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POSITIVITY OF VALUATIONS ON CONVEX BODIES AND INVARIANT VALUATIONS BY LINEAR ACTIONS

NGUYEN-BAC DANG AND JIAN XIAO

ABSTRACT. We define a notion of positivity on continuous and translation invariant valuations on convex bodies on a finite dimensional real vector space. We endow the valuation space generated by mixed volumes with a norm induced by the positive cone. This enables us to construct a continuous extension of the convolution operator on smooth valuations to the closure of that space. As an application, we prove a variant of Minkowski’s existence theorem. Furthermore, given a linear map, we generalize a theorem of Favre-Wulcan and Lin by proving that the eigenvalues of the linear map is related to the spectral radius of the induced linear operator on the space of valuations. Finally, given a linear action and under a natural strict log-concavity assumption on certain spectral radius of the induced linear operators on valuations, we study the positivity properties of the space of invariant valuations corresponding to the spectral radius of the operator.

CONTENTS

1. Introduction	1
1.1. Organization	6
Acknowledgements	6
2. Preliminaries	6
2.1. Convex valuations	6
2.2. McMullen’s grading decomposition	6
2.3. Alesker’s irreducibility theorem	7
2.4. Convolution and product of smooth valuations	8
3. Positive convex valuations	9
3.1. Positivity of valuations	9
3.2. Reverse Khovanskii-Teissier inequality	11
3.3. Norms on the space of valuations	14
3.4. An extension of the convolution operator	18
4. A variant of Minkowski’s existence theorem	19
4.1. Existence of the solutions	20
4.2. Compactness of the solution set	21
5. Dynamical degrees	22
5.1. Existence	22
5.2. Evaluation of dynamical degrees	27
5.3. A generalization: multiple dynamical degrees	30
6. Positivity of invariant convex valuations	31
6.1. Invariant classes in complex dynamics	31
6.2. Invariant convex valuations	32
References	37

1. INTRODUCTION

Let E be a Euclidian real vector space of dimension n , and let $\mathcal{K}(E)$ be the family of convex bodies (i.e., compact convex subsets) of E . We endow the space $\mathcal{K}(E)$ with the Hausdorff metric, that is, for

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any $K, L \in \mathcal{K}(E)$ the distance is defined by

$$d_H(K, L) = \min\{\varepsilon > 0 \mid K \subset L + \varepsilon \mathbf{B} \text{ \& } L \subset K + \varepsilon \mathbf{B}\},$$

where \mathbf{B} is the unit ball in E . A real (convex) valuation ϕ on E is a function $\phi : \mathcal{K}(E) \rightarrow \mathbb{R}$ such that

$$\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L)$$

for any $K, L \in \mathcal{K}(E)$ satisfying $K \cup L \in \mathcal{K}(E)$. Moreover, a valuation ϕ is called *translation invariant* if $\phi(K + t) = \phi(K)$ for any $K \in \mathcal{K}(E)$ and any $t \in E$, and it is called *continuous* if it is continuous with respect to the topology of $\mathcal{K}(E)$ given by the metric d_H . We denote by $\text{Val}(E)$ the Banach space of continuous, translation invariant valuations on E where the norm of $\phi \in \text{Val}(E)$ is given by:

$$(1) \quad \|\phi\| := \sup_{K \subset \mathbf{B}} |\phi(K)|,$$

where the supremum is taken over all convex bodies K contained in the unit ball \mathbf{B} .

A valuation $\phi \in \text{Val}(E)$ is called homogeneous of degree i , where $0 \leq i \leq n$, if for any $K \in \mathcal{K}(E)$ and any $\lambda \geq 0$, one has:

$$\phi(\lambda K) = \lambda^i \phi(K).$$

The subspace of $\text{Val}(E)$ of homogeneous valuations of degree i is denoted by $\text{Val}_i(E)$. By a theorem of McMullen (see [McM77]), there is a decomposition of $\text{Val}(E)$ in terms of $\text{Val}_i(E)$ given by:

$$\text{Val}(E) = \bigoplus_{i=0}^n \text{Val}_i(E).$$

The most basic examples of homogeneous valuations of degree i are given by the mixed volumes

$$K \mapsto V(L_1, \dots, L_{n-i}, K[i])$$

where $L_1, \dots, L_{n-i} \in \mathcal{K}(E)$ and the symbol $V(-)$ denotes the mixed volume of convex bodies, and $K[i]$ means that the convex body K is repeated i times in the expression of the mixed volume.

The space of valuations contains a dense subspace called the space of smooth valuations. We recall the definition of this space. The Lie group $\text{GL}(E)$ has a natural action on $\text{Val}(E)$:

$$\begin{aligned} \text{GL}(E) \times \text{Val}(E) &\rightarrow \text{Val}(E), \\ (g, \phi) &\mapsto g \cdot \phi, \end{aligned}$$

where $g \cdot \phi(K) := \phi(g^{-1}K)$ for any $K \in \mathcal{K}(E)$ (see [Ale01]). The valuation ϕ is called *smooth* if the map $g \mapsto g \cdot \phi$ is smooth. We denote by $\text{Val}^\infty(E)$ the subset of $\text{Val}(E)$ of smooth translation invariant valuations, and by $\text{Val}_i^\infty(E)$ the smooth translation invariant valuations which are homogeneous of degree i . Similar to the decomposition for $\text{Val}(E)$, one also has

$$\text{Val}^\infty(E) = \bigoplus_{i=0}^n \text{Val}_i^\infty(E).$$

We now introduce the following key notion of positivity for convex valuations. For any positive Radon measure μ on $\mathcal{K}(E)^{n-i}$ such that

$$\int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) < +\infty,$$

we define a valuation ϕ_μ given by

$$\phi_\mu(L) = \int_{\mathcal{K}(E)^{n-i}} V(L[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}).$$

Observe that the dominated convergence theorem ensures the fact that ϕ_μ is a continuous translation invariant valuation. Moreover, such a valuation is monotone in the sense that if $K \subset L \in \mathcal{K}(E)$ then $\phi(K) \leq \phi(L)$. Note that the linear map $\mu \rightarrow \phi_\mu$ is not injective.

A valuation $\phi \in \text{Val}_i(E)$ is said to be *positive* if there exists a measure μ as above such that $\phi = \phi_\mu$. We denote by $\mathcal{P}_i \subset \text{Val}_i(E)$ the set of positive homogeneous valuations of degree i .

Example 1.1. The set of positive linear combinations of mixed volumes of degree i is contained in \mathcal{P}_i .

Remark 1.2. We emphasize that the positivity notation introduced above is different from (and stronger than) the positivity in the traditional setting. In the traditional setting, a valuation $\phi \in \text{Val}(E)$ is called positive if $\phi(K) \geq 0$ for any $K \in \mathcal{K}(E)$. Many interesting results on this kind of positive valuations have been obtained by Parapatits-Wannerer [PW13] and Bernig-Fu [BF11]. Note that a monotone valuation must be positive in this traditional sense. There are valuations which are positive in the traditional sense but not monotone, e.g., Kazarnovskii pseudo-volume in hermitian integral geometry (see [BF11]), and there are also valuations which are monotone but not positive in our setting (see [Ber12, Section 5.5]).

By a polarization argument, a valuation $\phi \in \mathcal{P}_i$ defines a unique function on $\mathcal{K}(E)^i$:

$$\phi(L_1, \dots, L_i) = \frac{1}{i!} \left(\frac{\partial^i}{\partial t_1 \partial t_2 \dots \partial t_i} \right) \Big|_{t_1=\dots=t_i=0^+} \phi(t_1 L_1 + \dots + t_i L_i),$$

where L_1, \dots, L_i are convex bodies. If $L_1 = \dots = L_i = L$, then $\phi(L_1, \dots, L_i) = \phi(L)$.

We say that a valuation $\phi \in \mathcal{P}_i$ is *strictly positive* if there exists $\epsilon > 0$ such that

$$\phi(L_1, \dots, L_i) \geq \epsilon V(\mathbf{B}[n-i], L_1, \dots, L_i)$$

holds for any convex bodies L_1, \dots, L_i .

The convex cone \mathcal{P}_i generates a vector space $\mathcal{V}'_i \subset \text{Val}_i(E)$. For any $\phi \in \mathcal{V}'_i$, there is a signed Radon measure μ on $\mathcal{K}(E)^{n-i}$ such that its absolute value $|\mu|$ satisfies:

$$\int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d|\mu|(K_1, \dots, K_{n-i}) < +\infty.$$

The subspace \mathcal{V}'_i is endowed with an appropriate norm defined as follows.

Definition 1.3. For any $\phi \in \mathcal{V}'_i$, the norm $\|\cdot\|_{\mathcal{P}}$ is defined by

$$\|\phi\|_{\mathcal{P}} := \inf\{t \geq 0 \mid |\phi(L_1, \dots, L_i)| \leq tV(\mathbf{B}[n-i], L_1, \dots, L_i) \text{ for any } L_1, \dots, L_i \in \mathcal{K}(E)\}.$$

The fact that $\|\phi\|_{\mathcal{P}}$ is finite follows from the reverse Khovanskii-Teissier inequality [LX17] (see also Theorem 3.10). One of the main properties of the norm $\|\cdot\|_{\mathcal{P}}$ is that the subspace $\mathcal{V}'_i \cap \text{Val}_i^\infty(E)$ forms a dense subspace in \mathcal{V}'_i with respect to this norm (see Theorem 3.23).

Remark 1.4. The norm $\|\cdot\|_{\mathcal{P}}$ is inspired by complex geometry, the motivation is that the analogous notation for a cohomology class over a projective manifold measures the pseudo-effectivity of that class. In our setting, $\|\phi_\mu\|_{\mathcal{P}}$ measures the positivity of ϕ_μ .

Let $\mathcal{V}_i^{\mathcal{P}}$ be the completion of \mathcal{V}'_i with respect to the norm $\|\cdot\|_{\mathcal{P}}$. By definition, for any $L \subset \mathbf{B}$ we have $|\phi(L)| \leq \text{vol}(\mathbf{B})\|\phi\|_{\mathcal{P}}$, hence $\|\phi\| \leq \text{vol}(\mathbf{B})\|\phi\|_{\mathcal{P}}$. Thus there is a continuous injection

$$(\mathcal{V}_i^{\mathcal{P}}, \|\cdot\|_{\mathcal{P}}) \hookrightarrow (\text{Val}_i(E), \|\cdot\|).$$

A deep theorem of Alesker [Ale01] implies that the linear combinations of mixed volumes span a dense set in $\text{Val}(E)$. As a consequence, $\mathcal{V}_i^{\mathcal{P}}$ is dense in $\text{Val}_i(E)$ with respect to the norm $\|\cdot\|$. We do not know whether $\mathcal{V}_i^{\mathcal{P}}$ is dense in $\text{Val}_i(E)$ with respect to the norm $\|\cdot\|_{\mathcal{P}}$.

Besides the norm $\|\cdot\|_{\mathcal{P}}$, another norm $\|\cdot\|_{\mathcal{C}}$ induced by the cone structure is also defined on \mathcal{V}'_i . For any $\phi \in \mathcal{V}'_i$, $\|\phi\|_{\mathcal{C}}$ is given by

$$\|\phi\|_{\mathcal{C}} := \inf_{\phi = \phi_+ - \phi_-, \phi_{\pm} \in \mathcal{P}_i} (\phi_+(\mathbf{B}) + \phi_-(\mathbf{B})),$$

Its properties are also discussed in the paper (see Section 3.3.2). However, we do not know whether smooth valuations are dense in \mathcal{V}'_i for the topology induced by this cone norm.

Our first theorem shows that the convolution of valuations can be uniquely extended to $\mathcal{V}_i^{\mathcal{P}}$. Let us recall the convolution operation defined by Bernig-Fu [BF06] and studied further by Alesker [Ale11] on smooth valuations. By [BF06] (see also [Ale11]), there exists a unique continuous, symmetric bilinear map $*$ which is homogeneous of degree $-n$:

$$\begin{aligned} \text{Val}^\infty(E) \times \text{Val}^\infty(E) &\rightarrow \text{Val}^\infty(E), \\ (\phi, \varphi) &\mapsto \phi * \varphi, \end{aligned}$$

such that for any $K, L \in \mathcal{K}(E)$ with smooth and strictly convex boundary, one has that:

$$\text{vol}(\cdot + K) * \text{vol}(\cdot + L) = \text{vol}(\cdot + K + L) \in \text{Val}^\infty(E).$$

In particular, assume that $K_1, \dots, K_{n-i}, L_1, \dots, L_{n-j} \in \mathcal{K}(E)$ have smooth and strictly convex boundary, then

$$(2) \quad V(-; K_1, \dots, K_{n-i}) * V(-; L_1, \dots, L_{n-j}) = \frac{i!j!}{n!} V(-; K_1, \dots, K_{n-i}, L_1, \dots, L_{n-j}).$$

We can now state our first theorem (see Theorem 3.23 and Theorem 3.33).

Theorem A. *Fix two integers i, j such that $2n \geq i + j \geq n$. There exists a unique symmetric bilinear operator $\tilde{*} : \mathcal{V}_i^{\mathcal{P}} \times \mathcal{V}_j^{\mathcal{P}} \rightarrow \mathcal{V}_{i+j-n}^{\mathcal{P}}$ satisfying the following properties.*

- (1) *The operator $\tilde{*}$ is continuous with respect to the topology induced by the norm $\|\cdot\|_{\mathcal{P}}$.*
- (2) *The operator $\tilde{*}$ coincides with the convolution $*$ on $(\mathcal{V}_i^{\mathcal{P}} \cap \text{Val}_i^\infty(E)) \times (\mathcal{V}_j^{\mathcal{P}} \cap \text{Val}_j^\infty(E))$.*

A priori, the convolution is only well defined on the space of smooth valuations $\text{Val}^\infty(E)$ and one cannot extend it continuously to $\text{Val}(E)$. Theorem A allows us to extend the operation with respect to a finer topology than the one in $\text{Val}_i(E)$.

Bernig-Faifman and Alesker-Bernig (see [BF16], [AB12]) studied another extension on the generalized valuations satisfying specified conditions. The space of generalized valuations, denoted by $\text{Val}^{-\infty}(E)$, is defined to be the dual of $\text{Val}^\infty(E)$. However, it is unclear how one can compare these two extensions.

Our extension is closely related to equation (2). Indeed, if μ and ν are two Radon measures on $\mathcal{K}(E)^{n-i}$ and $\mathcal{K}(E)^{n-j}$ respectively so that their associated valuations ϕ_μ and ϕ_ν belong to \mathcal{V}_i' and \mathcal{V}_j' respectively, then the valuation $\phi_\mu \tilde{*} \phi_\nu \in \mathcal{V}_{i+j-n}'$ is a valuation associated to the measure:

$$\frac{i!j!}{n!} p_1^* \mu \otimes p_2^* \nu,$$

where $p_1 : \mathcal{K}(E)^{2n-i-j} \rightarrow \mathcal{K}(E)^{n-i}$ and $p_2 : \mathcal{K}(E)^{2n-i-j} \rightarrow \mathcal{K}(E)^{n-j}$ are the projections onto the first $n-i$ factors and the last $n-j$ factors respectively. The formula for the valuation $\phi_\mu \tilde{*} \phi_\nu$ is given by:

$$\phi_\mu \tilde{*} \phi_\nu(-) := \frac{i!j!}{n!} \int_{\mathcal{K}(E)^{2n-i-j}} V(-; K_1, \dots, K_{n-i}, K'_1, \dots, K'_{n-j}) d\mu(K_1, \dots, K_{n-i}) d\nu(K'_1, \dots, K'_{n-j}),$$

which is always well defined by Proposition 3.13.

Let $L_1, \dots, L_{n-1} \in \mathcal{K}(E)$ be convex bodies with non-empty interior, by Minkowski's existence theorem (see [Ale38]), there exists a unique (up to a translation) convex body $L \in \mathcal{K}(E)$ with non-empty interior such that

$$V(L_1, \dots, L_{n-1}, -) = V(L[n-1], -).$$

Our next result can be considered as a variant of Minkowski's existence theorem (see Theorem 4.1 and Proposition 4.2).

Theorem B. *For any $\psi \in \mathcal{P}_i$ strictly positive, then there is a constant $c > 0$ (depending only on ψ) and a convex body B with $\text{vol}(B) = 1$ such that*

$$\psi \tilde{*} V(B[i-1], -) = cV(B[n-1], -) \in \text{Val}_1(E).$$

Moreover, up to translations the solution set

$$S = \{B \in \mathcal{K}(E) \mid \psi \tilde{*} V(B[i-1], -) = cV(B[n-1], -), \text{vol}(B) = 1\}$$

is compact in $\mathcal{K}(E)$ endowed with the Hausdorff metric.

Remark 1.5. When $i = 1$, the previous Theorem is just a consequence of Minkowski's existence theorem [Ale38, Sch14] (see Example 3.7).

Our next results focus on linear actions on valuations. We are interested in the behaviour of the sequence $\{g^k \cdot \phi\}_{k=1}^\infty$ where $\phi \in \mathcal{V}_{n-i}^{\mathcal{P}}$ and $g \in \text{GL}(E)$. Given $g \in \text{GL}(E)$, $\phi \in \mathcal{P}_{n-i}$ and $\psi \in \mathcal{P}_i$ two strictly positive valuations, we define the i -th dynamical degree of g by

$$d_i(g) := \lim_{k \rightarrow \infty} ((g^k \cdot \phi) \tilde{*} \psi)^{1/k}.$$

The terminology “dynamical degree” comes from the study of dynamics of holomorphic maps, where these numbers are defined for rational self-maps on projective varieties. These two notions of dynamical degrees are closely related in the particular case of rational self-maps over toric varieties which preserve the torus action.

Note that g induces a linear operator (denoted by g_{n-i}) on the Banach space $(\mathcal{V}_{n-i}^{\mathcal{P}}, \|\cdot\|_{\mathcal{P}})$:

$$g_{n-i} : \mathcal{V}_{n-i}^{\mathcal{P}} \rightarrow \mathcal{V}_{n-i}^{\mathcal{P}}.$$

A direct application of the reverse Khovanski-Teissier inequality (see Theorem 3.10) and the method in [Dan17] shows that the number $d_i(g)$ is well-defined and is equal to the norm of the operator g_{n-i} . Our next theorem (see Theorem 5.9 and Theorem 5.16) relates the norm of g_{n-i} , the eigenvalues of g and the dynamical degrees.

Theorem C. *Given $g \in \mathrm{GL}(E)$, the dynamical degree $d_i(g)$ exists and is independent of the choices of the strictly positive valuations $\phi \in \mathcal{P}_{n-i}, \psi \in \mathcal{P}_i$. Moreover, assume that $\rho(g_{n-i})$ is the spectral radius of g_{n-i} and ρ_1, \dots, ρ_n are the eigenvalues of g satisfying*

$$|\rho_1| \geq |\rho_2| \geq \dots \geq |\rho_n|,$$

then the i -th dynamical degree $d_i(g) = \rho(g_{n-i}) = |\det g|^{-1} \prod_{k=1}^i |\rho_k|$.

Our proof relies on the observation that the dynamical degrees define continuous mappings from $\mathrm{GL}(E)$ to \mathbb{R} . We are then reduced to proving the Theorem C for diagonalizable matrices. Observe that our proof gives an alternative approach to the results of Lin (see [Lin12, Theorem 6.2]) and Favre-Wulcan (see [FW12, Corollary B]) which relied on Minkowski weights and integral geometry respectively.

We say that a valuation ϕ is $d_i(g)$ -invariant if it belongs to the eigenspace of eigenvalue $d_i(g)$ (i.e., $g \cdot \phi = d_i(g)\phi$).

By Alexandrov-Fenchel inequality or Theorem C, it is clear that the sequence of dynamical degrees $\{d_i(g)\}$ is log-concave. In particular, $d_i(g)^2 \geq d_{i+s}(g)d_{i-s}(g)$. Our last theorem (see Theorem 6.1) gives some positivity properties of invariant valuations under a natural strict log-concavity assumption on these numbers.

Theorem D. *Assume $2i \leq n$. Consider $g \in \mathrm{GL}(E)$. Then the following properties are satisfied.*

- (1) *There exists a non zero $d_i(g)$ -invariant valuation in $\overline{\mathcal{P}_{n-i}} \subset \mathcal{V}_{n-i}^{\mathcal{P}}$.*
- (2) *Assume that the strict log-concavity inequality is satisfied for some $s \leq \min(i, n-i)$:*

$$d_i(g)^2 > d_{i-s}(g)d_{i+s}(g),$$

then for any two $d_i(g)$ -invariant convex valuations $\phi_1, \phi_2 \in \mathcal{V}_{n-i}^{\mathcal{P}}$, we have

$$\phi_1 \tilde{*} \phi_2 = 0.$$

- (3) *Assume that*

$$d_1^2(g) > d_2(g),$$

then there exists a unique (up to a multiplication by a positive constant) $d_1(g)$ -invariant positive convex valuation $\phi \in \overline{\mathcal{P}_{n-1}} \subset \mathcal{V}_{n-1}^{\mathcal{P}}$. Moreover, ϕ lies in an extremal ray of $\overline{\mathcal{P}_{n-1}} \subset \mathcal{V}_{n-1}^{\mathcal{P}}$.

In the study of monomial maps, the conclusion of (3) implies also the existence of a unique invariant b -divisor class in the sense of [FW12]. The results (2) and (3) can be understood as the higher dimensional convex analog of a result by [BFJ08] for projective surfaces. Given a projective surface X and a dominant rational map f on it. Suppose that the dynamical degree $d_1(f)$ and $d_2(f)$ satisfy $d_1(f)^2 > d_2(f)$, Boucksom, Favre and Jonsson proved the existence and the uniqueness (up to scaling) of two nef Weil-classes θ^+ and θ^- which are $d_1(f)$ -invariant by f^* and f_* respectively. They proved also that the self-intersection $\theta^+ \cdot \theta^+$ is equal to zero.

