The free boundary Schur process and applications

Dan Betea, Jérémie Bouttier, Peter Nejjar, Mirjana Vuletić

The free boundary Schur process and applications

Dan Betea* Jérémie Bouttier† Peter Nejjar‡ Mirjana Vuletić§

October 18, 2017

Abstract

We investigate the free boundary Schur process, a variant of the Schur process introduced by Okounkov and Reshetikhin, where we allow the first and the last partitions to be arbitrary (instead of empty in the original setting). The pfaffian Schur process, previously studied by several authors, is recovered when just one of the boundary partitions is left free. We compute the correlation functions of the process in all generality via the free fermion formalism, which we extend with the thorough treatment of "free boundary states". For the case of one free boundary, our approach yields a new proof that the process is pfaffian. For the case of two free boundaries, we find that the process is not pfaffian, but a closely related process is. We also study three different applications of the Schur process with one free boundary: fluctuations of symmetrized last passage percolation models, limit shapes and processes for symmetric plane partitions, and for plane overpartitions. We finally obtain a few hypergeometric identities as byproducts of our approach.

1 Introduction

In this paper we introduce and study the free boundary Schur process, a random sequence of partitions which we now define. Recall that an (integer) *partition* λ is a nonincreasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots$ which vanishes eventually. Its *size* is $|\lambda| := \sum_{i \geq 1} \lambda_i$. For two partitions λ, μ such that $\lambda \subset \mu$ (i.e. $\lambda_i \geq \mu_i$ for all i), let $s_{\lambda/\mu}$ be the skew Schur function of shape λ/μ. We refer to [Mac95, Chapter 1] or [Sta99, Chapter 7] for general background on symmetric functions, and to Section 2.1.1 below for a summary of the relevant notions. Let us fix a nonnegative integer N, two nonnegative real numbers u and v, and two families $(\rho_k^+)_{1 \leq k \leq N}$ and $(\rho_k^-)_{1 \leq k \leq N}$ of specializations (which we can think of as collections of variables). To a sequence of partitions of the form

$$\mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \cdots \supset \mu^{(N-1)} \subset \lambda^{(N)} \supset \mu^{(N)} \tag{1.1}$$

we assign a weight

$$\mathcal{W}(\vec{\lambda}, \vec{\mu}) := u^{|\mu^{(0)}|} v^{|\mu^{(N)}|} \prod_{k=1}^{N} \left(s_{\lambda^{(k)}/\mu^{(k-1)}} \left(\rho_k^+\right) s_{\lambda^{(k)}/\mu^{(k)}} \left(\rho_k^-\right) \right). \tag{1.2}$$

The *partition function* $Z \equiv Z(u, v, \rho_1^+, \rho_1^-, \ldots, \rho_N^+, \rho_N^-)$ is the sum of weights of all sequences of the form (1.1). Under certain assumptions on the parameters $u, v, \rho_1^\pm, \ldots, \rho_N^\pm$ to be detailed in Section 2, the partition function is finite, and $\mathcal{W}/Z$ defines a probability distribution which is the *free boundary Schur process*.

For $u = v = 0$, we recover the original Schur process of Okounkov and Reshetikhin [OR03], which is such that the boundary partitions $\mu^{(0)}$ and $\mu^{(N)}$ are both equal to the empty (zero) partition $\emptyset$. For $u > 0$ and $v = 0$, only $\mu^{(N)}$ is constrained to be zero, and we recover the so-called pfaffian Schur process [BR05] up to the inessential change that, in this reference, $\mu^{(0)}$ is assumed to be the conjugate of an even partition — see Remark 2.9 below. Of course, the case $u = 0$ and $v > 0$ is equivalent by symmetry. The new situation

*IRIF, CNRS et Université Paris Diderot, Case 7014, F–75205 Paris Cedex 13, dan.betea@gmail.com
†Institut de Physique Théorique, Université Paris-Saclay, CEA, CNRS, F-91191 Gif-sur-Yvette, jeremie.bouttier@ipht.fr
Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France
‡IST Austria, 3400 Klosterneuburg, Austria, peter.nejjar@ist.ac.at
§Department of Mathematics, University of Massachusetts Boston, Boston, MA 02125, USA, mirjana.vuletic@umb.edu

considered in this paper is when $uv > 0$, i.e. when both boundaries are free. Note that the constant sequence equal to λ has weight $(uv)^{|\lambda|}$, therefore it is necessary to have $uv < 1$ for the partition function to be finite. Let us mention that, if we condition on having $\mu^{(0)} = \mu^{(N)}$, then we recover Borodin's periodic Schur process [Bor07]. Conversely, the free boundary Schur process of length N can be seen as a symmetrized version of the periodic Schur process of length $2N$.

The Schur process may be viewed as a simple point process on $\mathbb{Z}^2$ — see [OR03] or Section 2.1 below. As such, a natural question is to characterize the nature of this process. Okounkov and Reshetikhin showed that the original Schur process is determinantal, while Borodin and Rains proved that the pfaffian Schur process is, well, pfaffian [BR05]. Let us recall that a simple point process ξ on a discrete space $\mathcal{X}$ is said to be *pfaffian* if there exists a function K (called the *correlation kernel*) from $\mathcal{X} \times \mathcal{X}$ to the set of 2×2 complex-valued matrices, which is antisymmetric in the sense that $K(x, y)$ is the transpose of $-K(y, x)$ for all x, y, and such that, for any finite subset $U = \{x_1, \ldots, x_n\}$ of $\mathcal{X}$, we have

$$\mathrm{Prob}(U \subset \xi) = \mathrm{pf}[K(x_i, x_j)]_{1 \leq i,j \leq n}. \tag{1.3}$$

Here pf denotes the pfaffian and $[K(x_i, x_j)]_{1 \leq i,j \leq n}$ is the $2n \times 2n$ antisymmetric matrix formed by the blocks $K(x_i, x_j)$ — see Appendix A for more details. A determinantal point process corresponds to the particular case where the diagonal entries $K_{1,1}$ and $K_{2,2}$ of the kernel are identically zero; indeed the pfaffian in the right hand side (RHS) of (1.3) then reduces to the $n \times n$ determinant $\det_{1 \leq i,j \leq n} K_{1,2}(x_i, x_j)$. In this paper, we shall see that the free boundary Schur process is not pfaffian in general, but a closely related process is, and we will explicitly compute its correlation kernel. This situation is reminiscent of the periodic Schur process, which becomes determinantal only after a certain "shift-mixing" [Bor07].

Context and motivations. The problems we consider in this paper are part of an active area of research dubbed "integrable probability" — see for instance the exposition in [BG16]. A first major result in the area was the resolution of Ulam's problem by Baik, Deift and Johansson [BDJ99] who have shown that the longest increasing subsequence of a random permutation exhibits Tracy–Widom GUE fluctuations around its mean, and thus behaves like the largest eigenvalue of a large Gaussian Unitary Ensemble random matrix [TW94]. By the Robinson–Schensted correspondence, Ulam's problem is closely related to the so-called Plancherel measure on the set of partitions, and Okounkov [Oko01] realized that this measure is a particular instance of a *Schur measure*, whose determinantal correlations can be computed explicitly using the infinite wedge space (or free fermion) formalism. Asymptotics then reduce to simple saddle point analysis. Around the same time, a discrete version of the Plancherel measure, that of last passage percolation (LPP) in a rectangle with independent geometric weights, has been analyzed by Johansson [Joh00] using Schur measures and orthogonal polynomial techniques. Using asymptotics of Meixner polynomials, Johansson showed that this model belongs to the same universality class and that, in particular, the last passage time also fluctuates according to the Tracy–Widom GUE law.

The story continues with a series of papers by Baik and Rains [Rai00, BR99, BR01a, BR01b] studying longest increasing subsequences in random permutations subject to certain symmetry constraints (e.g. involutions). Upon poissonization the corresponding processes are pfaffian instead of determinantal, and new distributions like the Tracy–Widom GOE and GSE laws appear as fluctuations.

In parallel, Okounkov and Reshetikhin [OR03, OR07] introduced a time-dependent generalization of the Schur measure called the *Schur process*, using again the infinite wedge space formalism to prove its determinantal nature, and applied their result to analyze large plane partitions. As mentioned above, in the original setting, the process is constrained to start and end with the empty partition. Borodin and Rains [BR05] developed another approach to the Schur process via the Eynard–Mehta theorem; they treated similarly the *pfaffian Schur process*, which appeared implicitly in an earlier work of Sasamoto and Imamura [SI04], and corresponds in our language to having one free boundary and one empty boundary (alternatively it can be viewed as a symmetrized Schur process upon interpreting the free boundary as a reflection axis). In a different direction, Borodin [Bor07] considered the Schur process with periodic boundary conditions.

In this paper, we explore the "missing" type of boundary conditions, namely that of two free boundaries. Our main technical tool will be, as in [OR03], the infinite wedge space/free fermion formalism. Free boundaries are represented in this formalism as *free boundary states*, which were first introduced in [BCC17] in order to compute the partition function of free boundary steep tilings (an instance of free boundary Schur process to appear in Section 6). Here, we proceed to the next level of computing correlation functions, which requires understanding the interplay between free boundary states and fermionic operators. The determinantal nature of the original Schur process with empty boundary conditions results from Wick's theorem for free fermions. As we shall see, the adaptation of this theorem for free boundaries is not completely straightforward, and involves extended free boundary states which are not eigenvalues of the charge operator. A consequence of this is that the free boundary Schur process is neither determinantal nor pfaffian in general, but becomes pfaffian after we perform a certain random vertical shift of the point configuration, that translates in the point process language the "charge mixing" occurring in extended free boundary states. This phenomenon has some similarities with Borodin's shift-mixing for the periodic Schur process [Bor07], but the fermionic picture is rather different (as will be explained in a subsequent note, for periodic boundary conditions, Borodin's shift-mixing can be interpreted as the passage to the grand canonical ensemble, needed to apply Wick's theorem at finite temperature). In the case of a single free boundary, the shift goes away, and our approach yields a new derivation of the correlations functions of the pfaffian Schur process, alternative to that by Borodin and Rains [BR05] and the very recent one by Ghosal [Gho17] using Macdonald difference operators.

Among other recent developments related to the pfaffian Schur process, let us mention the work by Baik, Barraquand, Corwin and Suidan studying its applications to LPP in half-space [BBCS17b] and facilited TASEP [BBCS17a], and the work of Barraquand, Borodin, Corwin and Wheeler [BBCW17] introducing its Macdonald analogue. Here, we further investigate applications of the pfaffian Schur process by considering symmetric LPP thus complementing the results of [BR01a, BR01b, BBCS17b], as well as symmetric plane partitions and plane overpartitions — two models which can be rephrased in terms of lozenge and domino tilings, respectively. The fact that dimer models with free boundaries are related to pfaffians is not surprising. This was already observed for instance in [Ste90, CK11] via nonintersecting lattice paths. See also [DFR12, Pan15] for other limit shape results on tilings with free boundaries. Applications of the Schur process with two free boundaries will be investigated in a subsequent publication.

Outline. The paper is organized as follows: in Section 2 we list the main results of the paper, only introducing the basic concepts needed for the statements. It is divided in two parts: Section 2.1 leads to the two fundamental Theorems 2.7 and 2.10 stating that certain point processes associated with the free boundary Schur process have pfaffian correlations, while Section 2.2 deals with listing the applications we draw from the first. Section 3 is devoted to the proof of our two fundamental theorems via the machinery of free fermions. We also obtain in Theorem 3.14 an expression for the general multipoint correlation functions. Sections 4, 5 and 6 deal with asymptotic applications of Theorem 2.7 to models of symmetric last passage percolation, symmetric plane partitions and plane overpartitions respectively. Section 7 gathers some concluding remarks and perspectives. We list, in the Appendices, odds and ends we deemed too cumbersome to put in the main text. Of note is Appendix D which, besides some necessary computations for the main text, contains a few hypergeometric summation formulas we obtain as byproducts.

Throughout the text, we will denote by $[z^k]F(z)$ the coefficient of z^k in the formal power series $F(z)$. We will also use $\mathbb{Z}'$ for the set $\{i + \frac{1}{2} | i \in \mathbb{Z}\}$. $\mathcal{P}$ will symbolize the set of all partitions, themselves usually denoted by the letters λ and μ.

2 Main results

2.1 Correlation functions of the free boundary Schur process

2.1.1 Preliminary definitions and notations

Let us first introduce some definitions and notations related to symmetric functions that are needed to state our results in compact form. Let Sym be the algebra of symmetric functions, and let h_n (resp. p_n) be the complete homogeneous (resp. power sum) symmetric function of degree n. For two partitions $\lambda \supset \mu$, the skew Schur function $s_{\lambda/\mu}$ is given by $s_{\lambda/\mu} := \det_{1 \le i,j \le \ell(\lambda)} h_{\lambda_i - i + \mu_j - j}$ where $\ell(\lambda) := \max\{i : \lambda_i > 0\}$ is the length of λ.

A *specialization* ρ is an algebra homomorphism from Sym to the field $\mathbb{C}$ of complex numbers. It is uniquely determined by its values on the h_n's (or equivalently the p_n's), hence by the generating function

$$H(\rho;t) := \sum_{n\geq 0} h_n(\rho)t^n = \exp\left(\sum_{n\geq 1} \frac{p_n(\rho)t^n}{n}\right). \tag{2.1}$$

As is customary, for a symmetric function $f \in \mathsf{Sym}$, we will write $f(\rho)$ in lieu of $\rho(f)$. A specialization ρ is said *nonnegative* if $s_{\lambda/\mu}(\rho)$ is a nonnegative real number for any λ, μ. In view of (1.2), all specializations $\rho_1^{\pm}, \ldots, \rho_N^{\pm}$ should be nonnegative in order for the weight $\mathcal{W}(\vec{\lambda}, \vec{\mu})$ to be nonnegative. A necessary and sufficient condition [Tho64, AESW51] for ρ to be nonnegative is that its generating function be of the form

$$H(\rho;t) = e^{\gamma t} \prod_{i\geq 1} \frac{1+\beta_i t}{1-\alpha_i t} \tag{2.2}$$

where the *parameters* $\gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ form a summable collection of nonnegative real numbers (in particular, when $\gamma = \beta_1 = \beta_2 = \cdots = 0$, we recover the specialization in the variables $\alpha_1, \alpha_2, \ldots$). From (2.2), it appears that $H(\rho;t)$ and $H(\rho;t)^{-1}$ are both meromorphic functions of t.

For ρ, ρ' two specializations and s a complex number, we denote by $\rho \cup \rho'$ and $s\rho$ the specializations defined by

$$H(\rho \cup \rho';t) := H(\rho;t)H(\rho';t), \qquad H(s\rho;t) := H(\rho;st) \tag{2.3}$$

or equivalently

$$p_n(\rho \cup \rho') := p_n(\rho) + p_n(\rho'), \qquad p_n(s\rho) := s^n p_n(\rho), \qquad n \geq 1. \tag{2.4}$$

For nonnegative specializations, these operations have a transparent meaning on the parameters. Denoting by $\mathcal{P}$ the set of all partitions, we also define the (possibly infinite) quantities

$$H(\rho;\rho') := \sum_{\lambda\in\mathcal{P}} s_\lambda(\rho)s_\lambda(\rho'), \qquad \tilde{H}(\rho) := \sum_{\lambda\in\mathcal{P}} s_\lambda(\rho). \tag{2.5}$$

They are respectively associated with the so-called Cauchy and Littlewood identities [Sta99, Theorem 7.12.1 and Corollary 7.13.8], which amount to

$$H(\rho;\rho') = \exp\left(\sum_{n\geq 1} \frac{p_n(\rho)p_n(\rho')}{n}\right), \qquad \tilde{H}(\rho) = \exp\left(\sum_{n\geq 1}\left(\frac{p_{2n-1}(\rho)}{2n-1} + \frac{p_n(\rho)^2}{2n}\right)\right). \tag{2.6}$$

Note that the notation $H(\cdot;\cdot)$ is consistent: for ρ' the specialization in the single variable t, we have $H(\rho;\rho') = H(\rho;t)$. We also have the relations

$$H(\rho;\rho' \cup \rho'') = H(\rho;\rho')H(\rho;\rho''), \qquad \tilde{H}(\rho \cup \rho') = \tilde{H}(\rho)\tilde{H}(\rho')H(\rho;\rho'). \tag{2.7}$$

Proposition 2.1 (Folklore)**.** *Let ρ and ρ' be two nonnegative specializations of respective parameters $\gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ and $\gamma', \alpha_1', \beta_1', \alpha_2', \beta_2', \ldots$. Then $H(\rho, \rho')$ is finite if and only if $\alpha_i\alpha_j' < 1$ and $\beta_i\beta_j' < 1$ for all i, j, in which case it reads explicitly*

$$H(\rho;\rho') = e^{\gamma\gamma' + \gamma\sum_i(\alpha_i'+\beta_i') + \gamma'\sum_i(\alpha_i+\beta_i)} \prod_{i,j\geq 1} \frac{(1+\alpha_i\beta_j')(1+\beta_i\alpha_j')}{(1-\alpha_i\alpha_j')(1-\beta_i\beta_j')}. \tag{2.8}$$

And $\tilde{H}(\rho)$ is finite if and only if $\alpha_i < 1$ and $\beta_i < 1$ for all i, in which case it reads

$$\tilde{H}(\rho) = e^{\gamma + \frac{\gamma^2}{2} + \gamma\sum_i(\alpha_i+\beta_i)} \prod_i \frac{1}{(1-\alpha_i)(1-\beta_i)} \frac{\prod_{i,j}(1+\alpha_i\beta_j)}{\prod_{i<j}(1-\alpha_i\alpha_j)(1-\beta_i\beta_j)}. \tag{2.9}$$

2.1.2 Partition function

The computation of the partition function of the general free boundary Schur process was essentially carried out in [BCC17, Section 5.3]. In our current notation it is given as follows.

Proposition 2.2. *The partition function of the free boundary Schur process reads*

$$Z = \prod_{1\le k\le \ell\le N} H(\rho_k^+;\rho_\ell^-) \prod_{n\ge 1} \frac{\tilde H(u^{n-1}v^n\rho^+)\tilde H(u^n v^{n-1}\rho^-)H(u^{2n}\rho^+;v^{2n}\rho^-)}{1-u^n v^n} \tag{2.10}$$

where

$$\rho^\pm := \rho_1^\pm \cup \rho_2^\pm \cup \cdots \cup \rho_N^\pm. \tag{2.11}$$

It is finite if and only if $uv<1$ and all factors appearing in the RHS of (2.10) *are finite.*

Remark 2.3. For $u=v=0$, the second product in the RHS of (2.10) reduces to 1 and we recover the partition function of the original Schur process. For $u=1$ and $v=0$, the only nontrivial factor in the second product is $\tilde H(\rho^-)$ and we recover the partition function of the pfaffian Schur process [BR05, Proposition 3.2], up to slightly different conventions.

Let $\gamma^{(k\pm)}, \alpha_1^{(k\pm)}, \beta_1^{(k\pm)}, \alpha_2^{(k\pm)}, \beta_2^{(k\pm)}, \ldots$ denote the parameters of the nonnegative specializations $\rho_k^\pm$. We assume without loss of generality that $\alpha_1^{(k\pm)}$ (resp. $\beta_1^{(k\pm)}$) is the largest α-parameter (resp. β-parameter) of $\rho_k^\pm$. Then, by Proposition 2.1, the partition function Z is finite if and only if $uv<1$ and, for any k and ℓ such that $1\le k\le \ell\le N$, all the quantities $\alpha_1^{(k+)}\alpha_1^{(\ell-)}$, $\beta_1^{(k+)}\beta_1^{(\ell-)}$, $v\alpha_1^{(k+)}$, $v\beta_1^{(k+)}$, $u\alpha_1^{(k-)}$ and $u\beta_1^{(k-)}$ are strictly smaller than 1.

From now on, we assume that these conditions are satisfied. In preparation for the next section, we now state a technical lemma about the existence of a certain family of contours (on which we shall integrate).

Lemma 2.4. *For each $k=1,\ldots,N$, set*

$$\begin{aligned} a_k^+ &:= \max\left(\alpha_1^{(1+)}, \alpha_1^{(2+)}, \ldots, \alpha_1^{(k+)}, (uv)^2\alpha_1^{(k+1\,+)}, \ldots, (uv)^2\alpha_1^{(N+)}\right), \\ a_k^- &:= \max\left((uv)^2\alpha_1^{(1-)}, (uv)^2\alpha_1^{(2-)}, \ldots, (uv)^2\alpha_1^{(k-)}, \alpha_1^{(k+1\,-)}, \ldots, \alpha_1^{(N-)}\right), \end{aligned} \tag{2.12}$$

and define similarly $b_k^\pm$ by changing each α into a β in the definition. Then, there exists a sequence of nested contours $C_1^{(1)}, C_1^{(2)}, \ldots, C_N^{(1)}, C_N^{(2)}$ (with $C_1^{(1)}$ the outermost contour and $C_N^{(2)}$ the innermost one) such that, for all $k=1,\ldots,N$ and $e=1,2$, $C_k^{(e)}$ is contained in the annulus $\{z : v<|z|<u^{-1}\}$, encircles positively the real interval $[-b_k^-, a_k^-]$, but it neither crosses $(-\infty,-1/b_k^+]$ nor $[1/a_k^+,+\infty)$.

Proof. Observe that, by the above conditions on u,v and on the $\alpha_1^{(k\pm)}$, we have $\max(v,a_k^-) < \min(1/u, 1/a_k^+)$ for all k, and that those two quantities are decreasing functions of k. Therefore, we may pick a decreasing sequence $a_1^{(1)} > a_1^{(2)} > \cdots > a_N^{(1)} > a_N^{(2)}$ such that, for all k and e, $\max(v,a_k^-) < a_k^{(e)} < \min(1/u,1/a_k^+)$. Similarly, we may pick a decreasing sequence $b_1^{(1)} > b_1^{(2)} > \cdots > b_N^{(1)} > b_N^{(2)}$ such that $\max(v,b_k^-) < b_k^{(e)} < \min(1/u,1/b_k^+)$. Then, taking for each $C_k^{(e)}$ the circle of diameter $[-b_k^{(e)}, a_k^{(e)}]$ does the job. ☐

Remark 2.5. When $u,v,\alpha_1^{(k\pm)},\beta_1^{(k\pm)}$ are all smaller than 1, the contours can be simply taken as circles with center 0 and decreasing radii close to 1.

Remark 2.6. We may consider the situation where the specializations $\rho_k^\pm$ are not necessarily nonnegative and u,v are possibly complex numbers: we lose the probabilistic interpretation, but the partition function and the correlation functions may still be well-defined. We shall typically assume that $|u|,|v|<1$ and that $p_n(\rho_k^\pm) = O(r^n)$ with $r<1$ as $n\to\infty$ for each specialization, in which case all the factors $H,\tilde H$ in the expression (2.10) for the partition function are finite by (2.6). Contours may then be chosen as in Remark 2.5.

2.1.3 Point process and correlation functions

Following [OR03], we define the *point configuration* associated with a sample $(\vec{\lambda}, \vec{\mu})$ of the free boundary Schur process as

$$\mathfrak{S}(\vec{\lambda}) := \left\{ \left(i, \lambda_j^{(i)} - j + \frac{1}{2} \right), 1 \leq i \leq N, j \geq 1 \right\} \subset \mathbb{Z} \times \mathbb{Z}' \tag{2.13}$$

where $\mathbb{Z}' := \mathbb{Z} + 1/2$ (having half-integer ordinates makes formulas slightly more symmetric). This is a simple point process on $\mathbb{Z} \times \mathbb{Z}'$. Note that there is no loss of generality in considering only the partitions $\lambda^{(1)}, \ldots, \lambda^{(N)}$ in the definition of the point configuration, and this makes the forthcoming formulas more compact. One may study the statistics of the μ's by considering an auxiliary Schur process with increased length and zero specializations inserted where appropriate (as $s_{\lambda/\mu}(0) = \mathbb{1}_{\lambda=\mu}$).

One free boundary. Let us first discuss the previously known case of the pfaffian Schur process [BR05], obtained for $u = 0$. By homogeneity of the Schur functions, we may assume $v = 1$ without loss of generality.

Theorem 2.7. *For $u = 0$ and $v = 1$, $\mathfrak{S}(\vec{\lambda})$ is a pfaffian point process whose correlation kernel entries are given by*

$$\begin{aligned}
K_{1,1}(i,k;i',k') &= \frac{1}{(2\mathrm{i}\pi)^2} \oint_{C_i^{(1)}} \frac{\mathrm{d}z}{z^{k+1}} \oint_{C_{i'}^{(1)}} \frac{\mathrm{d}w}{w^{k'+1}} F(i,z)F(i',w)\kappa_{1,1}(z,w), \\
K_{1,2}(i,k;i',k') &= -K_{2,1}(i',k';i,k) = \frac{1}{(2\mathrm{i}\pi)^2} \oint_{C_i^{(1)}} \frac{\mathrm{d}z}{z^{k+1}} \oint_{C_{i'}^{(2)}} \frac{\mathrm{d}w}{w^{-k'+1}} \frac{F(i,z)}{F(i',w)} \kappa_{1,2}(z,w), \\
K_{2,2}(i,k;i',k') &= \frac{1}{(2\mathrm{i}\pi)^2} \oint_{C_i^{(2)}} \frac{\mathrm{d}z}{z^{-k+1}} \oint_{C_{i'}^{(2)}} \frac{\mathrm{d}w}{w^{-k'+1}} \frac{1}{F(i,z)F(i',w)} \kappa_{2,2}(z,w)
\end{aligned} \tag{2.14}$$

where the contours are as in Lemma 2.4 and

$$\begin{aligned}
F(i,z) &= \frac{\prod_{1\leq \ell \leq i} H(\rho_\ell^+; z)}{H(\rho^+; z^{-1}) \prod_{i \leq \ell \leq N} H(\rho_\ell^-; z^{-1})}, & \kappa_{1,1}(z,w) &= \frac{(z-w)\sqrt{zw}}{(z+1)(w+1)(zw-1)}, \\
\kappa_{1,2}(z,w) &= \frac{(zw-1)\sqrt{zw}}{(z+1)(w-1)(z-w)}, & \kappa_{2,2}(z,w) &= \frac{(z-w)\sqrt{zw}}{(z-1)(w-1)(zw-1)}.
\end{aligned} \tag{2.15}$$

Remark 2.8. The double contour integrals in (2.14) correspond to extracting coefficients in certain bivariate Laurent series. Note that only integer powers are involved since the $\sqrt{zw}$ in κ's is compensated by k, k' being half-integers. The conditions on the contours from Lemma 2.4 mean intuitively that, in each factor $F(i,z)^{\pm 1}$, a $H(\cdot; z)$ should be thought as a series in z and a $H(\cdot; z^{-1})$ as a series in z^{-1} (and similarly for w), while the $\kappa(z,w)$ should be thought as bivariate series in z^{-1} and w^{-1}. In $\kappa_{1,2}(z,w)$, the pole $1/(z-w)$ should be expanded as $\sum_{\ell \geq 0} w^k/z^{k+1}$ for $i \leq i'$, and as $-\sum_{\ell<0} w^k/z^{k+1}$ otherwise.

