf{r} - \mathbf{r}_A)\} := f(\mathbf{r}), \tag{4}$$

where this time, δ denotes a Dirac distribution on $\mathbb{T}^3$ (i.e. a 3D Dirac comb on $\mathbb{R}^3$, of period $2a \times 2a \times 2a$) and $\int_{\mathbb{T}^3} f(\mathbf{r})\mathrm{d}\mathbf{r} = 0$ in accordance with the Gauss theorem. Roughly speaking, the boundary value problem posed for the cube has been thereby redrafted by periodization for the whole space making it more convenient to solve.

2. Fundamental solution for the Laplacian

2.1. Integral representation

Let the 3-dimensional theta function,

$$\Theta_k\left(\frac{\pi\mathbf{r}}{2a} \,\middle|\, \mathrm{i}v\right) := \vartheta_k\left(\frac{\pi x}{2a} \,\middle|\, \mathrm{i}v\right)\vartheta_k\left(\frac{\pi y}{2a} \,\middle|\, \mathrm{i}v\right)\vartheta_k\left(\frac{\pi z}{2a} \,\middle|\, \mathrm{i}v\right), \quad \mathbf{r} = (x, y, z) \tag{5}$$

ϑ_k denoting the kth Jacobi theta function ($k = 1, 2, 3$ or 4) [9].

Thus, it is easy to verify that the fundamental solution for the Laplacian $\boldsymbol{\nabla}^2$ on the torus $\mathbb{T}^3$ i.e. one solution of the Poisson equation

$$\boldsymbol{\nabla}^2 G = -\delta(\mathbf{r}) + \frac{1}{8a^3}, \tag{6}$$

can be expressed by the following integral

$$G(\mathbf{r}) = \frac{1}{8\pi a}\int_0^{+\infty}\left\{\Theta_3\left(\frac{\pi\mathbf{r}}{2a}\,\middle|\,\mathrm{i}v\right) - 1\right\}\mathrm{d}v. \tag{7}$$

Indeed, recall that (5) is the exponentially convergent series

$$\Theta_3\left(\frac{\pi\mathbf{r}}{2a}\,\middle|\,\mathrm{i}v\right) = \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathrm{e}^{-\pi|\mathbf{k}|^2 v}\mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\mathbf{r}/a}, \quad \mathbf{k} = (n,\, m,\, p) \tag{8}$$

or also, by application of the Poisson summation formula, a sum of periodized Gaussians as,

$$\Theta_3\left(\frac{\pi\mathbf{r}}{2a}\,\middle|\,\mathrm{i}v\right) = \frac{1}{v^{3/2}}\sum_{\mathbf{k}\in\mathbb{Z}^3} \mathrm{e}^{-\pi(\mathbf{r}+2a\mathbf{k})^2/4a^2 v}. \tag{9}$$

In addition, (5) is solution of the heat equation $\boldsymbol{\nabla}^2\Theta_3 = (\pi/a^2)\,\partial_v\Theta_3$ such that in a distributional sense [9],

$$\lim_{v\to+\infty}\Theta_3 = 1 \quad \text{and} \quad \lim_{v\to 0+}\Theta_3 = 8a^3\,\delta(\mathbf{r}).$$

Moreover, one can show that the integral (7) is absolutely convergent on any compact not containing the origin. Hence, since

$$\boldsymbol{\nabla}^2 G = \frac{1}{8\pi a}\int_0^{+\infty}\boldsymbol{\nabla}^2\Theta_3\mathrm{d}v = \frac{1}{8a^3}\int_0^{+\infty}\partial_v\Theta_3\mathrm{d}v,$$

it follows well after integration the identity (6).

This fundamental solution is given up to an arbitrary constant (chosen equal to zero) and is of mean value 0, both conditions allowing to fix a zero potential reference at any point of the torus in the absence of current. In the context of electrostatics, the Poisson equation (6) shows that $G(\mathbf{r})$ may correspond to the electrostatic potential at the point $\mathbf{r}$ due to unit point charges at lattice points $2a\mathbf{k}$, all of 3-space being negatively and uniformly charged (of density $-1/8a^3$) for an overall electroneutrality, a such charge distribution being sometimes called a jellium crystal and $G(\mathbf{r})$ the jellium potential [10].

As already mentioned, the fundamental solution given by (7) is well-defined anywhere on the torus except at the origin $\mathbf{r} = \mathbf{0} \pmod{\Lambda}$ where it possesses a singularity. Indeed, since (see (9)),

$$\Theta_3(\mathbf{0}|\mathrm{i}v) = \frac{1}{v^{3/2}}\{1 + \mathcal{O}(\mathrm{e}^{-\pi/v})\}, \tag{10}$$

the integrand in (7) is $\mathcal{O}(v)^{-3/2}$ as v tends to zero and it ensues that the integral (7) is well divergent at the origin. Nevertheless, it is quite interesting for the rest of the paper to notice that *when* $\mathbf{r} = \mathbf{0}$ *it is possible to extract from the integral (7) a finite part in the Hadamard sense* denoted G_O and equal to [11],

$$G_O \coloneqq \frac{1}{8\pi a}\,{\text{⨍}}_0^{+\infty}\{\theta_3^3(\mathrm{i}v) - 1\}\,\mathrm{d}v = \frac{1}{8\pi a}\lim_{\varepsilon\to 0+}\left(\int_{\varepsilon^2}^{+\infty}\{\theta_3^3(\mathrm{i}v) - 1\}\,\mathrm{d}v - \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon)\right) \tag{11}$$

using here and in the sequel, as it is usual for these modular functions, the following notation

$$\vartheta_k(0|\mathrm{i}v) = \theta_k(\mathrm{i}v).$$

For the electrostatic problem, the regularization term $(1/4\pi\varepsilon a) + \mathcal{O}(\varepsilon)$ which removes the singularity from the jellium potential (7) at $\mathbf{r} = \mathbf{0}$ may be viewed as the 3-Coulomb potential, at the distance εa, due to a solitary unit charge located at the origin of $\mathbb{R}^3$, plus the (null) contribution of the uniformly charged 'jelly'. As a result, the finite part G_O may be thus interpreted as *the electrostatic potential due to the crystal and seen by the charge origin, its self-contribution being removed*, result which can be thus linked with the lattice sum arising from the related Poisson equation by summing all 3-Coulomb terms as follows

$$\frac{1}{8\pi a}\sum_{\mathbf{k}\in\mathbb{Z}^3}{}'\frac{1}{|\mathbf{k}|} = \frac{1}{8\pi a}\mathfrak{a}(1) \qquad \text{with} \qquad \mathfrak{a}(1) = \sum{}'(n^2 + m^2 + p^2)^{-1/2}, \tag{12}$$

the prime on the summation sign indicating omission of the term $\mathbf{k} = \mathbf{0}$ (the notation $\mathfrak{a}(1)$ is those introduced by Zucker, see [12, 13]).