Remark 1.6. We remark that Theorem C and Theorem D also hold for the norm $\|\cdot\|_{\mathcal{C}}$. As for Theorem A, as we do not know if the density result (Theorem 3.23) holds for the norm $\|\cdot\|_{\mathcal{C}}$, we have a slightly weaker version of Theorem A for this norm.

1.1. Organization. In Section 2, we give a brief review of valuations on convex sets. Section 3 devotes to the study of some positivity results of convex valuations, and the continuous extension of the convolution operator. In Section 4, using the convolution operator we study a generalization of Minkowski's existence theorem. In Section 5, we use the positivity results to study the dynamical degree and calculate its value. In Section 6, we study the positivity of invariant valuations under a natural strict log-concavity assumption on certain dynamical degrees.

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2. PRELIMINARIES

2.1. Convex valuations. We first give a brief overview of valuations on convex sets. The classical references are [MS83, McM93]. We also refer the reader to the more recent surveys [Ale07], [AF14] and [Ber12]. Our general reference for convexity is [Sch14].

Let E be a Euclidian real vector space of dimension n . We denote the family of non-empty compact convex subsets of E by $\mathcal{K}(E)$. Then $\mathcal{K}(E)$ has a natural topology induced by the Hausdorff metric defined as follows:

$$d_H(K, L) := \inf\{\varepsilon > 0 \mid K \subset L + \varepsilon\mathbf{B} \text{ \& } L \subset K + \varepsilon\mathbf{B}\},$$

where \mathbf{B} is the unit ball, where $K, L \in \mathcal{K}(E)$ and where $+$ is the Minkowski sum. By Blaschke selection theorem, $(\mathcal{K}(E), d_H)$ is a locally compact space. Moreover, by associating a convex set to its support function, $(\mathcal{K}(E), d_H)$ can be isometrically embedded into the function space $C^0(\mathbb{S}^{n-1})$ equipped with L^∞ -norm.

Definition 2.1. A functional $\phi : \mathcal{K}(E) \rightarrow \mathbb{R}$ is called a real *convex valuation* if

$$\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L)$$

whenever $K, L, K \cup L \in \mathcal{K}(E)$.

Remark 2.2. The convex valuation is just called valuation in classical literatures, here we follow the terminology of [Ale07] because the valuation theory has been extended to not necessarily convex sets on manifolds.

Definition 2.3. A convex valuation ϕ is called *continuous* if ϕ is continuous with respect to the Hausdorff metric d_H ; A convex valuation ϕ is called *translation-invariant* if $\phi(K + x) = \phi(K)$ for any $K \in \mathcal{K}(E)$ and any $x \in E$.

Let us denote by $\text{Val}(E)$ the space of translation-invariant continuous valuations. The linear space $\text{Val}(E)$ has the natural topology given by a sequence of semi-norms:

$$\|\phi\|_N = \sup_{K \subset \mathbf{B}_N} |\phi(K)|,$$

where \mathbf{B}_N is the ball of radius N . This sequence of semi-norms defines a Fréchet space structure on $\text{Val}(E)$. Actually, $\text{Val}(E)$ is a Banach space endowed with the norm $\|\cdot\|_1$.

2.2. McMullen's grading decomposition. We recall McMullen's decomposition of the space of valuations $\text{Val}(E)$.

Definition 2.4. A convex valuation ϕ is called α -homogeneous if $\phi(\lambda K) = \lambda^\alpha \phi(K)$ for any $\lambda \geq 0, K \in \mathcal{K}(E)$.

Let us denote by $\text{Val}_\alpha(E)$ the subspace of $\text{Val}(E)$ of α -homogeneous convex valuations. The following result is due to McMullen [McM77].

Theorem 2.5 (McMullen decomposition). *Let $n = \dim E$, then*

$$\text{Val}(E) = \bigoplus_{i=0}^n \text{Val}_i(E).$$

Furthermore, every valuation ϕ can be decomposed uniquely into even and odd parts

$$\phi = \phi^{\text{even}} + \phi^{\text{odd}},$$

where $\phi^{\text{even}}(-K) = \phi^{\text{even}}(K)$, $\phi^{\text{odd}}(-K) = -\phi^{\text{odd}}(K)$ for every $K \in \mathcal{K}(E)$. Thus we have the following decomposition

$$\text{Val}(E) = \bigoplus_{i=0, \dots, n; \epsilon \in \{\text{even}, \text{odd}\}} \text{Val}_i^\epsilon(E).$$

2.2.1. *Examples.* Let us present some examples of convex valuations:

- (1) The Euler characteristic χ which satisfies $\chi(K) = 1$ for every $K \in \mathcal{K}(E)$ is a constant valuation.
- (2) The Lebesgue measure $\text{vol}(\cdot)$ belongs to $\text{Val}_n(E)$.
- (3) For any convex body A , the function $\phi : \mathcal{K}(E) \rightarrow \mathbb{R}$ defined by $\phi(K) = \text{vol}(K + A)$ is in $\text{Val}(E)$.
- (4) Let $K_1, \dots, K_r \in \mathcal{K}(E)$ be convex bodies, then there is a polynomial relation

$$\text{vol}(t_1 K_1 + \dots + t_r K_r) = \sum_{i_1 + \dots + i_r = n} \frac{n!}{i_1! i_2! \dots i_r!} V(K_1[i_1], \dots, K_r[i_r]) t_1^{i_1} \dots t_r^{i_r},$$

where $t_i \geq 0$ and $K_j[i_j]$ denotes i_j copies of K_j and where the coefficient $V(K_1[i_1], \dots, K_r[i_r])$ denotes the *mixed volume*. Fix $A_1, \dots, A_{n-k} \in \mathcal{K}(E)$, then the function $\psi : \mathcal{K}(E) \rightarrow \mathbb{R}$ defined by

$$\psi(K) := V(K[k], A_1, \dots, A_{n-k})$$

belongs to $\text{Val}_k(E)$.

2.3. **Alesker's irreducibility theorem.** The group $\text{GL}(E)$ acts on $\text{Val}(E)$ by

$$(g \cdot \phi)(K) = \phi(g^{-1}K).$$

Note that $\text{Val}_i^{\text{even}}$ (resp. $\text{Val}_i^{\text{odd}}$) is invariant under this action.

Example 2.6. Assume that $\phi \in \text{Val}_i(E)$ is given by $\phi_{L_1, \dots, L_{n-i}}(K) := V(K[i], L_1, \dots, L_{n-i})$, then

$$\begin{aligned} (g \cdot \phi_{L_1, \dots, L_{n-i}})(K) &= V(g^{-1}(K)[i], L_1, \dots, L_{n-i}) \\ &= |\det g|^{-1} V(K[i], g(L_1), \dots, g(L_{n-i})) \\ &= |\det g|^{-1} \phi_{g(L_1), \dots, g(L_{n-i})}(K), \end{aligned}$$

which implies $g \cdot \phi_{L_1, \dots, L_{n-i}} = |\det g|^{-1} \phi_{g(L_1), \dots, g(L_{n-i})}$. In particular, if $|\det g| = 1$, then $g \cdot \phi_{L_1, \dots, L_{n-i}} = \phi_{g(L_1), \dots, g(L_{n-i})}$.

In the case of a general Radon measure μ on $\mathcal{K}(E)^{n-i}$ such that:

$$\int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], g(L_1), \dots, g(L_{n-i})) d\mu(L_1, \dots, L_{n-i}) < +\infty,$$

we have

$$g \cdot \phi_\mu(K) = \frac{1}{|\det g|} \int_{\mathcal{K}(E)^{n-i}} V(K[i], g(L_1), \dots, g(L_{n-i})) d\mu(L_1, \dots, L_{n-i}).$$

In particular, if we set $g \cdot \mu(L_1, \dots, L_{n-i}) = \mu(g^{-1}(L_1), \dots, g^{-1}(L_{n-i}))$, then $g \cdot \phi_\mu = \frac{1}{|\det g|} \phi_{g \cdot \mu}$.

Alesker's irreducibility theorem [Ale01] is one of the milestones of the modern development of convex valuation theory, it can be stated as follows:

Theorem 2.7 (Alesker's irreducibility theorem). *As a $\text{GL}(E)$ -module, the natural representation of $\text{GL}(E)$ on the space $\text{Val}_i^{\text{even}}(E)$ and $\text{Val}_i^{\text{odd}}(E)$ is irreducible for every $i = 0, 1, \dots, n$ (that is, there is no proper closed $\text{GL}(E)$ -invariant subspace).*

As an immediate consequence, the above irreducibility result implies McMullen's conjecture on mixed volumes: the valuations of the form $\phi(K) = \text{vol}(K + A)$ span a dense subspace in $\text{Val}(E)$; the mixed volumes span a dense subspace in $\text{Val}(E)$. Moreover, the above theorem also implies in the same way that the linear combinations of valuations of the form $\phi(K) = V(K[i], \Delta[n-i])$, where Δ is a simplex in E , are dense in the space $\text{Val}_i(E)$. Alesker's irreducibility theorem also enables us to define some explicit positive cones in $\text{Val}_i(E)$ with nice properties.

2.4. Convolution and product of smooth valuations.

Definition 2.8 (Alesker). A valuation $\phi \in \text{Val}(E)$ is called *smooth* if the map

$$\text{GL}(E) \rightarrow \text{Val}(E), g \mapsto g \cdot \phi$$

is smooth as a map from a Lie group to a Banach space.

As a smooth valuation ϕ induces a map $\text{GL}(E) \rightarrow \text{Val}(E)$ given by $g \mapsto g \cdot \phi \in \text{Val}(E)$. The space of smooth valuations can be endowed with the topology of C^∞ functions on $\text{GL}(E)$ with values in the Banach space $\text{Val}(E)$ (this is usually called the Garding topology). This topology is naturally stronger than the topology from $\text{Val}(E)$ since there is a continuous injection

$$\text{Val}^\infty(E) \hookrightarrow \text{Val}(E).$$

The space of smooth valuations is denoted by $\text{Val}^\infty(E)$, it is dense in $\text{Val}(E)$. Moreover, the representation of $\text{GL}(E)$ in $\text{Val}^\infty(E)$ is continuous (see e.g. [AF14]).

By McMullen's grading decomposition, we have

$$\text{Val}^\infty(E) = \bigoplus_{i=0, \dots, n; \epsilon \in \{\text{even}, \text{odd}\}} \text{Val}_i^{\epsilon, \infty}(E).$$

Example 2.9. Assume that $A_1, \dots, A_{n-i} \in \mathcal{K}(E)$ are strictly convex bodies with smooth boundary, then $\phi_{A_1, \dots, A_{n-i}}(-) = V(-[i]; A_1, \dots, A_{n-i})$ is in $\text{Val}_i^\infty(E)$.

Example 2.10 (G -invariant valuations). Let $G \subset \text{SO}(E)$ be a compact subgroup. Let $\text{Val}^G(E)$ be the subspace of $\text{Val}(E)$ of G -invariant convex valuations. By [Ale07, Proposition 2.6, 2.7] (see also [Ale04]), the space $\text{Val}^G(E)$ is finite dimensional if and only if G acts transitively on the unit sphere of E , and under the assumption that G acts transitively on the unit sphere of E one has $\text{Val}^G(E) \subset \text{Val}^\infty(E)$.

An crucial ingredient in recent development of valuation theory (or algebraic integral geometry) is the product structure introduced by Alesker [Ale04]. To define it, Alesker used his irreducibility theorem.

Definition 2.11 (Product). There exists a bilinear map

$$\text{Val}^\infty(E) \times \text{Val}^\infty(E) \rightarrow \text{Val}^\infty(E)$$

which is uniquely characterized by the following two properties:

- (1) continuity;
- (2) if $A, B \in \mathcal{K}(E)$ are strictly convex bodies with smooth boundary, then the product of $\phi_A(\cdot) = \text{vol}(\cdot + A)$, $\phi_B(\cdot) = \text{vol}(\cdot + B)$ is given by

$$\phi_A \cdot \phi_B(K) = \text{vol}_{V \times V}(\Delta(K) + (A \times B)),$$

where $\Delta : E \rightarrow E \times E$ is the diagonal embedding.

The product makes $\text{Val}^\infty(E)$ a commutative associative algebra with the unit given by the Euler characteristic.

Example 2.12. (see [Ale04, Proposition 2.2]) Assume that A_1, \dots, A_{n-k} and B_1, \dots, B_k are strictly convex bodies with smooth boundary, then

$$V(-; A_1, \dots, A_{n-k}) \cdot V(-; B_1, \dots, B_k) = \frac{k!(n-k)!}{n!} V(A_1, \dots, A_{n-k}, -B_1, \dots, -B_k) \text{vol}(-).$$

The convolution on $\text{Val}^\infty(E)$ was introduced by Bernig and Fu in [BF06].

Definition 2.13 (Convolution). There exists a bilinear map

$$\text{Val}^\infty(E) \times \text{Val}^\infty(E) \rightarrow \text{Val}^\infty(E)$$

which is uniquely characterized by the following two properties:

- (1) continuity;
- (2) if $A, B \in \mathcal{K}(E)$ are strictly convex bodies with smooth boundary, then the convolution of $\phi_A(\cdot) = \text{vol}(\cdot + A)$, $\phi_B(\cdot) = \text{vol}(\cdot + B)$ is given by

$$\phi_A * \phi_B(K) = \text{vol}(K + (A + B)).$$

The convolution makes $\text{Val}^\infty(E)$ a commutative associative algebra with the unit given by the Lebesgue measure.

The following formula of $*$ is important in its extension to arbitrary mixed volumes (see Section 3.4).

Example 2.14. (see [BF06, Corollary 1.3]) Assume that A_1, \dots, A_{n-k} and B_1, \dots, B_{n-l} are strictly convex bodies with smooth boundary, and $k + l \geq n$, then

$$V(-; A_1, \dots, A_{n-k}) * V(-; B_1, \dots, B_{n-l}) = \frac{k!l!}{n!} V(-; A_1, \dots, A_{n-k}, B_1, \dots, B_{n-l}).$$

The product and convolution of smooth valuations are dual to each other by Alesker's Fourier transform.

Theorem 2.15 (see [Ale11]). *There is an algebra isomorphism $\widehat{\cdot}: (\text{Val}^\infty(E), \cdot) \rightarrow (\text{Val}^\infty(E), *)$ such that*

$$\widehat{\phi \cdot \psi} = \widehat{\phi} * \widehat{\psi}, \quad \phi, \psi \in \text{Val}^\infty(E).$$

Remark 2.16. Comparing with the intersection theory in algebraic geometry, it is convenient to view $\text{Val}_i^\infty(E)$ as the group of numerical cycle classes of dimension i , then the convolution can be considered as the cup product of cohomology classes, the product can be considered as the intersection of cycles and Alesker-Fourier transform can be considered as Poincaré duality. In our setting, by Example 2.14 we find it convenient to apply convolution operation rather than product operation.

3. POSITIVE CONVEX VALUATIONS

3.1. Positivity of valuations. By Alesker's irreducibility theorem, we know that the mixed volumes span a dense subspace in $\text{Val}(E)$. Let $\phi \in \text{Val}_i(E)$, then for any $\varepsilon > 0$ there exist valuations given by mixed volumes and real numbers c_1, \dots, c_m such that

$$\|\phi - \sum_{k=1}^m c_k \psi_k\| \leq \varepsilon,$$

where $\psi_k(-) = V(-; K_1^k, \dots, K_{n-i}^k) \in \text{Val}_i(E)$ for some $K_1^k, \dots, K_{n-i}^k \in \mathcal{K}(E)$. This motivates the following definition for our positive cone.

For any positive Radon measure μ on $\mathcal{K}(E)^{n-i}$ such that

$$\int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) < +\infty,$$

Denote by ϕ_μ the map from $\mathcal{K}(E)$ to \mathbb{R} given by:

$$\phi_\mu(L) = \int_{\mathcal{K}(E)^{n-i}} V(L[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}),$$

where $L \in \mathcal{K}(E)$ is a convex body. We will see that for any Radon measure μ as above, the map ϕ_μ defines a continuous translation invariant valuation (see Lemma 3.4).

Definition 3.1. We define the convex cone $\mathcal{P}_i \subset \text{Val}_i(E)$ given by:

$$\mathcal{P}_i := \left\{ \phi_\mu \mid \phi_\mu(L) := \int_{\mathcal{K}(E)^{n-i}} V(L[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) \right\},$$

where μ is taken over the positive Radon measures on $\mathcal{K}(E)^{n-i}$ such that

$$\int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) < +\infty.$$

We call a valuation $\phi \in \text{Val}_i(E)$ *positive* if $\phi \in \mathcal{P}_i$.

It is clear that \mathcal{P}_i is a convex cone.

By a polarization argument, observe that a valuation $\phi \in \mathcal{P}_i$ defines a unique function on $\mathcal{K}(E)^i$:

$$\phi(L_1, \dots, L_i) = \frac{1}{i!} \frac{\partial^i}{\partial t_1 \partial t_2 \dots \partial t_i} (\phi(t_1 L_1 + \dots + t_i L_i))|_{t_1=\dots=t_i=0^+},$$

where L_1, \dots, L_i are convex bodies. In particular, $\phi(L, \dots, L) = \phi(L)$.

Definition 3.2. We say that a valuation $\phi \in \mathcal{P}_i$ is *strictly positive* if there exists $\varepsilon > 0$ such that:

$$\phi(L_1, \dots, L_i) \geq \varepsilon V(\mathbf{B}[n-i], L_1, \dots, L_i)$$

for any convex body $L_1, \dots, L_i \in \mathcal{K}(E)$.

Remark 3.3. The definition for “strict positivity” is inspired by the study of positivity properties of cohomology classes in complex geometry. The convex body \mathbf{B} can be viewed as a Kähler class, and the inequality defining strict positivity of ϕ_μ can be viewed as the pseudo-effectivity of $\phi_\mu - \varepsilon V(\mathbf{B}[n-i]; -)$.

We prove that the cone \mathcal{P}_i is well-defined, i.e., $\mathcal{P}_i \subset \text{Val}_i(E)$.

Lemma 3.4. For any Radon measure μ on $\mathcal{K}(E)^{n-i}$ such that:

$$\int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) < +\infty,$$

the valuation ϕ_μ defines a continuous and translation invariant valuation.

Proof. Let us first prove that the integral is well-defined. Take a convex body $L \in \mathcal{K}(E)$, there exists a constant $\lambda > 0$ such that $L \subset \lambda \mathbf{B}$. Since the mixed volume is monotone, we have:

$$\begin{aligned} \phi_\mu(L) &= \int_{\mathcal{K}(E)^{n-i}} V(L[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) \\ &\leq \lambda^i \int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d\mu(K_1, \dots, K_{n-i}) < +\infty. \end{aligned}$$

As $V(-; K_1, \dots, K_{n-i})$ is a translation invariant valuation for any $K_1, \dots, K_{n-i} \in \mathcal{K}(E)$, it is clear that ϕ_μ is also a translation invariant valuation. Let us prove that ϕ_μ is continuous. Assume that $d_H(L_k, L) \rightarrow 0$, we need to check that $\phi_\mu(L_k) \rightarrow \phi_\mu(L)$. This is a direct consequence of the dominated convergence theorem. \square

Definition 3.5. We denote by \mathcal{V}'_i the subspace generated by \mathcal{P}_i , i.e., $\mathcal{V}'_i = \mathcal{P}_i - \mathcal{P}_i$.

By Alesker’s density theorem, \mathcal{V}'_i is dense in $\text{Val}_i(E)$ (with respect to the norm $\|\cdot\|$).

Example 3.6. When μ is a finite linear combination of Dirac measures on $\mathcal{K}(E)^{n-i}$, then the associated valuation $\phi_\mu \in \mathcal{V}'_i$ is a linear combination of mixed volumes.

Example 3.7. Let us consider the positive cones \mathcal{P}_1 and \mathcal{P}_{n-1} :

- (1) By Minkowski’s existence theorem (see [Sch14]), if μ is a positive Borel measure on \mathbb{S}^{n-1} which is not concentrated on any great subsphere and has the origin as its center of mass, then μ is given by the surface area measure of a convex body with non-empty interior. In particular, for any $n-1$ convex bodies K_1, \dots, K_{n-1} with non-empty interior, up to a translation, there is a unique convex body K with non-empty interior such that

$$V(-; K_1, \dots, K_{n-1}) = V(-; K[n-1]).$$

By Minkowski’s existence theorem again, for any two convex bodies K, L , up to a translation, there exists a unique convex body M such that

$$V(-; K[n-1]) + V(-; L[n-1]) = V(-; M[n-1]).$$

We claim that the set of strictly positive elements in \mathcal{P}_1 is just

$$\{V(-; K[n-1]) \mid K \in \mathcal{K}(E) \text{ with non-empty interior}\}.$$

Thus the cone \mathcal{P}_1 can be viewed as a convex cone in the space of Borel measures on \mathbb{S}^{n-1} . To this end, let $\phi_\mu \in \mathcal{P}_1$, we show that it gives a bounded linear functional on $C^0(\mathbb{S}^{n-1})$ endowed with the norm $|\cdot|_\infty$. For any $f \in C^0(\mathbb{S}^{n-1})$, we have

$$\begin{aligned} \phi_\mu(f) &:= \int_{\mathcal{K}(E)^{n-1}} d\mu(A_1, \dots, A_{n-1}) \int_{\mathbb{S}^{n-1}} f dS(A_1, \dots, A_{n-1}) \\ &\leq |f|_\infty \int_{\mathcal{K}(E)^{n-1}} d\mu(A_1, \dots, A_{n-1}) \int_{\mathbb{S}^{n-1}} h_{\mathbf{B}} dS(A_1, \dots, A_{n-1}) \\ &= \phi_\mu(\mathbf{B})|f|_\infty, \end{aligned}$$

where $d\mu(A_1, \dots, A_{n-1})$ is the surface area associated to A_1, \dots, A_{n-1} and $h_{\mathbf{B}}$ is the support function of the unit ball which is equal to 1 on \mathbb{S}^{n-1} . Furthermore, if ϕ_μ is strictly positive, then by Minkowski's existence theorem there is a unique (up to a translation) convex body K_μ with non-empty interior such that $\phi_\mu = V(-; K_\mu[n-1])$.