Remark 2.9. Our expressions do not quite match those of [BR05, Theorem 3.3] mainly because Borodin and Rains impose that the "free boundary" is a partition whose conjugate has even parts. This change is inessential, and it is possible to go from one convention to another by a simple change of the boundary specialization. Actually, one can *interpolate* between the two by multiplying the weight (1.2) by an extra factor α^{oc} where oc denotes the number of odd columns of the Young diagram of $\mu^{(N)}$, and where α is a nonnegative parameter. With this extra weighting, Theorem 2.7 still holds provided that we modify the κ's into

$$\begin{aligned}
\kappa_{1,1}(z,w) &= \frac{(z-\alpha)(w-\alpha)(z-w)\sqrt{zw}}{(z^2-1)(w^2-1)(zw-1)}, & \kappa_{1,2}(z,w) &= \frac{(zw-1)(z-\alpha)\sqrt{zw}}{(z^2-1)(w-\alpha)(z-w)}, \\
\kappa_{2,2}(z,w) &= \frac{(z-w)\sqrt{zw}}{(z-\alpha)(w-\alpha)(zw-1)}
\end{aligned} \tag{2.16}$$

and that we slightly adapt the contours for $\alpha > 1$ — see Section 3.3.2 for more details. For $\alpha = 1$ we get back (2.15), while for $\alpha = 0$ we recover [BR05, Theorem 3.3] up to a simple change of variables.

Two free boundaries. We now turn to the general case of two free boundaries. Similarly to the periodic Schur process studied in [Bor07], the random point process $\mathfrak{S}(\vec{\lambda})$ is *neither* determinantal nor pfaffian in general, but a modification of it is. More precisely, let us fix an auxiliary real parameter t, and consider a $\mathbb{Z}$-valued random variable D_t independent of the Schur process, with law

$$\mathrm{Prob}(D_t = d) = \frac{t^{2d}(uv)^{2d^2}}{\theta_3(t^2;(uv)^4)}. \tag{2.17}$$

Here the normalization factor involves the Jacobi theta function $\theta_3(z;q) := \sum_{n\in\mathbb{Z}} q^{n^2/2} z^n$ — see Appendix B. We then consider the shifted point configuration

$$\mathfrak{S}_t(\vec{\lambda}) := \mathfrak{S}(\vec{\lambda}) + (0, 2D_t), \tag{2.18}$$

that is to say we move all points of $\mathfrak{S}(\vec{\lambda})$ vertically by a same shift $2D_t$. Note that, in contrast with the periodic Schur process [Bor07], we have to shift the point configuration by an *even* integer. As we shall see, the origin of this shift in the free fermion formalism is rather different.

Theorem 2.10. *The point process $\mathfrak{S}_t(\vec{\lambda})$ is pfaffian, and the entries of its correlation entries still have the form* (2.14) *with F and κ now given by*

$$\begin{aligned}
F(i,z) &= \frac{\prod_{1\le\ell\le i} H(\rho_\ell^+;z)}{\prod_{i\le\ell\le N} H(\rho_\ell^-;z^{-1})} \cdot \prod_{n\ge 1} \frac{H(u^{2n}v^{2n-2}\rho^-;z)H(u^{2n}v^{2n}\rho^+;z)}{H(u^{2n-2}v^{2n}\rho^+;z^{-1})H(u^{2n}v^{2n}\rho^-;z^{-1})},\\
\kappa_{1,1}(z,w) &= \frac{v^2}{tz^{1/2}w^{3/2}} \cdot \frac{((uv)^2;(uv)^2)_\infty^2}{(uz,uw,-\frac{v}{z},-\frac{v}{w};uv)_\infty} \cdot \frac{\theta_{(uv)^2}(\frac{w}{z})}{\theta_{(uv)^2}(u^2zw)} \cdot \frac{\theta_3\left((\frac{tzw}{v^2})^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)},\\
\kappa_{1,2}(z,w) &= \frac{w^{1/2}}{z^{1/2}} \cdot \frac{((uv)^2;(uv)^2)_\infty^2}{(uz,-uw,-\frac{v}{z},\frac{v}{w};uv)_\infty} \cdot \frac{\theta_{(uv)^2}(u^2zw)}{\theta_{(uv)^2}(\frac{w}{z})} \cdot \frac{\theta_3\left((\frac{tz}{w})^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)},\\
\kappa_{2,2}(z,w) &= \frac{tv^2}{z^{1/2}w^{3/2}} \cdot \frac{((uv)^2;(uv)^2)_\infty^2}{(-uz,-uw,\frac{v}{z},\frac{v}{w};uv)_\infty} \cdot \frac{\theta_{(uv)^2}(\frac{w}{z})}{\theta_{(uv)^2}(u^2zw)} \cdot \frac{\theta_3\left((\frac{tv^2}{zw})^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)}
\end{aligned} \tag{2.19}$$

where $(a_1,\dots,a_m;q)_\infty := \prod_{k=0}^\infty (1-a_1q^k)\cdots(1-a_mq^k)$ is the infinite q-Pochhammer symbol with multiple arguments, and $\theta_q(z) := (z;q)_\infty(q/z;q)_\infty$ is the "multiplicative" theta function — see Appendix B.

Several remarks are now in order:

1. We recover of course Theorem 2.7 for $u=0$ and $v=1$, as $D_t = 0$ hence $\mathfrak{S}(\vec{\lambda}) = \mathfrak{S}_t(\vec{\lambda})$.

2. Remark 2.8 still provides some "intuition" regarding the choice of contours: they should encircle certain poles of the integrands and not others, in order to pick the appropriate expansions of $H(\cdot;z)$ and $H(\cdot;z^{-1})$. More precisely, if $F(i,z)$ is in the numerator (respectively in the denominator), the z contour should encircle 0 and all the negative poles of F but none of the positive ones (respectively, 0 and all the positive zeros of F but none of the negative ones), as in [BBC+16].

   The main complication lies in the kernels $\kappa(z,w)$: they actually describe the free boundary Schur process of length $N=0$, for which $F\equiv 1$, and which is nothing but a single random partition drawn according to the $(uv)^{\mathrm{size}}$ measure. See also [Bor07, Corollary 2.6] for a related observation.

3. As in the case of one free boundary, $\kappa_{1,1}(z,w)$ and $\kappa_{2,2}(z,w)$ have a simple zero at $z=w$ while $\kappa_{1,2}(z,w)$ has a simple pole, due to the $\theta_{(uv)^2}(w/z)$ factor appearing in the numerator or denominator. Note that, according to Lemma 2.4 , the integration contours for z and w both lie in the annulus $\{v < |\cdot| < u^{-1}\}$, so we cannot hit any other zero of the factors $\theta_{(uv)^2}(w/z)$ and $\theta_{(uv)^2}(u^2zw)$, hence no other pole of $\kappa(z,w)$.

4. The fact that we have an arbitrary parameter t at our disposal allows to return to the correlation functions for the unshifted point process $\mathfrak{S}(\vec{\lambda})$. At a formal level, the n-point correlation function $\mathrm{Prob}\left(U\subset\mathfrak{S}(\vec{\lambda})\right)$ is equal to the coefficient of t^0 in $\theta_3(t^2;(uv)^4)\,\mathrm{Prob}\left(U\subset\mathfrak{S}_t(\vec{\lambda})\right)$, which can be

expressed as a contour integral over t, the correlation functions for $\mathfrak{S}_t(\vec{\lambda})$ being manifestly analytic functions of t. Actually, it is possible to obtain a general expression for the n-point correlation functions of both $\mathfrak{S}_t(\vec{\lambda})$ and $\mathfrak{S}(\vec{\lambda})$ in the form of a $2n$-fold contour integral — see Theorem 3.14.

5. It is still possible to introduce an extra weighting α^{oc} as in Remark 2.9, and possibly another one for the other free boundary $\mu^{(0)}$. This can be achieved by a simple modification of the boundary specializations ρ_1^+ and ρ_N^-, or of the kernels $\kappa(z,w)$. See Section 3.3.2.

2.2 Applications

We now present some applications of Theorem 2.7 to last passage percolation, symmetric plane partitions and plane overpartitions. We plan to present applications of Theorem 2.10 in a subsequent paper.

In our applications, all ρ_k^- are equal to the zero specialization. The weight (1.2) is then nonzero only for sequences (1.1) such that $\lambda^{(k)} = \mu^{(k)}$ for all k, which can be seen more simply as *ascending* sequences of partitions $\emptyset \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)}$. Furthermore, each specialization ρ_k^+ will be either a specialization in a single variable x_k (i.e. $H(\rho_k^+; z) = (1 - x_k z)^{-1}$) or its "dual" ($H(\rho_k^+; z) = 1 + x_k z$). Recall that, for a single variable x, we have $s_{\lambda/\mu}(x) = x^{|\lambda|-|\mu|}\mathbb{1}_{\lambda \succ \mu}$ where the notation $\lambda \succ \mu$ means that the skew shape λ/μ is a *horizontal strip* (i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$). Similarly, for the dual specialization $\bar{x}$, we have $s_{\lambda/\mu}(\bar{x}) = s_{\lambda'/\mu'}(x) = x^{|\lambda|-|\mu|}\mathbb{1}_{\lambda \succ' \mu}$ where the notation $\lambda \succ' \mu$ means that the skew shape λ/μ is a *vertical strip* (i.e. $\lambda_i - \mu_i \in \{0, 1\}$ for all i).

The *H-ascending Schur process* consists in taking only specializations in single variables. In that case, we obtain a measure over sequences of the form

$$\emptyset \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(N-1)} \prec \lambda^{(N)}. \tag{2.20}$$

The *HV-ascending Schur process* consists in taking alternatively a specialization in a single variable or a dual variable, to get a measure over sequences

$$\emptyset \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \cdots \prec \lambda^{(2M-1)} \prec' \lambda^{(2M)}, \qquad N = 2M. \tag{2.21}$$

In both cases, the unnormalized weight of a sequence will be

$$x_1^{|\lambda^{(1)}|} x_2^{|\lambda^{(2)}|-|\lambda^{(1)}|} \cdots x_N^{|\lambda^{(N)}|-|\lambda^{(N-1)}|}, \tag{2.22}$$

possibly with the extra weighting α^{oc} of Remark 2.9. For convenience, we state the following:

Proposition 2.11. *For the H- and HV-ascending Schur processes, the function F appearing in Theorem 2.7 reads respectively*

$$F_H(i,z) = \frac{\prod_{k=1}^{N}(1 - x_k/z)}{\prod_{k=1}^{i}(1 - x_k z)}, \qquad F_{HV}(i,z) = \frac{\prod_{k=1}^{\lfloor i/2 \rfloor}(1 + x_{2k} z)}{\prod_{k=1}^{\lceil i/2 \rceil}(1 - x_{2k-1} z)} \prod_{k=1}^{M} \frac{(1 - x_{2k-1}/z)}{(1 + x_{2k}/z)}. \tag{2.23}$$

In the next three subsections we describe the main results stated and proved in the applications parts of the paper: Sections 4, 5 and 6.

2.2.1 Symmetric last passage percolation

The *last passage percolation (LPP) time* through a symmetric $n \times n$ nonnegative (integer or real) valued matrix ω is the maximum sum one can collect over all up-right paths going through the matrix from the bottom left entry to the upper right entry. We note our matrices, if embedded in the plane, are symmetric around the $x = y$ diagonal. See Figure 1 for an example.

In the present work we consider the LPP time with symmetric, and (up to symmetry) independent geometric weights $\{\omega_{r,t}\}_{r,t \in \mathbb{Z}_{\geq 1}}$. These weights (i.e. random variables) are given by

$$\omega_{r,t} = \omega_{t,r} \sim \begin{cases} g(a_r a_t), & \text{if} \quad r \neq t, \\ g(\alpha a_r), & \text{if} \quad r = t \end{cases} \tag{2.24}$$

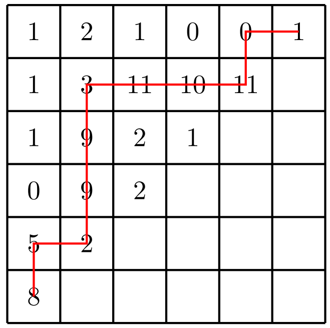

Figure 1: An example of a 6×6 symmetric matrix ω (only diagonal and above-diagonal elements shown) filled with nonnegative integers, and the last passage percolation time of 69 (the sum of the elements on the red path).

where $a_n, \alpha a_n \in (0,1), n \geq 1$, and $\mathrm{Prob}(g(q) = k) = q^k(1-q)$ for $k \in \mathbb{Z}_{\geq 0}$. For $(n_1, n_2), (r,t) \in \mathbb{Z}^2_{\geq 1}$ with $n_1 \leq r, n_2 \leq t$, consider up-right paths π from (n_1, n_2) to (r,t), i.e. $\pi = (\pi(0), \pi(1), \ldots, \pi(r - n_1 + t - n_2))$ with $\pi(0) = (n_1, n_2), \pi(r - n_1 + t - n_2) = (r,t)$ and $\pi(i) - \pi(i-1) \in \{(0,1),(1,0)\}$. The symmetric LPP time with geometric weights (2.24) is then defined to be

$$L_{(n_1,n_2)\to(r,t)} := \max_{\pi:(n_1,n_2)\to(r,t)} \sum_{(m,n)\in\pi} \omega_{m,n}. \tag{2.25}$$

Under the RSK bijection, LPP times become the largest part of integer partitions — see Figure 2 for a simulation. When considering $L_{(n_1,n_2)\to(r_l,t_l)}, l = 1, \ldots, k$ with (r_l, t_l) lying on a down-right path, these partitions form a Schur process with one free boundary, which is H-ascending for (r_l, t_l) lying on a horizontal line — see Section 4.1.1 for more details. Consequently, the event $\cap_{l=1}^k \{L_{(1,1)\to(r_l,t_l)} \leq s_l\}$ becomes the event that a point configuration (2.13) has no points in a set B. Such gap probabilities are given by Fredholm pfaffians since we have a pfaffian point process by Theorem 2.7 — see Appendix A for the definition of Fredholm pfaffians. This leads to the following theorem, which will be proven as Theorem 4.5 in Section 4.1.1.

Theorem 2.12. *Consider the LPP time* (2.25) *with weights* (2.24)*. Let $r_l, t_l \in \mathbb{Z}_{\geq 1}, r_l \leq t_l, l = 1, \ldots, k$ with $r_1 \leq \ldots \leq r_k, t_1 \geq \ldots \geq t_k$. Then*

$$\mathrm{Prob}\left(\bigcap_{l=1}^k \{L_{(1,1)\to(r_l,t_l)} \leq s_l\}\right) = \mathrm{pf}(J-K)_B, \tag{2.26}$$

where K, B are given in Theorem 4.5.

The identity (2.26) now allows us to extract asymptotics of the LPP time $L_{(1,1)\to(r,t)}$ as $r, t \to \infty$. The limiting fluctuations will, of course, depend on the end point (r,t) and the choice of parameters a_n, α. Here, we do not aim to exploit Theorem 2.12 in all possible directions. In the following theorem we fix $a_n = \sqrt{q}$ and choose α and the endpoint such that we are in a crossover regime, from which different limit laws can be recovered. The following Theorem will be proven as Theorem 4.6 in Section 4.1.1.

Theorem 2.13. *Consider the weights* (2.24) *with $a_j = \sqrt{q}, q \in (0,1), j \geq 1$ and $\alpha = 1 - 2vc_qN^{-1/3}$, where $c_q = \frac{1-\sqrt{q}}{q^{1/6}(1+\sqrt{q})^{1/3}}, v \in \mathbb{R}$. Let $u_1 > \cdots > u_k \geq 0$. Then*

$$\begin{aligned} \lim_{N\to\infty} \mathrm{Prob}\left(\bigcap_{i=1}^k \left\{L_{(1,1)\to(N-\lfloor u_i N^{2/3}\rfloor,N)} \leq \frac{2\sqrt{q}N}{1-\sqrt{q}} - u_i\frac{\sqrt{q}N^{2/3}}{1-\sqrt{q}} + c_q^{-1}s_iN^{1/3}\right\}\right) \\ = \mathrm{pf}(J - \chi_s K^v \chi_s)_{\{u_1,\ldots,u_k\}\times\mathbb{R}} \end{aligned} \tag{2.27}$$

where $\chi_s(u_i, x) = \mathbf{1}_{x>s_i}$ and K^v is defined in (4.10)*.*

Specializing to $k = 1$ we obtain in particular

$$\lim_{N\to\infty} \operatorname{Prob}\left(L_{(1,1)\to(N-\lfloor uN^{2/3}\rfloor,N)} \le \frac{2\sqrt{q}N}{1-\sqrt{q}} - u\frac{\sqrt{q}N^{2/3}}{1-\sqrt{q}} + c_q^{-1} s N^{1/3} \right) = \operatorname{pf}(J - K^v)_{(s,\infty)} =: F_{u,v}(s). \tag{2.28}$$

$F_{u,v}(s)$ performs a crossover between the classical distributions from random matrix theory. Namely, one has $F_{0,0} = F_{\mathrm{GOE}}$, $\lim_{v\to+\infty} F_{0,v}(s) = F_{\mathrm{GSE}}(s)$ and $\lim_{u\to+\infty} F_{u,v}(s - u^2 d_q^2) = F_{\mathrm{GUE}}(s)$, where $d_q = \frac{q^{1/6}}{2(1+\sqrt{q})^{2/3}}$ — see Section 4.0.1. Finally, as a small digression from the main topic of this paper, in Theorem 4.7 we consider the symmetric LPP with exponential weights, and give a short probabilistic proof that, in certain cases, away from the diagonal one observes the same fluctuations as in the unsymmetrized case, simply because the difference between the two rescaled LPP times converges to zero.

We point out that the symmetrized LPP has been studied by several authors. For $k = 1, u = 0$ Theorem 2.13 recovers results already obtained by Baik and Rains [BR99, BR01a, BR01b]. Furthermore, [SI04] considered, as half-space PNG, off-diagonal fluctuations as well as the transition $u \to +\infty$ mentioned above. For symmetric LPP with exponential weights, the same kind of crossover between $F_{\mathrm{GSE}}, F_{\mathrm{GUE}}, F_{\mathrm{GOE}}$ was obtained recently in [BBCS17b].

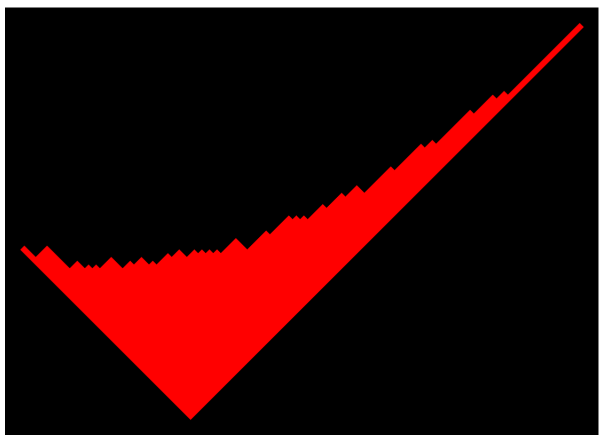

Figure 2: A random partition λ, in Russian notation, whose weight is proportional to $s_\lambda(q, \dots, q)$ (100 q's) for $q = 1/3$. The right arm (its first part) is, via the RSK correspondence, the last passage percolation time in a 100×100 symmetric matrix filled with otherwise i.i.d. geometric random numbers of parameter q^2.

2.2.2 Symmetric plane partitions

A *symmetric plane partition of length* N is a plane partition — i.e. an array of numbers $(\pi_{i,j})_{1\le i,j\le N}$ such that $\pi_{i,j} \ge \pi_{i,j+1}, \pi_{i,j} \ge \pi_{i+1,j}$, satisfying the symmetry condition $\pi_{i,j} = \pi_{j,i}$. It can be viewed as a symmetric pile of cubes stacked into the corner of a room or a lozenge tiling of the plane. See Figure 3 below for pictorial descriptions. A symmetric plane partition (more precisely, the half of it that determines the whole — Figure 3 on the left) can be sliced into ordinary partitions $\vec{\lambda} = (\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N)})$ with $\ell(\lambda^{(i)}) \le i$ using the simple formula $\lambda_k^{(i)} = \pi_{N-i+k,k}$ for $1 \le k \le i$. We choose a q^{Volume} measure, for $q \in (0, 1)$, which can be treated as an H-ascending Schur process for appropriately chosen (single variable) specializations.

In Section 5 we consider the large volume limit of symmetric plane partitions: $q = e^{-r} \to 1$ for scaling $ri \to \chi$ and $rk \to y$ where $(\chi, y) \in \mathbb{R}_+ \times \mathbb{R}$. A sample of a symmetric random plane for q close to 1 is given in Figure 4 (left, $rN \to \infty$), while one when $rN \to a$ is on the right. In both cases there are two distinct regions: the liquid region $\mathcal{L}$ where behavior is random, and the frozen region where behavior is deterministic. One can visualize this as in the figure using different colors for the three types of lozenges in the plane tiling.

The arctic curve, diving the liquid and frozen region is the zero locus of

$$D(\mathcal{X}, \mathcal{Y}) = -4(1 - \mathcal{A}\mathcal{Y})(\mathcal{X} - \mathcal{A}\mathcal{Y}) + (-1 - \mathcal{X} + \mathcal{Y} + \mathcal{A}^2\mathcal{Y})^2 \tag{2.29}$$

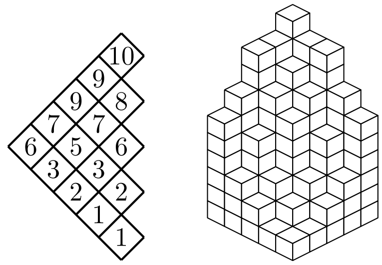

Figure 3: A symmetric plane partition: the numbers on the left determine the heights of cubes on the right.

where $\mathcal{X} = \exp(-\chi)$, $\mathcal{Y} = \exp(-y)$, and $\mathcal{A} = \exp(-a)$.

Case $\mathcal{A} = 0$ corresponds to symmetric plane partitions with no bound on the length, and the liquid region can be written in appropriate (u, v) coordinates as the amoeba of the polynomial $1 + u + v$ — see Section 5 for the definition of amoeba. As expected, one obtains the same liquid region as for non-symmetric plane partitions. The arctic shape for plane partitions was first obtained by Blöte, Hilhorst and Nienhuis [NHB84] in the physics literature and Cerf and Kenyon [CK01] in the mathematics literature. It was later rederived by Okounkov and Reshetikhin [OR03] using the Schur process.

The limit shape result can be derived from the following explicit formula for the density of particles.

Proposition 2.14. *For $(\chi, y) \in \mathcal{L}$ the density of particles is*

$$\rho(\chi, y) = \frac{\theta(\chi, y)}{\pi} \tag{2.30}$$

where $\theta(\chi, y) = \arg(z_\pm(\chi, y))$ and

$$z_\pm(\chi, y) = \frac{1 + \mathcal{X} - (1 + \mathcal{A}^2)\mathcal{Y} \pm \sqrt{D(\mathcal{X}, \mathcal{Y})}}{2(\mathcal{X} - \mathcal{A}\mathcal{Y})}. \tag{2.31}$$

To derive this result we start from the finite correlations given by Theorem 2.7, changing $i \to N - i$ for convenience, and perform steepest descent analysis to obtain the limiting behavior of the correlation kernel. In the limit we obtain a (incomplete beta) determinantal process for $\chi > 0$ and a pfaffian process for $\chi = 0$.