One can check that (11) and (12) are numerically identical: through an elementary numerical integration by means of the Mathematica software (fixing for instance, $\varepsilon = 0.01$), one finds thereby the value

$$\lim_{\varepsilon\to 0+}\left(\int_{\varepsilon^2}^{+\infty}\{\theta_3^3(\mathrm{i}v) - 1\}\,\mathrm{d}v - \frac{2}{\varepsilon}\right) = -2.837\ 295\ 727\ \ldots \equiv \mathfrak{a}(1) \tag{13}$$

which agrees up to the fifth decimal with the Zucker's result tabulated in his outstanding work [12]. We can hope to find a more accurate result by using more refined numerical integration procedure (see for instance [14]).

At first glance, it seems that there is a contradiction between the latter negative result and the sum of positive terms (12) but it is a long-known fact that such an infinite series is purely formal and cannot converge as it is (see

e.g. [10, 13, 15]). We see briefly below that the correct comprehension of (12) whose physical sense is undeniable must be found in analytic number theory using tools of complex analysis.

Let us indicate that the Hadamard finite-part (11) and relatives are already implicitly considered in the finite-size scaling theory for defining in an ad hoc way some Madelung-type constants describing different physical situations (geometry system, interparticle interaction, etc) (see [16, 17]).

2.2. Series representation

A standard approach (detailed in [10], see also [15]) is indeed to consider the following 3-dimensional zeta function

$$Z(s;\,\mathbf{u},\,\mathbf{v}) = \sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{\mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\mathbf{u}/a}}{|\mathbf{k}-\mathbf{v}/2a|^s} \tag{14}$$

with $\mathbf{u},\,\mathbf{v}\in\mathbb{T}^3$, which is an analytic function for $\Re s > 3$ (again, the prime means that any singularities are to be ignored in the sum). First, (14) is realized through the Mellin transform of the generalized 'theta function'

$$\Theta_g(\mathbf{u},\,\mathbf{v};\,\nu) = \sum_{\mathbf{v}\neq 2a\mathbf{k}} \mathrm{e}^{-\pi|\mathbf{k}-\mathbf{v}/2a|^2\nu}\mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\mathbf{u}/a}$$

as

$$\mathcal{M}(s;\,\mathbf{u},\,\mathbf{v}) \coloneqq \pi^{-s/2}\Gamma(s/2)Z(s;\,\mathbf{u},\,\mathbf{v}) = \int_0^{+\infty} \nu^{s/2-1}\Theta_g(\mathbf{u},\,\mathbf{v};\,\nu)\,\mathrm{d}\nu \tag{15}$$

and then, applying the Poisson summation to Θ_g leads to the reflection formula

$$Z(s;\,\mathbf{u},\,\mathbf{v}) = \mathrm{e}^{\mathrm{i}\pi\mathbf{u}.\mathbf{v}/2a^2}\pi^{s-3/2}\frac{\Gamma((3-s)/2)}{\Gamma(s/2)}Z(3-s;\,-\mathbf{v},\,\mathbf{u}), \tag{16}$$

where Γ is the Gamma function. In addition, it is well-known in analytic number theory (see e.g. [18]) that the function (15) has a meromorphic continuation in the entire s-plane beyond the line $\Re s = 3$, except for simple poles at $s = 0$ and $s = 3$.

As a result, taking $s = 2$, $\mathbf{u} = \mathbf{r}$ and $\mathbf{v} = \mathbf{0}$, it ensues for our purposes that $Z(2;\,\mathbf{r},\,\mathbf{0}) = \pi\, Z(1;\,\mathbf{0},\,\mathbf{r})$ whilst $\mathcal{M}(2;\,\mathbf{r},\,\mathbf{0})$ is identical to the integral (7) (up to the factor $1/8\pi a$). We have thereby established the following series representations for the fundamental solution in both forms,

$$G(\mathbf{r}) = \frac{1}{8\pi^2 a}\sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{\mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\mathbf{r}/a}}{|\mathbf{k}|^2} = \frac{1}{4\pi}\sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{1}{|\mathbf{r}-2a\mathbf{k}|}, \tag{17}$$

these equalities being in principle not everywhere pointwise, but understood through the analytic continuation of relevant zeta functions for any $\mathbf{r}\in\mathbb{T}^3$. Nevertheless, it is noteworthy that

- the first series (denoted Σ_1) is the formal Fourier series of G we may directly derived by solving (6) considering the usual trigonometric expansion for $\delta(\mathbf{r})$ on $\mathbb{T}^3$,

  $$\delta(\mathbf{r}) = \frac{1}{8a^3}\sum_{\mathbf{k}\in\mathbb{Z}^3} \mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\mathbf{r}/a}.$$

  As it is, Σ_1 is obviously divergent at the origin (mod Λ), otherwise it is conditionally convergent.

- While the second one (denoted Σ_2) simply corresponds to the electrostatic superposition of 3-Coulomb potentials at point $\mathbf{r}$ due to all lattice point charges if $\mathbf{r}\neq\mathbf{0}$, otherwise the self-potential of the single origin charge is removed.

So doing, notice from Σ_2 and (12) that one has here $G(\mathbf{0}) = G_O = (8\pi a)^{-1}\mathfrak{a}(1)$ showing, as already emphasized by Crandall and Buhler [10], that the analytic continuation leading to the previous results, by removing the singularity due to the origin charge, allots to the fundamental solution G when divergent the finite-part value (11) equal to the lattice sum (12).