(2) For \mathcal{P}_{n-1} , by the discussions in the proof of Theorem 3.23 and Theorem 6.1 we will see that

$$\mathcal{P}_{n-1} = \{V(-; K) \mid K \in \mathcal{K}(E)\}.$$

By the embedding theorem for convex bodies, \mathcal{P}_{n-1} can be also realized as a convex cone in the continuous function space $C^0(\mathbb{S}^{n-1})$, which is generated by support functions.

Remark 3.8. For the space $\text{Val}_{n-1}(E)$, we have McMullen's characterization [McM80]. Let $L(\mathbb{S}^{n-1})$ denote the space of the restriction of linear functions to the unit sphere, then there is an isomorphism between the quotient space $C^0(\mathbb{S}^{n-1})/L(\mathbb{S}^{n-1})$ and $\text{Val}_{n-1}(E)$. Thus for every $\phi \in \text{Val}_{n-1}(E)$, up to a linear function, there is a unique continuous function f_ϕ such that

$$\phi(K) = \int_{\mathbb{S}^{n-1}} f_\phi(x) dS(K^{n-1}; x),$$

where $dS(K^{n-1}; x)$ is the surface area measure of K . By the correspondences established in [LX17], the analogy of the space $\text{Val}_{n-1}(E)$ on a projective variety is the vector space of real numerical divisor classes, and the analogy of \mathcal{P}_{n-1} is the movable cone of divisor classes. As for \mathcal{P}_1 , its closure is the dual of the cone given by positive continuous functions, and its analogy in complex geometry is the movable cone of curve classes.

For the general space $\text{Val}_i(E) = \text{Val}_i^+(E) \oplus \text{Val}_i^-(E)$, we have the Klain-Schneider realizations (see e.g. [Ale01, Section 2], [Ale11, Section 2.4]). The space $\text{Val}_i^+(E)$ can be $\text{GL}(E)$ -equivalently realized as a subspace of the space of smooth sections of certain line bundle over the Grassmannian $\text{Gr}_i(E)$, and the space $\text{Val}_i^-(E)$ can be $\text{GL}(E)$ -equivalently realized as a subspace of the quotient of the space of smooth sections of certain line bundle over the partial flag space $\mathcal{F}_{i,i+1}(E)$. Thus by Klain-Schneider realizations, it seems possible to discuss positivity in the smooth section space of certain line bundles.

Remark 3.9. Another motivation for the definition of \mathcal{P}_k is the positive cone in $\text{Val}^{\text{SO}(n)}(E)$ – the space of $\text{SO}(n)$ -invariant valuations. By the definition in [Ber12, Section 5.5], a valuation ϕ is called positive if $\phi(K) \geq 0$ for all $K \in \mathcal{K}(E)$. By Hadwiger's theorem, a $\text{SO}(n)$ -invariant valuation ϕ is positive if and only if $\phi = \sum_k c_k \mu_k$, where $c_k \geq 0$ and μ_k is the k -th intrinsic volume. Thus $\mathcal{P}_k^{\text{SO}(n)} = \mathbb{R}_+ \mu_k$. In the setting of hermitian integral geometry, there are also similar results (see [BF11, Proposition 4.1]). It is interesting to give a characterization for valuations $\phi \in \text{Val}_i(E)$ satisfying $\phi(K) \geq 0$ for every $K \in \mathcal{K}(E)$.

3.2. Reverse Khovanskii-Teissier inequality. Consider two Radon measures μ, ν on $\mathcal{K}(E)^{n-i}$ and $\mathcal{K}(E)^{n-j}$ respectively. Let $\phi_\mu \in \mathcal{V}'_i, \phi_\nu \in \mathcal{V}'_j$ be their associated valuations. We define the valuation $\phi_\mu \tilde{*} \phi_\nu$ given by:

$$(3) \quad \phi_\mu \tilde{*} \phi_\nu(-) = \frac{i!j!}{n!} \int_{\mathcal{K}(E)^{2n-i-j}} V(-; A_1, \dots, A_{n-i}, B_1, \dots, B_{n-j}) d\mu(A) d\nu(B).$$

where $d\mu(A) := d\mu(A_1, \dots, A_{n-i}), d\nu(B) := d\nu(B_1, \dots, B_{n-j})$. We will see immediately that the integral in (3) is well defined, that is, for any $D \in \mathcal{K}(E)$, $\phi_\mu \tilde{*} \phi_\nu(D)$ is finite (see Corollary 3.14).

The following inequality is a key ingredient of our paper. It was proved for valuations given by mixed volumes in [LX17, Theorem 5.9]. In this section, we state it for valuations from the positive cones \mathcal{P}_i .

Theorem 3.10. *Let $\phi \in \mathcal{P}_k$ and $\psi \in \mathcal{P}_{n-k}$, then for any $K \in \mathcal{K}(E)$ we have*

$$\phi(K)\psi(K) \geq \text{vol}(K)\phi\tilde{*}\psi.$$

Proof. By definition, there exists two Radon measures μ and ν on $\mathcal{K}(E)^{n-k}$ and $\mathcal{K}(E)^k$ such that $\phi = \phi_\mu$ and $\psi = \phi_\nu$ respectively. By definition, $\phi_\mu\tilde{*}\phi_\nu$ is equal to

$$\phi_\mu\tilde{*}\phi_\nu = \frac{k!(n-k)!}{n!} \int_{\mathcal{K}(E)^n} V(A_1, \dots, A_{n-k}, B_1, \dots, B_k) d\mu(A_1, \dots, A_{n-k}) d\nu(B_1, \dots, B_k).$$

Claim: there is a constant $c > 0$ depending only on n, k such that

$$V(K[k]; A_1, \dots, A_{n-k})V(K[n-k]; B_1, \dots, B_k) \geq cV(A_1, \dots, A_{n-k}, B_1, \dots, B_k) \text{vol}(K).$$

The above inequality is just a slight generalization of [LX17, Theorem 5.9], and the proof is similar. We refer to [LX17, Section 5] for the details (see also [Xia17]). Let us give a sketch of the argument here. Without loss of generality, we can assume the A_l, B_l and K are open and have non-empty interior. We apply a result of [Gro90] and results from mass transport (see [Bre91, McC95]). Then after solving a real Monge-Ampère equation related to K , the desired geometric inequality of convex bodies can be reduced to an inequality for mixed discriminants – the mixed discriminants given by the Hessian of those convex functions defining the convex bodies. More precisely, as in [LX17] (see also [ADM99]) the inequality for mixed volumes is reduced to an inequality for integrals:

$$\begin{aligned} & \int_{\mathbb{R}^n} D(\nabla^2 f_{A_1}, \dots, \nabla^2 f_{A_{n-k}}, (\nabla^2 F_K)[k]) dx \int_{\mathbb{R}^n} D((\nabla^2 F_K)[n-k], \nabla^2 f_{B_1}, \dots, \nabla^2 f_{B_k}) dx \\ & \geq \frac{k!(n-k)!}{n!} \int_{\mathbb{R}^n} \det(\nabla^2 F_K) dx \int_{\mathbb{R}^n} D(\nabla^2 f_{A_1}, \dots, \nabla^2 f_{A_{n-k}}, \nabla^2 f_{B_1}, \dots, \nabla^2 f_{B_k}) dx, \end{aligned}$$

where ∇^2 is the Hessian operator, $D(-)$ denotes mixed discriminants, and f_{A_i}, f_{B_j}, F_K are convex functions obtained by the results in [Gro90] and [Bre91, McC95].

Let $M_K, M_1, \dots, M_{n-k}, M'_1, \dots, M'_k$ be the associated positive symmetric matrices given by $\nabla^2 F_K, \nabla^2 f_{A_1}, \dots, \nabla^2 f_{A_{n-k}}, \nabla^2 f_{B_1}, \dots, \nabla^2 f_{B_k}$ respectively. After an application of the Cauchy-Schwarz inequality

$$\left(\int |fg| dv \right)^2 \leq \left(\int |f|^2 dv \right) \left(\int |g|^2 dv \right)$$

to the left hand side of the above inequality for integrals, the pointwise inequality needed is:

$$D(M_K[k]; M_1, \dots, M_{n-k})D(M_K[n-k]; M'_1, \dots, M'_k) \geq \frac{k!(n-k)!}{n!} D(M_1, \dots, M_{n-k}, M'_1, \dots, M'_k) \det(M_K).$$

The above inequality for positive matrices is equivalent to an inequality for positive $(1, 1)$ -forms by replacing the positive matrices by positive $(1, 1)$ forms and the discriminants by wedge product of differential forms (see e.g. [Xia17, Section 2]). Assume that $M = [a_{i\bar{j}}]$ is a positive Hermitian matrix, then it determines a positive $(1, 1)$ form on \mathbb{C}^n given by:

$$M \mapsto \omega_M := \sqrt{-1} \sum_{i,j} a_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

By this correspondence, the pointwise inequality for discriminants is equivalent to

$$(\omega_{M_K}^k \wedge \omega_{M_1} \wedge \dots \wedge \omega_{M_{n-k}})(\omega_{M_K}^{n-k} \wedge \omega_{M'_1} \wedge \dots \wedge \omega_{M'_k}) \geq \frac{k!(n-k)!}{n!} \omega_{M_K}^n (\omega_{M_1} \wedge \dots \wedge \omega_{M_{n-k}} \wedge \omega_{M'_1} \wedge \dots \wedge \omega_{M'_k}).$$

Note that wedge products of positive $(1, 1)$ forms are Hermitian positive. More generally, assume that Φ is a Hermitian positive $(n-k, n-k)$ form, Ψ is a Hermitian positive (k, k) form and ω is a positive $(1, 1)$ form¹, then

$$(4) \quad (\Phi \wedge \omega^k)(\omega^{n-k} \wedge \Psi) \geq \frac{k!(n-k)!}{n!} (\Phi \wedge \Psi) \omega^n.$$

¹For the positivity of forms, we refer the reader to [Dem12b, Chapter 3] and [DELV11, Section 1]. In [DELV11, Definition 1.4], “Hermitian positive” is called semipositive.

Recall that a (l, l) form is Hermitian positive on the vector space \mathbb{C}^n if its associated Hermitian form on $\wedge^l \mathbb{C}^n$ is semipositive (see [DELV11, Definition 1.4]), that is, the coefficients of the (l, l) form give a semipositive Hermitian matrix on $\wedge^l \mathbb{C}^n$, here $\wedge^l \mathbb{C}^n$ is the l -th wedge product of \mathbb{C}^n . By taking some local coordinates, it is sufficient to check the above inequality when ω is given by the identity matrix. As Φ is Hermitian positive, then

$$\sum_{|J|=n-k} \left(\sum_{|I|=n-k} \Phi_{I,I} dz_J \wedge d\bar{z}_J - \Phi \right)$$

is also Hermitian positive. As Ψ is Hermitian positive and the cone generated by Hermitian positive (k, k) forms is dual to the cone generated by Hermitian positive $(n-k, n-k)$ forms (see [DELV11, Section 1]), we get

$$\left(\sum_{|J|=n-k} \left(\sum_{|I|=n-k} \Phi_{I,I} dz_J \wedge d\bar{z}_J - \Phi \right) \right) \wedge \Psi \geq 0,$$

which gives the desired pointwise inequality (4).

In summary, we finally obtain

$$\phi(K)\psi(K) \geq \text{vol}(K)\phi\tilde{*}\psi,$$

as required. \square

Remark 3.11. As for the terminology ‘‘reverse Khovanskii-Teissier inequality’’, it was used in [LX16]. The reason is that: the classical Khovanskii-Teissier inequality gives us a lower bound of $\phi\tilde{*}\psi$, but the above inequality gives us an upper bound:

$$\phi\tilde{*}\psi \leq \inf_{\text{vol}(K)=1} \phi(K)\psi(K).$$

See also [LX16] for a discussion in the abstract setting from the viewpoint of convex analysis. In complex geometry, as a corollary of Demailly’s holomorphic Morse inequality (see [Dem12a, Chapter 8]), the special case of the above inequality for divisor classes (when $k = 1$) was first noted by Siu [Siu93]. The inequality for general (k, k) classes was first noted in [Xia15]. The pointwise inequality for forms in the proof is a generalization of [Pop16], where the weak transcendental Morse inequality for $(1, 1)$ classes was proved with optimal estimate.

3.2.1. Bézout type inequality. Recently, inspired by Bézout bound in algebraic geometry, the second author [Xia17] noticed that the reverse Khovanskii-Teissier inequality can be used to obtain Bézout type inequality in convex geometry (see also [SZ16]). This can be also formulated using convolution.

Theorem 3.12 (see [Xia17], Theorem 1.1). *Let $\phi_i \in \mathcal{P}_{n-a_i}$ where $1 \leq i \leq r$ and $|a| := \sum_{i=1}^r a_i \leq n$, then there is a constant $c > 0$ depending only on n, a_1, \dots, a_r such that, for any $D \in \mathcal{K}(E)$ we have*

$$(\phi_1 \tilde{*} \dots \tilde{*} \phi_r)(D) \text{vol}(D)^{r-1} \leq c \prod_{i=1}^r \phi_i(D).$$

In particular, if $|a| = n$, then

$$(\phi_1 \tilde{*} \dots \tilde{*} \phi_r) \text{vol}(D)^{r-1} \leq c \prod_{i=1}^r \phi_i(D).$$

Proof. This follows directly from Theorem 3.10, as exactly in [Xia17, Theorem 1.1]. \square

Proposition 3.13. *The operator $\tilde{*}$ defined by the formula (3) induces a bilinear map $\tilde{*} : \mathcal{V}'_i \times \mathcal{V}'_j \rightarrow \mathcal{V}'_{i+j-n}$.*

Proof. This proposition follows immediately from the following Lemma 3.14 and Lemma 3.15. \square

Lemma 3.14. *For any $\phi_\mu \in \mathcal{P}_i, \psi_\nu \in \mathcal{P}_j$, the integral (3) defining $\phi_\mu \tilde{*} \psi_\nu$ is well defined.*

Proof. We only need to check that the integral defining $\phi_\mu \tilde{*} \psi_\nu(D)$ is well defined, when D has non-empty interior. This follows directly from Theorem 3.12. \square

It is possible that different Radon measures give the same valuations, we prove that $\phi_\mu \tilde{*} \phi_\nu$ is independent of the representations.

Lemma 3.15. *The valuation $\phi_\mu \tilde{*} \phi_\nu$ is independent of the choices of μ, ν .*

Proof. Consider Radon measures μ_1, μ_2 on $\mathcal{K}(E)^{n-i}$ and ν_1, ν_2 on $\mathcal{K}(E)^{n-j}$ respectively. Assume that $\phi_{\mu_1} = \phi_{\mu_2}, \phi_{\nu_1} = \phi_{\nu_2}$, we prove that $\phi_{\mu_1} \tilde{*} \phi_{\nu_1} = \phi_{\mu_2} \tilde{*} \phi_{\nu_2}$.

We need to verify that for any $L \in \mathcal{K}(E)$,

$$\begin{aligned} & \int_{\mathcal{K}(E)^{2n-i-j}} V(L[i+j-n]; A_1, \dots, A_{n-i}, B_1, \dots, B_{n-j}) d\mu_1(A) d\nu_1(B) \\ &= \int_{\mathcal{K}(E)^{2n-i-j}} V(L[i+j-n]; A_1, \dots, A_{n-i}, B_1, \dots, B_{n-j}) d\mu_2(A) d\nu_2(B). \end{aligned}$$

For any $t = (t_1, \dots, t_j) \in (\mathbb{R}^+)^j$, denote by $K_t = t_1 K_1 + \dots + t_j K_j$ where K_1, \dots, K_j are convex bodies. Since $\phi_{\nu_1} = \phi_{\nu_2}$, we have that $\phi_{\nu_1}(K_t) = \phi_{\nu_2}(K_t)$. Since $\phi_{\nu_i}(K_t)$ is a polynomial in t_1, \dots, t_j , the equality on the coefficients of the polynomial gives

$$\int_{\mathcal{K}(E)^{n-j}} V(K_1, \dots, K_j; B_1, \dots, B_{n-j}) d\nu_1(B) = \int_{\mathcal{K}(E)^{n-j}} V(K_1, \dots, K_j; B_1, \dots, B_{n-j}) d\nu_2(B).$$

In particular, this implies $\phi_{\mu_1} \tilde{*} \phi_{\nu_1} = \phi_{\mu_1} \tilde{*} \phi_{\nu_2}$. Similarly, $\phi_{\mu_1} \tilde{*} \phi_{\nu_2} = \phi_{\mu_2} \tilde{*} \phi_{\nu_2}$, hence $\phi_{\mu_1} \tilde{*} \phi_{\nu_1} = \phi_{\mu_2} \tilde{*} \phi_{\nu_2}$. \square

3.3. Norms on the space of valuations. The aim of this section is to define some norms on the space generated by \mathcal{P}_i . These norms are induced by the positive cone \mathcal{P}_i .

3.3.1. Positivity norm $\|\cdot\|_{\mathcal{P}}$. We define the norm $\|\cdot\|_{\mathcal{P}}$, for which we will show that the subspace $\mathcal{P}_i \cap \text{Val}^\infty(E)$ of smooth valuations is dense in \mathcal{V}'_i .

Definition 3.16. For any valuation $\phi \in \mathcal{V}'_i$, we define $\|\phi\|_{\mathcal{P}}$ by the following formula.

$$\|\phi\|_{\mathcal{P}} := \inf\{t \geq 0 \mid |\phi(L_1, \dots, L_i)| \leq tV(\mathbf{B}[n-i], L_1, \dots, L_i) \text{ for any } L_1, \dots, L_i \in \mathcal{K}(E)\}.$$

First we note that for any $\phi \in \mathcal{V}'_i$, $\|\phi\|_{\mathcal{P}}$ is well defined.

Proposition 3.17. *The map $\|\cdot\|_{\mathcal{P}} : \mathcal{V}'_i \rightarrow \mathbb{R}^+$ defines a norm on \mathcal{V}'_i .*

Proof. The only fact which is not straightforward is whether $\|\cdot\|_{\mathcal{P}_i}$ is well-defined. Consider $\phi \in \mathcal{V}'_i$, we prove that there exists a $t > 0$ such that

$$|\phi(L_1, \dots, L_i)| \leq tV(\mathbf{B}[n-i], L_1, \dots, L_i).$$

By definition, there exists a signed Radon measure μ on $\mathcal{K}(E)^{n-i}$ such that $\phi = \phi_\mu$. Consider the Hahn decomposition $\mu = \mu^+ - \mu^-$ of the measure μ so that $\phi_\mu = \phi_{\mu^+} - \phi_{\mu^-}$. One has that

$$|\phi(L_1, \dots, L_i)| \leq \phi_{\mu^+}(L_1, \dots, L_i) + \phi_{\mu^-}(L_1, \dots, L_i).$$

Let us find an upper bound for $\phi_{\mu^+}(L_1, \dots, L_i)$. By Theorem 3.10 we have

$$\begin{aligned} \phi_{\mu^+}(L_1, \dots, L_i) &= \int_{\mathcal{K}(E)^{n-i}} V(L_1, \dots, L_i, K_1, \dots, K_{n-i}) d\mu^+(K) \\ &\leq cV(\mathbf{B}[n-i], L_1, \dots, L_i) \int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], K_1, \dots, K_{n-i}) d\mu^+(K), \end{aligned}$$

where $c > 0$ depends only on $n, i, \text{vol}(\mathbf{B})$. Since $\phi_{\mu^+} \in \mathcal{P}_i$, we get

$$\phi_{\mu^+}(L_1, \dots, L_i) \leq tV(\mathbf{B}[n-i], L_1, \dots, L_i)$$

for some $t > 0$. Similar estimates also hold for ϕ_{μ^-} , this proves that $\|\phi\|_{\mathcal{P}} < +\infty$. \square

Remark 3.18. Observe that by homogeneity for L_1, \dots, L_i , we have

$$\|\phi\|_{\mathcal{P}} := \inf\{t \geq 0 \mid |\phi(L_1, \dots, L_i)| \leq tV(\mathbf{B}[n-i], L_1, \dots, L_i) \text{ for any } L_1, \dots, L_i \subset \mathbf{B}\}.$$

By the above remark, we get:

Proposition 3.19. *For any $\phi \in \mathcal{V}'_i$, $\|\phi\| \leq \text{vol}(\mathbf{B})\|\phi\|_{\mathcal{P}}$. Hence, there is a continuous injection:*

$$(\mathcal{V}'_i, \|\cdot\|_{\mathcal{P}}) \hookrightarrow (\text{Val}_i(E), \|\cdot\|).$$

Regarding the definition for $\|\cdot\|_{\mathcal{P}}$, we introduce the following positivity notation.