Let

$$i = \left\lfloor \frac{\chi}{r} \right\rfloor + \mathsf{i}, \qquad k = \left\lfloor \frac{y}{r} \right\rfloor + \mathsf{k}. \tag{2.32}$$

Theorem 2.15. *Let $(\chi, y) \in \mathcal{L}$. As $r \to 0+$ the rescaled process, with rescaling given by (2.32),*

- *for $\chi = 0$ converges to a pfaffian process with kernel*

$$\begin{aligned}
\mathsf{K}_{1,1}(\mathsf{i}, \mathsf{k}; \mathsf{i}', \mathsf{k}') &= \int_{\gamma_+} (1 - z)^{\mathsf{i}} \left(1 - \frac{1}{z}\right)^{\mathsf{i}'} z^{\mathsf{k}' - \mathsf{k} + 1} \frac{z - 1}{z + 1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \\
\mathsf{K}_{1,2}(\mathsf{i}, \mathsf{k}; \mathsf{i}', \mathsf{k}') &= \int_{\gamma_\pm} (1 - z)^{\mathsf{i} - \mathsf{i}'} z^{\mathsf{k}' - \mathsf{k} - 1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \\
\mathsf{K}_{2,2}(\mathsf{i}, \mathsf{k}; \mathsf{i}', \mathsf{k}') &= \int_{\gamma_-} (1 - z)^{-\mathsf{i}} \left(1 - \frac{1}{z}\right)^{-\mathsf{i}'} z^{\mathsf{k} - \mathsf{k}' - 1} \frac{1 + z}{1 - z} \frac{\mathrm{d}z}{2\pi \mathrm{i}}
\end{aligned} \tag{2.33}$$

  where γ_+ is taken if and only if $\mathsf{i} \geq \mathsf{i}'$. $\gamma_\pm$ is a contour from $z_-(\chi, y)$ to $z_+(\chi, y)$, γ_+ passing to the right of 0 and γ_- to left of 0;

- *for $\chi \neq 0$ converges to a determinantal process with kernel $\mathsf{K}_{1,2}$ as in (2.33).*

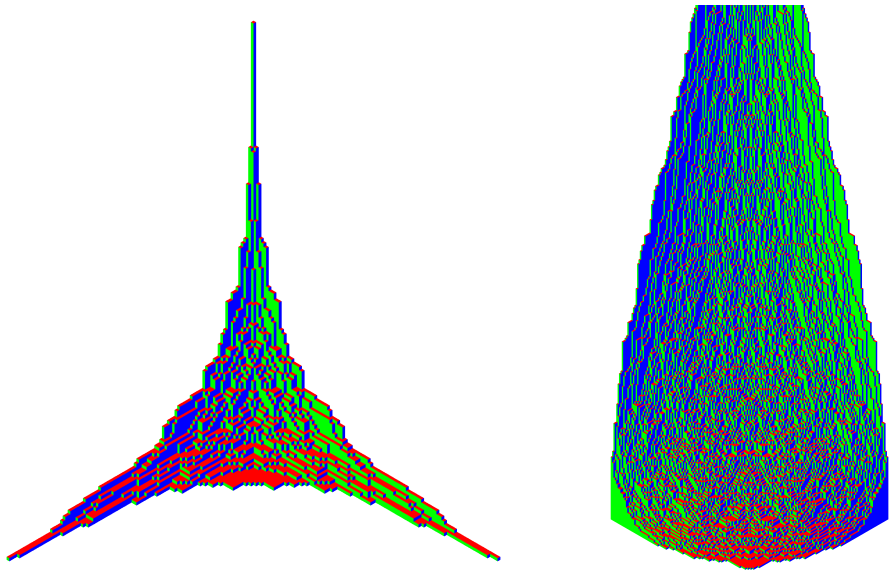

Figure 4: Large unbounded (left) and bounded (right) symmetric plane partitions.

2.2.3 Plane overpartitions

A *plane overpartition* is a plane partition where in each row the last occurrence of an integer can be overlined or not and all the other occurrences of this integer are not overlined, while in each column the first occurrence of an integer can be overlined or not and all the other occurrences of this integer are overlined. An example is given in the Figure 5. There is a natural measure one can study on plane overpartitions: the q^{Volume} measure, where the volume is given by the sum of all its entries.

A plane overpartition with the largest entry at most N and shape λ can be recorded as a sequence of partitions $\vec{\lambda} = (\emptyset \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \cdots \prec \lambda^{(2n-1)} \prec' \lambda^{(2N)} = \lambda)$ where $\lambda^{(i)}$ is the partition whose shape is formed by all fillings greater than $N - i/2$ with the convention that $\overline{k} = k - 1/2$. In this context, with the q^{Volume} measure considered, the sequence $\vec{\lambda}$ becomes an HV-ascending Schur process for appropriately chosen (single or dual variable) specializations.

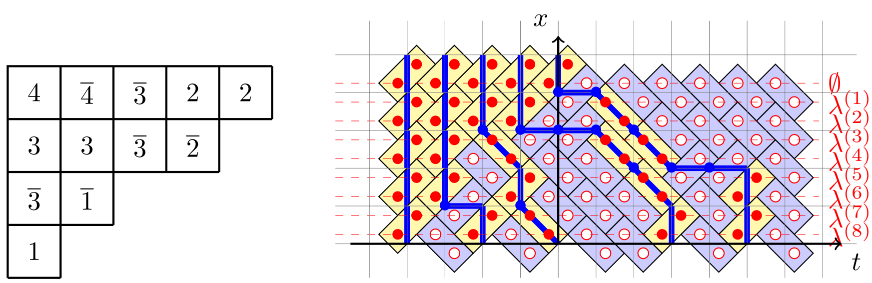

Figure 5: A plane overpartition (left) and its associated point configuration (right).

Plane overpartitions are in bijection with domino tilings — see Figure 5 (right). We hope that the "picture is worth a thousand words", but for more details see Section 6. In Section 6 we consider the large volume limit of plane overpartitions: $q = e^{-r} \to 1$ for scaling $ri \to \chi$ and $rk \to y$. We consider case $N = \infty$ only and leave $rN \to a$ for subsequent work. A sample for q close to 1 is given Figure 6.

The liquid region $\mathcal{L}$ is half of the amoeba of the polynomial $-1 + u + v + uv$ for the right choice of coordinates (u, v). The density in the liquid region is as in Proposition 2.14 for different $z_{\pm}$, with the explicit expression given in Section 6. This was originally obtained in [Vul07].

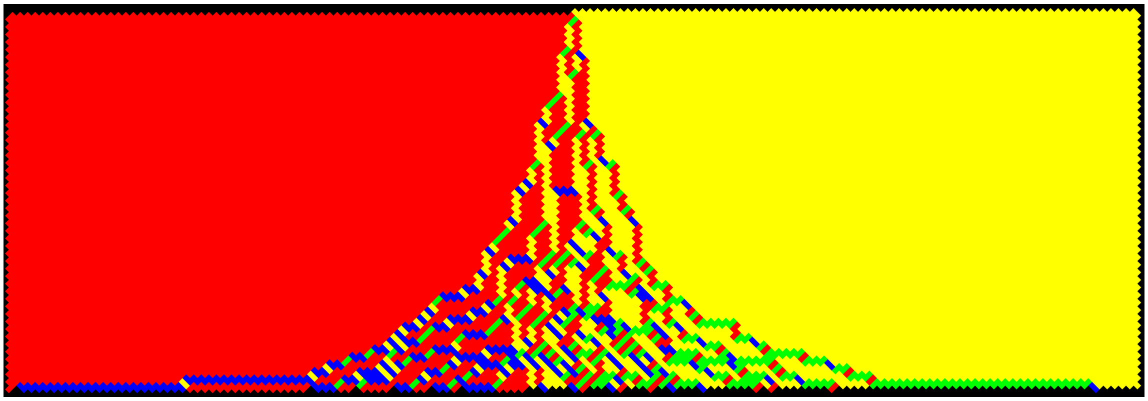

Figure 6: A large plane overpartition as a domino tiling.

We analyze the pfaffian local correlations given by Theorem 2.7 in the limit and obtain an analogue of Theorem 2.15.

Theorem 2.16. *Let $(\chi, y) \in \mathcal{L}$. As $r \to 0+$ the rescaled process, where the rescaling is given precisely in Section 6,*

- *for $\chi = 0$ converges to a pfaffian process with kernel*

$$\begin{aligned} \mathsf{K}_{1,1}(\mathsf{i},\mathsf{k};\mathsf{i}',\mathsf{k}') &= \int_{\gamma_+} \left(\frac{1-z}{1+z}\right)^{\mathsf{i}+\mathsf{i}'+1} z^{\mathsf{k}-\mathsf{k}'+1} \frac{\mathrm{d}z}{2\pi\mathrm{i}}, \\ \mathsf{K}_{1,2}(\mathsf{i},\mathsf{k};\mathsf{i}',\mathsf{k}') &= \int_{\gamma_\pm} \left(\frac{1-z}{1+z}\right)^{\mathsf{i}-\mathsf{i}'} z^{\mathsf{k}'-\mathsf{k}-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}}, \\ \mathsf{K}_{2,2}(\mathsf{i},\mathsf{k};\mathsf{i}',\mathsf{k}') &= \int_{\gamma_-} \left(\frac{1-z}{1+z}\right)^{-\mathsf{i}-\mathsf{i}'-1} z^{\mathsf{k}'-\mathsf{k}-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \end{aligned} \tag{2.34}$$

  where γ_+ is taken if and only if $\mathsf{i} \geq \mathsf{i}'$ and $\gamma_\pm$ are defined as in Theorem 2.15;

- *for $\chi \neq 0$ converges to a determinantal process with kernel $\mathsf{K}_{1,2}$ as in (2.34).*

3 Free fermions

This section is devoted to the proof of the results presented in Section 2.1 via the free fermion formalism.

3.1 Preliminaries

3.1.1 Notations and reminders

Here we recall the standard material which is useful for the study of the usual Schur process [OR03], following the notation conventions of [Oko01, Appendix A]. See also [JM83, MJD00, AZ13] and [Kac90, Chapter 14].

Admissible sets and partitions. Let us denote by $\mathbb{Z}' := \mathbb{Z} + 1/2$ the set of half-integers. We say that a subset S of $\mathbb{Z}'$ is *admissible* if it has a greatest element and its complement has a least element. Equivalently, we require $S_+ := S \setminus \mathbb{Z}'_{<0}$ and $S_- := \mathbb{Z}'_{<0} \setminus S$ to be both finite. We denote by $\mathcal{S}$ the set of admissible subsets. To each $S \in \mathcal{S}$ we may associate its *charge* $C(S)$ and its *energy* $H(S)$ defined by

$$C(S) := |S_+| - |S_-|, \qquad H(S) := \sum_{k \in S_+} k - \sum_{k \in S_-} k. \tag{3.1}$$

Clearly the energy is nonnegative and vanishes if and only if $S = \mathbb{Z}'_{\leq 0}$. The set of partitions being denoted $\mathcal{P}$, there is a well-known bijection between $\mathcal{S}$ and $\mathcal{P} \times \mathbb{Z}$ (the "combinatorial boson–fermion correspondence"): to each partition $\lambda \in \mathcal{P}$ and integer $c \in \mathbb{Z}$, we associate the admissible set

$$\mathfrak{S}(\lambda, c) := \{\lambda_i - i + 1/2 + c,\ i \geq 1\}. \tag{3.2}$$

It is not difficult to see that $\mathfrak{S}(\lambda, c)$ has charge c and energy $|\lambda| + c^2/2$.

Fock space and fermionic operators. The *fermionic Fock space*, denoted $\mathcal{F}$, is the infinite dimensional Hilbert space spanned by the orthonormal basis $|S\rangle, S \in \mathcal{S}$. Here we use the bra–ket notation and will denote by $\langle\cdot|$ dual vectors. We may think of a basis vector $|S\rangle$ as the semi-infinite wedge product

$$|S\rangle = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots \tag{3.3}$$

where $s_1 > s_2 > s_3 > \cdots$ are the elements of S, and $\{\underline{k},\ k \in \mathbb{Z}'\}$ is an orthonormal basis of some smaller "one-particle" vector space. For λ a partition and c an integer we introduce the shorthand notations

$$|\lambda, c\rangle := |\mathfrak{S}(\lambda, c)\rangle, \qquad |\lambda\rangle := |\lambda, 0\rangle, \qquad |c\rangle := |\emptyset, c\rangle. \tag{3.4}$$

The vector $|0\rangle$ is called the *vacuum*. The charge and energy naturally become diagonal operators acting on $\mathcal{F}$, which we still denote by C and H respectively. We also denote by R the *shift operator* such that $R|S\rangle = |S+1\rangle$ (i.e. all elements of S are incremented by 1).

We now define the *fermionic operators*: for $k \in \mathbb{Z}'$, let us define the operators ψ_k and ψ_k^* by

$$\psi_k|S\rangle := \begin{cases} 0, & \text{if } k \in S \\ (-1)^j|S \cup \{k\}\rangle, & \text{if } k \notin S \end{cases}, \qquad \psi_k^*|S\rangle := \begin{cases} (-1)^j|S \setminus \{k\}\rangle, & \text{if } k \in S \\ 0, & \text{if } k \notin S \end{cases} \tag{3.5}$$

where $j = |S \cap \mathbb{Z}'_{>k}|$. In the semi-infinite wedge picture, ψ_k corresponds to the exterior multiplication by $\underline{k}$ on the left, and ψ_k^* to its adjoint operator. They satisfy the canonical anticommutation relations

$$\{\psi_k, \psi_\ell^*\} = \delta_{k,\ell}, \qquad \{\psi_k, \psi_\ell\} = \{\psi_k^*, \psi_\ell^*\} = 0, \qquad k, \ell \in \mathbb{Z}' \tag{3.6}$$

where $\{a, b\} := ab + ba$. We also define the generating series

$$\psi(z) := \sum_{k \in \mathbb{Z}'} \psi_k z^k, \qquad \psi^*(w) := \sum_{k \in \mathbb{Z}'} \psi_k^* w^{-k}. \tag{3.7}$$

Observe that $\psi_k|0\rangle = \psi_{-k}^*|0\rangle = 0$ for $k < 0$. We now recall Wick's lemma in a form suitable for future generalizations — see for instance [BBC+16, Appendix B] for a proof.

Lemma 3.1 (Wick's lemma)**.** *Let Ψ be the vector space spanned by (possibly infinite linear combinations of) the ψ_k and ψ_k^*, $k \in \mathbb{Z}'$. For $\phi_1, \ldots, \phi_{2n} \in \Psi$, we have*

$$\langle 0|\phi_1 \cdots \phi_{2n}|0\rangle = \operatorname{pf} A \tag{3.8}$$

where A is the antisymmetric matrix defined by $A_{ij} := \langle 0|\phi_i\phi_j|0\rangle$ for $i < j$.

Bosonic and vertex operators. The *bosonic operators* α_n are defined by

$$\alpha_n := \sum_{k \in \mathbb{Z}'} \psi_{k-n}\psi_k^*, \qquad n = \pm 1, \pm 2, \ldots \tag{3.9}$$

and α_0 is the charge operator. We have $\alpha_n^* = \alpha_{-n}$, $\alpha_n|0\rangle = 0$ for $n > 0$, and the commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}, \qquad [\alpha_n, \psi(z)] = z^n\psi(z), \qquad [\alpha_n, \psi^*(w)] = -w^n\psi^*(w). \tag{3.10}$$

For ρ a specialization of the algebra of symmetric functions, we define the *vertex operators* $\Gamma_\pm(\rho)$ by

$$\Gamma_\pm(\rho) := \exp\left(\sum_{n\geq 1} \frac{p_n(\rho)\alpha_{\pm n}}{n}\right). \tag{3.11}$$

When x is a variable, we denote by $\Gamma_\pm(x)$ (resp. $\Gamma'_\pm(x)$) the vertex operators for the specialization in the single variable x (resp. its dual $\bar{x}$), for which $p_n(x) = x^n$ (resp. $p_n(\bar{x}) = (-1)^{n-1}x^n$). Clearly, $\Gamma_-(\rho)$ is the adjoint of $\Gamma_+(\rho)$ for any real ρ, and

$$\Gamma_+(\rho)|0\rangle = |0\rangle, \qquad \langle 0|\Gamma_-(\rho) = \langle 0|. \tag{3.12}$$

Given two specializations ρ, ρ', as $p_n(\rho \cup \rho') = p_n(\rho) + p_n(\rho')$, we have

$$\Gamma_+(\rho)\Gamma_+(\rho') = \Gamma_+(\rho \cup \rho') = \Gamma_+(\rho')\Gamma_+(\rho). \tag{3.13}$$

The commutation relations (3.10) and the Cauchy identity (2.6) imply that

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho) \tag{3.14}$$

while

$$\Gamma_\pm(\rho)\psi(z) = H(\rho; z^{\pm 1})\psi(z)\Gamma_\pm(\rho), \qquad \Gamma_\pm(\rho)\psi^*(w) = H(\rho; w^{\pm 1})^{-1}\psi^*(w)\Gamma_\pm(\rho). \tag{3.15}$$

These latter relations always make sense at a formal level; at an analytic level they require that the parameter of $H(\rho;\cdot)$ be within its disk of convergence. The crucial property of vertex operators is that skew Schur functions arise as their matrix elements, namely

$$\langle \lambda, c|\Gamma_+(\rho)|\mu, c'\rangle = \langle \mu, c'|\Gamma_-(\rho)|\lambda, c\rangle = \begin{cases} s_{\mu/\lambda}(\rho), & \text{if } c = c', \\ 0, & \text{otherwise.} \end{cases} \tag{3.16}$$

This results from (3.15), Wick's lemma and the Jacobi–Trudi identity.

Finally, we will use the fact that the fermionic operators can be reconstructed from the vertex operators and the charge and shift operators C and R (see e.g. [Kac90, Theorem 14.10]), a fact we will refer to as the *boson–fermion correspondence*.

Proposition 3.2. *We have:*

$$\begin{aligned} \psi(z) &= z^{C-\frac{1}{2}} R\, \Gamma_-(z)\Gamma'_+\left(-z^{-1}\right), \\ \psi^*(w) &= R^{-1} w^{-C+\frac{1}{2}}\Gamma'_-(-w)\Gamma_+\left(w^{-1}\right). \end{aligned} \tag{3.17}$$

3.1.2 Free boundary states and connection with the Schur process

Following [BCC17, Section 5.3], given two parameters u, v, we introduce the *free boundary states*

$$|\underline{v}\rangle := \sum_{\lambda\in\mathcal{P}} v^{|\lambda|}|\lambda\rangle, \qquad \langle \underline{u}| := \sum_{\lambda\in\mathcal{P}} u^{|\lambda|}\langle\lambda|. \tag{3.18}$$

Both are (respectively left and right) eigenvectors of the charge operator C with eigenvalue 0. For $u = v = 0$, we recover respectively the vacuum $|0\rangle$ and its dual $\langle 0|$. The following proposition generalizes (3.12) to arbitrary u, v.

Proposition 3.3 (Reflection relations)**.** *We have*

$$\Gamma_+(\rho)|\underline{v}\rangle = \tilde{H}(v\rho)\Gamma_-(v^2\rho)|\underline{v}\rangle, \qquad \langle\underline{u}|\Gamma_-(\rho) = \tilde{H}(u\rho)\Gamma_+(u^2\rho)\langle\underline{u}| \tag{3.19}$$

with $\tilde{H}$ defined as in (2.5)*.*

Proof. When projected on the standard basis, these relations amount to the identity [Mac95, I.5, Ex. 27(a), (3), p.93] (specialized at ρ), which is itself essentially a reformulation of the Littlewood identity. See also [BCC17, Proposition 10], which corresponds to the case where ρ is the specialization into a single variable t. It implies the general case by a standard argument. $\square$

Armed with all these definitions, we are now in position to make the connection with the free boundary Schur process. The remainder of this section is basically an adaptation of the arguments in [OR03] to the case of free boundaries, see also [BBC+16].

Proposition 3.4. *The partition of the free boundary Schur process is given by*

$$Z = \langle \underline{u}|\Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-)\cdots\Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-)|\underline{v}\rangle. \tag{3.20}$$

For U a finite subset of $\{1,\dots,N\}\times\mathbb{Z}'$, the probability $\varrho(U)$ that the point process $\mathfrak{S}(\vec{\lambda})$ contains U reads

$$\varrho(U) = \frac{Z_U}{Z} \tag{3.21}$$

where Z_U is obtained from the product in the rhs of (3.20) by inserting, for each $(i,k)\in U$, the operator $\psi_k\psi_k^*$ between $\Gamma_+(\rho_i^+)$ and $\Gamma_-(\rho_i^-)$. (If several points of U have the same abscissa, the $\psi_k\psi_k^*$ can be inserted in any order since they commute.)

Proof. This is a basic application of the transfer-matrix method: by (3.16), the vertex operators $\Gamma_\pm$ can be seen as transfer matrices for the Schur process, and the operator $\psi_k\psi_k^*$ "measures" whether there is a point at ordinate k (we have $\psi_k\psi_k^*|S\rangle = |S\rangle$ if $k\in S$ and 0 otherwise). $\square$

As a useful warm-up for the following, we may compute the partition function of the free boundary Schur process, using the method introduced in [BCC17, Section 5.3] (which we colloquially call *ping-pong*).

Proof of Proposition 2.2. In order to evaluate the product (3.20), the first step consists in "commuting" the Γ_+ to the right and the Γ_- to the left, using (3.14) and (3.13), to yield

$$Z = \prod_{1\le k\le \ell\le N} H(\rho_k^+;\rho_\ell^-)\times\langle \underline{u}|\Gamma_-(\rho^-)\Gamma_+(\rho^+)|\underline{v}\rangle \tag{3.22}$$

with $\rho^\pm$ as in (2.11). For $u=v=0$, i.e. for the original Schur process with "vacuum" boundary conditions, the rightmost factor is equal to 1 by (3.12). For general u, the reflection relations allow to write first

$$\langle \underline{u}|\Gamma_-(\rho^-)\Gamma_+(\rho^+)|\underline{v}\rangle = \tilde{H}(u\rho^-)\langle \underline{u}|\Gamma_+(\hat{\rho})|\underline{v}\rangle \tag{3.23}$$

with $\hat{\rho} = \rho^+\cup u^2\rho^-$. For $v=0$, i.e. for the pfaffian Schur process, the rightmost factor equals 1 and we are done. For $uv>0$, we need to use again the reflection relations infinitely many times to "bounce" the Γ back and forth:

$$\langle \underline{u}|\Gamma_+(\hat{\rho})|\underline{v}\rangle = \prod_{n\ge 1}\tilde{H}(u^{n-1}v^n\hat{\rho})\times\langle \underline{u}|\underline{v}\rangle. \tag{3.24}$$

Here we assume that $uv<1$ so that the argument of the "bouncing" Γ tends to the zero specialization as the number of reflections tends to infinity. From the definition of the free boundary states we have

$$\langle \underline{u}|\underline{v}\rangle = \sum_{\lambda\in\mathcal{P}}(uv)^{|\lambda|} = \prod_{n\ge1}\frac{1}{1-(uv)^n}. \tag{3.25}$$

Collecting all factors and rearranging them in a more symmetric manner (using the relation $\tilde{H}(\rho\cup\rho') = \tilde{H}(\rho)\tilde{H}(\rho')H(\rho;\rho')$ and other easy properties), we end up with the desired expression (2.10) for the partition function. The necessary and sufficient condition of finiteness is easy to check. $\square$

We may perform a similar manipulation to rewrite Z_U in (3.21), by playing ping-pong with the Γ's. The factors arising from commutations between Γ's or from reflection relations are the same as in Z, and thus cancel when we normalize to get the correlation function $\varrho(U)$. The fermionic operators ψ_k and ψ_k^* get "conjugated" by the Γ's crossing them. If we list the elements of U by increasing abscissa as $(i_1, k_1), \ldots, (i_n, k_n)$, then we end up with

$$\varrho(U) = \frac{\langle \underline{u} | \Psi_{k_1}(i_1) \Psi_{k_1}^*(i_1) \cdots \Psi_{k_n}(i_n) \Psi_{k_n}^*(i_n) | \underline{v} \rangle}{\langle \underline{u} | \underline{v} \rangle} \tag{3.26}$$

where

$$\Psi_k(i) := \mathrm{Ad}\left(\Gamma_+(\rho_i^{\rightarrow})\Gamma_-(\rho_i^{\leftarrow})^{-1}\right) \cdot \psi_k, \qquad \Psi_k^*(i) := \mathrm{Ad}\left(\Gamma_+(\rho_i^{\rightarrow})\Gamma_-(\rho_i^{\leftarrow})^{-1}\right) \cdot \psi_k^*. \tag{3.27}$$

Here Ad denotes the adjoint action

$$\mathrm{Ad}(A) \cdot B := ABA^{-1} \tag{3.28}$$

and the specializations $\rho_i^{\rightarrow}$ and $\rho_i^{\leftarrow}$ are given by

$$\rho_i^{\rightarrow} := \bigcup_{\ell=1}^{i} \rho_i^+ \cup \bigcup_{n \geq 1} \left(u^{2n} v^{2n-2} \rho^- \cup u^{2n} v^{2n} \rho^+\right), \qquad \rho_i^{\leftarrow} := \bigcup_{\ell=i}^{N} \rho_i^- \cup \bigcup_{n \geq 1} \left(u^{2n-2} v^{2n} \rho^+ \cup u^{2n} v^{2n} \rho^-\right). \tag{3.29}$$

The intuitive meaning of all this is the following: given a fermionic operator ψ_k inserted at position i in Z_U, the operator $\Gamma_+(\rho_i^{\rightarrow})$ corresponds to the product of all Γ_+'s that will cross it from left to right, similarly $\Gamma_-(\rho_i^{\leftarrow})$ corresponds to all Γ_-'s that will cross it from right to left. The operator $\Psi_k(i)$ is the operator resulting after all these commutations have been made. Note that the ordering of Γ's in Ad is irrelevant since they commute up to a scalar factor and that, by (3.15), $\Psi_k(i)$ (resp. $\Psi_k^*(i)$) is a linear combination of ψ's (resp. ψ^*'s).

In the case $u = v = 0$ [OR03], Okounkov and Reshetikhin were able to rewrite (3.26) as a determinant using Wick's lemma (we obtain a pfaffian from Lemma 3.1, but the matrix A has a specific block structure so its pfaffian reduces to a determinant of size n). From there, they could conclude that the point process $\mathfrak{S}(\vec{\lambda})$ is determinantal. This does not extend straightforwardly in the case of free boundaries ("naive" generalizations of Wick's lemma are false), and we will explain how to circumvent this problem in the next section.