3. Resistance calculation

3.1. Resistance between the vertices *O* and *A*

The fundamental solution (7) or (17) being now available, the potential V at any point $\mathbf{r}$ of the torus $\mathbb{T}^3$ (and thus, of the cube $\mathcal{C}$) may be exactly obtained from (4) via the triple convolution product

$$V(\mathbf{r}) = -G \overset{\mathbf{r}}{*} f = \frac{8I}{\sigma}\{G(\mathbf{r}) - G(\mathbf{r} - \mathbf{r}_A)\}, \tag{18}$$

i.e. using for instance the integral representation (7) and a translation property for Θ_3,

$$V(\mathbf{r}) = \frac{I}{\pi\sigma a}\int_0^{+\infty}\left\{\Theta_3\left(\frac{\pi\mathbf{r}}{2a}\,\middle|\,\mathrm{i}v\right) - \Theta_4\left(\frac{\pi\mathbf{r}}{2a}\,\middle|\,\mathrm{i}v\right)\right\}\mathrm{d}v. \tag{19}$$

It is easy to verify that (19) satisfies well to the Neumann boundary conditions. Since G is even, notice that holds the reflection formula $V(\mathbf{r}) = -V(\mathbf{r}_A - \mathbf{r})$.

In the light of the regularization procedure discussed previously, the voltage difference felt by the electric current flowing between the points O and A is thus equal to

$$V(\mathbf{r}_A) - V(\mathbf{0}) = -2V(\mathbf{0}) = \frac{16I}{\sigma}(G(\mathbf{r}_A) - G(\mathbf{0})) := \frac{2I}{\pi\sigma a}\mathfrak{M}, \tag{20}$$

where the constant $\mathfrak{M}$ depending on the formula used for G, is equally expressed as

- the finite-part integral

$$\begin{aligned}\mathfrak{M} &= \fint_0^{+\infty}\{\theta_4^3(\mathrm{i}v) - \theta_3^3(\mathrm{i}v)\}\,\mathrm{d}v = \lim_{\varepsilon\to 0+}\left(\int_{\varepsilon^2}^{+\infty}\{\theta_4^3(\mathrm{i}v) - \theta_3^3(\mathrm{i}v)\}\,\mathrm{d}v + \frac{2}{\varepsilon}\right)\\ &= 2.035\ 361\ 5\ \ \ldots,\end{aligned} \tag{21}$$

  numerical value obtained through a simple numerical integration,

- or the following first lattice sum

$$\mathfrak{M} = \frac{1}{\pi}\left(\sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{(-1)^{n+m+p}}{|\mathbf{k}|^2} - \sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{1}{|\mathbf{k}|^2}\right) = -\frac{2}{\pi}\sum_{n+m+p\in 2\mathbb{Z}+1}(n^2+m^2+p^2)^{-1} \tag{22}$$

- or the second one,

$$\begin{aligned}\mathfrak{M} &= 2\left(\sum_{\mathbf{k}\in\mathbb{Z}^3}\frac{1}{|2\mathbf{k}+\mathbf{1}|} - \sum_{\mathbf{k}\in\mathbb{Z}^3}{}'\frac{1}{|2\mathbf{k}|}\right)\\ &= 2\left(\sum_{\mathbf{k}\in(2\mathbb{Z}+1)^3}(n^2+m^2+p^2)^{-1/2} - \sum_{\mathbf{k}\in(2\mathbb{Z})^3}{}'(n^2+m^2+p^2)^{-1/2}\right).\end{aligned} \tag{23}$$

Considering the related Zucker's lattice sums as tabulated in [12], one thus obtains respectively

$$\mathfrak{M} = \frac{1}{\pi}(\mathfrak{d}(2) - \mathfrak{a}(2)) = \frac{3}{2}(\mathfrak{c}(1) - \mathfrak{a}(1)) = 2.035\ 361\ 509\ \ldots \tag{24}$$

and we can note that the agreement between all these numerical results is highly satisfactory. This constant $\mathfrak{M}$ is well-known in electrostatics of ionic crystals as the Madelung constant for the bcc crystal structure built with the cube $\mathcal{C}$ (for instance, a CsCl crystal [7, 13]). Moreover, $\mathfrak{M}$ is found positive meaning for consistency that the current must be considered flowing through the cube from A to O. Finally, denoting by $R = 1/\sigma a$ a standard resistance value (this is for instance the very known electric resistance between two opposite faces of the cube), the driving-point resistance (also called, two-point resistance) between the vertices O and A has exactly the value

$$R_{OA} = \frac{V(\mathbf{r}_A) - V(\mathbf{0})}{I} = \frac{2R}{\pi}\mathfrak{M} \approx 1.296 \times R. \tag{25}$$

For a silver cube of edge $a = 1$ cm, at 20 °C, $\sigma = 6.30 \times 10^7\ \Omega^{-1}\,\mathrm{m}^{-1}$ one finds $R_{OA} \approx 2.06 \times 10^{-6}\ \Omega$.

3.2. Other resistances

In a similar way, the driving-point resistance between the vertices O and B on the face diagonal, and between O and C on one edge are obtained by solving the Poisson equation (3) on the torus $\mathbb{T}^3$ with the current densities $\mathbf{J} = I\{\delta(\mathbf{r}) - \delta(\mathbf{r} - \mathbf{r}_B)\}$ and $\mathbf{J} = I\{\delta(\mathbf{r}) - \delta(\mathbf{r} - \mathbf{r}_C)\}$ respectively. Details of calculation are now omitted since similar to previous ones. Thus, defining the following Madelung-type constants

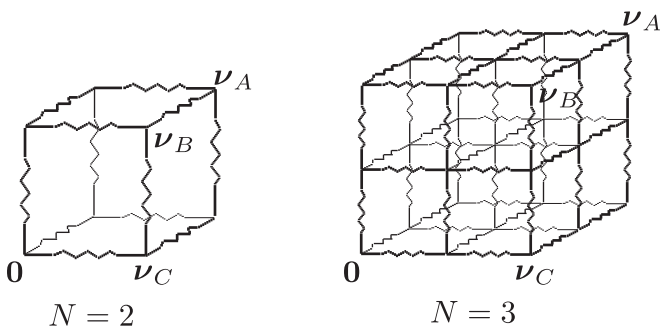

Figure 2. A N^3 cubic resistor network and nodes $\mathbf{0} = (0, 0, 0)$, $\boldsymbol{\nu}_A = (N-1, N-1, N-1)$, $\boldsymbol{\nu}_B = (N-1, 0, N-1)$, $\boldsymbol{\nu}_C = (N-1, 0, 0)$.