Definition 3.20. Let $\phi, \psi \in \mathcal{V}'_i$, we say that $\phi \preceq \psi$ (or equivalently, $\psi \succeq \phi$), if for any $L_1, \dots, L_i \in \mathcal{K}(E)$,

$$\phi(L_1, \dots, L_i) \leq \psi(L_1, \dots, L_i).$$

Using the terminology from complex geometry, $\phi \preceq \psi$ means that $\psi - \phi$ is pseudo-effective in some sense.

Lemma 3.21. Let $\psi \in \mathcal{P}_j, \phi_1, \phi_2 \in \mathcal{V}'_i$. Assume that $\phi_1 \succeq \phi_2$, then $\phi_1 \tilde{*} \psi \succeq \phi_2 \tilde{*} \psi$.

Proof. This follows directly from the definition. \square

We also note that the $\text{GL}(E)$ actions preserve the partial order \succeq .

Lemma 3.22. Let $g \in \text{GL}(E)$. Consider $\phi_1, \phi_2 \in \mathcal{V}'_i$ such that $\phi_1 \succeq \phi_2$, then $g \cdot \phi_1 \succeq g \cdot \phi_2$.

Next we show that the space of smooth valuations is dense in \mathcal{V}'_i with respect to the topology given by $\|\cdot\|_{\mathcal{P}}$.

Theorem 3.23. The space of finite sums of mixed volumes of convex bodies with strictly convex and smooth boundary is dense in \mathcal{V}'_i for the topology induced by the norm $\|\cdot\|_{\mathcal{P}}$. In particular, the space $\text{Val}_i^\infty(E) \cap \mathcal{V}'_i$ is dense in \mathcal{V}'_i for the topology induced by the norm $\|\cdot\|_{\mathcal{P}}$.

Proof. Since \mathcal{V}'_i is generated by \mathcal{P}_i , we are reduced to prove the density of smooth valuations in \mathcal{P}_i . We prove that the finite sums of mixed volumes of convex bodies with strictly convex and smooth boundary are dense in \mathcal{V}'_i .

We prove it in two steps.

Step 1: Let us first prove that the valuations in \mathcal{P}_i such that their associated measure has bounded support are dense in \mathcal{P}_i .

Take $\phi \in \mathcal{P}_i$ such that $\phi = \phi_\mu$ where μ is its associated positive Radon measure on $\mathcal{K}(E)^{n-i}$. For any integer $k > 0$, we consider the measure μ_k given by $\mu_k = \mu|_{B(0,k)}$, where

$$B(0, k) = \{(K_1, \dots, K_{n-i}) \in \mathcal{K}(E)^{n-i} \mid K_j \subset k\mathbf{B}, \forall 0 \leq j \leq n-i\} \subset \mathcal{K}(E)^{n-i}.$$

By construction, the measure μ_k has bounded support. (By Blaschke selection theorem, $B(0, k)$ is a compact set.) By the monotone convergence theorem, we have that:

$$\phi_{\mu_k}(L) = \int_{\mathcal{K}(E)^{n-i}} V(K_1, \dots, K_{n-i}, L[i]) d\mu_k(K_1, \dots, K_{n-i}) \rightarrow \phi(L).$$

Let us prove that $\|\phi_\mu - \phi_{\mu_k}\|_{\mathcal{P}}$ converges to zero as $k \rightarrow +\infty$. Fix some convex bodies L_1, \dots, L_i . By construction, one has that:

$$0 \leq \phi(L_1, \dots, L_i) - \phi_{\mu_k}(L_1, \dots, L_i).$$

Moreover, by Theorem 3.10 applied to $\phi' = V(K_1, \dots, K_{n-i}, -[i])$ and $\psi' = V(L_1, \dots, L_i, -[n-i])$ and to the convex body \mathbf{B} , there exists a constant $C > 0$ such that we have:

$$V(K_1, \dots, K_{n-i}, L_1, \dots, L_i) \leq C \frac{V(K_1, \dots, K_{n-i}, \mathbf{B}[i])}{\text{vol}(\mathbf{B})} V(\mathbf{B}[n-i], L_1, \dots, L_i).$$

Integrating on the previous inequality, one obtains:

$$\begin{aligned} & \phi(L_1, \dots, L_i) - \phi_{\mu_k}(L_1, \dots, L_i) \\ & \leq \frac{C}{\text{vol}(\mathbf{B})} \left(\int_{B(0,k)^c} V(K_1, \dots, K_{n-i}, \mathbf{B}[i]) d\mu(K_1, \dots, K_{n-i}) \right) V(\mathbf{B}[n-i], L_1, \dots, L_i), \end{aligned}$$

where $B(0, k)^c = \mathcal{K}(E)^{n-i} \setminus B(0, k)$. We have thus proved:

$$|(\phi - \phi_{\mu_k})(L_1, \dots, L_i)| \leq \frac{C}{\text{vol}(\mathbf{B})} (\phi(\mathbf{B}) - \phi_{\mu_k}(\mathbf{B})) V(\mathbf{B}[n-i], L_1, \dots, L_i),$$

for any convex bodies L_1, \dots, L_i . Since $\phi(\mathbf{B}) - \phi_{\mu_k}(\mathbf{B}) \rightarrow 0$ as $k \rightarrow +\infty$, we have that $\|\phi - \phi_{\mu_k}\|_{\mathcal{P}} \rightarrow 0$ as required.

Step 2: Suppose that $\phi = \phi_\mu \in \mathcal{P}_i$ is a valuation where μ is a Radon measure on $\mathcal{K}(E)^{n-i}$ whose support is bounded. We prove that ϕ can be approached by ϕ_k , where $\phi_k \in \mathcal{P}_i \cap \text{Val}_i^\infty(E)$ is a finite sum of mixed volumes given by convex bodies with strictly convex and smooth boundary.

Suppose that the support of μ is contained in $B(0, N)$ where $N > 0$ is an integer. For any $\epsilon > 0$, there exists a partition $\cup_{j=1}^m O_j$ of $B(0, N)$ such that for any $(K_1, \dots, K_{n-i}), (K'_1, \dots, K'_{n-i}) \in O_j$, one has:

$$(5) \quad d_H(K_j, K'_j) \leq \epsilon, \quad \forall 1 \leq j \leq n-i.$$

Since the valuations given by mixed volumes are monotone and since $\text{supp } \mu \subset B(0, N)$, there is a constant $C > 0$ (depending only on N, i) such that

$$(6) \quad |V(K_1, \dots, K_{n-i}, L_1, \dots, L_i) - V(K'_1, \dots, K'_{n-i}, L_1, \dots, L_i)| \leq C\epsilon V(\mathbf{B}[n-i], L_1, \dots, L_i).$$

Let us define the measure μ_ϵ given by

$$\mu_\epsilon := \sum_{j=1}^m \mu(O_j) \delta_{(K_1^j, \dots, K_{n-i}^j)},$$

where $(K_1^j, K_2^j, \dots, K_{n-i}^j) \in O_j$ satisfying that K_1^j, \dots, K_{n-i}^j are convex bodies with smooth and strictly convex boundary, and where $\delta_{(K_1^j, K_2^j, \dots, K_{n-i}^j)}$ is the dirac mass at the point $(K_1^j, K_2^j, \dots, K_{n-i}^j)$. Let us estimate the norm $\|\phi_{\mu_\epsilon} - \phi\|_{\mathcal{P}}$. Take $L_1, \dots, L_i \in \mathcal{K}(E)$. By definition, one has that

$$\begin{aligned} \phi_{\mu_\epsilon}(L_1, \dots, L_i) &= \sum_{j=1}^m \mu(O_j) V(K_1^j, \dots, K_{n-i}^j, L_1, \dots, L_i), \\ &= \sum_{j=1}^m \int_{O_j} V(K_1^j, \dots, K_{n-i}^j, L_1, \dots, L_i) d\mu(K_1, \dots, K_{n-i}). \end{aligned}$$

The difference $|\phi_{\mu_\epsilon}(L_1, \dots, L_i) - \phi(L_1, \dots, L_i)|$ is bounded by:

$$\begin{aligned} &|\phi_{\mu_\epsilon}(L_1, \dots, L_i) - \phi(L_1, \dots, L_i)| \\ &\leq \sum_{j=1}^m \left| \int_{O_j} (V(K_1^j, \dots, K_{n-i}^j, L_1, \dots, L_i) - V(K_1, \dots, K_{n-i}, L_1, \dots, L_i)) d\mu(K_1, \dots, K_{n-i}) \right| \\ &\leq \sum_{j=1}^m \int_{O_j} |V(K_1^j, \dots, K_{n-i}^j, L_1, \dots, L_i) - V(K_1, \dots, K_{n-i}, L_1, \dots, L_i)| d\mu(K_1, \dots, K_{n-i}). \end{aligned}$$

Applying (6) to the previous inequality, we obtain the following upper bound:

$$|\phi_{\mu_\epsilon}(L_1, \dots, L_i) - \phi(L_1, \dots, L_i)| \leq C\epsilon \sum_{j=1}^m \int_{O_j} V(\mathbf{B}[n-i], L_1, \dots, L_i) d\mu(K_1, \dots, K_{n-i}).$$

Hence,

$$|\phi_{\mu_\epsilon}(L_1, \dots, L_i) - \phi(L_1, \dots, L_i)| \leq C\epsilon V(\mathbf{B}[n-i], L_1, \dots, L_i) \mu(B(0, N)),$$

and this implies that $\|\phi_{\mu_\epsilon} - \phi\|_{\mathcal{P}} \leq C\epsilon \mu(B(0, N))$ is arbitrary small since $\mu(B(0, N))$ is finite.

We have thus proven that finite sums of mixed volumes of convex bodies with smooth and strictly convex boundary are dense in \mathcal{P}_i with respect to the norm $\|\cdot\|_{\mathcal{P}}$ as required. \square

A direct consequence is the following result:

Corollary 3.24. *The set of valuations $\{V(L; -[n-1]) \mid L \in \mathcal{K}(E)\}$ is dense in \mathcal{P}_{n-1} with respect to the topology given by $\|\cdot\|_{\mathcal{P}}$.*

3.3.2. *Cone norm* $\|\cdot\|_{\mathcal{C}}$. As \mathcal{V}'_i is generated by \mathcal{P}_i , it is naturally endowed a norm $\|\cdot\|_{\mathcal{C}}$ induced by the cone structure. This construction is inspired by the construction in algebraic geometry (see [Dan17]).

Definition 3.25. For any $\phi \in \mathcal{V}'_i$, we define $\|\phi\|_{\mathcal{C}}$ by the following formula:

$$\|\phi\|_{\mathcal{C}} := \inf_{\phi = \phi_+ - \phi_-, \phi_{\pm} \in \mathcal{P}_i} \phi_+(\mathbf{B}) + \phi_-(\mathbf{B}).$$

Here, the symbol \mathcal{C} stands for the fact that this norm is induced by the convex cone \mathcal{P}_i .

Remark 3.26. Equivalently, by the Jordan decomposition of signed measures we have $\|\phi_{|\mu}\|_{\mathcal{C}} := \phi_{|\mu}(\mathbf{B})$, where $|\mu|$ is the absolute value of a Radon measure μ on $\mathcal{K}(E)^{n-i}$.

Remark 3.27. By construction, if $\phi \in \mathcal{P}_i$, then $\|\phi\|_{\mathcal{C}} = \phi(\mathbf{B})$.

Lemma 3.28. *The function $\|\cdot\|_{\mathcal{C}}$ defined above is a norm on the space \mathcal{V}'_i .*

Proof. It is clear that:

- For any $c \in \mathbb{R}$ and any $\phi \in \mathcal{V}'_i$, we have $\|c\phi\|_{\mathcal{C}} = |c|\|\phi\|_{\mathcal{C}}$;
- For any $\phi, \psi \in \mathcal{V}'_i$, we have $\|\phi + \psi\|_{\mathcal{C}} \leq \|\phi\|_{\mathcal{C}} + \|\psi\|_{\mathcal{C}}$.

It remains to verify:

- If $\|\phi\|_{\mathcal{C}} = 0$, then $\phi = 0$.

To this end, take a sequence of decompositions $\phi = \phi_k^+ - \phi_k^-$ such that $\phi_k^+(\mathbf{B}) + \phi_k^-(\mathbf{B}) \rightarrow 0$. By the definition of the Banach structure on $\text{Val}(E)$ (see (1)), for any $K \subset \mathbf{B}$ we have

$$|\phi(K)| = |\phi_k^+(K) - \phi_k^-(K)| \leq \phi_k^+(\mathbf{B}) + \phi_k^-(\mathbf{B}) \rightarrow 0.$$

Hence, $\phi(K) = 0$ for any $K \subset \mathbf{B}$, implying $\phi = 0$. □

Proposition 3.29. *The set of positive valuations ϕ_{μ} , where μ has bounded support, is dense in \mathcal{P}_i with respect to the topology given by $\|\cdot\|_{\mathcal{C}}$.*

Proof. This is straightforward. □

3.3.3. *Comparison of two norms.*

Proposition 3.30. *For any $\phi \in \mathcal{V}'_i$, one has that $\|\phi\|_{\mathcal{P}} \leq C\|\phi\|_{\mathcal{C}}$ for some uniform constant $C > 0$. Hence, there is a continuous injection:*

$$(\mathcal{V}'_i, \|\cdot\|_{\mathcal{C}}) \hookrightarrow (\mathcal{V}'_i, \|\cdot\|_{\mathcal{P}}).$$

Proof. Consider $\phi \in \mathcal{V}'_i$ and assume that $\phi = \phi_+ - \phi_-$ where $\phi_+, \phi_- \in \mathcal{P}_i$. Fix some convex bodies L_1, \dots, L_i . One has that:

$$|\phi(L_1, \dots, L_i)| \leq |\phi_+(L_1, \dots, L_i)| + |\phi_-(L_1, \dots, L_i)|.$$

By Theorem 3.10 applied to $\phi' = \phi_{\pm}$, $\psi = V(L_1, \dots, L_i, -[n-i])$ and to the convex body \mathbf{B} respectively, there exists a uniform constant $C > 0$ such that:

$$\phi_{\pm}(L_1, \dots, L_i) \leq C\phi_{\pm}(\mathbf{B})V(\mathbf{B}[n-i], L_1, \dots, L_i).$$

In particular, this implies that:

$$|\phi(L_1, \dots, L_i)| \leq C(\phi_+(\mathbf{B}) + \phi_-(\mathbf{B}))V(\mathbf{B}[n-i], L_1, \dots, L_i).$$

By considering two sequences $\phi_{+,j}, \phi_{-,j} \in \mathcal{P}_i$ such that $\lim_j \phi_{+,j}(\mathbf{B}) + \phi_{-,j}(\mathbf{B}) = \|\phi\|_{\mathcal{C}}$, we obtain:

$$|\phi(L_1, \dots, L_i)| \leq C\|\phi\|_{\mathcal{C}}V(\mathbf{B}[n-i], L_1, \dots, L_i),$$

for any convex bodies L_1, \dots, L_i . By definition, we obtain:

$$\|\phi\|_{\mathcal{P}} \leq C\|\phi\|_{\mathcal{C}},$$

as required. □

Corollary 3.31. *One has the following sequence of continuous injections:*

$$(\mathcal{V}'_i, \|\cdot\|_{\mathcal{C}}) \hookrightarrow (\mathcal{V}'_i, \|\cdot\|_{\mathcal{P}}) \hookrightarrow (\text{Val}_i(E), \|\cdot\|).$$

Proof. This follows directly from Proposition 3.19 and Proposition 3.30. □

3.3.4. *Sub-multiplicity of norms.* We get the following sub-multiplicity result for the norms defined above. This will be important in the completion of the space \mathcal{V}'_i .

Lemma 3.32. *Let $\phi_\mu \in \mathcal{P}_i, \psi_\nu \in \mathcal{P}_j$, then there is $c > 0$ depending only on $i, j, n, \text{vol}(\mathbf{B})$ such that:*

- $\|\phi_\mu \tilde{*} \psi_\nu\|_{\mathcal{C}} \leq c \|\phi_\mu\|_{\mathcal{C}} \|\psi_\nu\|_{\mathcal{C}};$
- $\|\phi_\mu \tilde{*} \psi_\nu\|_{\mathcal{P}} \leq c \|\phi_\mu\|_{\mathcal{P}} \|\psi_\nu\|_{\mathcal{P}}.$

Proof. Let us first prove the first inequality. Note that

$$\|\phi_\mu \tilde{*} \psi_\nu\|_{\mathcal{C}} = \phi_\mu \tilde{*} \psi_\nu(\mathbf{B}) \leq c \phi_\mu(\mathbf{B}) \psi_\nu(\mathbf{B}) = c \|\phi_\mu\|_{\mathcal{C}} \|\psi_\nu\|_{\mathcal{C}},$$

where the second estimate follows from Theorem 3.12.

For the second inequality, let $L_1, \dots, L_{i+j-n} \in \mathcal{K}(E)$, we have

$$\begin{aligned} \phi_\mu \tilde{*} \psi_\nu(L_1, \dots, L_{i+j-n}) &= \frac{i!j!}{n!} \int_{\mathcal{K}(E)^{2n-i-j}} V(L_1, \dots, L_{i+j-n}, A_1, \dots, A_{n-i}, B_1, \dots, B_{n-j}) d\mu(A) d\nu(B) \\ &\leq c \|\phi_\mu\|_{\mathcal{P}} \int_{\mathcal{K}(E)^{n-j}} V(\mathbf{B}[n-i]; L_1, \dots, L_{i+j-n}, B_1, \dots, B_{n-j}) d\nu(B) \\ &\leq c \|\phi_\mu\|_{\mathcal{P}} \|\psi_\nu\|_{\mathcal{P}} V(\mathbf{B}[2n-i-j]; L_1, \dots, L_{i+j-n}). \end{aligned}$$

Thus, by definition $\|\phi_\mu \tilde{*} \psi_\nu\|_{\mathcal{P}} \leq c \|\phi_\mu\|_{\mathcal{P}} \|\psi_\nu\|_{\mathcal{P}}$. □

3.4. **An extension of the convolution operator.** Recall that for $\phi_\mu \in \mathcal{V}'_i, \psi_\nu \in \mathcal{V}'_j$, the formula for $\phi_\mu \tilde{*} \psi_\nu \in \mathcal{V}'_{i+j-n}$ is defined by

$$\phi_\mu \tilde{*} \psi_\nu(-) = \frac{i!j!}{n!} \int_{\mathcal{K}(E)^{2n-i-j}} V(-; A_1, \dots, A_{n-i}, B_1, \dots, B_{n-j}) d\mu(A) d\nu(B).$$

Let $\mathcal{V}_i^{\mathcal{C}}, \mathcal{V}_i^{\mathcal{P}}$ be the completions of the space \mathcal{V}'_i with respect to the norms $\|\cdot\|_{\mathcal{C}}$ and $\|\cdot\|_{\mathcal{P}}$ respectively.

In the following, we let $\gamma \in \{\mathcal{C}, \mathcal{P}\}$. We show that the operator $\tilde{*}$ extends continuously to the spaces \mathcal{V}_i^γ with respect to $\|\cdot\|_\gamma$.

Theorem 3.33. *With respect to $\|\cdot\|_\gamma$, the operator $\tilde{*} : \mathcal{V}'_i \times \mathcal{V}'_j \rightarrow \mathcal{V}'_{i+j-n}$ extends continuously to a bilinear operator*

$$\begin{aligned} \tilde{*} : \mathcal{V}_i^\gamma \times \mathcal{V}_j^\gamma &\rightarrow \mathcal{V}_{i+j-n}^\gamma \\ (\Phi, \Psi) &\mapsto \Phi \tilde{*} \Psi. \end{aligned}$$

Proof. We first consider the case when $\gamma = \mathcal{C}$. Assume that $\{\phi_k\} \subset \mathcal{V}'_i, \{\psi_k\} \subset \mathcal{V}'_j$ are Cauchy sequences with respect to the norm $\|\cdot\|_{\mathcal{C}}$, and $\phi_k \rightarrow \Phi, \psi_k \rightarrow \Psi$. We show that $\{\phi_k \tilde{*} \psi_k\} \subset \mathcal{V}'_{i+j-n}$ is also a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{C}}$.

As $\{\phi_k\}, \{\psi_k\}$ are Cauchy sequences, by the definition of the cone norm $\|\cdot\|_{\mathcal{C}}$, we have the following properties:

- (1) For any $\varepsilon > 0$ and for all k, l large enough, there exist decompositions

$$\phi_k - \phi_l = \phi^+ - \phi^-, \quad \psi_k - \psi_l = \psi^+ - \psi^-$$

such that $\phi^\pm \in \mathcal{P}_i, \psi^\pm \in \mathcal{P}_j$ and

$$\phi^+(\mathbf{B}) + \phi^-(\mathbf{B}) < \varepsilon, \quad \psi^+(\mathbf{B}) + \psi^-(\mathbf{B}) < \varepsilon.$$

- (2) There exist two decompositions

$$\phi_k = \phi_k^+ - \phi_k^-, \quad \psi_k = \psi_k^+ - \psi_k^-$$

such that $\phi_k^\pm \in \mathcal{P}_i, \psi_k^\pm \in \mathcal{P}_j$ and such that

$$\phi_k^+(\mathbf{B}) + \phi_k^-(\mathbf{B}) \leq C, \quad \psi_k^+(\mathbf{B}) + \psi_k^-(\mathbf{B}) \leq C$$

for a uniform constant $C > 0$.