3.2 Pfaffian correlations in the presence of free boundaries

3.2.1 Extended free boundary states

The starting point is to observe that a general basis vector $|\lambda\rangle$ of charge 0 can be written in the form

$$|\lambda\rangle = (-1)^{j_1 + \cdots + j_r + r/2} \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_r}^* |0\rangle \tag{3.30}$$

where $i_1 > \cdots > i_r$ are arbitrary positive half-integers, namely the elements of $\mathfrak{S}(\lambda, 0)_+$, and $j_1 > \cdots > j_r$ are arbitrary negative half-integers, namely the elements of $\mathfrak{S}(\lambda, 0)_-$. These numbers are closely related to the Frobenius coordinates of λ, and r is the size of the Durfee square of λ. Multiplying by $v^{|\lambda|} = v^{i_1 + \cdots + i_s - j_1 - \cdots - j_s}$ and summing over all possible pairs of sequences, we get

$$|\underline{v}\rangle = \left(\sum_{r \geq 0} \sum_{\substack{i_1 > \cdots > i_r > 0 \\ 0 > j_1 > \cdots > j_r}} \tilde{\psi}_{i_1}(v,t) \cdots \tilde{\psi}_{i_r}(v,t) \tilde{\psi}_{j_1}(v,t) \cdots \tilde{\psi}_{j_r}(v,t) \right) |0\rangle \tag{3.31}$$

with

$$\tilde{\psi}_i(v,t) = \begin{cases} t^{1/2} v^i \psi_i & \text{for } i \in \mathbb{Z}'_{>0}, \\ (-1)^{i+1/2} t^{-1/2} v^{-i} \psi_i^* & \text{for } i \in \mathbb{Z}'_{<0}. \end{cases} \tag{3.32}$$

Here t is a parameter which has no effect in (3.31) but will be useful in the following. The big sum looks nasty but it turns out that it can be generated in a rather elegant manner.

Proposition 3.5. *We have*

$$|\underline{v}\rangle = \Pi_0 e^{X(v,t)}|0\rangle \tag{3.33}$$

where Π_0 *denotes the projector onto the fermionic subspace of charge* 0*, and where*

$$X(v,t) := \sum_{\substack{(k,\ell)\in\mathbb{Z}'^2\\ k>\ell}} \tilde{\psi}_k(v,t)\tilde{\psi}_\ell(v,t). \tag{3.34}$$

Remark 3.6. For $|v|<1$, the sum in (3.34) is convergent in the space of bounded operators on $\mathcal{F}$.

Define the *extended free boundary state* $|\underline{v,t}\rangle$ as

$$|\underline{v,t}\rangle := e^{X(v,t)}|0\rangle. \tag{3.35}$$

Proposition 3.5 is then an immediate consequence of the following lemma.

Lemma 3.7. *The extended free boundary state decomposes in the canonical basis of* $\mathcal{F}$ *as*

$$|\underline{v,t}\rangle = \sum_{\substack{S\in\mathcal{S}\\ C(S)\ \textit{is even}}} t^{C(S)/2} v^{H(S)}|S\rangle. \tag{3.36}$$

Proof. We first observe that the $\tilde{\psi}_i$ all anticommute with one another, hence all the terms in $X(v,t)$ commute. Furthermore the square of each of these terms vanishes, hence

$$e^{X(v,t)} = \prod_{k>\ell}\left(1+\tilde{\psi}_k(v,t)\tilde{\psi}_\ell(v,t)\right). \tag{3.37}$$

By expanding this product, we obtain a sum over all possible finite sets $\{(k_1,\ell_1),\dots,(k_r,\ell_r)\}$ of pairs of elements of $\mathbb{Z}'$ such that $k_s>\ell_s$ for all s. Clearly, the contribution from sets containing twice the same element of $\mathbb{Z}'$ in different pairs vanishes, hence the sets with nonzero contribution can be identified with finite partial matchings $\{\{k_1,\ell_1\},\dots,\{k_r,\ell_r\}\}$ of $\mathbb{Z}'$. Let $M=\{m_1,\dots,m_{2r}\}$ be a subset of $\mathbb{Z}'$ with an even number $2r$ of elements: each one of the $(2r-1)!!$ perfect matchings of M will then arise exactly once in the sum. Using again the anticommutativity of the $\tilde{\psi}_i$, we see that all these perfect matchings have the same contribution up to a sign. It can be seen that all these contributions cancel with one another, except one, and we arrive at

$$\begin{aligned} e^{X(v,t)} &= \sum_{r\geq 0}\sum_{\substack{(m_1,\dots,m_{2r})\in\mathbb{Z}'^{2r}\\ m_1>\dots>m_{2r}}} \tilde{\psi}_{m_1}(v,t)\cdots\tilde{\psi}_{m_{2r}}(v,t)\\ &= \sum_{\substack{r',r''\geq 0\\ r'+r''\text{even}}}\sum_{\substack{i_1>\dots>i_{r'}>0\\ 0>j_1>\dots>j_{r''}}} \tilde{\psi}_{i_1}(v,t)\cdots\tilde{\psi}_{i_{r'}}(v,t)\tilde{\psi}_{j_1}(v,t)\cdots\tilde{\psi}_{j_{r''}}(v,t). \end{aligned} \tag{3.38}$$

(the second line is a simple relabelling of the sum, where we split the positive and negative indices). This expression can be compared with (3.31): we now have a sum over all admissible subsets S of $\mathbb{Z}'$ with an even charge (with $S_+=\{i_1,\dots,i_{r'}\}$ and $S_-=\{j_1,\dots,j_{r''}\}$). Multiplying by $|0\rangle$ and plugging back the definition of $\tilde{\psi}$, we obtain the wanted expression (3.36). ☐

By reindexing the sum in (3.36) as a sum over all partitions and all even charges, we can express $|\underline{v,t}\rangle$ in terms of $|\underline{v}\rangle$ and of the shift operator R, namely

$$|\underline{v,t}\rangle = \sum_{c\in 2\mathbb{Z}} t^{c/2} v^{c^2/2} R^c|\underline{v}\rangle. \tag{3.39}$$

Of course we may define a dual extended free boundary state

$$\langle\underline{u,t}| = \langle 0|e^{X(u,t)^*} = \sum_{\substack{S\text{ admissible}\\ C(S)\text{ even}}} t^{C(S)/2}u^{H(S)}\langle S| = \sum_{c\in 2\mathbb{Z}} t^{c/2}u^{c^2/2}\langle\underline{u}|R^{-c} \tag{3.40}$$

and its scalar product with $|\underline{v,t}\rangle$ reads

$$\langle \underline{u,t}|\underline{v,t}\rangle = \left(\sum_{c\in 2\mathbb{Z}} t^c (uv)^{c^2/2}\right)\langle \underline{u}|\underline{v}\rangle = \theta_3(t^2;(uv)^4)(uv;uv)_\infty^{-1} \tag{3.41}$$

which is finite for $uv<1$ (see Appendix B for reminders on theta functions and q-Pochhammer symbols).

Remark 3.8. The reflection relations (3.19) still hold when we replace $|\underline{v}\rangle$ by $|\underline{v,t}\rangle$ and $\langle\underline{u}|$ by $\langle\underline{u,t}|$ (this is immediate from (3.39), (3.41) and the commutation between R and the Γ's). The extended free boundary states also satisfy the remarkable *fermionic reflection relations*:

$$\psi(z)|\underline{v,t}\rangle = t^{-1}\frac{v-z}{v+z}\psi^*\left(\frac{v^2}{z}\right)|\underline{v,t}\rangle, \qquad \langle\underline{u,t}|\psi^*(w) = t^{-1}\frac{u-w^{-1}}{u-w^{-1}}\langle\underline{u,t}|\psi\left(\frac{1}{u^2w}\right). \tag{3.42}$$

These can be checked using the boson–fermion correspondence (3.17), or alternatively via an elementary calculation presented in Appendix C.

3.2.2 Wick's lemma for one free boundary

The notion of extended free boundary state yields a new proof that, for $u=0$, the point process $\mathfrak{S}(\vec{\lambda})$ is pfaffian – the original proof by Borodin and Rains [BR05] relied instead on a pfaffian analogue of the Eynard–Mehta theorem. Let us observe that, for $u=0$, we have $\langle\underline{u}|\underline{v}\rangle=1$ and the expression (3.26) for the correlation function $\varrho(U)$ can be rewritten as

$$\varrho(U) = \langle 0|\Psi_{k_1}(i_1)\Psi^*_{k_1}(i_1)\cdots\Psi_{k_n}(i_n)\Psi^*_{k_n}(i_n)|\underline{v,t}\rangle \tag{3.43}$$

(use $|\underline{v}\rangle = \Pi_0|\underline{v,t}\rangle$, and drop the projector Π_0 since we multiply on the left by a quantity of charge 0).

Lemma 3.9 (Wick's lemma for one free boundary)**.** *Let $\mathit{\Psi}$ be as before the vector space spanned by (possibly infinite linear combinations of) the ψ_k and ψ^*_k, $k\in\mathbb{Z}'$. For $\phi_1,\ldots,\phi_{2n}\in\mathit{\Psi}$, we have*

$$\langle 0|\phi_1\cdots\phi_{2n}|\underline{v,t}\rangle = \operatorname{pf} A \tag{3.44}$$

where A is the antisymmetric matrix defined by $A_{ij} := \langle 0|\phi_i\phi_j|\underline{v,t}\rangle$ for $i<j$.

Proof. Recall the definition (3.35) of $|\underline{v,t}\rangle$. We "commute" the operator $e^{X(v,t)}$ to the left, where it is absorbed by the left vacuum $\langle 0|$ (since $\langle 0|X(v,t)=0$), and get

$$\varrho(U) = \langle 0|\tilde{\phi}_1\cdots\tilde{\phi}_{2n}|0\rangle \tag{3.45}$$

where $\tilde{\phi}_i := e^{-X(v,t)}\phi_i e^{X(v,t)}$.

Now, the key observation is that, for any $\phi\in\mathit{\Psi}$, the commutator $[X(v,t),\phi]$ is also in $\mathit{\Psi}$ by the bilinearity of X and the canonical anticommutation relations. Therefore, $\mathit{\Psi}$ is stable under conjugation by $e^{-X(v,t)}$, and thus we may apply the usual Wick's lemma 3.1 to conclude that $\varrho(U)=\operatorname{pf} A$ with

$$A_{ij} = \langle 0|\tilde{\phi}_i\tilde{\phi}_j|0\rangle = \langle 0|\phi_i\phi_j|\underline{v,t}\rangle, \qquad i<j. \tag{3.46}$$

□

By applying Lemma 3.9 to (3.43), and being careful about the ordering between operators (note that $\{\Psi_k(i),\Psi_{k'}(i')\}=0$ for all i,i',k,k', while $\{\Psi_k(i),\Psi^*_{k'}(i')\}=0$ for $i=i'$ and $k\neq k'$), we find that $\mathfrak{S}(\vec{\lambda})$ is a pfaffian point process whose correlation kernel entries are given by

$$\begin{gathered} K_{1,1}(i,k;i',k') = \langle 0|\Psi_k(i)\Psi_{k'}(i')|\underline{v,t}\rangle, \qquad K_{2,2}(i,k;i',k') = \langle 0|\Psi^*_k(i)\Psi^*_{k'}(i')|\underline{v,t}\rangle, \\ K_{1,2}(i,k;i',k') = -K_{2,1}(i',k';i,k) = \begin{cases} \langle 0|\Psi_k(i)\Psi^*_{k'}(i')|\underline{v,t}\rangle, & \text{if } i\le i', \\ -\langle 0|\Psi^*_{k'}(i')\Psi_k(i)|\underline{v,t}\rangle, & \text{otherwise.}\end{cases}\end{gathered} \tag{3.47}$$

Note that the dependency on t is trivial ($K_{1,1}$, $K_{1,2}$ and $K_{2,2}$ are respectively proportional to t^{-1}, t^0 and t^1), and can be eliminated by row/column multiplications in the pfaffian, so that the point process $\mathfrak{S}(\vec{\lambda})$ is independent of t as it should.

To complete the proof of Theorem 2.7, we need to rewrite the entries of the correlation kernel in the form of double contour integrals. This will be done in Section 3.3, but before we discuss the case of two free boundaries.

3.2.3 Wick's lemma for two free boundaries

In the case $uv > 0$, when we rewrite (3.26) in terms of the extended free boundary states, it is no longer possible to "drop" the projectors Π_0 as was done in (3.43). But, from (3.39) and (3.40), we see that $\varrho(U)$ is proportional to the t^0 term in $\langle \underline{u,t}|\Psi_{k_1}(i_1)\Psi^*_{k_1}(i_1)\cdots\Psi_{k_n}(i_n)\Psi^*_{k_n}(i_n)|\underline{v,t}\rangle$. This quantity turns out to be pfaffian.

Lemma 3.10 (Wick's lemma for two free boundaries)**.** *Let Ψ be again the vector space spanned by (possibly infinite linear combinations of) the ψ_k and ψ^*_k, $k \in \mathbb{Z}'$. For $\phi_1, \dots, \phi_{2n} \in \Psi$ and $uv < 1$, we have*

$$\frac{\langle \underline{u,t}|\phi_1\cdots\phi_{2n}|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle} = \operatorname{pf} A \tag{3.48}$$

where A is the antisymmetric matrix defined by $A_{ij} = \langle \underline{u,t}|\phi_i\phi_j|\underline{v,t}\rangle / \langle \underline{u,t}|\underline{v,t}\rangle$ for $i < j$.

Proof. It is tempting to proceed as in the proof of Lemma 3.9, by commuting $e^{X(v,t)}$ to the left and similarly commuting $e^{X(u,t)^*}$ to the right, but those two quantities do not commute and it is unclear whether they have a nice quasi-commutation relation.

Instead, we again play ping-pong, but this time with fermionic operators. Let Ψ^+ (resp. Ψ^-) be the vector space spanned by the ψ_k and ψ^*_{-k} with $k < 0$ (resp. $k > 0$). We have $\Psi = \Psi^+ \oplus \Psi^-$ and, writing $\phi = \phi^+ + \phi^-$ for the associated decomposition of $\phi \in \Phi$, we have $\phi^+|0\rangle = 0$ and $\langle 0|\phi^- = 0$. Note that $X(v,t)$ is a bilinear combination of operators in Ψ^- only (which all anticommute with one another), and it follows that, for any $\phi \in \Psi$, we have

$$[\phi^-, X(v,t)] = 0, \qquad [\phi^+, X(v,t)] \in \Psi^-, \qquad [[\phi^+, X(v,t)], X(v,t)] = 0. \tag{3.49}$$

As a consequence we have

$$[\phi^+, e^{X(v,t)}] = [\phi^+, X(v,t)]e^{X(v,t)} \tag{3.50}$$

and hence

$$\phi^+|\underline{v,t}\rangle = [\phi^+, X(v,t)]|\underline{v,t}\rangle. \tag{3.51}$$

Similarly, we have the dual relations

$$\langle \underline{u,t}|\phi^- = \langle \underline{u,t}|[X(u,t)^*, \phi^-], \qquad [X(u,t)^*, \phi^-] \in \Psi^+. \tag{3.52}$$

We now establish (3.48) by induction on n. It is a tautology for $n = 1$. Let us assume it holds up to rank $n-1$. Let $\phi_1, \phi_2 \cdots, \phi_{2n}$ be elements of Ψ. We start by writing

$$\langle \underline{u,t}|\phi_1\phi_2\cdots\phi_{2n}|\underline{v,t}\rangle = \langle \underline{u,t}|\chi^{(1)}\phi_2\cdots\phi_{2n}|\underline{v,t}\rangle \tag{3.53}$$

with $\chi^{(1)} = \phi_1^+ + [X(u,t)^*, \phi_1^-] \in \Psi^+$, using (3.52). We then move $\chi^{(1)}$ to the right, using the fact that the anticommutators $\{\chi^{(1)}, \phi_i\}$ are all scalars, to get

$$\langle \underline{u,t}|\chi^{(1)}\phi_2\cdots\phi_{2n}|\underline{v,t}\rangle = \sum_{i=2}^{n}(-1)^i\{\chi^{(1)}, \phi_i\}\langle \underline{u,t}|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_{2n}|\underline{v,t}\rangle - \langle \underline{u,t}|\phi_2\cdots\phi_{2n}\chi^{(2)}|\underline{v,t}\rangle \tag{3.54}$$

with $\chi^{(2)} = [\chi^{(1)}, X(v,t)] \in \Psi^-$, using (3.51). Now we move $\chi^{(2)}$ in the rightmost term to the left, picking anticommutators on the way, until it hits $\langle \underline{u,t}|$ and can be transformed into $\chi^{(3)} = [X(u,t)^*, \chi^{(2)}] \in \Psi^+$, which we then move to the right, and so on. The χ's tend to zero as we iterate, since we pick at least a factor u or v on each iteration from the definition of X. Hence we arrive at

$$\langle \underline{u,t}|\phi_1\phi_2\cdots\phi_{2n}|\underline{v,t}\rangle = \sum_{i=2}^{n}(-1)^i\{\chi, \phi_i\}\langle \underline{u,t}|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_{2n}|\underline{v,t}\rangle \tag{3.55}$$

where $\chi = \chi^{(1)} - \chi^{(2)} + \chi^{(3)} - \cdots$. Applying this equality for $n = 2$ and $\phi_2 \to \phi_i$ we get that $\langle \underline{u,t}|\phi_1\phi_i|\underline{v,t}\rangle = \{\chi, \phi_i\}\langle \underline{u,t}|\underline{v,t}\rangle$, and hence, by applying the induction hypothesis, (3.55) can be rewritten as

$$\frac{\langle \underline{u,t}|\phi_1\phi_2\cdots\phi_{2n}|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle} = \sum_{i=2}^{n}(-1)^i A_{1i} \operatorname{pf} A^{(1i)} \tag{3.56}$$

where A is defined as in the proposition and $A^{(1i)}$ is its submatrix with the first and i-th rows and columns removed. We conclude by recognizing the RHS of (3.56) is the expansion of the pfaffian pf A with respect to the first row/column. □

Remark 3.11. Our proof relies on the fact that $X(v,t)$ and $X(u,t)^*$ are bilinear combinations of fermionic operators. The space of such (not necessarily charge-preserving) bilinear combinations, supplemented with the identity operator 1, forms a Lie algebra denoted D'_∞ which is an infinite dimensional analogue of the even dimensional orthogonal Lie algebra [JM83, §7]. It acts on the space Ψ of fermionic operators as the Lie algebra of the group of linear transformations preserving the canonical anticommutation relations, also known as fermionic Bogoliubov transformations. It would be interesting to exploit this fact to obtain a shorter proof of Proposition 3.10 for general u, v. Let us also mention that the generalized Wick theorem mentioned in [AZ13, Section 2.7] does not apply to our situation since it requires the preservation of charge, and as such implies determinantal (as opposed to pfaffian) correlations.

We now make the connection with the shifted process $\mathfrak{S}_t(\vec{\lambda})$ of Theorem 2.10 explicit. Set

$$\varrho_t(U) := \frac{\langle \underline{u,t} | \Psi_{k_1}(i_1)\Psi^*_{k_1}(i_1)\cdots\Psi_{k_n}(i_n)\Psi^*_{k_n}(i_n)|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}. \tag{3.57}$$

By (3.39) and (3.40), we have

$$\varrho_t(U) = \frac{1}{\langle \underline{u,t}|\underline{v,t}\rangle}\sum_{c\in 2\mathbb{Z}} t^c (uv)^{c^2/2}\langle \underline{u}|\Psi_{k_1-c}(i_1)\Psi^*_{k_1-c}(i_1)\cdots\Psi_{k_n-c}(i_n)\Psi^*_{k_n-c}(i_n)|\underline{v}\rangle \tag{3.58}$$

where we use the fact that $R^{-1}\psi_k R = \psi_{k-1}$ and hence $R^{-1}\Psi_k(i)R = \Psi_{k-1}(i)$ (R commutes with vertex operators). By (3.26) and (3.41), we get

$$\varrho_t(U) = \frac{1}{\theta_3(t^2;(uv)^4)}\sum_{c\in 2\mathbb{Z}} t^c (uv)^{c^2/2}\varrho\left(U-(0,c)\right) = \mathrm{Prob}\left(U\subset \mathfrak{S}_t(\vec{\lambda})\right). \tag{3.59}$$

In other words, $\varrho_t(U)$ is nothing but the correlation function for the point process $\mathfrak{S}_t(\vec{\lambda})$. By applying Lemma 3.10 to (3.57), and being again careful about the ordering between operators, we conclude that $\mathfrak{S}_t(\vec{\lambda})$ is indeed a pfaffian point process, and the entries of its correlation kernel read

$$\begin{gathered} K_{1,1}(i,k;i',k') = \frac{\langle \underline{u,t}|\Psi_k(i)\Psi_{k'}(i')|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}, \qquad K_{2,2}(i,k;i',k') = \frac{\langle \underline{u,t}|\Psi^*_k(i)\Psi^*_{k'}(i')|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}, \\ K_{1,2}(i,k;i',k') = -K_{2,1}(i',k';i,k) = \begin{cases} \frac{\langle \underline{u,t}|\Psi_k(i)\Psi^*_{k'}(i')|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}, & \text{if } i\le i', \\ -\frac{\langle \underline{u,t}|\Psi^*_{k'}(i')\Psi_k(i)|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}, & \text{otherwise.}\end{cases}\end{gathered} \tag{3.60}$$

Note that we recover (3.47) in the case $u=0$.

3.3 Contour integral representations of the correlation functions

3.3.1 Correlation kernels

Having proved the pfaffian nature of the point process $\mathfrak{S}_t(\vec{k})$ (which coincides with $\mathfrak{S}(\vec{k})$ in the case of one free boundary), the last step to establish Theorem 2.10 (and Theorem 2.7) is to show that the entries of the correlation kernel (3.60) match their announced expressions.

A simplifying assumption. For now, we assume that the specializations $\rho^{\pm}_k$ are such that the series $H(\rho_i^{\to};\cdot)$ and $H(\rho_i^{\leftarrow};\cdot)$ both converge in the disk $\{|\cdot|<R\}$ for some $R>\max(1,u,v)$. We will see at the end of this section how to lift this assumption.

Integral representation of $\Psi_k(i)$ and $\Psi^*_k(i)$. Following [OR03], we pass to the fermion generating functions $\psi(z)$ and $\psi^*(w)$ introduced in (3.7). Using (3.15), we get that

$$\begin{aligned}\Psi_k(i) &= [z^k]\operatorname{Ad}\left(\Gamma_+(\rho_i^{\rightarrow})\Gamma_-(\rho_i^{\leftarrow})^{-1}\right)\cdot\psi(z) = [z^k]\frac{H(\rho_i^{\rightarrow};z)}{H(\rho_i^{\leftarrow};z^{-1})}\psi(z) = [z^k]F(i,z)\psi(z),\\ \Psi^*_k(i) &= [w^{-k}]\operatorname{Ad}\left(\Gamma_+(\rho_i^{\rightarrow})\Gamma_-(\rho_i^{\leftarrow})^{-1}\right)\cdot\psi^*(w) = [w^{-k}]\frac{H(\rho_i^{\leftarrow};w^{-1})}{H(\rho_i^{\rightarrow};w)}\psi^*(w) = [w^{-k}]\frac{\psi^*(w)}{F(i,w)}\end{aligned} \tag{3.61}$$

where $[z^k]$ and $[w^{-k}]$ denotes coefficient extraction in the Laurent series to the right, and where we notice that $H(\rho_i^{\rightarrow};z)/H(\rho_i^{\leftarrow};z^{-1})$ is nothing but $F(i,z)$ as defined in Theorem 2.10. By our simplifying assumption, the function $z\mapsto F(i,z)$ and its inverse are both holomorphic in the annulus $\{R^{-1}<|z|<R\}$, and this is where we should take their Laurent expansion. Thus, we have the integral representations

$$\Psi_k(i) = \frac{1}{2\mathrm{i}\pi}\oint_C \frac{\mathrm{d}z}{z^{k+1/2}}F(i,z)\psi(z), \qquad \Psi^*_k(i) = \frac{1}{2\mathrm{i}\pi}\oint_C \frac{\mathrm{d}w}{w^{-k+1/2}}\frac{\psi(z)}{F(i,w)} \tag{3.62}$$

with C a positively oriented circle centered at 0 of radius $r\in(R^{-1},R)$. We may plug these into (3.60), and get the desired contour integral representation (2.14) (up to the choice of contours) with

$$\begin{aligned}\kappa_{1,1}(z,w) &= \frac{\langle u,t|\psi(z)\psi(w)|\underline{v,t}\rangle}{\langle u,t|\underline{v,t}\rangle},\\ \kappa_{2,2}(z,w) &= \frac{\langle \underline{u,t}|\psi^*(z)\psi^*(w)|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle},\end{aligned}\qquad \kappa_{1,2}(z,w) = \begin{cases}\frac{\langle \underline{u,t}|\psi(z)\psi^*(w)|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}, & \text{if } |z|>|w|,\\ -\frac{\langle \underline{u,t}|\psi^*(w)\psi(z)|\underline{v,t}\rangle}{\langle \underline{u,t}|\underline{v,t}\rangle}, & \text{if } |z|<|w|.\end{cases} \tag{3.63}$$

However, there are two possible convergence issues to consider:

- For $\kappa_{1,2}$ we should be careful that the product $\psi(z)\psi^*(w)$ (resp. $\psi^*(w)\psi(z)$) makes sense as operator on $\mathcal{F}$ only for $|z|>|w|$ (resp. $|z|<|w|$), as otherwise its diagonal entries are infinite. Thus, to obtain a correct double contour integral representation for $K_{1,2}(i,k;i',k')$, we should integrate z over a circle C of radius r, and w over a circle C' of radius r', with $r>r'$ if $i\le i'$ and $r<r'$ otherwise. Of course this choice of contours also works for $\kappa_{1,1}$ and $\kappa_{2,2}$ where the nesting condition is not necessary.