$$\begin{aligned}\mathfrak{M}' &:= \fint_0^{+\infty}\{\theta_4^2(\mathrm{i}\nu) - \theta_3^2(\mathrm{i}\nu)\}\theta_3(\mathrm{i}\nu)\mathrm{d}\nu = \frac{1}{\pi}\left(\sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{(-1)^{n+m}}{|\mathbf{k}|^2} - \sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{1}{|\mathbf{k}|^2}\right) = \frac{1}{\pi}(\mathfrak{c}(2) - \mathfrak{a}(2))\\ &= \sum_{\mathbf{k}\in\mathbb{Z}^3}((n+1/2)^2 + m^2 + (p+1/2)^2)^{-1/2} - \sum_{\mathbf{k}\in\mathbb{Z}^3}{}'(n^2+m^2+p^2)^{-1/2} = -(\mathfrak{b}(1) + \mathfrak{c}(1))\\ &= 2.254\ 775\ 9\ \ldots \end{aligned} \tag{26}$$

and

$$\begin{aligned}\mathfrak{M}'' &:= \fint_0^{+\infty}\{\theta_4(\mathrm{i}\nu) - \theta_3(\mathrm{i}\nu)\}\theta_3^2(\mathrm{i}\nu)\mathrm{d}\nu\\ &= \frac{1}{\pi}\left(\sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{(-1)^{n}}{|\mathbf{k}|^2} - \sum_{\mathbf{k}\in\mathbb{Z}^3}{}' \frac{1}{|\mathbf{k}|^2}\right) = \frac{1}{\pi}(\mathfrak{b}(2) - \mathfrak{a}(2))\\ &= \sum_{\mathbf{k}\in\mathbb{Z}^3}((n+1/2)^2 + m^2 + p^2)^{-1/2} - \sum_{\mathbf{k}\in\mathbb{Z}^3}{}'(n^2+m^2+p^2)^{-1/2} = -\frac{1}{2}(\mathfrak{b}(1) + 2\mathfrak{c}(1) + \mathfrak{d}(1))\\ &= 2.741\ 365\ 1\ \ldots \end{aligned} \tag{27}$$

for the first and the other configuration respectively, it is straightforward to derive the following resistances

$$R_{OB} = \frac{V(\mathbf{r}_B) - V(\mathbf{0})}{I} = \frac{2}{\pi}\frac{\mathfrak{M}'}{\sigma a} \approx 1.435 \times R \quad \text{and} \quad R_{OC} = \frac{V(\mathbf{r}_C) - V(\mathbf{0})}{I} = \frac{2}{\pi}\frac{\mathfrak{M}''}{\sigma a} \approx 1.745 \times R. \tag{28}$$

For a silver cube of edge $a = 1$ cm, at 20 °C, $\sigma = 6.30 \times 10^7\ \Omega^{-1}\,\mathrm{m}^{-1}$ one finds $R_{OB} \approx 2.28 \times 10^{-6}\ \Omega$ and $R_{OC} \approx 2.77 \times 10^{-6}\ \Omega$.

4. Cubic resistor network

Consider a $N \times N \times N$ cubic lattice formed from equal resistors $R(N)$ with nodes $\boldsymbol{\nu} = (n, m, p)$ and integers $n, m, p = 0, \ldots, N-1$ (see figure 2). We intend in this section to find the driving-point resistance between vertices of this cubic network, and to discuss when N tends to infinity, the continuum limit and conditions for the ohmic equivalence with the previous conducting cube $\mathcal{C}$.

4.1. Resistance between two diagonally opposite nodes

Let $R_{OA}^{(N)}$ the driving-point resistance between the diagonally opposite nodes $\mathbf{0} = (0, 0, 0)$ and $\boldsymbol{\nu}_A = (N-1, N-1, N-1) = (N-1)\mathbf{1}$.

Assuming a current I flowing from the node $\mathbf{0}$ to $\boldsymbol{\nu}_A$ and in the absence of other external current sources, the electric potential $V(\boldsymbol{\nu})$ at any node $\boldsymbol{\nu}$ can be obtained by applying the Ohm's and Kirchhoff's laws. Hence, the desired resistance is readily given by the ratio

$$R_{OA}^{(N)} = \frac{V(\mathbf{0}) - V(\boldsymbol{\nu}_A)}{I}.$$

This is a classical problem of multilinear algebra which has been recently solved by Wu [19] in an elegant manner using the formalism of dyadic algebra or, in a more abstract form, by Jafarizadeh *et al* [20] whose calculation is based on stratification of the underlying graph and the Stieltjes transform of the spectral distribution associated with the network.

Here, we prefer an alternative approach using the lattice Green's function method appearing in many problems of solid state physics [5, 6, 21], and which can be likened to a discrete formulation of problems

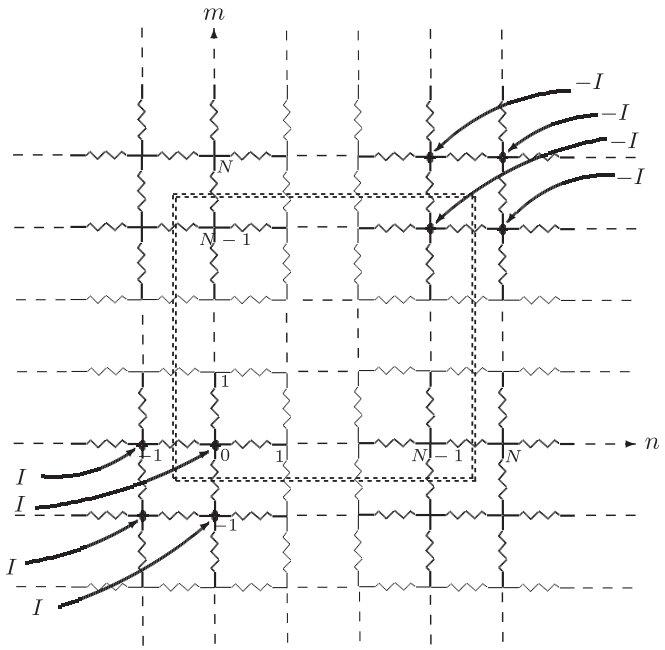

Figure 3. An infinite square lattice of identical resistors with an appropriate $2N$-periodic, even current distribution (here, even means symmetrical on either side of the axes $n = -1/2$ and $m = -1/2$). Such a current distribution ensures by the superposition theorem that no current can flow through the sides of the finite square network $\{0, \ldots, N-1\}^2$ surrounded in dashed lines.