We write $\phi_k \tilde{*} \psi_k - \phi_l \tilde{*} \psi_l$ as follows:

$$\begin{aligned}
 (7) \quad \phi_k \tilde{*} \psi_k - \phi_l \tilde{*} \psi_l &= \phi_k \tilde{*} (\psi_k - \psi_l) + (\phi_k - \phi_l) \tilde{*} \psi_l \\
 &= (\phi_k^+ - \phi_k^-) \tilde{*} (\psi^+ - \psi^-) + (\phi^+ - \phi^-) \tilde{*} (\psi_k^+ - \psi_k^-) \\
 &= (\phi_k^+ \tilde{*} \psi^+ + \phi_k^- \tilde{*} \psi^- + \phi^+ \tilde{*} \psi_k^+ + \phi^- \tilde{*} \psi_k^-) \\
 &\quad - (\phi_k^+ \tilde{*} \psi^- + \phi_k^- \tilde{*} \psi^+ + \phi^+ \tilde{*} \psi_k^- + \phi^- \tilde{*} \psi_k^+).
 \end{aligned}$$

This is a decomposition of $\phi_k \tilde{*} \psi_k - \phi_l \tilde{*} \psi_l$ as a difference of two elements in \mathcal{P}_{i+j-n} . By Lemma 3.32 applied to each term of (7), we get

$$(8) \quad \|\phi_k \tilde{*} \psi_k - \phi_l \tilde{*} \psi_l\|_{\mathcal{C}} \leq c' C \varepsilon,$$

where c' depends only on $\text{vol}(\mathbf{B}), i, j, n$. Thus $\{\phi_k \tilde{*} \psi_k\}$ must be a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathcal{C}}$.

Next, assume that $\{\phi'_k\}, \{\psi'_k\}$ are another two Cauchy sequences also satisfying $\phi'_k \rightarrow \Phi, \psi'_k \rightarrow \Psi$, we need to verify that the limits of $\{\phi'_k \tilde{*} \psi'_k\}$ and $\{\phi_k \tilde{*} \psi_k\}$ are the same, i.e.,

$$\lim_{k \rightarrow \infty} \|\phi'_k \tilde{*} \psi'_k - \phi_k \tilde{*} \psi_k\|_{\mathcal{C}} = 0.$$

Since $\|\phi'_k - \phi_k\|_{\mathcal{C}} \rightarrow 0$ and $\|\psi'_k - \psi_k\|_{\mathcal{C}} \rightarrow 0$, this follows from similar arguments as above.

In particular, the convolution of Φ, Ψ is defined by the following (well-defined) limit:

$$\Phi \tilde{*} \Psi := \lim_{k \rightarrow \infty} \phi_k \tilde{*} \psi_k \in \mathcal{V}_{i+j-n}^{\mathcal{C}} \subset \text{Val}_{i+j-n}(E).$$

Let us consider the case when $\gamma = \mathcal{P}$. We use the same notations as above. Assume that $\{\phi_k\} \subset \mathcal{P}_i - \mathcal{P}_i, \{\psi_k\} \subset \mathcal{P}_j - \mathcal{P}_j$ are Cauchy sequences with respect to the norm $\|\cdot\|_{\mathcal{P}}$, and $\phi_k \rightarrow \Phi, \psi_k \rightarrow \Psi$. We show that $\{\phi_k \tilde{*} \psi_k\} \subset \mathcal{P}_{i+j-n} - \mathcal{P}_{i+j-n}$ is also a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{P}}$.

As $\{\phi_k\}, \{\psi_k\}$ are Cauchy sequences, by the definition of the positivity norm $\|\cdot\|_{\mathcal{P}}$, we have the following properties:

(1) For any $\varepsilon > 0$ and for all k, l large enough,

$$\begin{aligned}
 -\varepsilon V(\mathbf{B}[n-i]; -) &\leq \phi_k - \phi_l \leq \varepsilon V(\mathbf{B}[n-i]; -) \\
 -\varepsilon V(\mathbf{B}[n-j]; -) &\leq \psi_k - \psi_l \leq \varepsilon V(\mathbf{B}[n-j]; -).
 \end{aligned}$$

(2) There exists $c > 0$ such that for all k we have

$$\begin{aligned}
 -cV(\mathbf{B}[n-i]; -) &\leq \phi_k \leq cV(\mathbf{B}[n-i]; -) \\
 -cV(\mathbf{B}[n-j]; -) &\leq \psi_k \leq cV(\mathbf{B}[n-j]; -)
 \end{aligned}$$

We write $\phi_k \tilde{*} \psi_k - \phi_l \tilde{*} \psi_l$ as follows:

$$\phi_k \tilde{*} \psi_k - \phi_l \tilde{*} \psi_l = \phi_k \tilde{*} (\psi_k - \psi_l) + (\phi_k - \phi_l) \tilde{*} \psi_l.$$

For any $L_1, \dots, L_{i+j-n} \in \mathcal{K}(E)$, as $\phi_k \tilde{*} (\psi_k - \psi_l)(L_1, \dots, L_{i+j-n})$ is computed by an integral, by the above properties it is easy to see that

$$|\phi_k \tilde{*} (\psi_k - \psi_l)(L_1, \dots, L_{i+j-n})| \leq c\varepsilon V(\mathbf{B}[2n-i-j]; L_1, \dots, L_{i+j-n}).$$

Hence, $\|\phi_k \tilde{*} (\psi_k - \psi_l)\|_{\mathcal{P}} \leq c\varepsilon$. Similarly, we also have $\|\psi_k \tilde{*} (\phi_k - \phi_l)\|_{\mathcal{P}} \leq c\varepsilon$.

The same argument shows that the limit

$$\Phi \tilde{*} \Psi := \lim_{k \rightarrow \infty} \phi_k \tilde{*} \psi_k \in \mathcal{V}_{i+j-n}^{\mathcal{P}} \subset \text{Val}_{i+j-n}(E)$$

is well defined, i.e., it is independent of the choices of the Cauchy sequences. \square

Remark 3.34. By Theorem 3.33, the results in Theorem 3.10 and Theorem 3.12 can be extended to valuations in the closure of the cones \mathcal{P}_i , with respect to the norms $\|\cdot\|_{\gamma}$.

4. A VARIANT OF MINKOWSKI'S EXISTENCE THEOREM

By the discussion in Example 3.7, the classical Minkowski's existence theorem shows that every strictly positive element in \mathcal{P}_1 is of the form $V(-; K[n-1])$. In this section, we discuss a generalization of this result, proving Theorem B.

4.1. Existence of the solutions.

Theorem 4.1. *For any strictly positive valuation $\psi \in \mathcal{P}_i$, there is a constant $c > 0$ (depending only on ψ) and a convex body B with $\text{vol}(B) = 1$ such that*

$$\psi \tilde{*} V(B[i-1]; -) = cV(B[n-1]; -) \in \text{Val}_1(E).$$

In the following proof, we denote by ϕ_B the valuation given by $\phi_B = V(B[i-1]; -[n-i+1])$ where B is a convex body.

Given $\psi \in \mathcal{P}_i$, by scaling the convex set B , Theorem 4.1 implies that the functional equation (with unknown $B \in \mathcal{K}(E)$):

$$(\psi - V(B[n-i]; -)) \tilde{*} \phi_B = 0 \in \text{Val}_1(E), \text{ where } \text{vol}(B) > 0$$

always admits a solution.

Proof. The proof is inspired by the method in [LX16]². We consider the following variational problem:

$$c := \inf_{M \in \mathcal{K}(E), \text{vol}(M)=1} \psi(M).$$

Claim 1: Let $\{M_l\}$ be a minimizing sequence, that is, $\text{vol}(M_l) = 1$ and $\psi(M_l) \searrow c$, then we prove that up to some translations, the sequence $\{M_l\}$ is compact in $(\mathcal{K}(E), d_H)$.

Since $\psi \in \mathcal{P}_i$ is strictly positive, there exists an $\epsilon > 0$ such that:

$$\psi(L_1, \dots, L_i) \geq \epsilon V(\mathbf{B}[n-i], L_1, \dots, L_i)$$

for any convex body L_1, \dots, L_i . In particular, one has that

$$V(K[n-i], M[i]) \leq \psi(M)$$

for any convex body M where $K = 1/\epsilon^{n-i} \mathbf{B}$. Then there is a uniform constant $d > 0$ such that

$$V(K[n-i]; M_l[i]) \leq d$$

for the minimizing sequence M_l .

By Alexandrov-Fenchel's inequality, we have

$$V(K[n-i]; M_l[i]) \geq V(K[n-1], M_l)^{n-i/n-1} \text{vol}(M_l)^{i-1/n-1} = V(K[n-1], M_l)^{n-i/n-1},$$

where the last equality follows from $\text{vol}(M_l) = 1$. In particular, $V(K[n-1], M_l)$ is uniformly bounded above. Let $r_l > 0$ be the minimal number such that $M_l \subset r_l K$ (up to a translation). Or equivalently, $1/r_l$ is the maximal number such that $M_l/r_l \subset K$ (up to a translation). By the Diskant inequality,

$$\begin{aligned} 1/r_l &\geq \frac{V(K[n-1], M_l)^{\frac{1}{n-1}} - \left(V(K[n-1], M_l)^{\frac{n}{n-1}} - \text{vol}(K) \text{vol}(M_l)^{\frac{1}{n-1}} \right)^{\frac{1}{n}}}{\text{vol}(M_l)^{\frac{1}{n-1}}} \\ &\geq \frac{\text{vol}(K)}{nV(K[n-1], M_l)}, \end{aligned}$$

where the last inequality follows from the generalized binomial formula (see also [LX17]). We get³

$$(9) \quad r_l \leq nV(K[n-1], M_l) / \text{vol}(K).$$

Thus the sequence r_l is uniformly bounded above. Then Blaschke selection theorem implies that, up to translations, the sequence M_l has an accumulation point $B \in \mathcal{K}(E)$ with $\text{vol}(B) = 1$. In particular,

$$c = \psi(B) = \inf_{M \in \mathcal{K}(E), \text{vol}(M)=1} \psi(M).$$

Claim 2: For any $N \in \mathcal{K}(E)$, we have

$$(10) \quad \frac{n!}{i!(n-i+1)!} \psi \tilde{*} \phi_B(N) - \psi(B)V(B[n-1], N) \geq 0,$$

²It was realized in [LX17] that the same ideas had previously appeared in the classical work of Alexandrov [Ale38].

³This can also be obtained by applying Theorem 3.10.

and

$$(11) \quad \frac{n!}{i!(n-i+1)!} \psi^* \tilde{\phi}_B(B) - \psi(B)V(B[n-1], B) = 0.$$

Note that, since the minimal of the variational problem is achieved at $M = B$, for any $t \geq 0$ and any convex body N , we have

$$\psi \left(\frac{B + tN}{\text{vol}(B + tN)^{1/n}} \right) \geq \psi(B).$$

Calculating the right derivative at $t = 0$ implies

$$\frac{n!}{i!(n-i+1)!} \psi^* \tilde{\phi}_B(N) - \psi(B)V(B[n-1], N) \geq 0.$$

The equality (11) for B follows from the minimal property of B .

Claim 3: There is a convex body L with non-empty interior such that

$$\frac{n!}{i!(n-i+1)!} \psi^* \tilde{\phi}_B(-) = V(L[n-1], -).$$

By the discussion in Example 3.7, this is a direct consequence of Minkowski's existence theorem since $\psi^* \tilde{\phi}_B \in \mathcal{P}_1$ is strictly positive.

Now we can finish the proof of our theorem. By Claim 2 and 3, we have

$$V(L[n-1], N) - \psi(B)V(B[n-1], N) \geq 0$$

for any $N \in \mathcal{K}(E)$. Let $N = L$, we get

$$\text{vol}(L) = V(L[n-1], L) \geq \psi(B)V(B[n-1], L) \geq \psi(B) \text{vol}(B)^{n-1/n} \text{vol}(L)^{1/n}.$$

Thus $\text{vol}(L)^{n-1/n} \geq \psi(B) \text{vol}(B)^{n-1/n}$. On the other hand, let $N = B$, the equality in Claim 2 implies

$$V(L[n-1], B) = \psi(B) \text{vol}(B) \geq \text{vol}(L)^{n-1/n} \text{vol}(B)^{1/n}.$$

Thus $V(L[n-1], B) = \text{vol}(L)^{n-1/n} \text{vol}(B)^{1/n}$, which implies the $L = \psi(B)^{1/n-1} B$. Then we get

$$\frac{n!}{i!(n-i+1)!} \psi^* \tilde{\phi}_B(-) = V(L[n-1], -) = \psi(B)V(B[n-1], -).$$

This finishes the proof of the result. \square

4.2. Compactness of the solution set. In Minkowski's existence theorem, up to some translation, the solution is unique. In the generalized case, we show that the (normalized) solution set of the functional equation (with unknown $B \in \mathcal{K}(E)$)

$$(\psi - V(B[n-i]; -)) \tilde{\phi}_B = 0 \in \text{Val}_1(E), \text{ where } \text{vol}(B) = 1, \phi_B(-) = V(-; B[i-1]),$$

is compact in $(\mathcal{K}(E), d_H)$.

Proposition 4.2. *Given any strictly positive valuation $\psi \in \mathcal{P}_i$, up to translations, the set of normalized solutions of the above equation is compact.*

Proof. Fix a convex body L with non-empty interior. Since $\text{vol}(B) = 1$, similar to the argument in Theorem 4.1, by Blaschke selection theorem and the Diskant inequality it is sufficient to show that $V(B; L[n-1])$ is uniformly bounded above.

To this end, note that

$$V(B[n-1], L) \geq V(B, L[n-1])^{1/n-1} \text{vol}(B)^{n-2/n-1},$$

thus it is sufficient to prove the upper bound for $V(B[n-1], L)$. By the functional equation, we get

$$\frac{n!}{i!(n-i+1)!} (\psi^* \tilde{\phi}_B)(L) = V(B[n-1], L).$$

Assume that ψ is given by the measure μ , then

$$\begin{aligned} \frac{n!}{i!(n-i+1)!}(\psi \tilde{*} \phi_B)(L) &= \int_{\mathcal{K}(E)^{n-i}} V(B[i-1], L; A_1, \dots, A_{n-i}) d\mu(A_1, \dots, A_{n-i}) \\ &\leq cV(B[i-1], L[n-i+1]) \int_{\mathcal{K}(E)^{n-i}} V(L[i], A_1, \dots, A_{n-i}) d\mu(A_1, \dots, A_{n-i}), \end{aligned}$$

where the second inequality follows from Theorem 3.10, and $c > 0$ depends only on $n, i, \text{vol}(L)$. Then it is sufficient to give an upper bound for $V(B[i-1], L[n-i+1])$.

Since ψ is strictly positive,

$$1 = \text{vol}(B) = \frac{n!}{i!(n-i+1)!}(\psi \tilde{*} \phi_B)(B) \geq c'V(L[n-i]; B[i]),$$

thus $V(B[i], L[n-i])$ is uniformly bounded above. On the other hand, since $\text{vol}(B) = 1$, the Alexandrov-Fenchel inequality implies that

$$V(B[i], L[n-i]) \geq V(B[i-1], L[n-i+1])^{n-i/n-i+1}.$$

Thus $V(B[i-1], L[n-i+1])$ is uniformly bounded above, which implies the compactness of the solution set. \square

Remark 4.3. By the above proof, it is clear that the compactness result holds whenever the $\text{vol}(B)$ has a uniformly positive lower bound.

Remark 4.4. Using the same argument as in Theorem 4.1 and Proposition 4.2, one can get the following analogy in complex geometry (see also [LX16, Section 5]).

Let X be a compact Kähler manifold of dimension n . Assume that $\Theta \in H^{k,k}(X, \mathbb{R})$ is a strictly positive (k, k) class in the sense that for some Kähler class ω the class $\Theta - \omega^k$ contains some positive (k, k) current. Let

$$c = \inf_{A \text{ Kähler}, \text{vol}(A)=1} (\Theta \cdot A^{n-k}).$$

Then there is a decomposition

$$\Theta \cdot B^{n-k-1} = cB^{n-1} + \mathcal{N},$$

where B is big and nef satisfying $\text{vol}(B) = 1$, $\mathcal{N} \cdot N \geq 0$ for any nef class N and $\mathcal{N} \cdot B = 0$. Moreover, the set of the (normalized) solutions B is compact.

In particular, if any big nef class is Kähler, we must have $\mathcal{N} = 0$, thus on Kähler manifolds satisfying this condition, for any strictly positive (k, k) class Θ , there is a Kähler class B such that

$$(\Theta - B^k) \cdot B^{n-k-1} = 0.$$

Note that this holds for Abelian varieties and generic hyperkähler manifolds.

Assume that X is a smooth Abelian variety or generic hyperkähler manifold, and assume $2k \leq n$. By Hodge theory, we have the primitive decomposition with respect to the Kähler class B :

$$\Theta - B^k = P_k \oplus B \cdot \Gamma,$$

where $P_k \in H^{k,k}(X, \mathbb{R})$ is the primitive class (i.e., $B^{n-2k+1} \cdot P_k = 0$), and Γ is a $(k-1, k-1)$ class. In particular, if $n = 4, k = 2$, then $(\Theta - B^2) \cdot B = 0$. Hence, *up to a primitive class*, every strictly positive $(2, 2)$ class is equal to B^2 for some Kähler class B .

5. DYNAMICAL DEGREES

5.1. Existence. Recall that $\text{GL}(E)$ has a natural action on $\text{Val}(E)$, which is defined by

$$(g \cdot \phi)(K) = \phi(g^{-1}K).$$

The space $\text{Val}_i(E)$ is fixed by this action. Furthermore, by Example 2.6, the map $\phi \mapsto g \cdot \phi$ maps the positive cone \mathcal{P}_i to \mathcal{P}_i .

Definition 5.1 (Degree). Given $\psi \in \mathcal{P}_i$ and $\phi \in \mathcal{P}_{n-i}$ strictly positive, the $(n-i)$ -th degree of $g \in \text{GL}(E)$ with respect to ϕ, ψ is defined by

$$\text{deg}_{n-i}(g) = (g \cdot \psi) \tilde{*} \phi.$$

We are interested in the sequence $\{\text{deg}_{n-i}(g^p)\}_p$.

Definition 5.2 (Dynamical degree). Given $g \in \text{GL}(E)$, $\psi \in \mathcal{P}_i$ and $\phi \in \mathcal{P}_{n-i}$ strictly positive, the $(n-i)$ -th dynamical degree of g is defined by

$$\begin{aligned} d_{n-i}(g) &:= \lim_{k \rightarrow \infty} \text{deg}_{n-i}(g^k)^{1/k} \\ &= \lim_{k \rightarrow \infty} ((g^k \cdot \psi) \tilde{*} \phi)^{1/k}. \end{aligned}$$

Remark 5.3. In the study of the dynamics of a holomorphic map $f : X \rightarrow X$ where X is a projective variety, one can similarly define a degree:

$$\text{deg}_k(f) = \int_X f^* \omega^k \wedge \omega^{n-k},$$

where ω is a Kähler class on X . Similarly, we can study the asymptotic behaviour of the sequence $\text{deg}_{n-i}(f^k)$, $k \in \mathbb{N}$, and the i -th dynamical degree of f is defined similarly.

Our first fundamental result is that the $(n-i)$ -th dynamical degree exists, that is, the limit defining $d_{n-i}(g)$ exists, and $d_{n-i}(g)$ is independent of the choices of $\psi \in \mathcal{P}_i, \phi \in \mathcal{P}_{n-i}$.

5.1.1. *Sub-multiplicity estimate.* In order to prove the existence of $d_{n-i}(g)$, we first establish the following sub-multiplicity estimate for degrees.

Lemma 5.4. Consider $\phi \in \mathcal{P}_{n-i}$ and $\psi \in \mathcal{P}_i$ are given by

$$\psi(-) = V(-; B[n-i]) \in \mathcal{P}_i, \phi(-) = V(-; B[i]) \in \mathcal{P}_{n-i},$$

where $B \in \mathcal{K}(E)$ has non-empty interior. We consider the $n-i$ -th degree deg_{n-i} given by ϕ, ψ . Assume $f, g \in \text{GL}(E)$, then there is a constant $C > 0$ depending only on $\text{vol}(B), n, i$ such that

$$\text{deg}_{n-i}(fg) \leq C \text{deg}_{n-i}(f) \text{deg}_{n-i}(g).$$

In particular, given $g \in \text{GL}(E)$, the sequence $\{\log(C \text{deg}_{n-i}(g^k))\}_{k=1}^{\infty}$ is subadditive, that is,

$$\log(C \text{deg}_{n-i}(g^{k+l})) \leq \log(C \text{deg}_{n-i}(g^k)) + \log(C \text{deg}_{n-i}(g^l)), \text{ for any } k, l \in \mathbb{N}.$$

Proof. For any convex body B , let us denote by ϕ_B and ψ_B given by $\phi_B = V(B[i], -[n-i])$ and $\psi_B = V(B[n-i], -[i])$.

Since $\text{deg}_{n-i}(-)$ is given by ψ_B and ϕ_B , we get

$$\begin{aligned} \text{deg}_{n-i}(f) \text{deg}_{n-i}(g) &= ((f \cdot \psi_B) \tilde{*} \phi_B) ((g \cdot \psi_B) \tilde{*} \phi_B) \\ (12) \quad &= |\det(fg)|^{-1} (\psi_{f(B)} \tilde{*} \phi_B) (\psi_{g(B)} \tilde{*} \phi_B) \\ &= |\det(fg)|^{-1} |\det f|^{-1} (\psi_{f(B)} \tilde{*} \phi_B) (\psi_{fg(B)} \tilde{*} \phi_{f(B)}). \end{aligned}$$

Note that there exists a constant $c' > 0$ such that $(\psi_{f(B)} \tilde{*} \phi_B) (\psi_{fg(B)} \tilde{*} \phi_{f(B)}) = c' \phi_B(f(B)) \psi_{fg(B)}(f(B))$. By Theorem 3.10, there is a uniform constant $c > 0$ such that

$$\begin{aligned} (\psi_{f(B)} \tilde{*} \phi_B) (\psi_{fg(B)} \tilde{*} \phi_{f(B)}) &\geq c \text{vol}(f(B)) (\psi_{fg(B)} \tilde{*} \phi_B) \\ (13) \quad &= c |\det f| |\det fg| \text{vol}(B) ((fg \cdot \psi_B) \tilde{*} \phi_B) \\ &= c |\det f| |\det fg| \text{vol}(B) \text{deg}_{n-i}(fg). \end{aligned}$$

Thus, (12) and (13) imply that

$$\text{deg}_{n-i}(fg) \leq C \text{deg}_{n-i}(f) \text{deg}_{n-i}(g),$$

where $C = 1/(c \text{vol}(B)) > 0$ and this finishes the proof of the sub-multiplicity estimate. \square

Remark 5.5. In the study of complex dynamics, the analogous estimate for rational self-maps is obtained in [DS05b, DS04] using the theory of positive currents. The above simple proof is inspired by [Dan17].