- The second issue is specific to the case of free boundaries: we should make sure that the action of $\psi(z)$ or $\psi^*(z)$ on the extended free boundary states is well-defined. It can be seen that this requires $v<|z|<u^{-1}$. Intuitively speaking, the probability that $|\underline{v}\rangle$ and $|\underline{v,t}\rangle$ (resp. $\langle\underline{u}|$ and $\langle\underline{u,t}|$) have an "excitation" at level k decays as $v^{|k|}$ (resp. $u^{|k|}$), and the action of $\psi(z)$ and $\psi^*(z)$ does not blow up if and only if $|z|^{-1}<v$ (resp. $|z|<u$). Thus, for the double contour integrals to make sense, we should take the radii r,r' between v and u^{-1} (which is possible since $R>\max(u,v)$).

Remark 3.12. Using the canonical anticommutation relations, it is possible to see that there is a single meromorphic function $\kappa_{1,2}(z,w)$ with a pole at $z=w$, and that its two expressions in (3.63) correspond to Laurent expansions in different annuli. Furthermore, the fermionic reflection relations (3.42) imply that

$$\kappa_{1,2}(z,w) = t\frac{v+w}{v-w}\kappa_{1,1}\left(z,\frac{v^2}{w}\right) = t^{-1}\frac{z-v}{z+v}\kappa_{2,2}\left(\frac{v^2}{z},w\right). \tag{3.64}$$

Evaluation of fermionic propagators. We now turn to the evaluation of the κ's. In the case of one free boundary ($u=0$), the computations are rather easy, for instance to compute $\kappa_{1,2}(z,w)=\langle 0|\psi(z)\psi^*(w)|\underline{v,t}\rangle$ for $|z|>|w|$ we use (3.36) and notice that the only $|S\rangle$ that contribute have charge 0 and correspond to "hook" partitions. Taking into account signs and the special contribution from $|0\rangle$, we get

$$\kappa_{1,2}(z,w) = \sum_{k\in\in\mathbb{Z}'_{<0}}(-1)^{k+1/2}\frac{z^k}{v^k}\sum_{\ell\in\mathbb{Z}'_{>0}}\frac{w^\ell}{v^\ell} + \sum_{k\in\mathbb{Z}'_{<0}}\frac{z^k}{w^k} = \frac{(zw-v^2)\sqrt{zw}}{(z+v)(w-v)(z-w)}. \tag{3.65}$$

For $v=1$, we obtain the expression announced in Theorem 2.7, and we leave the reader derive similarly the expressions for $\kappa_{1,1}(z,w)$ and $\kappa_{2,2}(z,w)$.

We may adapt this approach to the case of two free boundaries, but the computations become involved — see Appendices C and D. A simpler approach is to use Proposition 3.2 to rewrite the fermions in terms of vertex operators. For instance, to evaluate $\kappa_{1,2}(z,w)$ for $|z|>|w|$, we write

$$\begin{aligned}\langle \underline{u,t}|\psi(z)\psi^*(w)|\underline{v,t}\rangle &= \langle \underline{u,t}|(z/w)^{C-1/2}\Gamma_-(z)\Gamma'_+\left(-z^{-1}\right)\Gamma'_-(-w)\Gamma_+\left(w^{-1}\right)|\underline{v,t}\rangle \\ &= \sqrt{\frac{w}{z}}\left(\sum_{c\in 2\mathbb{Z}}(tz/w)^c (uv)^{c^2/2}\right)\langle \underline{u}|\Gamma_-(z)\Gamma'_+\left(-z^{-1}\right)\Gamma'_-(-w)\Gamma_+\left(w^{-1}\right)|\underline{v}\rangle\end{aligned} \tag{3.66}$$

where we use (3.39), (3.40) and the commutation of vertex operators with R and C. The last factor on the second line can be identified with the partition function of a certain free boundary Schur process of length 2, compare with Proposition 3.4. Using Propositions 2.1 and 2.2 to evaluate this partition function, and recognizing several Pochhammer symbols and theta functions, we arrive at

$$\langle \underline{u,t}|\psi(z)\psi^*(w)|\underline{v,t}\rangle = \sqrt{\frac{w}{z}}\theta_3\left((tz/w)^2;(uv)^4\right)\frac{((uv)^2;(uv)^2)^2_\infty \theta_{(uv)^2}(u^2zw)\langle \underline{u}|\underline{v}\rangle}{(uv,uz,-uw,-vz^{-1},vw^{-1};uv)_\infty \theta_{(uv)^2}(w/z)}. \tag{3.67}$$

Upon dividing by the normalization (3.41), we obtain the expression of $\kappa_{1,2}(z,w)$ announced in Theorem 2.10. The announced expressions of $\kappa_{1,1}(z,w)$ and $\kappa_{2,2}(z,w)$ can be checked using (3.64) and simple manipulations of Pochhammer symbols and theta functions.

Conclusion of the proof of Theorem 2.10. We now have established that, under our simplifying assumption, the entry $K_{e,e'}(i,k;i',k')$ of the correlation kernel of $\mathfrak{S}_t(\vec{k})$ has the contour integral representation (2.14) with F and the κ's given by (2.19), and where the integration contours $C_i^{(e)}, C_{i'}^{(e')}$ are circles centered at 0 with radii $r,r'\in(\max(R^{-1},v),\min(R,u))$. Observe that these contours satisfy certain hypotheses of Lemma 2.4:

- they should be appropriately nested for $e\neq e'$,
- they are both contained in the annulus $\{v<|\cdot|<u^{-1}\}$,
- $C_i^{(e)}$ encircles positively the interval $[-b_i^-,a_i^-]$ but does not cross $(-\infty,-1/b_i^+]$ nor $[1/a_i^+,+\infty)$ (since $a_i^\pm, b_i^\pm$ are all smaller than R^{-1}), and similarly for $C_{i'}^{(e')}$.

We are now free to deform these contours into any pair $C_i^{(e)}, C_{i'}^{(e')}$ belonging to a family satisfying the hypotheses of Lemma 2.4. Indeed, we do not meet any poles of the integrand by doing so: the poles and zeros of $F(i,z)$ belong to the intervals $(-\infty,-\min(1/u,1/b_i^+)]$, $[-\max(v,b_i^-),\max(v,a_i^-)]$ and $[\min(1/u,1/a_i^+),+\infty)$, which we can avoid during the deformation, while, as noted in item 3 below Theorem 2.10, the only possible pole of $\kappa_{e,e'}(z,w)$ when z,w are in the annulus $\{v<|\cdot|<u^{-1}\}$ is for $e\neq e'$ and $z=w$, and is taken care of by the nesting condition. This establishes Theorem 2.10 under our simplifying assumption.

To lift this hypothesis, we use an analytic continuation argument along the same line as in [BBC+16, Section 4.4]. Let us consider arbitrary parameters $u,v,\rho_k^\pm$ such that the partition function Z is finite, and fix some contours $C_1^{(1)},C_1^{(2)},\dots,C_N^{(1)},C_N^{(2)}$ satisfying the assumptions of Lemma 2.4. The basic idea is to multiply the specializations $\rho_k^\pm$ by some extra parameter s, keeping the contours fixed: let us denote by $K^{(s)}_{e,e'}(i,k;i',k')$ the quantities given by (2.14) and (2.19) after such modification (note that the only dependency on s is in the function F). It is not difficult to see that the $K^{(s)}$ are analytic functions of s in a neighborhood of the interval $[0,1]$, as the poles of F do not cross the integration contours. For s small enough, $K^{(s)}$ is the correlation kernel of the free boundary Schur process as the simplifying assumption holds. On the other hand, the partition function and correlation functions are also analytic functions of s in a neighborhood of $[0,1]$, hence $K^{(s)}$ remains a valid correlation kernel up to $s=1$, which concludes the proof of Theorem 2.10.

3.3.2 Variations on free boundary states

In this section, we explain how to handle the extra weighting α^{oc} mentionned in Remark 2.9. Recall that $|\underline{v}\rangle = \sum_\lambda v^{|\lambda|}|\lambda\rangle$. We define the following companion boundary vectors:

$$\begin{aligned} |\underline{v^{ec}}\rangle &= \sum_{\lambda:\ \lambda \text{ has even columns}} v^{|\lambda|}|\lambda\rangle, \quad |\underline{v^{\alpha,\ oc}}\rangle = \sum_\lambda \alpha^{\#\text{ of odd columns of }\lambda} v^{|\lambda|}|\lambda\rangle, \\ |\underline{v^{er}}\rangle &= \sum_{\lambda:\ \lambda \text{ has even rows}} v^{|\lambda|}|\lambda\rangle, \quad |\underline{v^{\beta,\ or}}\rangle = \sum_\lambda \beta^{\#\text{ of odd rows of }\lambda} v^{|\lambda|}|\lambda\rangle \end{aligned} \tag{3.68}$$

where ec, er, oc, or stand for respectively *even columns, even rows, odd columns, odd rows*, and where α, β are parameters. Note that $|\underline{v^{1,\ oc}}\rangle = |\underline{v^{1,\ or}}\rangle = |\underline{v}\rangle$ while $|\underline{v^{0,\ oc}}\rangle = |\underline{v^{ec}}\rangle$ and $|\underline{v^{0,\ or}}\rangle = |\underline{v^{er}}\rangle$. Analogously we may define covectors $\langle \underline{u^{ec}}|$, etc.

Proposition 3.13. *We have*

$$|\underline{v^{\alpha,\ oc}}\rangle = \Gamma_-(\alpha v)|\underline{v^{ec}}\rangle, \qquad |\underline{v^{\beta,\ or}}\rangle = \Gamma'_-(\beta v)|\underline{v^{er}}\rangle. \tag{3.69}$$

Proof. The first (resp. second) identity results from the fact that any partition can be decomposed uniquely into a partition with even columns (resp. rows), and an horizontal (resp. vertical) strip. ☐

Noting that $\Gamma_-(x)\Gamma'_-(-x) = 1$ for any single variable x, we deduce that the modified free boundary states can be expressed in terms of the original one as

$$|\underline{v^{\alpha,\ oc}}\rangle = \Gamma_-(\alpha v)\Gamma'_-(-v)|\underline{v}\rangle, \qquad |\underline{v^{\beta,\ or}}\rangle = \Gamma'_-(\beta v)\Gamma_-(-v)|\underline{v}\rangle. \tag{3.70}$$

Replacing $|\underline{v}\rangle$ by $|\underline{v^{\alpha,\ oc}}\rangle$ in Proposition 3.4, we obtain the partition function and correlation functions for the free boundary Schur process with the extra weight α^{oc} counting the number of odd columns of $\mu^{(N)}$. By (3.70), we readily see that such modification amounts to replacing the boundary specialization ρ_N^- by the specialization $\rho_N^-[\alpha, v]$ such that

$$H(\rho_N^-[\alpha, v]; t) = \frac{1 - tv}{1 - \alpha t v} H(\rho_N^-; t). \tag{3.71}$$

In other words, we are "adding" a specialization in the single variable αv, and "subtracting" (in the sense of plethystic negation) a variable v. Note that $\rho_N^-[\alpha, v]$ is a priori not nonnegative. Still, this allows to deduce easily the partition function and correlations functions of the modified process from Proposition 2.2 and Theorem 2.10. Because of the added extra variable αv, in view of the discussion after Proposition 2.2, the finiteness condition is slightly modified: we must add the constraints $\alpha u v < 1$ and $\alpha \alpha_1^{(k+)} v < 1$ for all k (such constraint is only relevant for $\alpha > 1$). Note that the "subtraction" of v does not create additional constraints. Lemma 2.4 is easily adapted by replacing $\alpha_1^{(N-)}$ by $\max(\alpha v, \alpha_1^{(N-)})$. In (2.19), changing ρ_N^- into $\rho_N^-[\alpha, v]$ produces some extra factors in $F(i, z)$. These factors will appear in the contour integral representation (2.14), once for $F(i, z)$ and once for $F(i', w)$. We may conventionally choose to absorb them in a redefinition of the κ's, and keep F unchanged: this yields the announced expression (2.16) for $u = 0$ and $v = 1$. It is not difficult to see that such a redefinition of the κ's amounts to replacing in (3.63) $|\underline{v, t}\rangle$ by

$$|\underline{v^{\alpha,\ oc}, t}\rangle := \Gamma_-(\alpha v)\Gamma'_-(-v)|\underline{v, t}\rangle. \tag{3.72}$$

Of course, we could perform a similar trick to introduce instead an extra weight β^{or} counting the number of odd rows of $\mu^{(N)}$. Let us record the corresponding redefinition of the κ's for $u = 0$ and $v = 1$:

$$\begin{aligned} \kappa_{1,1}(z, w) &= \frac{\sqrt{zw}(z - w)}{(zw - 1)(z + \beta)(w + \beta)}, \qquad \kappa_{1,2}(z, w) = \frac{\sqrt{zw}(zw - 1)(w + \beta)}{(z - w)(w^2 - 1)(z + \beta)}, \\ \kappa_{2,2}(z, w) &= \frac{\sqrt{zw}(z - w)(z + \beta)(w + \beta)}{(z^2 - 1)(w^2 - 1)(zw - 1)}. \end{aligned} \tag{3.73}$$

The derivation of the modified κ's for general u, v, possibly with another weight counting odd rows or columns in the other boundary partition $\mu^{(0)}$, is left to the reader.

3.3.3 General correlation functions

It turns out that the free fermion formalism used in this section allows to derive an explicit $2n$-fold contour integral representation for the general n-point correlation function of both $\mathfrak{S}(\vec{\lambda})$ and $\mathfrak{S}_t(\vec{\lambda})$:

Theorem 3.14. *Let $U = \{(i_1, k_1), \ldots, (i_n, k_n)\}$ be a finite subset of $\{1, \ldots, N\} \times \mathbb{Z}'$, with $i_1 \leq \cdots \leq i_n$. The probability $\varrho(U)$ that the point process $\mathfrak{S}(\vec{\lambda})$ contains U reads*

$$\varrho(U) = \frac{1}{(2\mathrm{i}\pi)^{2n}} \oint \cdots \oint \left(\prod_{j=1}^{n} \frac{\mathrm{d}z_j \mathrm{d}w_j}{z_j^{k_j+1} w_j^{-k_j+1}} \cdot \frac{F(i_j, z_j)}{F(i_j, w_j)} \right) \Phi(z_1, \ldots, z_n; w_1, \ldots, w_n) \tag{3.74}$$

where, for each $j = 1, \ldots, n$, z_j and w_j are integrated over the respective contours $C_{i_j}^{(1)}$ and $C_{i_j}^{(2)}$ satisfying the conditions from Lemma 2.4, and where F is as in Theorem 2.10 while

$$\begin{aligned} \Phi(z_1, \ldots, z_n; w_1, \ldots, w_n) = \sqrt{\frac{w_1 \cdots w_n}{z_1 \cdots z_n}} \frac{((uv)^2; (uv)^2)_\infty^{2n}}{\prod_{i=1}^{n} (uz_i, -uw_i, -vz_i^{-1}, vw_i^{-1}; uv)_\infty} \\ \frac{\prod_{i,j=1}^{n} \theta_{(uv)^2}(u^2 z_i w_j)}{\prod_{1 \leq i \leq j \leq n} \theta_{(uv)^2}(w_j/z_i) \prod_{1 \leq i < j \leq n} \theta_{(uv)^2}(z_j/w_i)} \prod_{1 \leq i < j \leq n} \frac{\theta_{(uv)^2}(z_j/z_i)\theta_{(uv)^2}(w_j/w_i)}{\theta_{(uv)^2}(u^2 z_i z_j)\theta_{(uv)^2}(u^2 w_i w_j)}. \end{aligned} \tag{3.75}$$

The probability $\varrho_t(U)$ that the point process $\mathfrak{S}_t(\vec{\lambda})$ contains U admits the same expression, upon replacing $\Phi(z_1, \ldots, z_n; w_1, \ldots, w_n)$ by

$$\Phi_t(z_1, \ldots, z_n; w_1, \ldots, w_n) = \theta_3\left(\left(t \frac{z_1 \cdots z_n}{w_1 \cdots w_n} \right)^2 ; (uv)^4 \right) \frac{\Phi(z_1, \ldots, z_n; w_1, \ldots, w_n)}{\theta_3(t^2; (uv)^4)}. \tag{3.76}$$

Proof. We start from the fermionic representation (3.26) of $\varrho(U)$, and plug in the contour integral representation (3.62) of $\Psi_k(i)$ and $\Psi_k^*(i)$: we obtain the $2n$-fold contour integral (3.74) with

$$\Phi(z_1, \ldots, z_n; w_1, \ldots, w_n) = \frac{\langle \underline{u} | \psi(z_1)\psi^*(w_1) \cdots \psi(z_n)\psi^*(w_n) | \underline{v} \rangle}{\langle \underline{u} | \underline{v} \rangle}. \tag{3.77}$$

This quantity may be evaluated by the same strategy as for the fermionic propagators in Section 3.3.1, by using the boson–fermion correspondence (Proposition 3.2) to rewrite the ψ/ψ^* in terms of vertex operators (and R and C operators that are immediately factored out). We recognize the partition function of a certain free boundary Schur process of length $2N$, which after some massaging yields (3.75). The discussion of integration contours is easily adapted from that in Section 3.3.1.

For $\varrho_t(U)$, we proceed in the same way starting from the fermionic representation (3.57), which amounts to replacing Φ by

$$\Phi_t(z_1, \ldots, z_n; w_1, \ldots, w_n) = \frac{\langle \underline{u, t} | \psi(z_1)\psi^*(w_1) \cdots \psi(z_n)\psi^*(w_n) | \underline{v, t} \rangle}{\langle \underline{u, t} | \underline{v, t} \rangle}. \tag{3.78}$$

By (3.39) and (3.40), we see that Φ_t differs from Φ by a simple charge-related factor, leading to (3.76). $\square$

Remark 3.15. By Wick's lemma for two free boundaries (Lemma 3.10), we have

$$\Phi_t(z_1, \ldots, z_n; w_1, \ldots, w_n) = \mathrm{pf} \begin{bmatrix} \kappa_{1,1}(z_i, z_j) & \kappa_{1,2}(z_i, w_j) \\ -\kappa_{1,2}(z_j, w_i) & \kappa_{2,2}(w_i, w_j) \end{bmatrix}_{1 \leq i,j \leq n} \tag{3.79}$$

where the κ's are as in Theorem 2.10. Plugging in the explicit expression for Φ_t given in Theorem 3.14, we obtain a remarkable pfaffian identity. In the case $u = v = 0$, this identity reduces to the well-known Cauchy determinant. In the case $u = 0$, $v = 1$ of Theorem 2.7, we obtain an identity equivalent to Schur's pfaffian $\mathrm{pf}_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{x_i + x_j} = \prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{x_i + x_j}$ (the equivalence goes as follows: substitute the expression (2.15) for the κ's, pull out the trivial row/column factors and take $x_{2i-1} = \frac{z_i - 1}{z_i + 1}$, $x_{2i} = \frac{1 - w_i}{w_i + 1}$ for $i = 1, \ldots, n$). In the general case, our pfaffian identity seems to be related to an identity due to Okada [Oka06], see Appendix D.

4 Symmetric Last Passage Percolation

In this section we consider the last passage percolation (LPP) time with symmetric and up to symmetry independent geometric weights. For $(a_n)_{n\geq 1}\in(0,1)^{\mathbb{N}}$, $\alpha\in(0,\inf\{\frac{1}{a_n},n\geq 1\})$ and $r,t\in\mathbb{Z}_{\geq 1}$ these weights are given by

$$\omega_{r,t}=\omega_{t,r}\sim\begin{cases} g(a_r a_t), & \text{if} \quad r\neq t,\\ g(\alpha a_r), & \text{if} \quad r=t\end{cases} \tag{4.1}$$

where $\mathrm{Prob}(g(q)=k)=q^k(1-q)$ for $k\in\mathbb{Z}_{\geq 0}$.

For $(k,l),(m,n)\in\mathbb{Z}^2_{\geq 1}$ with $k\leq m,l\leq n$, consider up-right paths π from (k,l) to (m,n), i.e. $\pi=(\pi(0),\pi(1),\dots,\pi(m-k+n-l))$ with $\pi(0)=(k,l),\pi(m-k+n-l)=(m,n)$ and $\pi(i)-\pi(i-1)\in\{(0,1),(1,0)\}$. The symmetric LPP time with geometric weights (4.1) is then defined to be

$$L_{(k,l)\to(m,n)}:=\max_{\pi:(k,l)\to(m,n)}\sum_{(r,t)\in\pi}\omega_{r,t} \tag{4.2}$$

where the maximum is taken over all up-right paths from (k,l) to (m,n). Note that we have the recursion

$$L_{(k,l)\to(m,n)}=\max\{L_{(k,l)\to(m-1,n)},L_{(k,l)\to(m,n-1)}\}+\omega_{m,n}. \tag{4.3}$$

4.0.1 Definition of distribution functions

We start by defining the distribution functions which will appear later. The distributions defined below are mostly given in terms of contour integrals, which is why we make the following definition.

Definition 4.1. For $\varphi\in[0,2\pi]$ and $z\in\mathbb{R}$ denote by $\mathfrak{G}_z^{\varphi}=\{z+|s|e^{\mathrm{sgn}(s)\mathrm{i}\varphi},s\in\mathbb{R}\}$ the infinite curve oriented from $z+\infty e^{-\mathrm{i}\varphi}$ to $z+\infty e^{\mathrm{i}\varphi}$. If f is a function and $V\subset\mathbb{C}$, we denote by γ_V any counterclockwise oriented simple closed curve containing all elements of V in its interior and excluding all poles of f that are not elements of V.

Definition 4.2. Let $u_1>u_2>\cdots>u_k\geq 0$ and $a,b\in\{1,\dots,k\}$. For $v\in\mathbb{R}$ we define

$$K^v_{1,1}(u_a,\xi;u_b,\xi')=\frac{1}{(2\pi\mathrm{i})^2}\int_{\mathfrak{G}_1^{\pi/3}}\mathrm{d}Z\int_{\mathfrak{G}_1^{\pi/3}}\mathrm{d}W\frac{(Z-W)(W+2v)(Z+2v)}{4ZW(Z+W)}\frac{e^{Z^3/3-Z^2u_a-Z\xi}}{e^{-W^3/3+W^2u_b+W\xi'}} \tag{4.4}$$

and $K^v_{1,2}=K^{v,1}_{1,2}+K^{v,2}_{1,2}$, where

$$K^{v,1}_{1,2}(u_a,\xi;u_b,\xi')=\frac{1}{(2\pi\mathrm{i})^2}\int_{\mathfrak{G}_{A_Z}^{\pi/3}}\mathrm{d}Z\int_{\mathfrak{G}_{A_W}^{2\pi/3}}\mathrm{d}W\frac{(Z+W)(Z+2v)}{2(W+2v)Z(Z-W)}\frac{e^{Z^3/3-Z^2u_a-\xi Z}}{e^{W^3/3-W^2u_b-W\xi'}}, \tag{4.5}$$

with $A_Z>A_W>-2v,A_Z>0$. For $a\leq b$, we have $K^{v,2}_{1,2}(u_a,\xi;u_b,\xi')=0$, and if $a>b$, then

$$K^{v,2}_{1,2}(u_a,\xi;u_b,\xi')=\frac{-1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}Ze^{Z^2(u_b-u_a)+Z(\xi'-\xi)} \tag{4.6}$$

with $\mathrm{i}\mathbb{R}$ oriented with increasing imaginary part. Finally, we define $K^v_{2,2}=K^{v,1}_{2,2}+K^{v,2}_{2,2}$ through

$$K^{v,1}_{2,2}(u_a,\xi;u_b,\xi')=\frac{1}{(2\pi\mathrm{i})^2}\int_{\mathfrak{G}_{B_3}^{2\pi/3}}\mathrm{d}Z\int_{\mathfrak{G}_{B_4}^{2\pi/3}}\mathrm{d}W\frac{Z-W}{(W+2v)(Z+2v)(Z+W)}\frac{e^{-Z^3/3+u_aZ^2+Z\xi}}{e^{W^3/3-u_bW^2-W\xi'}}, \tag{4.7}$$

with $B_3>-2v>B_4,B_3<-B_4$.

We define $K^{v,2}_{2,2}$ for $u_a=u_b=0$ to be

$$K^{v,2}_{2,2}(0,\xi;0,\xi')=\frac{e^{8v^3/3-2v\xi'}}{2\pi\mathrm{i}}\int_{\mathfrak{G}_{C_1}^{2\pi/3}}\mathrm{d}Z\frac{e^{-Z^3/3+Z\xi}}{Z-2v}+\mathrm{sgn}(\xi'-\xi)e^{-2v|\xi-\xi'|} \tag{4.8}$$

with $C_1 < 2v$, whereas if $u_a + u_b > 0$

$$\begin{aligned} K^{v,2}_{2,2}(u_a,\xi;u_b,\xi') = &\frac{e^{8v^3/3+4v^2u_b-2v\xi'}}{2\pi\mathrm{i}} \int_{\mathfrak{G}_{B_2}^{2\pi/3}} \mathrm{d}Z \frac{1}{Z-2v} e^{-Z^3/3+Z^2u_a+\xi Z} \\ &- \frac{1}{2\pi\mathrm{i}} \oint_{\mathfrak{G}_{B_1}^{2\pi/3}} \mathrm{d}Z \frac{2Z}{(Z+2v)(Z-2v)} e^{Z^2(u_a+u_b)+Z(\xi-\xi')} \end{aligned} \tag{4.9}$$

with $B_1 > 2|v|$, $B_2 > 2v$. We can now define the following antisymmetric kernel, note that we introduce the prefactor $d_q = \frac{q^{1/6}}{2(1+\sqrt{q})^{2/3}}$ in its definition so we do not have to insert it later:

$$K^v(u_a,\xi;u_b,\xi') = \begin{pmatrix} K^v_{1,1}(d_qu_a,\xi;d_qu_b,\xi') & K^v_{1,2}(d_qu_a,\xi;d_qu_b,\xi') \\ -K^v_{1,2}(d_qu_b,\xi';d_qu_a,\xi) & K^v_{2,2}(d_qu_a,\xi;d_qu_b,\xi') \end{pmatrix}. \tag{4.10}$$

Definition 4.3. The Tracy–Widom GUE distribution is given by

$$F_{\mathrm{GUE}}(s) = \mathrm{pf}(J - \hat{K}_{\mathcal{A}_2})_{(s,\infty)} \tag{4.11}$$

with $\hat{K}_{\mathcal{A}_2}(x,y) = \begin{pmatrix} 0 & K_{\mathcal{A}_2}(x,y) \\ -K_{\mathcal{A}_2}(y,x) & 0 \end{pmatrix}$ and

$$K_{\mathcal{A}_2}(x,y) = \frac{1}{(2\pi\mathrm{i})^2} \int_{\mathfrak{G}_{-1}^{2\pi/3}} \mathrm{d}z \int_{\mathfrak{G}_{1}^{\pi/3}} \mathrm{d}w \frac{e^{w^3-wx}}{e^{z^3-zy}} \frac{1}{w-z} \tag{4.12}$$

is the Airy kernel.