discussed in the previous sections. Indeed, according to the superposition theorem of current distributions, the physical situation is identical to those for an *infinite* 3-dimensional cubic lattice with the following current distribution, even and $2N$-periodic along each lattice principal direction of vector $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ (notice that: $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \mathbf{1}$),

$$J(\boldsymbol{\nu}) = I\left\{\sum_{\mathbf{k}\in(2\mathbb{Z})^3} \Delta(\boldsymbol{\nu} - N\mathbf{k}) - \sum_{\mathbf{k}\in(2\mathbb{Z}+1)^3} \Delta(\boldsymbol{\nu} - N\mathbf{k})\right\}, \tag{29}$$

where Δ denotes the octuplet,

$$\begin{aligned}\Delta(\boldsymbol{\nu}) = {} & \delta(\boldsymbol{\nu}) + \delta(\boldsymbol{\nu} + \mathbf{e}_1) + \delta(\boldsymbol{\nu} + \mathbf{e}_2) + \delta(\boldsymbol{\nu} + \mathbf{e}_1 + \mathbf{e}_2) \\ & + \delta(\boldsymbol{\nu} + \mathbf{e}_3) + \delta(\boldsymbol{\nu} + \mathbf{e}_1 + \mathbf{e}_3) + \delta(\boldsymbol{\nu} + \mathbf{e}_2 + \mathbf{e}_3) + \delta(\boldsymbol{\nu} + \mathbf{1})\end{aligned} \tag{30}$$

and δ is the Kronecker unit impulsion,

$$\delta(\boldsymbol{\nu} - \boldsymbol{\nu}_0) = \begin{cases} 1 & \boldsymbol{\nu} = \boldsymbol{\nu}_0 \\ 0 & \text{otherwise} \end{cases}.$$

Figure 3 displays for simplicity the corresponding two dimensional configuration where the periodicity and evenness of current distributions ensure that no current can enter or exit from the sides of the primitive cell $\{0, \ldots, N-1\}^2$ of side-length N. Thus, the problem posed for the infinite lattice is equivalent to the Neumann problem for a single square $N \times N$ resistor network.

Thus, we are led to solve on the 3-dimensional discrete torus $(\mathbb{Z}/2N\mathbb{Z})^3$ the difference equation

$$\begin{aligned}\boldsymbol{\nabla}'^2 V(\boldsymbol{\nu}) \coloneqq {} & \sum_{i=1}^{3} \{V(\boldsymbol{\nu} \pm \mathbf{e}_i) - V(\boldsymbol{\nu})\} = -R(N)J(\boldsymbol{\nu}) \\ & = -R(N)I\{\Delta(\boldsymbol{\nu}) - \Delta(\boldsymbol{\nu} + N\mathbf{1})\},\end{aligned} \tag{31}$$

where the finite-difference operator $\boldsymbol{\nabla}'^2$ is the so-called lattice Laplacian, the latter equation being thus the discrete analog of the Poisson equation (4).

The related lattice Green function $G'(\boldsymbol{\nu})$ i.e. the solution on $(\mathbb{Z}/2N\mathbb{Z})^3$ of equation (to be compared with (6))

$$\boldsymbol{\nabla}'^2 G' = -\delta(\boldsymbol{\nu}) + \frac{1}{(2N)^3} \tag{32}$$

Table 1. Some exact values of the two-point resistance $R_{OA}^{(N)}$ in units of $R(N)$.

N	2	3	4	5	6
$\frac{R_{OA}^{(N)}}{R(N)} = \frac{\alpha(N)}{N}$	$\frac{5}{6}$	$\frac{43}{42}$	$\frac{401}{357}$	$\frac{1\,888\,231}{1\,593\,834}$	$\frac{92\,427\,683}{75\,366\,720}$

is easily obtained using the discrete Fourier transform and shift properties, knowing that

$$\delta(\boldsymbol{\nu}) = \frac{1}{(2N)^3} \sum_{\mathbf{k}\in\{-N,\,\ldots,\,N-1\}^3} \mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\boldsymbol{\nu}/N}, \quad \boldsymbol{\nu} \in \{-N,\,\ldots,\,N-1\}^3.$$

Hence, one finds the well-known result (see e.g. [5] and references therein) given up to a constant,

$$G'(\boldsymbol{\nu}) = \frac{1}{16N^3} \sum_{\mathbf{k}=(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3}{}' \frac{\mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\boldsymbol{\nu}/N}}{3 - \cos(n\pi/N) - \cos(m\pi/N) - \cos(p\pi/N)}, \tag{33}$$

the prime on the summation indicating as usual that the term $\mathbf{k} = \mathbf{0}$ must be omitted.

Therefore, from (31), the electric potential at any node $\boldsymbol{\nu} \in (\mathbb{Z}/2N\mathbb{Z})^3$ is derived as

$$\begin{aligned} V(\boldsymbol{\nu}) = R(N)I\{&G'(\boldsymbol{\nu}) + G'(\boldsymbol{\nu}+\mathbf{e}_1) + G'(\boldsymbol{\nu}+\mathbf{e}_2) + G'(\boldsymbol{\nu}+\mathbf{e}_1+\mathbf{e}_2) \\ &+ G'(\boldsymbol{\nu}+\mathbf{e}_3) + G'(\boldsymbol{\nu}+\mathbf{e}_1+\mathbf{e}_3) + G'(\boldsymbol{\nu}+\mathbf{e}_2+\mathbf{e}_3) + G'(\boldsymbol{\nu}+\mathbf{1}) \\ &- G'(\boldsymbol{\nu}+N\mathbf{1}) - G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{e}_1) - G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{e}_2) - G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{e}_1+\mathbf{e}_2) \\ &- G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{e}_3) - G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{e}_1+\mathbf{e}_3) - G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{e}_2+\mathbf{e}_3) - G'(\boldsymbol{\nu}+N\mathbf{1}+\mathbf{1})\}, \end{aligned}$$