Lemma 5.6 (Fekete lemma). *For every subadditive sequence $\{a_k\}_{k=1}^\infty$, the limit $\lim_{k \rightarrow \infty} \frac{a_k}{k}$ exists and*

$$\lim_{k \rightarrow \infty} \frac{a_k}{k} = \inf_{k \geq 1} \frac{a_k}{k}.$$

Theorem 5.7. *Given $g \in \mathrm{GL}(E)$, the dynamical degree $d_{n-i}(g)$ exists and is independent of the choices of strictly positive $\psi \in \mathcal{P}_i, \phi \in \mathcal{P}_{n-i}$.*

Proof. If $\psi = V(-; \mathbf{B}[n-i]), \phi = V(-; \mathbf{B}[i])$, the existence of $d_{n-i}(g)$ follows directly from Fekete's lemma.

For the independence on ψ, ϕ , we first note that

$$\psi \preceq \|\psi\|_{\mathcal{P}} V(\mathbf{B}[n-i]; -), \quad \phi \preceq \|\phi\|_{\mathcal{P}} V(\mathbf{B}[i]; -),$$

which follow from the definition of $\|\cdot\|_{\mathcal{P}}$. Applying Lemma 3.22 implies:

$$g^k \cdot \psi \preceq \|\psi\|_{\mathcal{P}} g^k \cdot V(\mathbf{B}[n-i]; -).$$

Moreover, Lemma 3.21 yields:

$$(g^k \cdot \psi) \tilde{*} \phi \leq \|\psi\|_{\mathcal{P}} g^k \cdot V(\mathbf{B}[n-i]; -) \tilde{*} \phi.$$

Then we get:

$$(14) \quad (g^k \cdot \psi) \tilde{*} \phi \leq \|\psi\|_{\mathcal{P}} \|\phi\|_{\mathcal{P}} (g^k \cdot V(\mathbf{B}[n-i]; -)) \tilde{*} V(\mathbf{B}[i]; -).$$

On the other hand, by the strict positivity of ψ, ϕ , there is a constant $C > 0$ depending only on ψ, ϕ such that

$$(15) \quad C(g^k \cdot V(\mathbf{B}[n-i]; -)) \tilde{*} V(\mathbf{B}[i]; -) \leq (g^k \cdot \psi) \tilde{*} \phi.$$

Thus, the inequalities (14), (15) imply that $d_{n-i}(g)$ does not depend on the choices of $\phi \in \mathcal{P}_{n-i}$ and $\psi \in \mathcal{P}_i$. \square

5.1.2. *Norms of linear operators.* Let $g \in \mathrm{GL}(E)$, then by Example 2.6 it induces a linear operator (denoted by g_i):

$$g_i : \mathcal{V}'_i \rightarrow \mathcal{V}'_i.$$

In the following, let $\gamma \in \{\mathcal{C}, \mathcal{P}\}$.

We first show that g_i extends to a map:

$$g_i : \mathcal{V}_i^\gamma \rightarrow \mathcal{V}_i^\gamma.$$

Lemma 5.8. *Let $g \in \mathrm{GL}(E)$. Assume that $\|\phi_k - \phi\|_\gamma \rightarrow 0$, then $\|g \cdot \phi_k - g \cdot \phi\|_\gamma \rightarrow 0$.*

Proof. For the norm $\|\cdot\|_{\mathcal{C}}$, it is obvious.

We only need to deal with the norm $\|\cdot\|_{\mathcal{P}}$. By definition, we have

$$|(\phi_k - \phi)(L_1, \dots, L_i)| \leq \|\phi_k - \phi\|_{\mathcal{P}} V(\mathbf{B}[n-i]; L_1, \dots, L_i),$$

which implies

$$\begin{aligned} |g \cdot (\phi_k - \phi)(L_1, \dots, L_i)| &\leq \|\phi_k - \phi\|_{\mathcal{P}} V(\mathbf{B}[n-i]; g^{-1}(L_1), \dots, g^{-1}(L_i)) \\ &= \|\phi_k - \phi\|_{\mathcal{P}} \frac{1}{|\det g|} V(g(\mathbf{B})[n-i]; L_1, \dots, L_i). \end{aligned}$$

On the other hand, by Theorem 3.10 we have

$$V(g(\mathbf{B})[n-i]; L_1, \dots, L_i) \leq cV(g(\mathbf{B})[n-i]; \mathbf{B}[i])V(\mathbf{B}[n-i]; L_1, \dots, L_i),$$

where $c > 0$ depends only on $n, i, \mathrm{vol}(\mathbf{B})$. Hence,

$$\|g \cdot (\phi_k - \phi)\|_{\mathcal{P}} \leq c \frac{1}{|\det g|} V(g(\mathbf{B})[n-i]; \mathbf{B}[i]) \|\phi_k - \phi\|_{\mathcal{P}}.$$

This finishes the proof of the result. \square

Next we show that the dynamical degree $d_{n-i}(g)$ is just the spectral radius of this operator.

Theorem 5.9. *Let $g \in \mathrm{GL}(E)$ and let g_i be the induced operator on \mathcal{V}_i^γ , then the following equality is satisfied:*

$$d_{n-i}(g) = \|g_{n-i} : \mathcal{V}_i^{\mathcal{P}} \rightarrow \mathcal{V}_i^{\mathcal{P}}\| = \|g_{n-i} : \mathcal{V}_i^{\mathcal{C}} \rightarrow \mathcal{V}_i^{\mathcal{C}}\|,$$

where the symbol $\|g_{n-i} : \mathcal{V}_i^{\mathcal{P}} \rightarrow \mathcal{V}_i^{\mathcal{P}}\|$ and $\|g_{n-i} : \mathcal{V}_i^{\mathcal{C}} \rightarrow \mathcal{V}_i^{\mathcal{C}}\|$ denotes the norm of the operator g_{n-i} on $\mathcal{V}_i^{\mathcal{P}}$ and $\mathcal{V}_i^{\mathcal{C}}$ respectively.

Proof. For simplicity, since each space is endowed with its appropriate norm, we denote by $\|g_{n-i}\|_{\mathcal{P}}$ and $\|g_{n-i}\|_{\mathcal{C}}$ the norm of the operator g_{n-i} on $\mathcal{V}_i^{\mathcal{P}}$ and $\mathcal{V}_i^{\mathcal{C}}$ respectively. We need to verify the equality

$$d_{n-i}(g) = \lim_{k \rightarrow \infty} \|g_i^k\|_{\gamma}^{1/k}.$$

We first consider the case when $\gamma = \mathcal{C}$.

Let $\phi_{\mathbf{B}} = V(\mathbf{B}[n-i]; -)$, by definition we get

$$\begin{aligned} \|g^k \cdot \phi_{\mathbf{B}}\|_{\mathcal{C}} &= (g^k \cdot \phi_{\mathbf{B}})(\mathbf{B}) = V(g^k(\mathbf{B})[n-i], \mathbf{B}[i]) / |\det g|^k, \\ \|\phi_{\mathbf{B}}\|_{\mathcal{C}} &= \phi_{\mathbf{B}}(\mathbf{B}) = V(\mathbf{B}[n-i], \mathbf{B}[i]). \end{aligned}$$

This implies that

$$(16) \quad \|g_i^k\|_{\mathcal{C}} \geq \frac{V(g^k(\mathbf{B})[n-i], \mathbf{B}[i])}{|\det g|^k \mathrm{vol}(\mathbf{B})}.$$

On the other hand, take a sequence $\phi_l \in \mathcal{P}_i - \mathcal{P}_i$ such that $\|\phi_l\|_{\mathcal{C}} = 1$ and $\|g^k \cdot \phi_l\|_{\mathcal{C}} \rightarrow \|g_i^k\|_{\mathcal{C}}$ as $l \rightarrow \infty$. For l_0 large enough, we have $\|g_i^k\|_{\mathcal{C}} \leq 2\|g^k \cdot \phi_{l_0}\|_{\mathcal{C}}$. Assume that $\phi_{l_0} = \phi_{l_0}^+ - \phi_{l_0}^-$ is a decomposition for ϕ_{l_0} , then

$$\|g_i^k\|_{\mathcal{C}} \leq 2(g^k \cdot \phi_{l_0}^+(\mathbf{B}) + g^k \cdot \phi_{l_0}^-(\mathbf{B})).$$

For the term $g^k \cdot \phi_{l_0}^+(\mathbf{B})$, by Theorem 3.10 there is a constant $c > 0$ depending only on $n, i, \mathrm{vol}(\mathbf{B})$ such that

$$\begin{aligned} g^k \cdot \phi_{l_0}^+(\mathbf{B}) &= \int_{\mathcal{K}(E)^{n-i}} V(g^{-k}(\mathbf{B})[i], A_1, \dots, A_{n-i}) d\mu_{l_0}^+(A_1, \dots, A_{n-i}) \\ &\leq cV(g^{-k}(\mathbf{B})[i], \mathbf{B}[n-i]) \int_{\mathcal{K}(E)^{n-i}} V(\mathbf{B}[i], A_1, \dots, A_{n-i}) d\mu_{l_0}^+(A_1, \dots, A_{n-i}) \\ &= cV(g^{-k}(\mathbf{B})[i], \mathbf{B}[n-i]) \phi_{l_0}^+(\mathbf{B}). \end{aligned}$$

Similarly,

$$(17) \quad g^k \cdot \phi_{l_0}^-(\mathbf{B}) \leq cV(g^{-k}(\mathbf{B})[i], \mathbf{B}[n-i]) \phi_{l_0}^-(\mathbf{B}).$$

Since $\|\phi_{l_0}\|_{\mathcal{C}} = 1$, we get

$$(18) \quad \|g_i^k\|_{\mathcal{C}} \leq 2cV(g^{-k}(\mathbf{B})[i], \mathbf{B}[n-i]).$$

Next we consider the case when $\gamma = \mathcal{P}$. Note that $\|\phi_{\mathbf{B}}\|_{\mathcal{P}} = 1$. By the definition for $\|g^k \cdot \phi_{\mathbf{B}}\|_{\mathcal{P}}$, we have $g^k \cdot \phi_{\mathbf{B}}(\mathbf{B}) \leq \|g^k \cdot \phi_{\mathbf{B}}\|_{\mathcal{P}} \mathrm{vol}(\mathbf{B})$, hence

$$\|g^k \cdot \phi_{\mathbf{B}}\|_{\mathcal{P}} \geq V(\mathbf{B}[n-i], g^{-k}(\mathbf{B})[i]) / \mathrm{vol}(\mathbf{B}).$$

This implies that

$$(19) \quad \|g_i^k\|_{\mathcal{P}} \geq \frac{V(g^k(\mathbf{B})[n-i], \mathbf{B}[i])}{|\det g|^k \mathrm{vol}(\mathbf{B})}.$$

On the other hand, take a sequence ϕ_l such that $\|\phi_l\|_{\mathcal{P}} = 1$ and $\|g^k \cdot \phi_l\|_{\mathcal{P}} \rightarrow \|g_i^k\|_{\mathcal{P}}$ as $l \rightarrow \infty$. For l_0 large enough, we have $\|g_i^k\|_{\mathcal{P}} \leq 2\|g^k \cdot \phi_{l_0}\|_{\mathcal{P}}$. For any L_1, \dots, L_i ,

$$\begin{aligned} |g^k \cdot \phi_{l_0}(L_1, \dots, L_i)| &= |\phi_{l_0}(g^{-k}(L_1), \dots, g^{-k}(L_i))| \\ &\leq \|\phi_{l_0}\|_{\mathcal{P}} V(\mathbf{B}[n-i], g^{-k}(L_1), \dots, g^{-k}(L_i)). \end{aligned}$$

Applying $\|\phi_{l_0}\|_{\mathcal{P}} = 1$ and Theorem 3.10 yields a uniform constant $c' > 0$ such that

$$(20) \quad \|g_i^k\|_{\mathcal{P}} \leq c'V(g^k(\mathbf{B})[n-i], \mathbf{B}[i]) / |\det g|^k.$$

In summary, by (16), (18), (19), (20) and taking the limits, we obtain the desired equality

$$d_{n-i}(g) = \lim_{k \rightarrow \infty} \|g_i^k\|_\gamma^{1/k}.$$

□

5.1.3. *Log-concavity.* By Theorem 5.7, the definition of $d_{n-i}(g)$ is independent of the choices of ψ, ϕ . A direct consequence of this result is the following:

Proposition 5.10. *For any $g \in \mathrm{GL}(E)$, the sequence $\{d_i(g)\}$ is log-concave, that is, for $1 \leq i \leq n-1$*

$$d_i(g)^2 \geq d_{i-1}(g)d_{i+1}(g).$$

Proof. By Theorem 5.7, we get

$$\begin{aligned} d_i(g) &= \lim_{k \rightarrow \infty} (|\det g^k|^{-1} V(g^k(B)[i], B[n-i]))^{1/k} \\ &= |\det g|^{-1} \lim_{k \rightarrow \infty} (V(g^k(B)[i], B[n-i]))^{1/k}, \end{aligned}$$

where B is a fixed convex body with non-empty interior. Then the log-concavity property follows immediately from the Alexandrov-Fenchel inequality for mixed volumes. □

5.1.4. *Relative version.* In the study of dynamics of a holomorphic map that preserves some fibration, it is useful to consider a relative version of dynamical degrees. We have a corresponding picture for convex valuations. Let S be a subspace of dimension m , and assume that $l : S \rightarrow E$ is the embedding. Assume that $g \in \mathrm{GL}(E)$ fixes the subspace S , equivalently, there is a map $f \in \mathrm{GL}(S)$ such that $g \circ l = l \circ f$.

Definition 5.11. Assume that $\psi \in \mathcal{P}_i(E), \phi \in \mathcal{P}_{n-i+m}(E)$ are strictly positive, and let $\tau_B = V(-; B[m]) \in \mathrm{Val}_{n-m}(E)$, where $B \in \mathcal{K}(S)$ satisfies $\mathrm{vol}_S(B) > 0$, then the $(n-i)$ -th relative degree of g is defined by

$$\mathrm{reldeg}_{n-i}(g) = (g \cdot \psi) \tilde{*} \phi \tilde{*} \tau_B.$$

Definition 5.12. The $(n-i)$ -th relative dynamical degree of g is defined by

$$\mathrm{reld}_{n-i}(g) = \lim_{k \rightarrow \infty} (\mathrm{reldeg}_{n-i}(g^k))^{1/k}.$$

Similar to Theorem 5.7, we have:

Theorem 5.13. *The relative dynamical degree $\mathrm{reld}_{n-i}(g)$ exists and is independent of the choices of $\psi \in \mathcal{P}_i, \phi \in \mathcal{P}_{n-i}$ (which are strictly positive), and $B \in \mathcal{K}(S)$ with non-empty interior.*

Proof. The proof is similar to Theorem 5.7, so we omit the details. The only extra ingredient is the following reduction formula for mixed volumes (see [Sch14, Theorem 5.3.1]).

Lemma 5.14. *Let k be an integer satisfying $1 \leq k \leq n-1$, let $H \subset \mathbb{R}^n$ be a k -dimensional linear subspace and let $L_1, \dots, L_k, K_1, \dots, K_{n-k}$ be convex bodies with $L_i \subset H$ for $i = 1, \dots, k$. Then*

$$\binom{n}{k} V(L_1, \dots, L_k, K_1, \dots, K_{n-k}) = V_H(L_1, \dots, L_k) V_{H^\perp}(p_{H^\perp}(K_1), \dots, p_{H^\perp}(K_{n-k})),$$

where $V_H(\cdot)$ and $V_{H^\perp}(\cdot)$ denote the mixed volume in H and H^\perp , and $p_{H^\perp} : \mathbb{R}^n \rightarrow H^\perp$ is the projection map.

□

Remark 5.15. Similar to the complex geometry setting (see e.g. [DN11], [Dan17]), one could also establish a product formula between the dynamical degrees and the relative dynamical degrees.

5.2. Evaluation of dynamical degrees. In this section, we give a formula for $d_{n-i}(g)$ using the eigenvalues of g . The key point is the formula

$$d_{n-i}(g) = |\det g|^{-1} \lim_{k \rightarrow \infty} (V(g^k(B)[n-i], B[i]))^{1/k}.$$

Theorem 5.16. *Let $g \in \mathrm{GL}(E)$, and assume that ρ_1, \dots, ρ_n are the eigenvalues of g satisfying*

$$|\rho_1| \geq |\rho_2| \geq \dots \geq |\rho_n|,$$

then the $(n-i)$ -th dynamical degree $d_{n-i}(g) = |\det g|^{-1} \prod_{k=1}^{n-i} |\rho_k|$.

It is clear that we only need to check the equality

$$\widehat{d}_{n-i}(g) := \lim_{k \rightarrow \infty} (V(g^k(B)[n-i], B[i]))^{1/k} = \prod_{k=1}^{n-i} |\rho_k|.$$

Remark 5.17. In the study of dynamics of monomial maps, the above formula was first obtained in [Lin12, FW12]. The proof of [Lin12] is algebraic, and the proof of [FW12] applies some ideas from integral geometry. We present a different (and simpler) approach to the calculation of $d_i(g)$, by using positivity results.

5.2.1. Simple case: $d_1(g)$. We first discuss the simple calculation for $d_1(g)$. We need to verify the formula

$$\lim_{k \rightarrow \infty} V(g^k(B), B[n-1])^{1/k} = |\rho_1(g)|.$$

By Theorem 5.7, for any $L, M \in \mathcal{K}(E)$ with non-empty interior, we have

$$d_1(g) = |\det g|^{-1} \lim_{k \rightarrow \infty} V(g^k(L), M[n-1])^{1/k}.$$

First, we prove $d_1(g) \leq |\det g|^{-1} |\rho_1(g)|$. To this end, we fix a Euclidean structure on E and assume that $0 \in L$ is an interior point. Then for any point $x \in \partial L$ we have $|g(x)| \leq \|g\| \|x\|$, thus

$$g(L) \subset c \|g\| \mathbf{B}$$

where \mathbf{B} is the unit ball and $c = \max_{x \in \partial L} |x|$. In particular, applying the observation to g^k implies

$$g^k(L) \subset c \|g^k\| \mathbf{B}.$$

Thus,

$$\begin{aligned} d_1(g) &\leq |\det g|^{-1} \lim_{k \rightarrow \infty} (\|g^k\|^{1/k} V(c \mathbf{B}, M[n-1]))^{1/k} \\ &= |\det g|^{-1} |\rho_1(g)|. \end{aligned}$$

Next, we prove the reverse inequality $d_1(g) \geq |\det g|^{-1} |\rho_1(g)|$. For any k , we can take a unit vector x_k such that $|g^k(x_k)| = \|g^k\|$. We take $L = 2\mathbf{B}$ and take $M = \mathbf{B}$. Then the segment $S_k := [0, x_k] \subset L$, yielding

$$V(g^k(S_k), M[n-1]) \leq V(g^k(L), M[n-1]).$$

Note that

$$V(g^k(S_k), M[n-1]) = \|g^k\| V(\|g^k\|^{-1} g^k(S_k), M[n-1]).$$

Since $\|g^k\|^{-1} g^k(S_k)$ is a unit vector, Lemma 5.14 implies

$$V(\|g^k\|^{-1} g^k(S_k), M[n-1]) = n^{-1} V_{g^k(S_k)^\perp}(M).$$

Since $M = \mathbf{B}$, the volume $V_{g^k(S_k)^\perp}(M)$ is a constant, thus

$$V(g^k(S_k), M[n-1]) = c \|g^k\|.$$

Then taking the limit implies

$$\begin{aligned} d_1(g) &= |\det g|^{-1} \lim_{k \rightarrow \infty} V(g^k(L), M[n-1])^{1/k} \\ &\geq |\det g|^{-1} \lim_{k \rightarrow \infty} (c \|g^k\|)^{1/k} = |\det g|^{-1} |\rho_1(g)|. \end{aligned}$$

In summary, we get the formula $d_1(g) = |\det g|^{-1} |\rho_1(g)|$.

5.2.2. *General case.* For the general case, the idea is as follows:

- (1) Prove the formula for diagonalizable matrices over \mathbb{C} with distinct eigenvalues;
- (2) Show that $d_{n-i}(\cdot)$ is a continuous function over $\mathrm{GL}(E)$;
- (3) For an arbitrary $g \in \mathrm{GL}(E)$, approximate g using diagonalizable matrices over \mathbb{C} with distinct eigenvalues and apply the continuity of $d_{n-i}(\cdot)$.