Definition 4.4. We define $F_{u,v}$ through

$$F_{u,v}(s) = \mathrm{pf}(J - K^v(u,\xi;u,\xi'))_{(s,\infty)}. \tag{4.13}$$

The F_{GOE} and F_{GSE} distributions which appear in the following can be defined through Fredholm pfaffians — see e.g. Lemmas 2.6, 2.7 in [BBCS17b], but their explicit form will not be needed later and hence we omit giving it. $F_{u,v}$ interpolates between various distribution functions. First, we have $F_{0,0}(s) = F_{\mathrm{GOE}}(s)$; the equivalence of $F_{0,0}$ with existing definitions of F_{GOE} was checked in e.g. Lemma 2.6 of [BBCS17b]. It follows from our Theorem 4.6 and (4.26) of [BR99] that $F_{0,v}(s) = F^{\boxslash}(s;v)$, where $F^{\boxslash}$ is defined in Definition 4 of [BR01b]. This and (2.33) in [BR01b] imply $\lim_{v\to+\infty} F_{0,v}(s) = F_{\mathrm{GSE}}(s)$. Finally, if $v = v(u)$ is such that $u + 2v(u) \to +\infty$ for $u \to +\infty$, then $\lim_{u\to+\infty} F_{u,v}(s - u^2d_q^2) = F_{\mathrm{GUE}}(s)$. This follows from the convergence of K^v (under conjugation) to $\hat{K}_{\mathcal{A}_2}$ and dominated convergence.

4.1 Results

The first result we present is a formula for the multipoint distribution of LPP times along down right paths.

Theorem 4.5. *Consider the LPP time 4.2 with weights 4.1. Let $r_l, t_l \in \mathbb{Z}_{\geq 1}, r_l \leq t_l, l = 1,\ldots,k$ with $r_1 \leq \ldots \leq r_k, t_1 \geq \ldots \geq t_k$. Then*

$$\mathrm{Prob}\left(\bigcap_{l=1}^k \{L_{(1,1)\to(r_l,t_l)} \leq s_l\}\right) = \mathrm{pf}(J-K)_B \tag{4.14}$$

where $B = \{(i,x) \in \{1,\ldots,k\} \times \mathbb{Z}' : x > s_i - 1/2\}$ is equipped with the counting measure. The kernel K is given by

$$\begin{aligned} K_{1,1}(a,x_a;b,x_b) = &\frac{1}{(2\pi\mathrm{i})^2} \oint \mathrm{d}z \oint \mathrm{d}w \frac{z-w}{(z^2-1)(w^2-1)(zw-1)} z^{-x_a+1/2} w^{-x_b+1/2} (1-\alpha/z)(1-\alpha/w) \\ &\times \prod_{i=1}^{r_a} \frac{1}{1-a_iz} \prod_{i=1}^{r_b} \frac{1}{1-a_iw} \prod_{i=1}^{t_a} (1-a_i/z) \prod_{i=1}^{t_b} (1-a_i/w) \end{aligned} \tag{4.15}$$

for counterclockwise oriented circle contours around 0 *satisfying* $\alpha, 1 < |z|, |w| < \min_{i=1,\dots,t_1} \frac{1}{a_i}$*;*

$$\begin{aligned} K_{1,2}(a,x_a;b,x_b) =& \frac{1}{(2\pi\mathrm{i})^2} \oint \mathrm{d}z \oint \mathrm{d}w \frac{zw-1}{(z-w)(z^2-1)} z^{-x_a+1/2} w^{x_b-3/2} \frac{1-\alpha/z}{1-\alpha/w} \\ &\times \prod_{i=1}^{r_a} \frac{1}{1-a_i z} \prod_{i=1}^{r_b}(1-a_i w) \prod_{i=1}^{t_a}(1-a_i/z) \prod_{i=1}^{t_b} \frac{1}{1-a_i/w} \end{aligned} \tag{4.16}$$

for counterclockwise oriented circle contours around 0 *with* $\max_{i=1,\dots,t_1} a_i, \alpha < |z|, |w| < \min_{i=1,\dots,t_1} \frac{1}{a_i}$ *and, if* $a \le b$, $|w| < |z| > 1$*, and, if* $b < a$, $|z| < |w| > 1$*,; and finally*

$$\begin{aligned} K_{2,2}(a,x_a;b,x_b) =& \frac{1}{(2\pi\mathrm{i})^2} \oint \mathrm{d}z \oint \mathrm{d}w z^{x_a-3/2} w^{x_b-3/2} \frac{z-w}{zw-1} \frac{1}{(1-\alpha/z)(1-\alpha/w)} \\ &\times \prod_{i=1}^{r_a}(1-a_i z) \prod_{i=1}^{r_b}(1-a_i w) \prod_{i=1}^{t_a} \frac{1}{1-a_i/z} \prod_{i=1}^{t_b} \frac{1}{1-a_i/w} \end{aligned} \tag{4.17}$$

for counterclockwise oriented circle contours around 0 *with* $\alpha, \max_{i=1,\dots,t_1} a_i < |z|, |w| < \min_{i=1,\dots,t_1} \frac{1}{a_i}$ *and* $1 < |zw|$.

The following Theorem will be obtained from the previous one by asymptotic analysis.

Theorem 4.6. *Consider the weights* (4.1) *with* $a_j = \sqrt{q}, q \in (0,1), j \ge 1$ *and* $\alpha = 1 - 2vc_q N^{-1/3}$*, where* $c_q = \frac{1-\sqrt{q}}{q^{1/6}(1+\sqrt{q})^{1/3}}, v \in \mathbb{R}$*. Let* $u_1 > \dots > u_k \ge 0$*. Then*

$$\begin{aligned} \lim_{N\to\infty} \mathrm{Prob}\left(\bigcap_{i=1}^{k} \left\{ L_{(1,1)\to(N-\lfloor u_i N^{2/3} \rfloor, N)} \le \frac{2\sqrt{q}N}{1-\sqrt{q}} - u_i \frac{\sqrt{q}N^{2/3}}{1-\sqrt{q}} + c_q^{-1} s_i N^{1/3} \right\} \right) =& \\ = \mathrm{pf}(J - \chi_s K^v \chi_s)_{\{u_1,\dots,u_k\}\times\mathbb{R}}& \end{aligned} \tag{4.18}$$

where $\chi_s(u_i, x) = \mathbf{1}_{x > s_i}$ *and* K^v *is defined in* (4.10).

As a small digression from the main topic of this paper and section, we consider now the symmetric LPP time for exponential weights. If the weights on the diagonal are not too large, we show that away from the diagonal the symmetrized LPP has the same fluctuations as the unsymmetrized one. By a soft probabilistic argument, we show that the difference between the unsymmetrized and symmetrized LPP times goes to zero. We thus obtain a very short proof which also applies to non solvable models which can be sandwiched between the symmetrized and unsymmetrized LPP times. What our argument does not provide is the phase transition of the diagonal weights where the observed convergence breaks down, see Remark 4.8. Let $\omega_{r,t}^{\mathrm{exp,sym}}, r, t \in \mathbb{Z}$ be independent and distributed as

$$\omega_{r,t}^{\mathrm{exp,sym}} \sim \begin{cases} \exp(1), & \text{if} \quad r < t, \\ 0, & \text{otherwise.} \end{cases} \tag{4.19}$$

Denote by $L_{A\to B}^{\mathrm{exp,sym}}$ the LPP time from A to B in this model ($A, B \in \mathbb{Z}^2$). Denote by $L_{A\to B}^{\mathrm{exp}}$ the LPP time from A to B in the model where the $\omega_{r,t}, r, t \in \mathbb{Z}$ are i.i.d., and $\omega_{r,t} \sim \exp(1)$. Let $\hat\omega_{r,t}, r, t \in \mathbb{Z}$ be random variables and $\hat L_{A\to B}$ the LPP time from A to B in the model with weights $\hat\omega_{r,t}$. We impose that for all $1 \le m \le n$ we have

$$L_{0\to(m,n)}^{\mathrm{sym,exp}} \le \hat L_{0\to(m,n)} \le L_{0\to(m,n)}^{\mathrm{exp}}. \tag{4.20}$$

Note that, except for (4.20), no restriction on the law of the $\hat\omega_{r,t}, r, t \in \mathbb{Z}$ was made, so one can hardly hope to find an explicit formula for the distribution of the $\hat L_{0\to(m,n)}$. Nevertheless, we have the following result, with $\mathcal{A}_2$ denoting the Airy$_2$ process (see e.g. [BP08]).

Theorem 4.7. *Let* $\kappa \in (0,1)$ *and* $u_1 < u_2 < \ldots < u_k$*. With* $\eta_i = \kappa + 2u_i(\kappa(1+\sqrt{\kappa}))^{2/3} t^{-1/3}$ *let* $P_i = (\eta_i t, t)$*. Then*

$$\lim_{t\to\infty} \mathrm{Prob}\left(\bigcap_{i=1}^{k} \left\{ \hat L_{0\to P_i} \le t(1+\sqrt{\eta_i})^2 + \kappa^{-1/6}(1+\sqrt{\kappa})^{-4/3} s_i t^{1/3} \right\} \right) = \mathrm{Prob}\left(\bigcap_{i=1}^{k} \left\{ \mathcal{A}_2(u_i) \le s_i \right\} \right). \tag{4.21}$$

Remark 4.8. As was already mentioned, further asymptotic regimes can be considered. It was shown in Theorem 1.4 of [BBCS17b] that the convergence of the rescaled $L^{\mathrm{sym,exp}}_{0\to(\kappa N,N)}, \kappa < 1$, to F_{GUE} still holds if in (4.19), we put $\omega^{\mathrm{exp,sym}}_{r,r} \sim \exp(\tau)$, for $\tau > \frac{\sqrt{\kappa}}{1+\sqrt{\kappa}}$, whereas Theorem 4.7 only shows this for $\tau \geq 1$. For the geometric LPP time $L_{(1,1)\to(\kappa N,N)}$ with weights (4.1) and $a_n = \sqrt{q}$ the convergence to F_{GUE} should hold as long as $\alpha < \frac{1+\sqrt{\kappa q}}{\sqrt{\kappa}+\sqrt{q}}$, but we shall not pursue this direction. Finally, as was already obtained in [BR99], $L_{(1,1)\to(N,N)}$ (rescaled) converges to F_{GSE} for $\alpha < 1$. We note that this case is more involved than the others, see Section 5 of [BBCS17b] for the discussion and solution of the arising difficulties in the exponential case.

4.1.1 Proofs of Theorems 4.5, 4.6, 4.7

We start by proving Theorem 4.5. The symmetric LPP time (4.2) is a marginal of a Schur process with an *even columns* free boundary partition. This can be seen using the framework developed in [BBB+15] which we mostly follow and refer to for further references. For a word $w = (w_1, \dots, w_n) \in \{\prec, \succ\}^n$ and a sequence of partitions $\vec{\lambda} = (\emptyset = \lambda^{(0)}, \dots, \lambda^{(n)})$ we say that $\vec{\lambda}$ is $w-$interlaced if $\lambda^{(i-1)} w_i \lambda^{(i)}, i = 1, \dots, n$ and we define $w^\star = (w^\star_n, \dots, w^\star_1) \in \{\prec, \succ\}^n$ by imposing $w^\star_i \neq w_i$. Furthermore, given w, we set $\Gamma_i = \Gamma_+$ (resp. $\Gamma_i = \Gamma_-$) if w_i equals $\prec$ (resp. $\succ$), and we define $w^{\mathrm{sym}} = w \cdot w^\star$ where $\cdot$ means concatenation. For $w \in \{\prec, \succ\}^n$, we label the elements of $\{i : \Gamma_i = \Gamma_+\}$ as $i_1 \leq \ldots \leq i_m$ and those of $\{i : \Gamma_i = \Gamma_-\}$ as $j_{n-m} \leq \ldots \leq j_1$. We now define $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ by setting

$$\mathfrak{s}_{i_k} = a_k, \quad k = 1, \dots, m; \quad \mathfrak{s}_{j_k} = a_k, \quad k = 1, \dots, n-m \tag{4.22}$$

with the $(a_n)_{n\geq 1}$ from (4.1).

To w we associate an encoded shape: we construct a down-right path $\hat{\pi} = (\hat{\pi}(0), \dots, \hat{\pi}(n)), \hat{\pi}(0) = (1, \#\{i : w_i =\succ\} + 1), \hat{\pi}(n) = (\#\{i : w_i =\prec\} + 1, 1)$ of unit steps by setting $\hat{\pi}(i+1) - \hat{\pi}(i) = (1, 0)$ if w_{i+1} equals $\prec$, and $\hat{\pi}(i+1) - \hat{\pi}(i) = (0, -1)$ otherwise. This path can be seen as the boundary of a Young diagram drawn in French convention, denoted by $sh(w)$; the bottom left corner of this Young diagram is located at $(1, 1)$ — see Figure 7 left. For fixed λ, μ we have the bijective local growth rule

$$\mathcal{T}_{\mathrm{loc}} : \{\kappa : \kappa \prec \lambda, \kappa \prec \mu\} \times \mathbb{Z}_{\geq 0} \to \{\nu : \nu \succ \lambda, \nu \succ \mu\} \tag{4.23}$$

where $\nu := \mathcal{T}_{\mathrm{loc}}((\kappa, k))$ is given by

$$\begin{aligned} \nu_1 &= \max\{\lambda_1, \mu_1\} + k, \\ \nu_i &= \max\{\lambda_i, \mu_i\} + \min\{\lambda_{i-1}, \mu_{i-1}\} - \kappa_{i-1}, \quad i \geq 2 \end{aligned} \tag{4.24}$$

and one has

$$|\nu| + |\kappa| = |\mu| + |\lambda| + k. \tag{4.25}$$

Let $\{G_{r,t}, r, t \geq 1\}$ be nonnegative integers with $G_{r,t} = G_{t,r}$ ($G_{r,t}$ is a possible realization of the $\omega_{r,t}$ from (4.1)). We now recursively construct partitions $\lambda^{r,t}, (r, t) \in \mathbb{Z}^2_{\geq 1}$ as follows: for $(r, t) \in \{(1, k) : k \geq 1\} \cup \{(k, 1) : k \geq 1\}$ we set $\lambda^{r,t} = \emptyset$. Given $\lambda^{r,t} =: \kappa, \lambda^{r+1,t} =: \mu, \lambda^{r,t+1} =: \lambda$ we define

$$\lambda^{r+1,t+1} := \mathcal{T}_{\mathrm{loc}}((\kappa, G_{r,t})). \tag{4.26}$$

Note that, since $G_{r,t} = G_{t,r}$ and $\mathcal{T}_{\mathrm{loc}}$ is symmetric in λ and μ, we have that $\lambda^{r,t} = \lambda^{t,r}$.

Given $w^{\mathrm{sym}} \in \{\prec, \succ\}^{2n}$ and the corresponding down-right path $\hat{\pi} = (\hat{\pi}(0), \dots, \hat{\pi}(2n))$, we denote

$$\lambda^{(m)} = \lambda^{\hat{\pi}(m)}, m = 0, \dots, 2n \tag{4.27}$$

and note that we have $\lambda^{(n+k)} = \lambda^{(n-k)}, k = 0, \dots, n$ — see Figure 7 right.

The following Proposition is an elementary induction on $|sh(w^{\mathrm{sym}})|$ (see Theorem 3.2 of [BBB+15] for a similar proof), which we omit carrying out (the partition function Z_w appearing in (4.29) was computed in (2.10)).

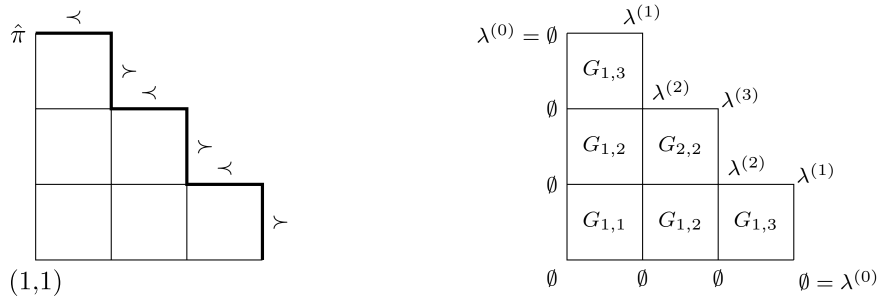

Figure 7: Left: The encoded shape $sh(w^{\mathrm{sym}}) = (3, 2, 1)$ for $w = (\prec, \succ, \prec)$ together with the down-right path $\hat{\pi} = ((1, 4), (2, 4), (2, 3), (3, 3), (3, 2), (4, 2), (4, 1))$. Right: The partitions $\lambda^{(0)}, \ldots, \lambda^{(3)}, \ldots, \lambda^{(0)}$ associated to the points of $\hat{\pi}$ are constructed recursively using the empty partitions on the boundary, the $G_{r,t}$ and $\mathcal{T}_{\mathrm{loc}}$.

Proposition 4.9. *Let $w \in \{\prec, \succ\}^n, w^{\mathrm{sym}} = w \cdot w^\star$ and $\mathfrak{s}_i, i = 1, \ldots, 2n$, be the variables (4.22) for w^{sym}. With the symmetric weights (4.1) we obtain by (4.27) a probability distribution on*

$$\{\vec{\lambda} : \vec{\lambda} = (\emptyset = \lambda^{(0)}, \ldots, \lambda^{(n-1)}, \lambda^{(n)}, \lambda^{(n-1)}, \ldots, \lambda^{(0)}), \vec{\lambda} \textit{ is } w^{\mathrm{sym}}\textit{-interlaced}\} \tag{4.28}$$

which is given by

$$\mathrm{Prob}(\{\vec{\lambda}\}) = \frac{1}{Z_w} \langle \lambda^{(n)} | \Gamma_-(\{\alpha\}) | \underline{1}_{ec} \rangle \cdot \prod_{i=1}^{n} \langle \lambda^{(i-1)} | \Gamma_i(\{\mathfrak{s}_i\}) | \lambda^{(i)} \rangle. \tag{4.29}$$

Now we can proof Theorem 4.5.

Proof of Theorem 4.5. Note that, by construction, $L_{(1,1)\to(m,n)} = \lambda_1^{m+1,n+1}$, where $\lambda_1^{m+1,n+1}$ is from (4.26) and $G_{r,t} = \omega_{r,t}$. Consequently, the LPP times become a gap probability for the point process $\mathfrak{S}(\vec{\lambda})$ where $\vec{\lambda}$ is distributed as (4.29). By (A.15), the left hand side of (4.14) is given as a Fredholm pfaffian, and the corresponding correlation functions were computed in Theorem 2.7,(2.16) and Section 3.3.2, leading to the identity (4.14). ☐

Proof of Theorem 4.6. We have to show the convergence, as $N \to \infty$, of the Fredholm pfaffian provided by Theorem 4.5. By Proposition 4.10, the kernel K from Theorem 4.5 converges pointwise to K^v, and by Proposition 4.11 we can apply Lemma A.3 which yields an integrable upper bound, allowing us to apply dominated convergence to show the convergence of the Fredholm pfaffian. ☐

Proof of Theorem 4.7. We show it first for $\hat{L}_{0\to P_i} = L^{\mathrm{sym,exp}}_{0\to P_i}$. Set

$$L^{\mathrm{exp,resc}}_{0\to P_i} = \frac{L^{\mathrm{exp}}_{0\to P_i} - t(1+\sqrt{\eta_i})^2}{\kappa^{-1/6}(1+\sqrt{\kappa})^{-4/3} t^{1/3}}, \qquad L^{\mathrm{exp,sym,resc}}_{0\to P_i} = \frac{L^{\mathrm{exp,sym}}_{0\to P_i} - t(1+\sqrt{\eta_i})^2}{\kappa^{-1/6}(1+\sqrt{\kappa})^{-4/3} t^{1/3}}. \tag{4.30}$$

Denote the k-dimensional random vectors

$$X_n = \left(L^{\mathrm{exp,resc}}_{0\to P_i}, i = 1, \ldots, k\right), \qquad \tilde{X}_n = \left(L^{\mathrm{sym,exp,resc}}_{0\to P_i}, i = 1, \ldots, k\right). \tag{4.31}$$

We use the notion of convergence in distribution and convergence in probability to 0 for $\mathbb{R}^k$ valued random variables — see e.g. the beginning of Section 3.1 in [FN17]. By Theorem 2 of [BP08], (4.21) holds for $\hat{L}_{0\to P_i} = L^{\mathrm{exp}}_{0\to P_i}$. Thus by Lemma 3.5 of [CFP10], the proof is finished once we show that $X_n - \tilde{X}_n$ converges to 0 in probability. For this it suffices to show that $L^{\mathrm{exp,resc}}_{0\to P_i} - L^{\mathrm{sym,exp,resc}}_{0\to P_i}$ converges to 0 in probability for arbitrary i. I.e., for any $\delta > 0$

$$\lim_{t\to\infty} \mathrm{Prob}(L^{\mathrm{exp}}_{0\to P_i} - L^{\mathrm{sym,exp}}_{0\to P_i} > \delta t^{1/3}) = 0. \tag{4.32}$$

By slow decorrelation [CFP12], we have that for any $\nu \in (2/3,1), \delta > 0$

$$\lim_{t\to\infty} \mathrm{Prob}(L^{\exp}_{0\to P_i} - L^{\exp}_{0\to(\kappa t^\nu + ut^{\nu-1/3},t^\nu)} - L^{\exp}_{(\kappa t^\nu+ut^{\nu-1/3},t^\nu)\to P_i} > \delta t^{1/3}/2) = 0. \tag{4.33}$$

Hence it suffices to check that $L^{\exp,\mathrm{resc}}_{0\to(\kappa t^\nu+ut^{\nu-1/3},t^\nu)} + L^{\exp,\mathrm{resc}}_{(\kappa t^\nu+ut^{\nu-1/3},t^\nu)\to P_i} - L^{\mathrm{sym},\exp,\mathrm{resc}}_{0\to P_i}$ converges to 0 in probability. Since transversal fluctuations are $\mathcal{O}(t^{2/3})$ (see Theorem 3.5 in [Nej17]) and $\nu > 2/3$, we have that the probability that the maximizer of $L^{\exp}_{(\kappa t^\nu+ut^{\nu-1/3},t^\nu)\to P_i}$ contains a point of the diagonal vanishes as $t\to\infty$. This means

$$\lim_{t\to\infty} \mathrm{Prob}(L^{\exp}_{(\kappa t^\nu+ut^{\nu-1/3},t^\nu)\to P_i} \neq L^{\mathrm{sym},\exp}_{(\kappa t^\nu+ut^{\nu-1/3},t^\nu)\to P_i}) = 0. \tag{4.34}$$

By subadditivity of $L^{\mathrm{sym},\exp}$ it thus suffices to show that for any $\delta > 0$

$$\lim_{t\to\infty} \mathrm{Prob}(L^{\exp}_{0\to(\kappa t^\nu+ut^{\nu-1/3},t^\nu)} - L^{\mathrm{sym},\exp}_{0\to(\kappa t^\nu+ut^{\nu-1/3},t^\nu)} > \delta t^{1/3}/2) = 0 \tag{4.35}$$

which for $\ell_1 = t^\nu$ is the same as

$$\lim_{\ell_1\to\infty} \mathrm{Prob}(L^{\exp}_{0\to(\kappa\ell_1+u\ell_1^{1-1/3\nu},\ell_1)} - L^{\mathrm{sym},\exp}_{0\to(\kappa\ell_1+u\ell_1^{1-1/3\nu},\ell_1)} > \delta\ell_1^{1/3\nu}/2) = 0. \tag{4.36}$$

Note that from (4.32) to (4.36) we improved the bound from $\delta t^{1/3}$ to $\delta\ell_1^{1/3\nu}/2$. Let $k > 0$ be such that $\nu^{-k}/3 > 1$ and define $\ell_j = \ell_{j-1}^\nu, j \geq 2$. We can now repeat the steps from (4.32) to (4.36) with ℓ_1 instead of t. Recursively, we thus see that (4.32) follows from

$$\lim_{\ell_k\to\infty} \mathrm{Prob}(L^{\exp}_{0\to(\kappa\ell_k+u\ell_k^{x_k},\ell_k)} - L^{\mathrm{sym},\exp}_{0\to(\kappa\ell_k+u\ell_k^{x_k},\ell_1)} > 2^{-k}\delta\ell_k^{\nu^{-k}/3}) = 0 \tag{4.37}$$

with $x_1 := 2/3, x_{j+1} := (\nu + x_j - 1)/\nu, j \geq 1$. But since $\nu^{-k}/3 > 1$, we can simply bound

$$\lim_{\ell_k\to\infty} \mathrm{Prob}(L^{\exp}_{0\to(\kappa\ell_k+u\ell_k^{x_k},\ell_k)} - L^{\mathrm{sym},\exp}_{0\to(\kappa\ell_k+u\ell_k^{x_k},\ell_1)} > \delta\ell_k^{\nu^{-k}/3}2^{-k}) \leq \lim_{\ell_k\to\infty} \mathrm{Prob}(L^{\exp}_{0\to(\ell_k,\ell_k)} > \delta 2^{-k}\ell_k^{\nu^{-k}/3}) = 0. \tag{4.38}$$

Now setting $\hat{X}_n = \left(\hat{L}^{\mathrm{resc}}_{0\to P_i}, i = 1,\dots,k\right)$ we have that componentwise

$$X_n - \hat{X}_n \leq X_n - \tilde{X}_n \tag{4.39}$$

and hence $X_n - \hat{X}_n$ converges to 0 in probability. By Lemma 3.5 of [CFP10] and since componentwise $X_n - \hat{X}_n \geq 0$ the result follows. ☐

4.1.2 Asymptotics

In the following asymptotics, we will ignore the integer parts in (4.40) when it leads to no confusion.