i.e. after simplification,

$$\begin{aligned} V(\boldsymbol{\nu}) = &\frac{R(N)I}{16N^3} \sum_{\mathbf{k}=(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3}{}' \frac{\mathrm{e}^{\mathrm{i}\pi\mathbf{k}.\boldsymbol{\nu}/N}(1-(-1)^{n+m+p})}{3 - \cos(n\pi/N) - \cos(m\pi/N) - \cos(p\pi/N)} \\ &\times (1 + \mathrm{e}^{\mathrm{i}\pi n/N} + \mathrm{e}^{\mathrm{i}\pi m/N} + \mathrm{e}^{\mathrm{i}\pi p/N} + \mathrm{e}^{\mathrm{i}\pi(n+m)/N} + \mathrm{e}^{\mathrm{i}\pi(n+p)/N} + \mathrm{e}^{\mathrm{i}\pi(m+p)/N} + \mathrm{e}^{\mathrm{i}\pi(n+m+p)/N}). \end{aligned} \tag{34}$$

Since $V(\boldsymbol{\nu}_A) = V((N-1)\mathbf{1}) = -V^*(\mathbf{0})$, it is thus immediate to deduce that

$$\begin{aligned} V(\mathbf{0}) - V(\boldsymbol{\nu}_A) &= 2\Re V(\mathbf{0}) \\ &= \frac{R(N)I}{4N^3} \sum_{\substack{(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3 \\ n+m+p \text{ odd}}} \frac{1}{3 - \cos(n\pi/N) - \cos(m\pi/N) - \cos(p\pi/N)} \\ &\quad\times (1 + \cos(\pi n/N) + \cos(\pi m/N) + \cos(\pi p/N) + \cos(\pi(n+m)/N) \\ &\quad + \cos(\pi(n+p)/N) + \cos(\pi(m+p)/N) + \cos(\pi(n+m+p)/N)) \end{aligned}$$

or using a trigonometric identity for the latter term in brackets,

$$V(\mathbf{0}) - V(\boldsymbol{\nu}_A) = \frac{R(N)I}{N}\alpha(N), \tag{35}$$

where

$$\alpha(N) = \frac{2}{N^2} \sum_{\substack{(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3 \\ n+m+p \text{ odd}}} \frac{\cos(\pi n/2N)\cos(\pi m/2N)\cos(\pi p/2N)\cos(\pi(n+m+p)/2N)}{3 - \cos(n\pi/N) - \cos(m\pi/N) - \cos(p\pi/N)}. \tag{36}$$

Notice that the latter voltage difference is positive in agreement with the fact that, above, the current has been chosen flowing into the network from the node $\mathbf{0}$ to $\boldsymbol{\nu}_A$. Thus, the expected two-point resistance $R_{OA}^{(N)}$ is

$$R_{OA}^{(N)} = \frac{R(N)}{N}\alpha(N) \tag{37}$$

and some exact values for $2 \leqslant N \leqslant 6$ are listed in table 1. In particular, we actually found the value 5/6 for $N = 2$ which is the well-known solution of the '12 resistors cube' problem.

Further numerical tests show also that $\alpha(N)/N$, $N \geqslant 2$, thus also $\alpha(N)$, is a positive increasing sequence. Moreover, by the Rayleigh's Monotonicity Law [22], the following exact bounds can be obtained for $N \geqslant 2$,

$$\frac{5}{6}\,\frac{1}{2^{N-2}} \leqslant \frac{\alpha(N)}{N} \leqslant \frac{5}{6}(N-1). \tag{38}$$

Indeed, let us remark that $\alpha(N)/N$ is a *normalized* two-point resistance between O and A for any $N \geqslant 2$ (which amounts equivalently to set $R(N) = 1$ for all N). Then, cutting three adjacent faces of the cubic network except the 12 resistors cell at the common vertex (see figure 4) yields to

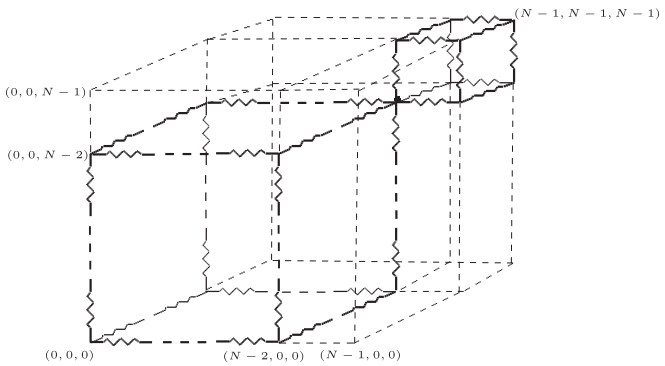

Figure 4. Cutting is electrical edge deletion and the two-point resistance between the nodes (0, 0, 0) and $(N-1, N-1, N-1)$ increases.

$$\frac{\alpha(N)}{N} \leqslant \frac{\alpha(N-1)}{N-1} + \frac{5}{6}, \quad N \geqslant 2,$$

thus the upper bound holds recalling that $\alpha(2)/2 = 5/6$. Conversely, shorting the same latter faces and diminishing if necessary all resistance values to $1/2$ leads to

$$\frac{\alpha(N)}{N} \geqslant \frac{1}{2}\,\frac{\alpha(N-1)}{N-1}, \quad N \geqslant 2,$$

thus the lower bound holds.