Lemma 5.18. *Assume $g \in \mathrm{GL}(E)$ is diagonalizable over \mathbb{C} , and assume g has distinct eigenvalues. Then $\widehat{d}_k(g) = \prod_{i=1}^k |\rho_i(g)|$.*

Proof. After a change of basis, we could assume that the matrix form of g takes its real Jordan canonical form. Since g has distinct eigenvalues, its real Jordan canonical form can be written as

$$\begin{pmatrix} J_1 & & & & & & \\ & \ddots & & & & & \\ & & J_s & & & & \\ & & & \lambda_{s+1} & & & \\ & & & & \ddots & & \\ & & & & & & \lambda_n \end{pmatrix},$$

where $J_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ corresponds to the non-real eigenvalue $\lambda_i = a_i + \sqrt{-1}b_i$, and $\lambda_{s+1}, \dots, \lambda_n$ are real eigenvalues.

In order to calculate the dynamical degree of g , we consider the following convex body

$$K_{\mathbf{r}} = D_{r_1} \times \dots \times D_{r_s} \times I_{r_{s+1}} \times \dots \times I_{r_n},$$

where D_{r_i} is a disk of radius r_i , and I_{r_j} is a segment of length r_j with 0 as its center.

For any $\gamma, \tau \geq 0$, we have $\gamma K_{\mathbf{r}} + \tau K_{\mathbf{t}} = K_{\gamma \mathbf{r} + \tau \mathbf{t}}$. In particular, this gives an explicit formula for $\mathrm{vol}(\gamma K_{\mathbf{r}} + \tau K_{\mathbf{t}})$. On the other hand, note that

$$\mathrm{vol}(K_{\gamma \mathbf{r} + \tau \mathbf{t}}) = \mathrm{vol}(\gamma K_{\mathbf{r}} + \tau K_{\mathbf{t}}) = \sum_k \frac{n!}{k!(n-k)!} V(K_{\mathbf{r}}[k], K_{\mathbf{t}}[n-k]) \gamma^k \tau^{n-k}.$$

By comparing the coefficients, we get the explicit formula for $V(K_{\mathbf{r}}^k, K_{\mathbf{t}}^{n-k})$ for any \mathbf{r}, \mathbf{t} . Here, we omit the detailed computations.

Next we take $\mathbf{r} = \mathbf{t} = (1, \dots, 1)$ and compute $V(g^p(K_{\mathbf{1}})[k], K_{\mathbf{1}}[n-k])$. To this end, we note that

$$g^p(K_{\mathbf{1}}) = K_{\mathbf{r}_p},$$

where $\mathbf{r}_p = (|\lambda_1|^p, \dots, |\lambda_s|^p, |\lambda_{s+1}|^p, \dots, |\lambda_n|^p)$. Then a direct computation shows that

$$\widehat{d}_k(g) = \prod_{i=1}^k |\rho_i(g)|.$$

□

Remark 5.19. The calculations in Lemma 5.18 are inspired by the calculations in [FW12, Section 5.1], where the authors did the computations for diagonalizable maps over \mathbb{R} and also gave a remark for diagonalizable maps over \mathbb{C} .

Next we show that the dynamical degree function

$$d_k : \mathrm{GL}(E) \rightarrow \mathbb{R}, \quad g \mapsto d_k(g)$$

is continuous.

Theorem 5.20. *The dynamical degree $d_k(\cdot)$ is a continuous function on $\mathrm{GL}(E)$. More precisely, let $\{g_l\}_{l \geq 1}, g \in \mathrm{GL}(E)$ endowed with the topology induced by the L^2 -norm of $E \times E$, then*

$$\lim_{g_l \rightarrow g} d_k(g_l) = d_k(g).$$

Proof. It is sufficient to prove that

$$\lim_{g_l \rightarrow g} \widehat{d}_k(g_l) = \widehat{d}_k(g).$$

By Theorem 5.7, the dynamical degree is independent of the choices of ϕ, ψ . In the following we take $K = \mathbf{B}$ to be the unit ball with 0 as its center. We have

$$\widehat{d}_k(g) = \lim_{k \rightarrow \infty} V(g^p(K)[k], K[n-k])^{1/p}.$$

We first prove $\lim_{l \rightarrow \infty} \widehat{d}_k(g_l) \geq \widehat{d}_k(g)$. We consider the *inradius* of $g_l^p(K)$ relative to $g^p(K)$, which is defined by

$$r(g_l^p(K), g^p(K)) := \max\{\lambda > 0 \mid \lambda g^p(K) \subset g_l^p(K) \text{ up to some translation}\}.$$

Applying the Diskant inequality to $g_l^p(K), g^p(K)$, we get

$$r(g_l^p(K), g^p(K)) \geq \frac{\text{vol}(g_l^p(K))}{nV(g_l^p(K)[n-1], g^p(K))}.$$

We next estimate the mixed volume $V(g_l^p(K)[n-1], g^p(K))$. Note that

$$\begin{aligned} V(g_l^p(K)[n-1], g^p(K)) &= |\det(g_l)|^p V(K[n-1], (g_l^{-p} \circ g^p)(K)) \\ &= |\det(g_l)|^p \int_{\mathbb{S}^{n-1}} h_{(g_l^{-p} \circ g^p)(K)}(x) dS(K^{n-1}; x), \end{aligned}$$

where $h_{(g_l^{-p} \circ g^p)(K)}$ is the support function of the convex body $(g_l^{-p} \circ g^p)(K)$, and $dS(K^{n-1}; \cdot)$ is the surface area measure. For any linear map $A : E \rightarrow E$, by the definition of support function we have

$$h_{A(K)}(x) = \max\{x \cdot y \mid y \in A(K)\} = \max\{A^T x \cdot y \mid y \in K\}.$$

Thus $h_{A(K)}(x) = h_K(A^T x)$. Since $K = \mathbf{B}$, we have $h_K(x) = h_{\mathbf{B}}(x) = |x|$. Then we get

$$\begin{aligned} h_{(g_l^{-p} \circ g^p)(K)}(x) &= h_K((g_l^{-p} \circ g^p)^T x) = |(g_l^{-p} \circ g^p)^T x| \\ &\leq \|g_l^{-p} \circ g^p\| |x| = \|g_l^{-p} \circ g^p\| h_K(x). \end{aligned}$$

Applying the above inequality to $V(g_l^p(K)[n-1], g^p(K))$ implies

$$V(g_l^p(K)[n-1], g^p(K)) \leq |\det(g_l)|^p (\|g_l^{-p} \circ g^p\|) \text{vol}(K).$$

Then we have

$$\begin{aligned} V(g_l^p(K)[k], K[n-k])^{1/p} &\geq r(g_l^p(K), g^p(K))^{k/p} V(g^p(K)[k], K[n-k])^{1/p} \\ &\geq \left(\frac{\text{vol}(g_l^p(K))}{nV(g_l^p(K)[n-1], g^p(K))} \right)^{k/p} V(g^p(K)[k], K[n-k])^{1/p} \\ &\geq \left(\frac{|\det(g_l)|^p \text{vol}(K)}{n|\det(g_l)|^p (\|g_l^{-p} \circ g^p\|) \text{vol}(K)} \right)^{k/p} V(g^p(K)[k], K[n-k])^{1/p} \\ &= (\|g_l^{-p} \circ g^p\|^{1/p})^{-k} n^{-k/p} V(g^p(K)[k], K[n-k])^{1/p}. \end{aligned}$$

Lemma 5.21. *For any sequence g_l converging to g , we have*

$$\lim_{l \rightarrow +\infty} \lim_{p \rightarrow +\infty} \|g_l^{-p} \circ g^p\|^{1/p} \leq 1.$$

Proof. We only need to consider the action of $g_l^{-p} \circ g^p$ on invariant subspaces. Assume that $\|x\| = 1$ and $x \in \ker(g - \lambda I)^b$, where b is the multiplicity of the eigenvalue λ . By assumption, we have that:

$$g^p(x) \in \ker(g - \lambda I)^b.$$

By considering the Jordan form of g , there exists a constant $C > 0$ (independent of x , as $\|x\| = 1$) such that:

$$\|g^p(x)\| \leq Cp^b |\lambda|^p.$$

Since g_l converges to g , $g^p(x)$ is in the union of invariant subspaces of g_l which correspond to the eigenvalues converging to λ . Thus for any fixed $\delta > 0$, there exists l_δ such that when $l \geq l_\delta$, we have

$$\|g_l^{-p} \circ g^p(x)\| \leq C' p^{b'} (|\lambda| - \delta)^{-p} |\lambda|^p,$$

where C', b' are uniform constants by considering Jordan forms. Taking the limits gives

$$\lim_{l \rightarrow +\infty} \lim_{p \rightarrow +\infty} \|g_l^{-p} \circ g^p\|^{1/p} \leq 1.$$

□

Using the above lemma, we get $\widehat{d}_k(g) \leq \lim_{l \rightarrow \infty} \widehat{d}_k(g_l)$.

Similarly, by studying the inradius of $g^p(K)$ relative to $g_l^p(K)$, we get $\widehat{d}_k(g) \geq \lim_{l \rightarrow \infty} \widehat{d}_k(g_l)$. This finishes the proof of the continuity. □

Remark 5.22. The complex analog of Theorem 5.20 implies the following interesting continuity result for dynamical degrees of holomorphic maps:

Let X be a compact Kähler manifold of dimension n . Assume that f_l, f are dominated holomorphic self-maps of X , and assume that the induced actions

$$f_l^*, f^* : H^{1,1}(X, \mathbb{R}) \rightarrow H^{1,1}(X, \mathbb{R})$$

satisfy $\lim_{l \rightarrow \infty} f_l^* = f^*$, then $\lim_{l \rightarrow \infty} d_k(f_l) = d_k(f)$ holds for any k .

To our knowledge, the previous result is that: if the induced actions on $H^{k,k}(X, \mathbb{R})$ satisfies $\lim_{l \rightarrow \infty} f_l^* = f^*$, then $\lim_{l \rightarrow \infty} d_k(f_l) = d_k(f)$.

Now we can finish the proof of Theorem 5.16.

Proof of Theorem 5.16. It is sufficient to prove $\widehat{d}_k(g) = \prod_{i=1}^k |\rho_i(g)|$. Assume that $f \in \text{GL}(E)$ is diagonalizable over \mathbb{C} and has distinct eigenvalues. For any fixed $g \in \text{GL}(E)$, we consider the path

$$g_t := (1-t)f + tg.$$

By linear algebra (see e.g. [Har95]), there is a sequence g_l such that each g_l has distinct eigenvalues (thus it is diagonalizable over \mathbb{C}), and $\lim_{l \rightarrow \infty} g_l = g$. (Note that this density statement is not true for diagonalizable matrices over \mathbb{R} .)

Since the eigenvalues depend continuously on the entries of a matrix, we get $\lim_{l \rightarrow \infty} |\rho_i(g_l)| = |\rho_i(g)|$. Applying $\widehat{d}_k(g_l) = \prod_{i=1}^k |\rho_i(g_l)|$ and Theorem 5.20 yields

$$\widehat{d}_k(g) = \lim_{l \rightarrow \infty} \widehat{d}_k(g_l) = \lim_{l \rightarrow \infty} \prod_{i=1}^k |\rho_i(g_l)| = \prod_{i=1}^k |\rho_i(g)|.$$

□

5.3. A generalization: multiple dynamical degrees. Actually we can show the existence of some kind of dynamical degrees of multiple linear actions. By the previous discussions for dynamical degrees, for simplicity, we only consider valuations of the type $V(-; B[i])$.

Definition 5.23. Let $g_1, g_2, \dots, g_k \in \text{GL}(E)$, and let $B \in \mathcal{K}(E)$ be a convex body with non-empty interior. Then we define the degree $\text{deg}(g_1, \dots, g_k)$ as

$$\text{deg}(g_1, \dots, g_k) = (g_1 \cdot \psi_B) \tilde{*} \dots \tilde{*} (g_k \cdot \psi_B) \tilde{*} \phi_B,$$

where $\psi_B = V(-; B)$ and $\phi_B = V(-; B[n-k])$. In particular, if $g_1 = \dots = g_k = g$, then we get the k -th degree $\text{deg}_k(g)$ (up to some scaling).

Proposition 5.24. *If we define the dynamical degree of g_1, \dots, g_k as*

$$d(g_1, \dots, g_k) := \limsup_{p \rightarrow \infty} \text{deg}(g_1^p, \dots, g_k^p)^{1/p},$$

then $d(g_1, \dots, g_k)$ exists and does not depend on the choices of B . Moreover, $d(g_1, \dots, g_k)$ is bounded above by $\prod_{i=1}^k d_k(g_i)$.

Proof. This follows directly from Theorem 3.12. □

5.3.1. *An application to Laurent system.* We give an application to the solution set of a Laurent system. First recall the famous Bernstein-Khovanskii-Kushnirenko theorem (see e.g. [Ber75], [Kho78], [Kus76]). Let $V = \mathbb{R}^n$. We identify \mathbb{Z}^n with the Laurent monomials, i.e., to each integral point $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we associate the monomial $x^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$. A Laurent polynomial $P(x) = \sum c_{\mathbf{a}} x^{\mathbf{a}}$ is a finite linear combination of Laurent monomials with coefficients $c_{\mathbf{a}} \in \mathbb{C}$. The support of a Laurent polynomial $P(x) = \sum c_{\mathbf{a}} x^{\mathbf{a}}$ is defined as

$$\text{supp}(P) := \{\mathbf{a} \in \mathbb{Z}^n \mid c_{\mathbf{a}} \neq 0\}.$$

We denote the convex hull of a finite set $I \subset \mathbb{Z}^n$ by $\Delta_I \subset V$. For each finite set $I \subset \mathbb{Z}^n$, we associate the linear subspace of Laurent polynomials: $L_I = \{P \mid \text{supp}(P) \subset I\}$.

Theorem 5.25 (Bernstein-Khovanskii-Kushnirenko theorem). *Let I_1, \dots, I_n be finite sets of \mathbb{Z}^n . Let $N(I_1, \dots, I_n)$ be the number of solutions in $(\mathbb{C}^*)^n$ of a general system of Laurent polynomial equations $P_1 = P_2 = \dots = P_n = 0$ with $P_i \in L_{I_i}$, then*

$$N(I_1, \dots, I_n) = n!V(\Delta_{I_1}, \dots, \Delta_{I_n}).$$

The group $\text{GL}(n, \mathbb{Z})$ has a natural action on \mathbb{Z}^n , which in turn induces an action on the Laurent polynomials:

$$P(x) = \sum c_{\mathbf{a}} x^{\mathbf{a}} \mapsto (g \cdot P)(x) := \sum c_{\mathbf{a}} x^{g(\mathbf{a})},$$

where $g \in \text{GL}(n, \mathbb{Z})$. It is natural to ask the asymptotic behaviour of the number of solutions under this induced action. Note that we have $g(\Delta_I) = \Delta_{g(I)}$.

Proposition 5.26. *Let I_1, \dots, I_n be finite sets of \mathbb{Z}^n , and let $g_i \in \text{GL}(n, \mathbb{Z})$ with $1 \leq i \leq k$. Let $N(p, g_1, \dots, g_k)$ be the number of solutions in $(\mathbb{C}^*)^n$ of a general system of Laurent polynomial equations $P_1 = P_2 = \dots = P_n = 0$ with $P_i \in L_{g_i^p(I_i)}$ for $i \leq k$ and $P_j \in L_{I_j}$ for $j \geq k+1$, then the limit*

$$\limsup_{p \rightarrow +\infty} \frac{1}{p} \log N(p, g_1, \dots, g_k)$$

exists. In particular, the function $N(\cdot, g_1, \dots, g_k)$ defined over positive integers has polynomial growth.

Proof. Fix a convex body $L \subset V$ with non-empty interior. Then there exists a constant $c > 0$ such that $\Delta_{I_i} \subset cL$ (up to some translation) for any i . This implies

$$\begin{aligned} N(p, g_1, \dots, g_k) &= n!V(\Delta_{g_1^p(I_1)}, \dots, \Delta_{g_k^p(I_k)}, \Delta_{I_{k+1}}, \dots, \Delta_{I_n}) \\ &= n!V(g_1^p(\Delta_{I_1}), \dots, g_k^p(\Delta_{I_k}), \Delta_{I_{k+1}}, \dots, \Delta_{I_n}) \\ &\leq n!c^n V(g_1^p(L), \dots, g_k^p(L), L[n-k]). \end{aligned}$$

Applying Proposition 5.24 gives the desired result. \square

Remark 5.27. In the complex geometry setting, for holomorphic self-maps of a compact Kähler manifold, the multiple dynamical degrees control how the multiple maps separate the orbits.

6. POSITIVITY OF INVARIANT CONVEX VALUATIONS

In this section, we focus on the space $\text{Val}(E)$ and the positive cones defined in this space.

Let $\phi_{\mu} \in \mathcal{V}'_{n-i}$, recall that the action of $g \in \text{GL}(E)$ on ϕ_{μ} (see Example 2.6) is given by

$$g \cdot \phi_{\mu} = \frac{1}{|\det g|} \phi_{g \cdot \mu}.$$

6.1. Invariant classes in complex dynamics. To motivate the discussions, we first recall some facts from complex dynamics. Let X be a compact Kähler manifold of dimension n , and let $f \in \text{Aut}(X)$ be a holomorphic automorphism of X . Positive invariant classes and invariant currents play an important role in the study of dynamics of f . We consider the following positive cone in $H^{k,k}(X, \mathbb{R})$:

$$\mathcal{P}_k = \{\{\Theta\} \in H^{k,k}(X, \mathbb{R}) \mid \Theta \text{ is a smooth positive } (k, k) \text{ form}\}.$$

It is clear that \mathcal{P}_k is convex. We denote its closure in $H^{k,k}(X, \mathbb{R})$ by $\overline{\mathcal{P}}_k$. It is clear that $\overline{\mathcal{P}}_k$ is a closed convex cone with non-empty interior, satisfying $\overline{\mathcal{P}}_k - \overline{\mathcal{P}}_k = H^{k,k}(X, \mathbb{R})$. Since f^* preserves $\overline{\mathcal{P}}_k$, the Perron-Frobenius theorem implies that there exists an eigenclass $\Gamma_k \in \overline{\mathcal{P}}_k \setminus \{0\}$ such that

$$f^*\Gamma_k = d_k\Gamma_k,$$

where d_k is the spectral radius of f^* on $H^{k,k}(X, \mathbb{R})$. Moreover, d_k is equal to the k -th dynamical degree of f (see e.g. [DS05a]).

6.2. Invariant convex valuations. In this section, we prove a general Theorem (see Theorem 6.1) which will imply Theorem D.

Let $g \in \mathrm{GL}(E)$, $\phi \in \mathrm{Val}_{n-i}(E)$, we say that ϕ is *invariant* (or $d_i(g)$ -*invariant*) if $g \cdot \phi = d_i(g)\phi$.

By Proposition 5.10, the sequence of dynamical degrees $d_i(g)$ is log-concave. In particular, we have

$$d_i(g)^2 \geq d_{i+s}(g)d_{i-s}(g)$$

whenever $i \pm s$ are well defined.

As in [FW12, Section 6, 7], suppose that $d_i(g)^2 > d_{i+1}(g)d_{i-1}(g)$, then the authors show how to obtain a $d_i(g)$ -invariant valuation by methods from dynamics. We focus on the positivity properties of such invariant valuations, but under a weaker condition. Note that by log-concavity,

$$d_{i+1}(g)d_{i-1}(g) \geq d_{i+s}(g)d_{i-s}(g).$$

Thus the condition $d_i(g)^2 > d_{i-s}(g)d_{i+s}(g)$ is in general much weaker than the condition $d_i(g)^2 > d_{i-1}(g)d_{i+1}(g)$.

We show that positive invariant valuations have very weak positivity, if this kind of strict log concavity assumption holds. In the following, let $\gamma \in \{\mathcal{C}, \mathcal{P}\}$.

Theorem 6.1. *Assume that $2i \leq n$, and $g \in \mathrm{GL}(E)$. Then the following properties are satisfied.*

- (1) *The subspace of $d_i(g)$ -invariant valuations in $\mathrm{Val}_{n-i}(E)$ is non trivial.*
- (2) *Assume that the strict log-concavity inequality is satisfied for $s \leq \min(i, n-i)$:*

$$d_i(g)^2 > d_{i-s}(g)d_{i+s}(g),$$

then for any two $d_i(g)$ -invariant convex valuations $\psi_1, \psi_2 \in \mathcal{V}_{n-i}^\gamma$ we have

$$\psi_1 \tilde{*} \psi_2 = 0.$$

- (3) *Assume that*

$$d_1^2(g) > d_2(g),$$

then there exists a unique (up to a multiplication by a positive constant) $d_1(g)$ -invariant positive convex valuation $\psi \in \overline{\mathcal{P}_{n-1}}^\gamma$ (the closure of \mathcal{P}_{n-1} in the topology given by $\|\cdot\|_\gamma$), and ψ lies in an extremal ray of $\overline{\mathcal{P}_{n-1}}^\gamma$.

Proof. Let us prove statement (1). Up to a conjugation by an element of $\mathrm{GL}(E)$, we are reduced to the problem of finding a ρ^{i-n} -invariant valuation in \mathcal{P}_{n-i} for $0 \leq i \leq n$, where ρ is the spectral radius of g in each of the following cases:

- (a) The matrix of g in the canonical basis has Jordan form and the only eigenvalue of g is $\rho \in \mathbb{R}$.
- (b) One has that $n = 2$ and $g = \rho \mathrm{Id} \circ h$ where h is in the orthogonal group and where $\rho \in]0, +\infty[$.