Proposition 4.10. *Set $a_j = \sqrt{q}$ for a $q \in (0,1)$ and let $u_1 > \cdots > u_k \geq 0$ and $c_q = \frac{1-\sqrt{q}}{q^{1/6}(1+\sqrt{q})^{1/3}}, d_q = \frac{q^{1/6}}{2(1+\sqrt{q})^{2/3}}$. Set $\alpha = 1 - 2vc_qN^{-1/3}, v \in \mathbb{R}$. We take $t_l = N, r_l = N - \lfloor u_l N^{2/3}\rfloor, l = 1,\dots,k$ and*

$$\begin{aligned} x_a &= \lfloor \frac{2\sqrt{q}}{1-\sqrt{q}}N\rfloor - \lfloor u_a N^{2/3}\frac{\sqrt{q}}{1-\sqrt{q}}\rfloor + \lfloor \xi N^{1/3}\rfloor - 1/2, \\ x_b &= \lfloor \frac{2\sqrt{q}}{1-\sqrt{q}}N\rfloor - \lfloor u_b N^{2/3}\frac{\sqrt{q}}{1-\sqrt{q}}\rfloor + \lfloor \xi' N^{1/3}\rfloor - 1/2. \end{aligned} \tag{4.40}$$

We then have for the kernels K from Theorem 4.5 and K^v from (4.10):

$$\begin{aligned} &\lim_{N\to\infty} N^{2/3}e^{-N^{2/3}\ln(1-\sqrt{q})(u_a+u_b)}c_q^{-1}K_{1,1}(a,x_a;b,x_b) = c_q K^v_{1,1}(d_q u_a, c_q\xi; d_q u_b, c_q\xi'), \\ &\lim_{N\to\infty} N^{1/3}e^{N^{2/3}\ln(1-\sqrt{q})(u_b-u_a)}K_{1,2}(a,x_a;b,x_b) = c_q K^v_{1,2}(d_q u_a, c_q\xi; d_q u_b, c_q\xi'), \\ &\lim_{N\to\infty} e^{N^{2/3}\ln(1-\sqrt{q})(u_b+u_a)}c_q K_{2,2}(a,x_a;b,x_b) = c_q K^v_{2,2}(d_q u_a, c_q\xi; d_q u_b, c_q\xi'). \end{aligned} \tag{4.41}$$

Proof. We start with $K_{1,2}$. We get for $a \leq b$

$$
\begin{aligned}
K_{1,2}(a,x_a;b,x_b) &= \frac{1}{(2\pi\mathrm{i})^2}\oint_{1/\sqrt{q}>|z|>\alpha,1}\mathrm{d}z\oint_{1/\sqrt{q},|z|>|w|>\alpha,\sqrt{q}}\mathrm{d}w\frac{z-\alpha}{w-\alpha}\frac{zw-1}{w(z-w)(z-1)(z+1)} \qquad (4.42)\\
&\times\frac{e^{Nf_1(z)+N^{2/3}u_af_2(z)-N^{1/3}f_3(z)\xi}}{e^{Nf_1(w)+N^{2/3}u_bf_2(w)-N^{1/3}f_3(w)\xi'}}\\
&= \frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_{-1,0,1,w}}\mathrm{d}z\oint_{\gamma_{\sqrt{q},\alpha}}\mathrm{d}w\frac{z-\alpha}{w-\alpha}\frac{zw-1}{w(z-w)(z-1)(z+1)} \qquad (4.43)\\
&\times\frac{e^{Nf_1(z)+N^{2/3}u_af_2(z)-N^{1/3}f_3(z)\xi}}{e^{Nf_1(w)+N^{2/3}u_bf_2(w)-N^{1/3}f_3(w)\xi'}}
\end{aligned}
$$

where we have set

$$
\begin{aligned}
f_1(z) &= \ln(1-\sqrt{q}/z)-\ln(1-\sqrt{q}z)-\ln(z)\frac{2\sqrt{q}}{1-\sqrt{q}},\\
f_2(z) &= \ln(z)\frac{\sqrt{q}}{1-\sqrt{q}}+\ln(1-\sqrt{q}z),\\
f_3(z) &= \ln(z).
\end{aligned}
\tag{4.44}
$$

One readily computes $f_1(1) = f_1'(1) = f_1''(1) = 0, f_1'''(1) = 2c_q^{-3}$ and $f_2(1) = \ln(1-\sqrt{q}), f_2'(1) = 0, f_2''(1) = -\frac{\sqrt{q}}{(1-\sqrt{q})^2}$, also $d_q = -f_2''(1)c_q^2/2$.

Let us briefly outline the strategy for the asymptotics. We choose contours $\gamma_{-1,0,1,w}$ for z and $\gamma_{\sqrt{q},\alpha}$ for w such that they pass (almost) through the critical point 1. Furthermore, for $\delta > 0$ small (but independent of N) we want $\Re(f_1(z)) \leq -c_{0,0}$ and $-\Re(f_1(w)) \leq -c_{0,0}$ for some $c_{0,0} > 0$ (depending on δ) for $z \in \gamma^{\delta}_{-1,0,1,w}, w \in \gamma^{\delta}_{\sqrt{q},\alpha}$ where we define $\gamma^{\delta}_{-1,0,1,w} = \{z \in \gamma_{-1,0,1,w} : |z-1| \leq \delta\}, \gamma^{\delta}_{\sqrt{q},\alpha} = \{w \in \gamma_{\sqrt{q},\alpha} : |w-1| \leq \delta\}$.

Given this, the integral over $(\gamma^{\delta}_{-1,0,1,w} \cup \gamma^{\delta}_{\sqrt{q},\alpha})^c$ will vanish. On $\gamma^{\delta}_{-1,0,1,w} \cup \gamma^{\delta}_{\sqrt{q},\alpha}$ we use Taylor to obtain $Nf_1(1+Zc_qN^{-1/3}) = Z^3/3+\mathcal{O}(Z^4N^{-1/3})$. For this reason we want $(\gamma^{\delta}_{-1,0,1,w}-1)N^{1/3}$ respectively $(\gamma^{\delta}_{\sqrt{q},\alpha}-1)N^{1/3}$ to lie (up to a part of length $\mathcal{O}(1)$) in $\{z \in \mathbb{C} : \Re(z^3) < 0\}$ respectively $\{w \in \mathbb{C} : \Re(w^3) > 0\}$.

We first note that $\Re(f_1(e^{\mathrm{i}s})) = 0$ for all s. Furthermore, we compute

$$
L(z) = zf_1'(z) = \frac{\sqrt{q}(1+\sqrt{q})}{1-\sqrt{q}}\frac{(1-z)^2}{(\sqrt{q}-z)(-1+\sqrt{q}z)}. \tag{4.45}
$$

Now

$$
\Im(L(e^{\mathrm{i}s})) = r\Im((1-\cos(s)-\mathrm{i}\sin(s))^2(\sqrt{q}-\cos(s)+\mathrm{i}\sin(s))(-1+\sqrt{q}\cos(s)-\sqrt{q}\mathrm{i}\sin(s))) \tag{4.46}
$$

where r is an irrelevant real number. From (4.46), an elementary computation shows

$$
\Im(L(e^{\mathrm{i}s})) = 0. \tag{4.47}
$$

Next we treat $\Re(L(e^{\mathrm{i}s})) = L(e^{\mathrm{i}s})$. We have

$$
\Re(L(z)) = \frac{\sqrt{q}(1+\sqrt{q})}{1-\sqrt{q}}\frac{1}{|(\sqrt{q}-z)(-1+\sqrt{q}z)|^2}\Re((1-z)^2(\sqrt{q}-\bar{z})(-1+\sqrt{q}\bar{z})) \tag{4.48}
$$

Again a simple computation reveals from (4.48)

$$
\Re((1-e^{\mathrm{i}s})^2(\sqrt{q}-e^{-\mathrm{i}s})(-1+\sqrt{q}e^{-\mathrm{i}s})) = 2(-1+\cos(s))(1-2\sqrt{q}\cos(s)+\sqrt{q}^2), \tag{4.49}
$$

which is negative on $(0,2\pi)$ and strictly decreasing on $[0,\pi]$. This implies

$$
L(e^{\mathrm{i}s}) = \frac{\sqrt{q}(1+\sqrt{q})}{1-\sqrt{q}}\frac{2(-1+\cos(s))}{|\sqrt{q}-e^{\mathrm{i}s}|^2}. \tag{4.50}
$$

Let now $\varepsilon > 0$, and $\phi \in [0, 2\pi]$. Then by Taylor approximation for some $t_s \in [0,1]$

$$\begin{aligned}\Re(f_1(e^{\mathrm{i}s}+\varepsilon e^{\mathrm{i}\phi})) &= \Re(f_1(e^{\mathrm{i}s})) + \Re(f_1'(e^{\mathrm{i}s})\varepsilon e^{\mathrm{i}\phi}) + \frac{1}{2}\Re(f_1''(e^{\mathrm{i}s}+t_s\varepsilon e^{\mathrm{i}\phi})\varepsilon^2 e^{2\mathrm{i}\phi}) \\ &= L(e^{\mathrm{i}s})\varepsilon\cos(\phi - s) + \frac{\varepsilon^2}{2}\Re(f_1''(e^{\mathrm{i}s}+t_s\varepsilon e^{\mathrm{i}\phi})e^{2\mathrm{i}\phi}).\end{aligned} \tag{4.51}$$

Take $\varepsilon = -\varepsilon_1 L(e^{\mathrm{i}s})$ for $\varepsilon_1 > 0$ small and $\phi = \pi + s$ respectively $\phi = s$ to get

$$\begin{aligned}&\Re(f_1((1+\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s})) \geq L(e^{\mathrm{i}s})^2\varepsilon_1/2, \\ &\Re(f_1((1-\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s})) \leq -L(e^{\mathrm{i}s})^2\varepsilon_1/2.\end{aligned} \tag{4.52}$$

Choose now the contour

$$\gamma_{\sqrt{q},\alpha}(s) = (1+N^{-1/3}(2|v|c_q+1/2))(1+\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s}, \quad s \in [0,2\pi]. \tag{4.53}$$

The prefactor $(1+N^{-1/3}(2|v|c_q+1/2))$ makes sure that α lies inside the contour. Furthermore, by (4.52) and (4.50), for any $\delta > 0$ independent of N there is a $c_{0,0} > 0$ for which we have $-\Re(f_1(w)) \leq -c_{0,0}$ for $w \in \gamma_{\sqrt{q},\alpha} \setminus \gamma^{\delta}_{\sqrt{q},\alpha} = \{w \in \gamma_{\sqrt{q},\alpha} : |w-1| > \delta\}$. To choose $\gamma_{-1,0,1,w}$ consider first

$$\tilde{\gamma}_{-1,0,1,w}(s) = (1+N^{-1/3}(2|v|c_q+1))(1-\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s}, \quad s \in [0,2\pi]. \tag{4.54}$$

By (4.52) and (4.50), $\Re(f_1(\tilde{\gamma}_{-1,0,1,w}(s)))$ behaves as desired, but $\Re(\tilde{\gamma}_{-1,0,1,w}(s)) < 1+N^{-1/3}(2|v|c_q+1))$ if $\varepsilon_1 L(e^{\mathrm{i}s})(-1+\cos(s))^{-1}$ is too small, i.e. the contour may not cross the positive real axis with the right angles. We thus do a local modification. Choose $\eta_1 > 0$ small and let $\eta_2 > 0$ be the number such that $e^{\mathrm{i}\arcsin(\eta_1)} + \eta_2 \in \{(1-\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s}, s \in [0,2\pi]\}$, and let $e^{\mathrm{i}\arcsin(\eta_1)} + \eta_2 = (1-\varepsilon_1 L(e^{\mathrm{i}s_{\eta_2}}))e^{\mathrm{i}s_{\eta_2}}$. Let $M > 1$ and define

$$\begin{aligned}&\gamma^{loc,1}_{-1,0,1,w}(s) = 1+\mathrm{i}s+s/M,\ s\in[0,\eta_1] \quad \text{and} \\ &\gamma^{loc,2}_{-1,0,1,w}(s) = e^{\mathrm{i}\arcsin(\eta_1)} + s,\ s \in [\eta_2, 1+\eta_1/M - \cos(\arcsin(\eta_1))].\end{aligned} \tag{4.55}$$

Choosing M sufficiently large and η_1 sufficiently small we get by Taylor approximation around 1 and $e^{\arcsin(\eta_1)\mathrm{i}}$ that for some constants $d_1, d_2 > 0$

$$\Re(f_1(\gamma^{loc,1}_{-1,0,1,w}(s))) \leq -s^3 d_1 \quad \text{and} \quad \Re(f_1(\gamma^{loc,2}_{-1,0,1,w}(s))) \leq -\eta_1^2 s d_2. \tag{4.56}$$

Define finally $\gamma^{loc,3}_{-1,0,1,w} = \{(1-\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s}, s \in [s_{\eta_2}, \pi]\}$.

Since $\Re(f_1(z)) = \Re(f_1(\bar{z}))$ it suffices to define $\gamma_{-1,0,1,w}$ on the upper half plane. We set

$$\gamma_{-1,0,1,w} = (1+N^{-1/3}(2|v|c_q+1))(\gamma^{loc,1}_{-1,0,1,w} + \gamma^{loc,2}_{-1,0,1,w} + \gamma^{loc,3}_{-1,0,1,w}) \tag{4.57}$$

where the + means we concatenate the curves such that $\gamma_{-1,0,1,w}$ is counterclockwise oriented. On the lower half plane, we simply take the image of (4.57) under complex conjugation. We now choose $\delta < \eta_1$ such that $\gamma^{\delta}_{-1,0,1,w} \subset \{1+\mathrm{i}s+|s|/M, s \in [-\eta_1,\eta_1]\}$.

By virtue of (4.56), (4.52), (4.50) for $z \in \gamma_{-1,0,1,w} \setminus \gamma^{\delta}_{-1,0,1,w} = \{z \in \gamma_{-1,0,1,w} : |z-1| > \delta\}$ we have $\Re(f_1(z)) \leq -c_{0,1}$ for a $c_{0,1} > 0$ and we can take $c_{0,0} = c_{0,1}$. We choose Σ to be the part of the contours where $z \notin \gamma^{\delta}_{-1,0,1,w}$ and/or $w \notin \gamma^{\delta}_{\sqrt{q},\alpha}$. Then the integral (4.43) is on

$$\Sigma \cup (\gamma^{\delta}_{-1,0,1,w} \cup \gamma^{\delta}_{\sqrt{q},\alpha}) = \gamma_{-1,0,1,w} \cup \gamma_{\sqrt{q},\alpha}. \tag{4.58}$$

On Σ, there is a $c_{0,0} > 0$ such that $\Re(f_1(z)) < -c_{0,0}$ and/or $-\Re(f_1(w)) < -c_{0,0}$, and also $\Re(f_1(z)), -\Re(f_1(w)) < c_{0,0}/4$. Furthermore, we can bound $\left|\frac{z-\alpha}{w(w-\alpha)}\frac{zw-1}{(z-w)(z^2-1)}\right| < C(\delta)N^{1/3}$ where $C(\delta)$ is a constant depending on δ. So overall we may bound

$$N^{1/3}e^{N^{2/3}f_2(1)(u_b-u_a)}\left|\int_{\Sigma}\mathrm{d}z\mathrm{d}w\frac{zw-1}{(z^2-1)(z-w)}\frac{z-\alpha}{w(w-\alpha)}\frac{e^{Nf_1(z)+N^{2/3}u_af_2(z)-N^{1/3}f_3(z)\xi}}{e^{Nf_1(w)+N^{2/3}u_bf_2(w)-N^{1/3}f_3(w)\xi'}}\right| \leq Ce^{-Nc_{0,0}/2} \tag{4.59}$$

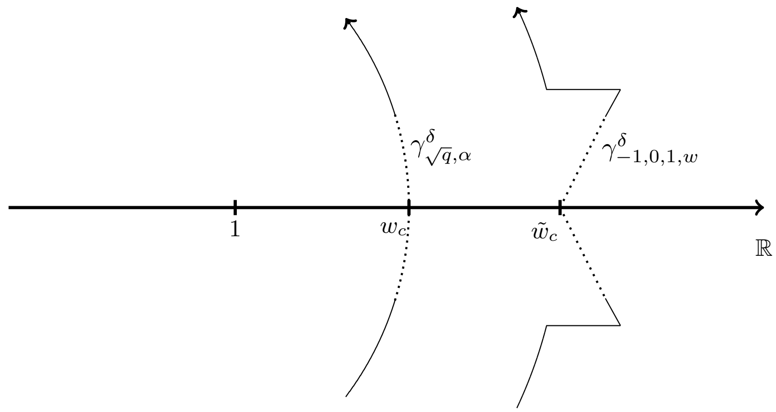

Figure 8: Parts of the contours $\gamma_{\sqrt{q},\alpha}$ and $\gamma_{-1,0,1,w}$ from (4.53),(4.57) close to where they cross the positive real axis in $w_c = 1 + N^{-1/3}(2|v|c_q + 1/2)$, $\tilde{w}_c = w_c + N^{-1/3}/2$. The curve segments $\gamma^{\delta}_{\sqrt{q},\alpha}, \gamma^{\delta}_{-1,0,1,w}$ are shown in dotted lines.

for an $C > 0$.

For the integral on $\gamma^{\delta}_{-1,0,1,w} \cup \gamma^{\delta}_{\sqrt{q},\alpha}$, we do the change of variable

$$z = 1 + Zc_qN^{-1/3} \quad w = 1 + Wc_qN^{-1/3}. \tag{4.60}$$

Use now Taylor and define $F(Z,W)$ via

$$\begin{aligned} &e^{N(f_1(z)-f_1(w))}e^{N^{2/3}(u_af_2(z)-u_bf_2(w))}e^{N^{1/3}(\xi' f_3(w)-\xi f_3(z))} \\ &= e^{N^{2/3}(u_af_2(1)-u_bf_2(1))}e^{Z^3/3+Z^2u_ac_q^2f_2''(1)/2-\xi Zc_q-W^3/3-W^2u_bc_q^2f_2''(1)/2+Wc_q\xi'} \\ &\times e^{\mathcal{O}(Z^4N^{-1/3})+\mathcal{O}(Z^3N^{-1/3})+\mathcal{O}(Z^2N^{-1/3})+\mathcal{O}(W^4N^{-1/3})+\mathcal{O}(W^3N^{-1/3})+\mathcal{O}(W^2N^{-1/3})} \\ &= F(Z,W)e^{N^{2/3}(u_af_2(1)-u_bf_2(1))} \end{aligned} \tag{4.61}$$

and denote $\tilde{F}(Z,W) = e^{Z^3/3+Z^2u_ac_q^2f_2''(1)/2-\xi Zc_q-W^3/3-W^2u_bc_q^2f_2''(1)/2+Wc_q\xi'}$. To control the contribution from the error terms in (4.61), we use the inequality $|e^x - 1| \leq e^{|x|}|x|$.

With the change of variable (4.60), we have to control

$$\begin{aligned} &\Big|\frac{c_q}{(2\pi\mathrm{i})^2}\int_{(\gamma^{\delta}_{-1,0,1,w}-1)\frac{N^{1/3}}{c_q}}\mathrm{d}Z\int_{(\gamma^{\delta}_{\sqrt{q},\alpha}-1)\frac{N^{1/3}}{c_q}}\mathrm{d}W\frac{Z+2v}{W+2v}\frac{Z+W+ZWN^{-1/3}c_q}{(1+Wc_qN^{-1/3})(Z-W)Z(2+Zc_qN^{-1/3})} \\ &\qquad\qquad\times(F(Z,W)-\tilde{F}(Z,W))\Big| \leq \\ &\frac{c_q}{(2\pi)^2}\int_{(\gamma^{\delta}_{-1,0,1,w}-1)\frac{N^{1/3}}{c_q}}|\mathrm{d}Z|\int_{(\gamma^{\delta}_{\sqrt{q},\alpha}-1)\frac{N^{1/3}}{c_q}}|\mathrm{d}W|\left|\frac{Z+2v}{W+2v}\frac{Z+W+ZWN^{-1/3}c_q}{(Z-W)Z}\tilde{F}(Z,W)\right| \\ &\times e^{|\mathcal{O}(Z^4N^{-1/3})+\mathcal{O}(Z^3N^{-1/3})+\mathcal{O}(Z^2N^{-1/3})+\mathcal{O}(W^4N^{-1/3})+\mathcal{O}(W^3N^{-1/3})+\mathcal{O}(W^2N^{-1/3})|} \\ &\times|\mathcal{O}(Z^4N^{-1/3})+\mathcal{O}(Z^3N^{-1/3})+\mathcal{O}(Z^2N^{-1/3})+\mathcal{O}(W^4N^{-1/3})+\mathcal{O}(W^3N^{-1/3})+\mathcal{O}(W^2N^{-1/3})| \leq \\ &\frac{c_q}{(2\pi)^2}\int_{(\gamma^{\delta}_{-1,0,1,w}-1)\frac{N^{1/3}}{c_q}}|\mathrm{d}Z|\int_{(\gamma^{\delta}_{\sqrt{q},\alpha}-1)\frac{N^{1/3}}{c_q}}|\mathrm{d}W|\left|\frac{Z+2v}{W+2v}\frac{Z+W+ZWN^{-1/3}c_q}{(Z-W)Z}\frac{e^{Z^3(1+\chi_1)/3-\xi Z(1+\chi_2)c_q}}{e^{W^3(1+\chi_3)/3-W(1+\chi_4)c_q\xi'}}\right| \end{aligned} \tag{4.62}$$

$$\times\left|\frac{e^{Z^2(1+\chi_5)u_ac_q^2f_2''(1)/2}}{e^{W^2(1+\chi_6)u_bc_q^2f_2''(1)/2}}\right|N^{-1/3}|\mathcal{O}(Z^4)+\mathcal{O}(Z^3)+\mathcal{O}(Z^2)+\mathcal{O}(W^4)+\mathcal{O}(W^3)+\mathcal{O}(W^2)| \tag{4.63}$$

where the $\chi_i \in \mathbb{R}, i = 1, \ldots, 6$ can be taken as small in absolute value as desired by taking δ small. Now for large N, the $|e^{Z^3(1+\chi_1)/3}e^{-W^3(1+\chi_3)/3}|$ term dominates the integral. At the integration boundary, it is of order $e^{-\mathcal{O}(\delta^3 N)}$. This easily implies that $N^{1/3} \times (4.63) \times (4.62)$ remains bounded as $N \to \infty$. Consequently, $(4.63) \times (4.62)$ vanishes like $N^{-1/3}$. We can thus take $\tilde{F}(Z,W)$ instead of $F(Z,W)$ and only make an error of $\mathcal{O}(N^{-1/3})$. By doing so, we are left with

$$\frac{c_q}{(2\pi\mathrm{i})^2}\int_{(\gamma^\delta_{-1,0,1,w}-1)\frac{N^{1/3}}{c_q}} \mathrm{d}Z \int_{(\gamma^\delta_{\sqrt{q},\alpha}-1)\frac{N^{1/3}}{c_q}} \mathrm{d}W \frac{Z+2v}{W+2v}\frac{Z+W+ZWc_qN^{-1/3}}{(Z-W)Z}\frac{(2+Zc_qN^{-1/3})^{-1}}{1+WN^{-1/3}c_q}\tilde{F}(Z,W). \tag{4.64}$$

Finally, in (4.64) we can extend the curves to infinity (inside $\{z \in \mathbb{C} : \Re(z^3) < 0\}$ and $\{w \in \mathbb{C} : \Re(w^3) > 0\}$) and thus only make an error $e^{-\mathcal{O}(N)}$.

We can then deform the contours to be as in (4.5) without errors.

To summarize, we have shown that

$$N^{1/3}e^{f_2(1)N^{2/3}(u_b-u_a)}K_{1,2}(a,x_a;b,x_b) = \frac{c_q}{(2\pi\mathrm{i})^2}\int_{\mathfrak{G}^{\pi/3}_{A_Z}} \mathrm{d}Z \int_{\mathfrak{G}^{2\pi/3}_{A_W}} \mathrm{d}W \frac{(Z+W+ZWc_qN^{-1/3})(Z+2v)}{(W+2v)Z(Z-W)} \tag{4.65}$$

$$\times \frac{e^{Z^3/3+Z^2u_ac_q^2f_2''(1)/2-\xi Zc_q-W^3/3-W^2u_bc_q^2f_2''(1)/2+Wc_q\xi'}}{(1+WN^{-1/3}c_q)(2+Zc_qN^{-1/3})} \tag{4.66}$$

$$+\mathcal{O}(N^{-1/3}+e^{-Nc_{0,0}/2}+e^{-\mathcal{O}(N)}), \tag{4.67}$$

implying

$$\lim_{N\to\infty} N^{1/3}e^{f_2(1)N^{2/3}(u_b-u_a)}K_{1,2}(a,x_a;b,x_b) = \frac{c_q}{(2\pi\mathrm{i})^2}\int_{\mathfrak{G}^{\pi/3}_{A_Z}} \mathrm{d}Z \int_{\mathfrak{G}^{2\pi/3}_{A_W}} \mathrm{d}W \frac{(Z+W)(Z+2v)}{2(W+2v)Z(Z-W)} \times e^{Z^3/3+Z^2u_ac_q^2f_2''(1)/2-\xi Zc_q-W^3/3-W^2u_bc_q^2f_2''(1)/2+Wc_q\xi'} \tag{4.68}$$

for $A_Z > A_W > -2v, A_Z > 0$.