What precedes show that by summing over expanding cubes, the series (36) is $\mathcal{O}(N^2)$ at most as N increases, and obviously $\lim_{N\to+\infty}\alpha(N)$ does not exist by such a summation method. But, notice that for any $N \geqslant 2$,

$$0 < \alpha(N) < \frac{10}{\pi^2} \sum_{\substack{(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3 \\ n+m+p \text{ odd}}} (n^2+m^2+p^2)^{-1} \tag{39}$$

since

$$1 - \cos x > \frac{x^2}{5} \quad \text{for} \quad 0 < |x| \leqslant \pi.$$

Taking the limit $N \to +\infty$, the series in the RHS of (39) is still not convergent, but we have seen in section 3.1 that such an infinite lattice sum, considered as the analytic continuation evaluation of a relevant zeta function, is related to the Madelung constant (22). As a result, *it appears reasonable to conjecture that (36) by an appropriate rearrangement of terms may retain a finite value as N increases, viz.*

$$\lim_{N\to+\infty} \alpha(N) \sim -\frac{2}{\pi}\mathfrak{M} < 0. \tag{40}$$

Surprisingly, this limit value is negative. The change of sign made to $\alpha(+\infty)$ and originating in the analytic continuation machinery implemented for making convergent (36) as N tends to $+\infty$, means that the current asymptotically must flow, for mathematical consistency, from the node $\boldsymbol{\nu}_A$ to $\mathbf{0}$ i.e. $V(\mathbf{0}) - V(\boldsymbol{\nu}_A) < 0$ (or I changed to $-I$). Nevertheless, from a physical point of view, it is fortunate that such an adjustment does not affect the definition of the driving-point resistance $R_{OA}^{(N)}$. Recall that a similar result was found in section 3.1 concerning the conducting cube $\mathcal{C}$. Therefore, from (37), one writes,

$$\lim_{N\to+\infty} R_{OA}^{(N)} \sim \frac{2}{\pi}\mathfrak{M} \times \lim_{N\to+\infty} \frac{R(N)}{N}. \tag{41}$$

Assume that $R(N) = \mathcal{O}(N^\beta)$ as $N \to +\infty$. Hence, if $\beta < 1$ or $\beta > 1$, the two-point resistance $R_{OA}^{(N)}$ tends to 0 or $+\infty$ respectively.

More interesting is the case $\beta = 1$ where $R_{OA}^{(N)}$ has asymptotically a *finite nonzero value*. Thus, if we set $R(N) = r_0 N$ for N sufficiently large, $r_0 > 0$ a constant, it ensues that,

$$\lim_{N\to+\infty} R_{OA}^{(N)} \sim \frac{2}{\pi}\mathfrak{M} r_0. \tag{42}$$

Comparing (42) and (25) shows that the cubic resistor network formed from equal resistor $R(N) = N/\sigma a$ is asymptotically 'ohmically' equivalent to an homogeneous conducting cube of edge a and conductivity σ.

Since σ^{-1} is a resistance per unit length, the resistance $R(N)$ connecting two nodes is thus the resistance for a segment of length a/N which can be viewed as a fictitious distance separating these two nodes. This is consistent

Table 2. Some exact values of two-point resistances $R_{OB}^{(N)}$ and $R_{OC}^{(N)}$ in units of $R(N)$.

N	2	3	4	5	6
$\frac{R_{OB}^{(N)}}{R(N)} = \frac{\alpha'(N)}{N}$	$\frac{3}{4}$	$\frac{34}{35}$	$\frac{12\,403}{11\,424}$	$\frac{26\,702\,084}{23\,110\,593}$	$\frac{20\,760\,877\,553}{17\,267\,675\,040}$
$\frac{R_{OC}^{(N)}}{R(N)} = \frac{\alpha''(N)}{N}$	$\frac{7}{12}$	$\frac{179}{210}$	$\frac{11\,411}{11\,424}$	$\frac{16\,762\,677}{15\,407\,062}$	$\frac{57\,233\,476\,829}{49\,884\,394\,560}$

with the fact that each edge of the cubic network has N nodes (thus $N-1$ resistances) whose fictitious total length is asymptotically a i.e. the actual side of the equivalent cube $\mathcal{C}$.

4.2. Other resistances

The driving-point resistance between the nodes $\mathbf{0} = (0,\,0,\,0)$ and $\boldsymbol{\nu}_B = (N-1,\,0,\,N-1) = (N-1)(\mathbf{e}_1 + \mathbf{e}_3)$ on the face diagonal, and between $\mathbf{0}$ and $\boldsymbol{\nu}_C = (N-1,\,0,\,0) = (N-1)\mathbf{e}_1$ on one edge of the network are similarly obtained by solving the difference equation (31) on the discrete torus $(\mathbb{Z}/2N\mathbb{Z})^3$ with the current distributions $J(\boldsymbol{\nu}) = I\{\Delta(\boldsymbol{\nu}) - \Delta(\boldsymbol{\nu} + N\mathbf{e}_1 + N\mathbf{e}_3)\}$ and $J(\boldsymbol{\nu}) = I\{\Delta(\boldsymbol{\nu}) - \Delta(\boldsymbol{\nu} + N\mathbf{e}_1)\}$ respectively, Δ being the octuplet (30).

Hence, analogously to the previous section, it is direct to show that on the one hand,

$$R_{OB}^{(N)} = \frac{V(\mathbf{0}) - V(\boldsymbol{\nu}_B)}{I} = \frac{R(N)}{N}\alpha'(N) \tag{43}$$

with

$$\alpha'(N) = \frac{2}{N^2} \sum_{\substack{(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3 \\ n+p \text{ odd}}} \frac{\cos(\pi n/2N)\cos(\pi m/2N)\cos(\pi p/2N)\cos(\pi(n+m+p)/2N)}{3 - \cos(n\pi/N) - \cos(m\pi/N) - \cos(p\pi/N)}, \tag{44}$$

and on the other hand,

$$R_{OC}^{(N)} = \frac{V(\mathbf{0}) - V(\boldsymbol{\nu}_C)}{I} = \frac{R(N)}{N}\alpha''(N) \tag{45}$$

with

$$\alpha''(N) = \frac{2}{N^2} \sum_{\substack{(n,m,p)\in\{-N,\,\ldots,\,N-1\}^3 \\ n \text{ odd}}} \frac{\cos(\pi n/2N)\cos(\pi m/2N)\cos(\pi p/2N)\cos(\pi(n+m+p)/2N)}{3 - \cos(n\pi/N) - \cos(m\pi/N) - \cos(p\pi/N)}. \tag{46}$$

Table 2 give some exact values of these two-point resistances for $2 \leqslant N \leqslant 6$.