Suppose we are in the case (a). Fix $i \leq n$. Let (e_1, \dots, e_n) be the canonical basis of E , let B be the unit ball in E and denote by $E_i = \mathrm{Vect}(e_1, \dots, e_i)$. Consider $B_i := B \cap E_i$ and consider the valuation given by:

$$\phi_i(L) := V(B_i[i], L[n-i]).$$

Let us compute $g \cdot \phi_i(L)$ for $L \in \mathcal{K}(E)$:

$$g \cdot \phi_i(L) = V(B_i[i], g^{-1}(L)[n-i]) = \frac{1}{|\det(g)|} V(g(B_i)[i], L[n-i]).$$

By the projection formula for mixed volumes (Lemma 5.14), since B_i is contained in a subspace of dimension i and since g leaves the subspace E_i invariant, we have:

$$g \cdot \phi_i(L) := \frac{1}{\rho^n} \binom{n}{i}^{-1} \mathrm{vol}_{E_i}(g(B_i)) \mathrm{vol}_{E_i^\perp}(p_i(L)),$$

where $p_i : E \rightarrow E_i^\perp$ is the orthogonal projection onto E_i^\perp . Since $|\det(g|_{E_i})| = \rho^i$, we have that:

$$g \cdot \phi_i = \rho^{i-n} \phi_i,$$

as required.

Suppose we are in the case (b). Then $g = \rho \text{Id} \circ h$ where h is an element of the orthogonal group. If $i = 0$ then the valuation vol is ρ^2 -invariant and if $i = 2$, then the trivial valuation constant equal to 1 is ρ^0 -invariant. Let us find a valuation in \mathcal{P}_1 which is ρ -invariant. There exists a ball K in E such that $h(K) = K$. Consider the valuation $\phi \in \mathcal{P}_1$ given by:

$$\phi(L) := V(K, L),$$

for any $L \in \mathcal{K}(E)$. We have that:

$$g \cdot \phi(L) = \frac{1}{\rho^2} V(g(K), L) = \frac{1}{\rho^2} V(\rho(K), L) = \frac{1}{\rho} V(K, L) = \rho^{-1} \phi(L),$$

as required.

Let us show how statement (1) follows from the previous arguments. Take $g \in \text{GL}(E)$. By construction, there exists a decomposition of E into:

$$E = \oplus E_k,$$

where each E_k is a g -invariant subspace such that $g|_{E_k}$ satisfies condition (a) or (b). Denote by $\lambda_k = \rho(g|_{E_k})$. On each subspace, there exists a convex body $B_k \subset E_k$ such that the valuation given by $V(B_k[j], -[\dim E_k - j])$ is $\lambda_k^{j-\dim E_k}$ -invariant. Considering a well-chosen valuation of the form

$$\phi(L) = V(B_1[i_1], \dots, B_k[i_k], L[n - i]),$$

where $i_1 + \dots + i_k = i$, gives the required invariant valuation.

Let us prove statement (2). First note that it is sufficient to prove

$$\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B = 0,$$

where $\phi_B(-) = V(-; B[n - 2i])$ and $B \in \mathcal{K}(E)$ is a convex body with non-empty interior and smooth boundary.

Note also that if $\psi \in \text{Val}_{n-i}(E)$ is $d_i(g)$ -invariant, then for any $c \neq 0$ and $K \in \mathcal{K}(E)$ we have:

$$\begin{aligned} ((cg) \cdot \psi)(K) &= \psi((cg)^{-1}(K)) = c^{i-n} \psi(g^{-1}(K)) \\ &= c^{i-n} (g \cdot \psi)(K) = c^{i-n} d_i(g) \psi(K) \\ &= d_i(cg) \psi(K), \end{aligned}$$

thus $(cg) \cdot \psi = d_i(cg) \psi$. In particular, ψ is g -invariant if and only if it is cg -invariant. Without loss of generality, to simplify the notations, we can assume that $|\det g| = 1$.

We first consider the case for $\mathcal{V}_{n-i}^{\mathcal{C}}$.

For $j \in \{1, 2\}$, since $\psi_j \in \mathcal{V}_{n-i}^{\mathcal{C}}$, we can take a sequence $\psi_{j,l} = \psi_{j,l}^+ - \psi_{j,l}^-$ such that $\lim_{l \rightarrow \infty} \psi_{j,l} = \psi_j$ and

$$(21) \quad \psi_{j,l}^+(B) + \psi_{j,l}^-(B) \leq c$$

for some uniform constant $c > 0$, where $\psi_{j,l}^+, \psi_{j,l}^- \in \mathcal{P}_{n-i}$.

Since $\psi_1, \psi_2 \in \mathcal{V}_{n-i}^{\mathcal{C}}$ are invariant valuations, we have

$$\begin{aligned} g^k \cdot (\psi_1 \tilde{*} \psi_2) \tilde{*} \phi_B &= (g^k \cdot \psi_1) \tilde{*} (g^k \cdot \psi_2) \tilde{*} \phi_B \\ &= d_i(g)^{2k} \psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B. \end{aligned}$$

The expansion of $(\psi_{1,l} \tilde{*} \psi_{2,l}) \tilde{*} \phi_B$ gives:

$$\begin{aligned} \psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B &= \lim_{l \rightarrow \infty} (\psi_{1,l} \tilde{*} \psi_{2,l}) \tilde{*} \phi_B \\ &= \lim_{l \rightarrow \infty} \left(\psi_{1,l}^+ \tilde{*} \psi_{2,l}^+ + \psi_{1,l}^- \tilde{*} \psi_{2,l}^- - \psi_{1,l}^+ \tilde{*} \psi_{2,l}^- - \psi_{1,l}^- \tilde{*} \psi_{2,l}^+ \right) \tilde{*} \phi_B. \end{aligned}$$

Since $\psi_{j,l}^+, \psi_{j,l}^-$ are positive, we get:

$$(22) \quad d_i(g)^{2k}(\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B) \leq \liminf_{l \rightarrow \infty} g^k \cdot \left(\psi_{1,l}^+ \tilde{*} \psi_{2,l}^+ + \psi_{1,l}^- \tilde{*} \psi_{2,l}^- \right) \tilde{*} \phi_B.$$

Applying Theorem 3.10 to $\phi := g^k \cdot (\psi_{1,l}^+ \tilde{*} \psi_{2,l}^+)$ (respectively $g^k \cdot (\psi_{1,l}^- \tilde{*} \psi_{2,l}^-)$), $\psi := \phi_B \in \text{Val}_{2i}(E)$ and the convex body $K := g^k(B)$, we obtain

$$(23) \quad \text{vol}(g^k(B)) \left(g^k \cdot (\psi_{1,l}^\epsilon \tilde{*} \psi_{2,l}^\epsilon) \tilde{*} \phi_B \right) \leq g^k \cdot (\psi_{1,l}^\epsilon \tilde{*} \psi_{2,l}^\epsilon)(g^k(B)) \phi_B(g^k(B)),$$

where $\epsilon \in \{+, -\}$. On the other hand, by Theorem 3.12, we have

$$\begin{aligned} \phi_B(g^k(B)) &= V(g^k(B)[2i], B[n-2i]) \\ &= V(g^k(B)[i+s], g^k(B)[i-s], B[n-2i]) \\ &\leq C_1 V(g^k(B)[i+s], B[n-i-s]) V(g^k(B)[i-s], B[n-i+s]), \end{aligned}$$

where $C_1 > 0$ is a constant which depends only on B , i and n .

By (21) and Theorem 3.10, we also have

$$(24) \quad \liminf_{l \rightarrow \infty} ((\psi_{1,l}^+ \tilde{*} \psi_{2,l}^+ + \psi_{1,l}^- \tilde{*} \psi_{2,l}^-) \tilde{*} \phi_B) \leq C_3$$

for some constant $C_3 > 0$. Note that there exists a constant $C_2 > 0$ such that

$$(25) \quad g^k \cdot (\psi_{1,l}^\epsilon \tilde{*} \psi_{2,l}^\epsilon)(g^k(B)) = C_2 (\psi_{1,l}^\epsilon \tilde{*} \psi_{2,l}^\epsilon) \tilde{*} \phi_B.$$

By (22), (23), (24), (25) and the estimate for $\phi_B(g^k(B))$, we deduce that there exists a uniform constant $C_4 > 0$ such that

$$(26) \quad d_i(g)^{2k}(\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B) \leq C_4 V(g^k(B)[i+s], B[n-i-s]) V(g^k(B)[i-s], B[n-i+s]).$$

Next we consider the case for $\mathcal{V}_{n-i}^{\mathcal{P}}$.

We take approximations $\psi_{j,l}$ such that $\lim_{l \rightarrow \infty} \psi_{j,l} = \psi_j$ and

$$|\psi_{j,l}(L_1, \dots, L_{n-i})| \leq c V(\mathbf{B}[n-i]; L_1, \dots, L_{n-i})$$

for some uniform constant $c > 0$, and any L_1, \dots, L_{n-i} . As we are reduced to the situation $|\det g| = 1$, this implies

$$(27) \quad |g^k \cdot \psi_{j,l}(L_1, \dots, L_{n-i})| \leq c V(g^k(\mathbf{B})[n-i]; L_1, \dots, L_{n-i}).$$

By the definition of $\tilde{*}$ and (27), we get

$$g^k \cdot (\psi_{1,l} \tilde{*} \psi_{2,l}) \tilde{*} \phi_B \leq c^2 V(g^k(\mathbf{B})[2i], B[n-2i]).$$

Then the same arguments as above shows that

$$(28) \quad d_i(g)^{2k}(\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B) \leq C_5 V(g^k(\mathbf{B})[i+s], B[n-i-s]) V(g^k(\mathbf{B})[i-s], B[n-i+s]).$$

In summary, if $\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B > 0$, after taking k -th root of the above inequality (26) or (28) and letting k tend to infinity, we get

$$d_i(g)^2 \leq d_{i+s}(g) d_{i-s}(g).$$

This contradicts with our assumption. Thus,

$$\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B \leq 0.$$

Since the valuations $-\psi_1$ is also invariant, the previous argument holds and we also have:

$$(-\psi_1) \tilde{*} \psi_2 \tilde{*} \phi_B \leq 0.$$

Hence, we must have $\psi_1 \tilde{*} \psi_2 \tilde{*} \phi_B = 0$.

Finally we prove the statement (3). Suppose $i = n - 1$ (thus the assumption is $d_1(g)^2 > d_2(g)$).

We claim that

$$\overline{\mathcal{P}_{n-1}}^{\mathcal{P}} = \overline{\mathcal{P}_{n-1}}^{\mathcal{C}} = \mathcal{P}_{n-1}$$

and that any valuation $\phi \in \overline{\mathcal{P}_{n-1}}^{\mathcal{P}}$ is of the form $V(L; -[n-1])$ for some $L \in \mathcal{K}(E)$.

Take $\phi \in \overline{\mathcal{P}_{n-1}^{\mathcal{P}}}$. By Corollary 3.24, there exists a sequence of valuations $\phi_j = V(L_j, -[n-1])$ such that $\|\phi_j - \phi\|_{\mathcal{P}} \rightarrow 0$. Then we have that $V(L_j, \mathbf{B}[n-1])$ is uniformly bounded above. By Diskant's inequality (similar to the estimate (9)), the convex bodies L_j (up to some translations) are bounded. We can thus extract a subsequence of L_j (up to some translations) converging to a convex body L . In particular, $\phi = V(L, -[n-1])$ as required.

Next let $\psi \in \overline{\mathcal{P}_{n-1}^{\mathcal{C}}}$, we prove that ψ is also of the form $\psi(-) = V(L; -[n-1])$. As $\mathcal{P}_{n-1} \subset \overline{\mathcal{P}_{n-1}^{\mathcal{P}}}$, any valuation in \mathcal{P}_{n-1} is of the form $V(L; -)$. Hence there exists a sequence of convex bodies $L_k \in \mathcal{K}(E)$ such that

$$\|V(L_k; -[n-1]) - \psi\|_{\mathcal{C}} \rightarrow 0.$$

This implies that $V(L_k; \mathbf{B}[n-1])$ is uniformly bounded above. Then the same argument as in the previous step shows that $\psi = V(L; -[n-1])$ for some $L \in \mathcal{K}(E)$, as required.

This finishes the proof of the claim.

Now we have $\psi_1(-) = V(-[n-1]; K)$ and $\psi_2(-) = V(-[n-1]; L)$ for some $K, L \in \mathcal{K}(E)$. Then $\psi_i \tilde{*} \psi_j \tilde{*} \phi_B = 0$ for $i, j \in \{1, 2\}$ implies

$$V(K, L, B[n-2]) = V(K[2], B[n-2]) = V(L[2], B[n-2]) = 0.$$

In particular,

$$V(K, L, B[n-2]) = V(K[2], B[n-2])V(L[2], B[n-2]).$$

Now the uniqueness result follows from [Sch14, Theorem 7.6.8], which we present below as a lemma.

Lemma 6.2. *If the equality holds in*

$$V(K, L, C_1, \dots, C_{n-2}) \geq V(K[2], C_1, \dots, C_{n-2})V(L[2], C_1, \dots, C_{n-2}),$$

where C_1, \dots, C_{n-2} are smooth convex bodies with non-empty interior, then K, L are homothetic.

As in our setting, B is smooth, this immediately proves the uniqueness of invariant valuations.

The proof for the extremal ray property also follows from the above lemma. Assume that $\psi \in \overline{\mathcal{P}_{n-1}^{\gamma}}$ is invariant and can be written as

$$\psi = \phi_1 + \phi_2,$$

where $\phi_1 = V(-; K_1), \phi_2 = V(-; K_2)$. We need to verify that ϕ_1, ϕ_2 are proportional. The vanishing of $\psi \tilde{*} \psi \tilde{*} \phi_B$ is equivalent to

$$V(K_1, K_2, B[n-2]) = V(K_1[2], B[n-2]) = V(K_2[2], B[n-2]) = 0,$$

which yields that K_1, K_2 are homothetic. Thus ψ must lie in an extremal ray of the cone $\overline{\mathcal{P}_{n-1}^{\gamma}} \subset \mathcal{V}_{n-1}^{\gamma}$. \square

6.2.1. Weak closedness. The above argument for Theorem 6.1 (3) shows that the cone \mathcal{P}_{n-1} is closed with respect to the topology given by $\|\cdot\|_{\mathcal{P}}$. Actually, this cone is also weakly closed in the following sense. Observe that for any convex body $K \in \mathcal{K}(E)$, the evaluation map induces a continuous linear form on $\mathcal{V}_k^{\mathcal{P}}$:

$$\text{ev}_K : \mathcal{V}_k^{\mathcal{P}} \rightarrow \mathbb{R}, \phi \mapsto \phi(K).$$

The continuity of ev_K follows from

$$|\phi(K)| \leq \|\phi\|_{\mathcal{P}} V(\mathbf{B}[n-k], K[k]).$$

Consider the weak topology, which is the coarsest topology on $\mathcal{V}_k^{\mathcal{P}}$ such that the evaluation maps ev_K are continuous. We first note that the weak topology contains a countable basis of neighborhoods. Consider the finite intersection of neighborhoods of the form:

$$U = \left\{ \phi \in \mathcal{V}_k^{\mathcal{P}} \mid \left| \phi(P) - \sum_{i=1}^N a_i V(P_{1,i}, \dots, P_{n-k,i}, P[k]) \right| < b \right\},$$

where $a_i, b \in \mathbb{Q}$, $N \in \mathbb{N}$ and where P and $P_{j,i}$ are rational polytopes in E . By construction U is an open set of $\mathcal{V}_k^{\mathcal{P}}$ for the weak topology. The fact that such subset U defines a basis of neighborhoods results from the density of rational polytopes inside $\mathcal{K}(E)$.

Proposition 6.3. *The cone $\mathcal{P}_{n-1} \subset \mathcal{V}_{n-1}^{\mathcal{P}}$ is closed with respect to the weak topology. In particular, one has the following equality:*

$$\overline{\mathcal{P}_{n-1}}^{\mathcal{P}} = \mathcal{P}_{n-1} = \overline{\mathcal{P}_{n-1}}^w,$$

where $\overline{\mathcal{P}_{n-1}}^w$ is the closure of the cone \mathcal{P}_{n-1} with respect to the weak topology and where $\overline{\mathcal{P}_{n-1}}^{\mathcal{P}}$ is the closure of the cone \mathcal{P}_{n-1} with respect to the norm $\|\cdot\|_{\mathcal{P}}$.

Proof. Since the space $\mathcal{V}_{n-1}^{\mathcal{P}}$ endowed with the weak topology is first countable, every point $\phi \in \mathcal{V}_{n-1}^{\mathcal{P}}$ in the weak closure of the cone \mathcal{P}_{n-1} is the weak limit of a sequence $\phi_j \in \mathcal{P}_{n-1}$. Recall that every valuation in \mathcal{P}_{n-1} is of the form $V(M; -[n-1])$ for some convex body $M \in \mathcal{K}(E)$ and one can write each ϕ_j as $\phi_j = V(L_j; -[n-1])$ where $L_j \in \mathcal{K}(E)$. Since ϕ_j converges weakly to ϕ , this implies that:

$$\phi_j(\mathbf{B}) = V(L_j, \mathbf{B}[n-1]) \rightarrow \phi(\mathbf{B}),$$

as j tend to $+\infty$. In particular, the sequence $\{V(L_j, \mathbf{B}[n-1])\}_j$ is bounded. By Diskant's inequality, there exists a subsequence of the sequence $\{L_j\}_{j \in \mathbb{N}}$ (up to translations), which converges to a convex body L . Hence, we have that $\phi = V(L; -[n-1])$ for some $L \in \mathcal{K}(E)$ and $\phi \in \mathcal{P}_{n-1}$ as required. \square

Remark 6.4. We are not sure about the weak closedness of \mathcal{P}_k when $k \neq n-1$.

Remark 6.5. In general the invariant valuations are not smooth. The invariant valuations in [FW12] are given by the volume of a projection onto a linear subspace. By the reduction formula for mixed volumes, they are given by mixed volumes, which are elements in \mathcal{P}_i .

Remark 6.6. For any $g \in \mathrm{GL}(E)$, the action of g satisfies $g(\mathcal{P}_i) \subset \mathcal{P}_i$. Recall that in functional analysis we have the famous Krein-Rutman theorem:

Let X be a Banach space, and let $\mathcal{C} \subset X$ be a closed convex cone such that $\mathcal{C} - \mathcal{C}$ is dense in X . Let $T : X \rightarrow X$ be a non-zero compact operator satisfying $T(\mathcal{C}) \subset \mathcal{C}$, and assume that its spectral radius $\rho(T)$ is strictly positive. Then there is an eigenvector $u \in \mathcal{C} \setminus \{0\}$ such that $T(u) = \rho(T)u$.

If X is of finite dimension, then this is the Perron-Frobenius theorem, which is very useful to construct invariant classes in complex dynamics. In our setting, in general the induced linear operator by g is not compact. However, if we consider the finite dimensional space $\mathrm{Val}^G(E)$ where $G \subset \mathrm{SO}(E)$ is a compact subgroup acting transitively on the unit sphere of E , and consider appropriate cones in this space, then we can apply the result directly.

Remark 6.7. We remark that the same vanishing result also holds true for the dynamics of dominated holomorphic maps. Furthermore, by Hodge theory (see e.g. [Voi07]), the extremal ray property holds true for invariant $(1,1)$ classes. More precisely, using the notations in Section 6.1, we have:

Let X be a compact Kähler manifold of dimension n . Let $f : X \rightarrow X$ be a dominated holomorphic self-map. Assume $2k \leq n$. If $d_k^2 > d_{k+s}d_{k-s}$, then for any Kähler class ω and any invariant positive classes $\Theta_1, \Theta_2 \in \overline{\mathcal{P}}_k \subset H^{k,k}(X, \mathbb{R})$ we have

$$\Theta_1 \cdot \Theta_2 \cdot \omega^{n-2k} = 0.$$

Moreover, if $d_1^2 > d_2$, then the non-zero invariant class $\Theta \in \overline{\mathcal{P}}_1$ is unique (up to some scaling) and lies in an extremal ray of $\overline{\mathcal{P}}_1$.

The proof of $\Theta_1 \cdot \Theta_2 \cdot \omega^{n-2k} = 0$ is the same as in Theorem 6.1, where we apply the reverse Khovanskii-Teissier inequality in complex geometry [LX17]. For the uniqueness and extremity of $\Theta \in \overline{\mathcal{P}}_1$, we decompose $\Theta_i, i = 1, 2$ as follows:

$$\Theta_i = a_i \omega + P_i,$$

where $a_i \in \mathbb{R}$, and P_i is a primitive class, i.e., $\omega^{n-1} \cdot P_i = 0$. Since $\Theta_i \cdot \Theta_j \cdot \omega^{n-2} = 0$ for $i, j \in \{1, 2\}$, both P_1 and P_2 can not be zero. Moreover, combining with $\omega^{n-1} \cdot P_i = 0$ implies

$$\begin{aligned} P_1^2 \cdot \omega^{n-2} &= -a_1^2 \omega^n, \\ P_2^2 \cdot \omega^{n-2} &= -a_2^2 \omega^n, \\ P_1 \cdot P_2 \cdot \omega^{n-2} &= -a_1 a_2 \omega^n. \end{aligned}$$

Thus the matrix $[P_i \cdot P_j \cdot \omega^{n-2}]_{i,j}$ is degenerate. By Hodge-Riemann bilinear relations, we have $P_1 = cP_2$ for some non-zero constant c . Then we get $a_1^2 = c^2 a_2^2$. We claim $a_1 = ca_2$, which then implies $\Theta_1 = c\Theta_2$. If some $a_i = 0$, then this is clear; otherwise, if $a_1 = -ca_2$, by considering $\Theta_1 - c\Theta_2$ we get that ω is also an invariant class, which is impossible by the vanishing result. Thus we finish the proof of the uniqueness result. The extremity property follows from the same argument.

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