Again, we can reasonably conjecture that

$$\lim_{N\to+\infty} \alpha'(N) \sim -\frac{2}{\pi}\mathfrak{M}' \quad \text{and} \quad \lim_{N\to+\infty} \alpha''(N) \sim -\frac{2}{\pi}\mathfrak{M}'', \tag{47}$$

leading to the same conclusions as above viz. if $R(N) = N/\sigma a$ for N sufficiently large,

$$\lim_{N\to+\infty} R_{OB}^{(N)} \sim \frac{2}{\pi}\frac{\mathfrak{M}'}{\sigma a} \quad \text{and} \quad \lim_{N\to+\infty} R_{OC}^{(N)} \sim \frac{2}{\pi}\frac{\mathfrak{M}''}{\sigma a} \tag{48}$$

which show, comparing with (28), that the continuum limit of such a cubic resistor network is exactly the conducting cube $\mathcal{C}$.

5. Perspectives and conclusions

First, all results presented in this paper both for the cubic medium, the finite cubic resistor network and its continuum limit may be easily generalized in any spatial dimensionality $n \geqslant 3$. The case $n = 2$ (i.e. in a 3-dimensional approach, the electrical resistance between two edges of a square prism) is more delicate to study owing to the existence of logarithmic singularities for the potentials and the impossibility to fix a zero potential reference at infinity for a 2D Coulomb potential. Nevertheless, these difficulties may be overcome as already discussed in [7, 23].

Secondly, we leave it to the reader to adjust the calculations of sections 2 and 3 when considered a cuboid, for instance a cuboid $a \times a \times b$. In that case, setting the axial ratio $\alpha = a/b$ and defining the Madelung-type constant

$$\mathfrak{M}(\alpha) := \lim_{\varepsilon \to 0+} \left(\int_{\varepsilon^2}^{+\infty} \{\theta_3^2(\mathrm{i}\nu)\theta_3(\mathrm{i}\alpha^2\nu) - \theta_4^2(\mathrm{i}\nu)\theta_4(\mathrm{i}\alpha^2\nu)\}\,\mathrm{d}\nu - \frac{2}{\alpha\varepsilon} \right), \tag{49}$$

it is not difficult to derive the following two-point resistance between the vertices O and $A = (a, a, b)$,

$$R_{OA}(\alpha) = \frac{2}{\pi}\frac{\mathfrak{M}(\alpha)}{\sigma b}$$

allowing some interesting extensions and applications regarding the asymptotics $\alpha \gg 1$ (case of a wire of length b and of square cross-section $a^2 \ll 1$) or $\alpha \ll 1$ (case of a square sheet of side a and thickness $b \ll 1$).

At last, as already mentioned in Introduction, the application of computational methods presented in this work should allow to generalize the van der Pauw technique to 3-dimensional samples.

We shall not pursue here any further these questions which will be the subject of future works.

Acknowledgments

The author is grateful to the reviewers for their constructive comments and suggestions.

ORCID iDs

Malik Mamode https://orcid.org/0000-0003-0868-8762

References

[1] Weiss J D, Kaplar R J and Kambour K E 2008 A derivation of the van der Pauw formula from electrostatics *Solid-State Electron.* **52** 91–8
[2] van der Pauw L J 1958 A method of measuring the resistivity and Hall coefficient on lamellae of arbitrary shape *Philips Tech. Rev.* **20** 220–4
[3] Kasl C and Hoch M J R 2005 Effects of sample thickness on the van der Pauw technique for resistivity measurements *Rev. Sci. Instrum.* **76** 033907
[4] Chaikin P M and Lubenskii T C 1995 *Principles of Condensed Matter Physics* (Cambridge: Cambridge University Press)
[5] Cserti J 2000 Application of the lattice Green's function for calculating the resistance of an infinite network of resistors *Am. J. Phys.* **68** 896–906
[6] Asad J H 2013 Exact evaluation of the resistance in an infinite face-centered cubic network *J. Stat. Phys.* **150** 1177–82
[7] Mamode M 2016 Computation of the Madelung constant for hypercubic crystal structures in any dimension *J. Math. Chem.* **55** 734–51
[8] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 1 (New York: McGraw-Hill)
[9] Olver F W J, Lozier D W, Boisvert R F and Clark C W (ed) 2010 *NIST Handbook of Mathematical Functions* (New York, NY: Cambridge University Press) print companion to [24]
[10] Crandall R E and Buhler J P 1987 Elementary function expansions for Madelung constants *J. Phys. A: Math. Gen.* **20** 5497
[11] Blanchet L and Faye G 2000 Hadamard regularization *J. Math. Phys.* **41** 7675–714
[12] Zucker I J 1975 Madelung constants and lattice sums for invariant cubic lattice complexes and certain tetragonal structures *J. Phys. A: Math. Gen.* **8** 1734
[13] Zucker I J 1976 Functional equations for poly-dimensional zeta functions and the evaluation of Madelung constants *J. Phys. A: Math. Gen.* **9** 499
[14] Monegato G 1994 Numerical evaluation of hypersingular integrals *J. Comput. Appl. Math.* **50** 9–31
[15] Crandall R E 1998 Fast evaluation of Epstein zeta functions (http://www.reed.edu/physics/faculty/crandall/papers/epstein.pdf)
[16] Chamati H and Tonchev N S 2000 Exact results for some Madelung-type constants in the finite-size scaling theory *J. Phys. A: Math. Gen.* **33** L167
[17] Brankov J G and Tonchev N S 1988 On the finite-size scalling equation for the spherical model *J. Stat. Phys.* **52** 143–59
[18] Apostol T M 1976 *Introduction to Analytic Number Theory* (Berlin: Springer)
[19] Wu F Y 2004 Theory of resistor networks: the two-point resistance *J. Phys. A: Math. Gen.* **37** 6653
[20] Jafarizadeh M A, Sufiani R and Jafarizadeh S 2007 Calculating two-point resistances in distance-regular resistor networks *J. Phys. A: Math. Theor.* **40** 4949
[21] Inawashiro S, Horiguchi T, Katsura S, Morita T and Abe Y 1971 Lattice Green's function. Introduction *J. Math. Phys.* **12** 892–5
[22] Doyle P G and Snell L 1984 *Random Walks and Electric Networks* (*Carus Mathematical Monographs* Series 22) (Washington, DC: The Mathematical Association of America) 83–149
[23] Mamode M 2014 Fundamental solution of the Laplacian on flat tori and boundary value problems for the planar Poisson equation in rectangles *Bound. Value Problems* **1** 221
[24] NIST Digital Library of Mathematical Functions. Online companion to [9], see http://dlmf.nist.gov/