If now $a > b$, we have the condition $|z| < |w|$ on our contours. Deforming them so as to equal (4.57), (4.53), we pick up an extra residue, which equals

$$N^{1/3}e^{f_2(1)N^{2/3}(u_b-u_a)}\frac{-1}{2\pi\mathrm{i}}\oint_{\gamma_0}\mathrm{d}z e^{N^{2/3}f_2(z)(u_a-u_b)+N^{1/3}f_3(z)(\xi'-\xi)}z^{-1}. \tag{4.69}$$

For $\gamma_0(s) = e^{\mathrm{i}s}$ the contribution of the integral over γ_0^δ in (4.69) clearly vanishes. Note $u_a - u_b < 0$. On γ_0^δ we do the same change of variable $z = 1+Zc_qN^{-1/3}$. Next we Taylor-expand $f_2(1+Zc_qN^{-1/3})$, $f_3(1+Zc_qN^{-1/3})$ as before, and control the contribution from the remainder terms as before. Sending then $N \to \infty$ shows that (4.69) converges to

$$\frac{-c_q}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}Z e^{Z^2c_q^2f_2''(1)(u_a-u_b)/2}e^{Zc_q(\xi'-\xi)}, \tag{4.70}$$

with $\mathrm{i}\mathbb{R}$ oriented with increasing imaginary part.

Next we come to $K_{1,1}(a,x_a;b,x_b)$. We have

$$K_{1,1}(a,x_a;b,x_b) = \frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_{-1,0,1}}\mathrm{d}z\oint_{\gamma_{-1,0,1}}\mathrm{d}w e^{N(f_1(z)+f_1(w))+N^{2/3}(u_af_2(z)+u_bf_2(w))} \times \frac{e^{-N^{1/3}(f_3(z)\xi+f_3(w)\xi')}(z-w)}{(z^2-1)(w^2-1)(zw-1)}(z-\alpha)(w-\alpha). \tag{4.71}$$

We can choose the contour (4.57) for both z and w in (4.71). Redoing all the steps made for $K_{1,2}$ we obtain

$$\lim_{N\to\infty} N^{2/3}e^{-N^{2/3}f_2(1)(u_a+u_b)}K_{1,1}(a,x_a;b,x_b) = \frac{c_q^2}{(2\pi\mathrm{i})^2}\int_{\mathfrak{G}^{\pi/3}_1}\mathrm{d}Z\int_{\mathfrak{G}^{\pi/3}_1}\mathrm{d}W\frac{(Z-W)(W+2v)(Z+2v)}{4ZW(Z+W)} \times e^{Z^3/3+Z^2u_ac_q^2f_2''(1)/2-\xi Zc_q+W^3/3+W^2u_bc_q^2f_2''(1)/2-Wc_q\xi'}. \tag{4.72}$$

Finally, we come to $K_{2,2}(a, x_a; b, x_b)$. We have

$$K_{2,2}(a,x_a;b,x_b) = \frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_{\sqrt{q},\alpha}}\mathrm{d}z\oint_{\gamma_{\sqrt{q},\alpha,1/z}}\mathrm{d}w e^{-N(f_1(z)+f_1(w))-N^{2/3}(u_a f_2(z)+u_b f_2(w))+N^{1/3}(f_3(z)\xi+f_3(w)\xi')} \times \frac{z-w}{zw(z-\alpha)(w-\alpha)(zw-1)}. \tag{4.73}$$

We can choose the contour for z to also contain $1/\alpha$, e.g. $\gamma_{\sqrt{q},\alpha}(s) = (1 + N^{-1/3}(2|v|c_q + 1/2))e^{\mathrm{i}s}$. Note that on the unit circle, $\Re(-f_2(z)), \Re(-f_2(1/z))$ are maximal at $z = 1$ and decrease until they reach $z = -1$.

We consider first the case $u_a + u_b > 0$. We start with the residue of w at $1/z$. It equals

$$\frac{1}{2\pi\mathrm{i}}\oint_{(1+N^{-1/3}(2|v|c_q+1/2))e^{\mathrm{i}s}}\mathrm{d}z\frac{z^2-1}{z(z-\alpha)(-\alpha)(z-1/\alpha)}e^{-N^{2/3}(f_2(z)u_a+f_2(1/z)u_b)}e^{N^{1/3}(f_3(z)\xi-f_3(z)\xi')}. \tag{4.74}$$

Repeating the steps of the asymptotics for $K_{1,2}$, one then obtains

$$\lim_{N\to\infty} e^{N^{2/3}f_2(1)(u_a+u_b)}(4.74) = \frac{-1}{2\pi\mathrm{i}}\oint_{\mathfrak{S}_{B_1}^{2\pi/3}}\mathrm{d}Z\frac{2Z}{(Z+2v)(Z-2v)}e^{-Z^2 f_2''(1)c_q^2(u_a+u_b)/2+Zc_q(\xi-\xi')} \tag{4.75}$$

with $B_1 > 2|v|$.

Next we consider the contribution from the pole at $w = \alpha$, given by

$$\frac{e^{N^{2/3}f_2(1)(u_a+u_b)}}{2\pi\mathrm{i}}\oint_{(1+N^{-1/3}(2|v|c_q+1/2))e^{\mathrm{i}s}}\mathrm{d}z\frac{e^{-N(f_1(z)+f_1(\alpha))-N^{2/3}(f_2(z)u_a+f_2(\alpha)u_b)+N^{1/3}(f_3(z)\xi+f_3(\alpha)\xi')}}{z\alpha(z\alpha-1)} \tag{4.76}$$

We may deform the contour $(1 + N^{-1/3}(2|v|c_q + 1/2))e^{\mathrm{i}s}$ in (4.76) to

$$(1+N^{-1/3}(2|v|c_q+1/2))(1+\varepsilon_1 L(e^{\mathrm{i}s}))e^{\mathrm{i}s}, s\in[0,2\pi]. \tag{4.77}$$

without errors. By the same asymptotic analysis performed for $K_{1,2}$ we then see that

$$\lim_{N\to\infty}(4.76) = \frac{e^{8v^3/3-2v^2c_q^2f_2''(1)u_b-2vc_q\xi'}}{2\pi\mathrm{i}}\int_{\mathfrak{S}_{B_2}^{2\pi/3}}\mathrm{d}Z\frac{1}{Z-2v}e^{-Z^3/3-Z^2u_ac_q^2f_2''(1)/2+\xi Zc_q} \tag{4.78}$$

with $B_2 > 2v$.

We compute the remaining term

$$\lim_{N\to\infty}\frac{e^{N^{2/3}f_2(1)(u_a+u_b)}}{(2\pi\mathrm{i})^2}\oint_{\gamma_{\sqrt{q},\alpha}}\mathrm{d}z\oint_{\gamma_{\sqrt{q}}}\mathrm{d}w e^{-N(f_1(z)+f_1(w))-N^{2/3}(u_af_2(z)+u_bf_2(w))+N^{1/3}(f_3(z)\xi+f_3(w)\xi')} \times\frac{z-w}{zw(z-\alpha)(w-\alpha)(zw-1)}. \tag{4.79}$$

As contours, we can choose (4.77) for $\gamma_{\sqrt{q},\alpha}$, and $\frac{1-N^{-1/3}(4|v|c_q+1)}{1+N^{-1/3}(2|v|c_q+1/2)} \times (4.77)$ for $\gamma_{\sqrt{q}}$ (note that with this choice, $\gamma_{\sqrt{q},\alpha}$ has no point of $\gamma_{\sqrt{q}}$ in its interior, and vice versa).

With this choice, we can now proceed exactly as before. This then yields that

$$(4.79) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathfrak{S}_{B_3}^{2\pi/3}}\mathrm{d}Z\int_{\mathfrak{S}_{B_4}^{2\pi/3}}\mathrm{d}W\frac{e^{-Z^3/3-u_aZ^2c_q^2f_2''(1)/2+Zc_q\xi}}{e^{W^3/3+u_bW^2c_q^2f_2''(1)/2-Wc_q\xi'}}\frac{Z-W}{(W+2v)(Z+2v)(Z+W)}. \tag{4.80}$$

for $B_3 > -2v, B_4 < -2v$ and $B_3 < -B_4$.

For $u_a = u_b = 0$, we proceed as follows. Writing the integer parts, the residue of $w = 1/z$ is given by

$$\frac{1}{2\pi\mathrm{i}}\oint_{(1+N^{-1/3}(2|v|c_q+1/2))e^{\mathrm{i}s}}\mathrm{d}z\frac{z^2-1}{(-\alpha)(z-1/\alpha)(z-\alpha)}z^{\lfloor N^{1/3}\xi\rfloor-\lfloor N^{1/3}\xi'\rfloor-1}. \tag{4.81}$$

Now, if $\lfloor N^{1/3}\xi\rfloor - \lfloor N^{1/3}\xi'\rfloor \leq -1$, we can send $z\to\infty$ in (4.81) which shows (4.81) $=0$. If $\lfloor N^{1/3}\xi\rfloor - \lfloor N^{1/3}\xi'\rfloor > 0$, (4.81) equals the sum of the residues at $z=\alpha, 1/\alpha$. and if $\lfloor N^{1/3}\xi\rfloor - \lfloor N^{1/3}\xi'\rfloor = 0$, (4.81) equals the sum of the residues at $z=0,\alpha,1/\alpha$. Thus

$$\begin{aligned}\lim_{N\to\infty}(4.81) &= \lim_{N\to\infty}\Big(-\mathbf{1}_{\{\frac{\lfloor N^{1/3}\xi\rfloor-\lfloor N^{1/3}\xi'\rfloor}{N^{1/3}}>0\}}\left(e^{-2vc_q\frac{\lfloor N^{1/3}\xi\rfloor-\lfloor N^{1/3}\xi'\rfloor}{N^{1/3}}}+e^{2vc_q\frac{\lfloor N^{1/3}\xi\rfloor-\lfloor N^{1/3}\xi'\rfloor}{N^{1/3}}}\right)\\ &\quad-\mathbf{1}_{\{\lfloor N^{1/3}\xi\rfloor-\lfloor N^{1/3}\xi'\rfloor=0\}}\alpha^{-1}\Big)\\ &= -\mathbf{1}_{\{\xi-\xi'>0\}}(e^{-2vc_q(\xi-\xi')}+e^{2vc_q(\xi-\xi')})-\mathbf{1}_{\{\xi=\xi'\}}.\end{aligned}\tag{4.82}$$

The contribution (4.76) from the pole at $w=\alpha$ is as before, but here we compute separately the residue $z=1/\alpha$ in (4.76), which equals

$$\frac{1}{\alpha}e^{f_3(\alpha)N^{1/3}(\xi'-\xi)}\to_{N\to\infty} e^{2vc_q(\xi-\xi')}.\tag{4.83}$$

What thus remains to compute from (4.76) is the limit of

$$\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_{\sqrt{q}}}\mathrm{d}z\frac{1}{z\alpha(z\alpha-1)}e^{-N(f(\alpha)+f(z))}e^{N^{1/3}\xi' f_3(\alpha)}e^{N^{1/3}\xi f_3(z)},\tag{4.84}$$

which equals

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{G}_{C_1}^{2\pi/3}}\mathrm{d}Z e^{-Z^3/3+Z\xi c_q}\frac{e^{8v^3/3-2vc_q\xi'}}{Z-2v}\tag{4.85}$$

with $C_1<2v$.

So in total, the sum of the residues of w in $1/z$ and α converges to

$$(4.85)+\mathrm{sgn}(\xi'-\xi)e^{-2vc_q|\xi-\xi'|}.\tag{4.86}$$

The remaining term is exactly (4.80) for $u_a=u_b=0$. □

The following proposition will provide the required integrable upper bound.

Proposition 4.11. . *Let c_q, K, x_a, x_b, α be as in Proposition 4.10. Let $L>0$ and $\xi,\xi'>-L$ and $a\leq b$. Then there is an N_0 and a $C>0$ (which may depend on L, u_a, u_b, but not on ξ,ξ') such that for $N>N_0$*

$$N^{1/3}e^{N^{2/3}\ln(1-\sqrt{q})(u_b-u_a)}|K_{1,2}(a,x_a;b,x_b)|\leq Ce^{-c\xi+d\xi'},\tag{4.87}$$

$$N^{2/3}e^{-N^{2/3}\ln(1-\sqrt{q})(u_b+u_a)}|K_{1,1}(a,x_a;b,x_b)|\leq Ce^{-c\xi-c\xi'},\tag{4.88}$$

$$e^{N^{2/3}\ln(1-\sqrt{q})(u_b+u_a)}|K_{2,2}(a,x_a;b,x_b)|\leq Ce^{d\xi+d\xi'},\tag{4.89}$$

with $c=2|v|c_q+3/4, d=2|v|c_q+2/3$.

Proof. We assume first that $\xi,\xi'\geq 0$. To show (4.88), note that the proof of pointwise convergence easily implies that for $\xi=\xi'=0$ and the contour (4.57) for $\gamma_{-1,0,1}$ one has

$$\begin{aligned}N^{2/3}\frac{|K_{1,1}(a,x_a;b,x_b)|}{e^{N^{2/3}f_2(1)(u_a+u_b)}}\leq\frac{N^{2/3}}{e^{N^{2/3}f_2(1)(u_a+u_b)}}\oint_{\gamma_{-1,0,1}}|\mathrm{d}z|\oint_{\gamma_{-1,0,1}}|\mathrm{d}w|\left|e^{N(f_1(z)+f_1(w))+N^{2/3}(u_af_2(z)+u_bf_2(w))}\right|\\ \times\left|\frac{(z-\alpha)(w-\alpha)(z-w)}{(z^2-1)(w^2-1)(zw-1)}\right|\leq C\end{aligned}\tag{4.90}$$

where f_1,f_2 are as in the proof of Proposition 4.10. If now $\xi,\xi'\geq 0$, this creates an extra factor

$$|e^{-N^{1/3}(\ln(z)\xi+\ln(w)\xi')}|.\tag{4.91}$$

For the contour (4.57), we have that for $z, w \in \gamma_{-1,0,1}$ $|z|, |w| \geq 1 + N^{-1/3}(2|v|c_q + 1)$. Hence we can bound for N large enough $|e^{-N^{1/3}(\ln(z)\xi + \ln(w)\xi')}| \leq e^{-c(\xi+\xi')}$.

Next we come to (4.87). As contours we choose again $\gamma_{\sqrt{q},\alpha} = (4.53), \gamma_{-1,0,1,w} = (4.57)$. The proof of pointwise convergence easily implies that for $\xi = \xi' = 0$ we have a constant upper bound. If now $\xi, \xi' \geq 0$, then we get an extra factor which equals

$$|e^{N^{1/3}\ln(w)\xi' - N^{1/3}\ln(z)\xi}|. \tag{4.92}$$

Now for $w \in \gamma_{\sqrt{q},\alpha}$, we have $|w| \leq 1 + N^{-1/3}(2|v|c_q + 1/2)$. Consequently, for N large enough, we have $N^{1/3}\ln(|w|) \leq 2|v|c_q + 2/3$. Since for $z \in \gamma_{-1,0,1,w}$ we have $|z| \geq 1 + N^{-1/3}(2|v|c_q + 1)$ we get $-N^{1/3}\ln(|z|) \leq -(2|v|c_q + 3/4)$. This implies that $(4.92) \leq e^{-c\xi + d\xi'}$. As for $K_{2,2}$, consider the case $u_a + u_b > 0$ first. In (4.76), by definition of α and choice of contour, we obtain $|(4.76)| \leq Ce^{d(\xi+\xi')}$. Equally, one obtains $|(4.74)|e^{N^{2/3}\ln(1-\sqrt{q})(u_b+u_a)} \leq e^{d(\xi+\xi')}$. Finally, (4.79) can be bounded by $e^{d(\xi+\xi')}$ as well, simply by the choice of contours. The case $u_a + u_b = 0$ is treated similarly.

Finally, if $\xi, \xi' \in [-L, 0]$, the proof(s) of pointwise convergence give an upper bound C, where the constant C may depend on L. If e.g. $\xi \in [-L, 0], \xi' > 0$ we multiply the bound obtained in the case $\xi \leq 0, \xi' = 0$ by the bound obtained for $\xi' \geq 0, \xi = 0$, finishing the proof. $\square$

5 Symmetric plane partitions

A *free boundary plane partition of length* N is an array $(\pi_{i,j})_{1\leq j\leq i\leq N}$ of nonnegative integers satisfying the properties

$$\pi_{i,j} \geq \pi_{i+1,j}, \qquad \pi_{i,j} \geq \pi_{i,j+1} \tag{5.1}$$

for all meaningful i, j. Its *volume* is the sum of its entries:

$$|\pi| = \sum_{1\leq j\leq i\leq N} \pi_{i,j}. \tag{5.2}$$

Clearly a free boundary plane partition of length N is half of a *symmetric plane partition* with base in the square $N \times N$, which by definition is an array $(\pi_{i,j})_{1\leq i,j\leq N}$ satisfying the above constraints plus the symmetry constraint $\pi_{i,j} = \pi_{j,i}$. An example of length 5 and volume 79 is depicted in Figure 9.

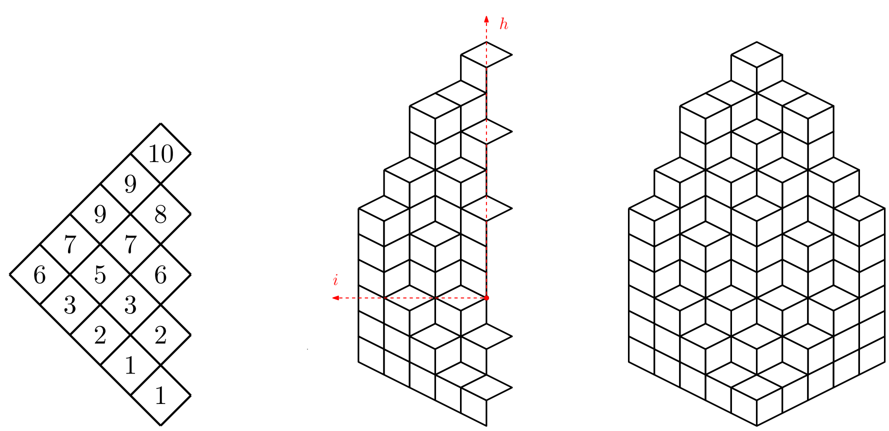

Figure 9: On the left: a free boundary plane partition π of length 5, with $\pi_{1,1} = 10, \pi_{4,2} = 5, \ldots,$ corresponding to the sequence of interlacing partitions $\emptyset \prec (6) \prec (7,3) \prec (9,5,2) \prec (9,7,3,1) \prec (10,8,6,2,1)$; in the middle: a free right boundary stack of cubes depicting π via the heights of the horizontal lozenges; on the right: the corresponding symmetric plane partition with base contained in a 5×5 square.

Let us fix a real parameter $0 < q < 1$. We will study the asymptotics of free boundary plane partitions weighted according to their volume; that is, distributed as

$$\text{Prob}(\pi) \propto q^{|\pi|} \tag{5.3}$$

in the limit $q \to 1$ and $N \to \infty$.

Free boundary plane partitions π of length N weighted by their volume are in bijection with the H-ascending Schur process on

$$\vec{\lambda} = (\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N)}) \tag{5.4}$$

with parameters $x_i = q^{N+1-i}$ (the free boundary is $\lambda^{(N)}$) via the following identification:

$$\lambda_k^{(i)} = \pi_{N-i+k,k}, \ 1 \leq k \leq i. \tag{5.5}$$

In this setting we have

$$\mathrm{Prob}(\vec{\lambda}) \propto q^{\sum_{1 \leq i \leq N} |\lambda^{(i)}|}. \tag{5.6}$$

The sequence of partitions $\vec{\lambda}$, itself induced by π, gives a point process via the identification $\lambda^{(i)} \mapsto \{k_s^{(i)} = \lambda_s^{(i)} - s + \frac{1}{2}\}$.

It turns out the computations and formulas are simpler if we reverse time. That is, if we introduce a new sequence of partitions $\mu^{(i)}$ defined by

$$\mu^{(i)} = \lambda^{(N-i)} \tag{5.7}$$

so that our Schur process becomes $\emptyset = \mu^{(N)} \prec \cdots \prec \mu^{(0)}$.

This is a pfaffian process whose matrix correlation kernel is given in Proposition 2.11. Here we restate the result for the chosen specializations. We work in the reversed time coordinates induced by the partitions $\mu^{(i)}$.

Theorem 5.1. *The point process induced by π via the sequence of partitions $\mu^{(i)}$ is pfaffian with 2×2 correlation kernel given by*

$$K(i,k;i',k') = \begin{bmatrix} K_{1,1}(i,k;i',k') & K_{1,2}(i,k;i',k') \\ -K_{1,2}(i',k';i,k) & K_{2,2}(i,k;i',k') \end{bmatrix} \tag{5.8}$$

where

$$\begin{aligned} K_{1,1}(i,k;i',k') &= \frac{1}{(2\pi \mathrm{i})^2} \int_{C_z} \int_{C_w} F(N-i,z) F(N-i',w) \frac{1}{z^{k+1} w^{k'+1}} \frac{\sqrt{zw}(z-w)}{(z+1)(w+1)(zw-1)} \mathrm{d}w \mathrm{d}z, \\ K_{1,2}(i,k;i',k') &= \frac{1}{(2\pi \mathrm{i})^2} \int_{C_z} \int_{C_w} \frac{F(N-i,z)}{F(N-i',w)} \frac{w^{k'-1}}{z^{k+1}} \frac{\sqrt{zw}(zw-1)}{(z+1)(w-1)(z-w)} \mathrm{d}w \mathrm{d}z, \\ K_{2,2}(i,k;i',k') &= \frac{1}{(2\pi \mathrm{i})^2} \int_{C_z} \int_{C_w} \frac{1}{F(N-i,z)F(N-i',w)} z^{k-1} w^{k'-1} \frac{\sqrt{zw}(z-w)}{(z-1)(w-1)(zw-1)} \mathrm{d}w \mathrm{d}z, \end{aligned} \tag{5.9}$$

the contours are counterclockwise oriented circles centered at the origin of radii slightly larger than 1 with the additional constraint that for $K_{1,2}$ C_z surrounds C_w if and only if $i' \leq i$ and finally where we have denoted

$$F(N-i,z) := \frac{(q/z;q)_N}{(q^{i+1}z;q)_{N-i}}. \tag{5.10}$$

Remark 5.2. Every particle in our process corresponds to a *horizontal* lozenge in the middle picture from Figure 9, and we imagine tiling the floor of the room with infinitely many horizontal lozenges going down. Some authors (e.g., [OR03]) prefer to write the kernel in terms of heights h of these lozenges. For a given particle at position (i,k), the coordinate of the corresponding lozenge (in the axes depicted, with the origin at the hidden corner of the box) is $(i,h) = (i, k - \frac{i}{2})$. We opt to work throughout with the ordinate k of particle positions, but the reader interested in the lozenge picture has only to keep in mind $k = h + \frac{i}{2}$.

We now pick a real number $\mathsf{a} > 0$. We zoom in around a point (i, k) in the point process induced by $\mu^{(i)}$ for large N, i, k as q approaches 1. Precisely, we are interested in the asymptotic regime $r \to 0+$ with

$$\begin{aligned} q = e^{-r} &\to 1, \\ rN &\to \mathsf{a}, \\ ri &\to \chi, \\ rk &\to y. \end{aligned} \tag{5.11}$$

The coordinate system (χ, y) represents our macroscopic coordinates, with $y \in \mathbb{R}$ and $0 \le \chi \le \mathsf{a}$. The positive real number a plays the role of a boundary parameter. We furthermore introduce macroscopic exponential coordinates (and exponential boundary parameter)

$$\mathcal{X} = \exp(-\chi), \qquad \mathcal{Y} = \exp(-y), \qquad \mathcal{A} = \exp(-\mathsf{a}). \tag{5.12}$$

Notice $0 < \mathcal{A} < 1, \mathcal{A} \le \mathcal{X} \le 1$. We first focus on the case $0 < \chi < \mathsf{a}$ and thus $\mathcal{A} < \mathcal{X} < 1$.

Analyzing the kernel asymptotically in the regime (5.11) (following [OR03]) can be reduced (more details will be given below) to the analysis of the z integrand $F(N-i, z)/z^{k+1/2}$.

We first note in the limit $r \to 0+$, the zeros of $F(N-i, z)$ accumulate in the set $\mathcal{Z} = [0, 1]$ while the poles in the set $\mathcal{P} = [\frac{1}{\mathcal{X}}, \frac{1}{\mathcal{A}}]$. Moreover as $r \to 0+$ we have

$$\frac{F(N-i, z)}{z^{k+1/2}} \sim \exp \frac{1}{r} S(z; \chi, y) \tag{5.13}$$

where

$$S(z; \chi, y) = \mathrm{Li}_2(\mathcal{A}/z) - \mathrm{Li}_2(1/z) + \mathrm{Li}_2(\mathcal{X}z) - \mathrm{Li}_2(\mathcal{A}z) - y \log z. \tag{5.14}$$

To apply the method of steepest descent to S, we look for its critical points: that is, solutions of $z \frac{\partial}{\partial z} S(z; \chi, y) = 0$. There are two of them:

$$z_\pm = z_\pm(\chi, y) = \frac{1 + \mathcal{X} - (1 + \mathcal{A}^2)\mathcal{Y} \pm \sqrt{D(\mathcal{X}, \mathcal{Y})}}{2(\mathcal{X} - \mathcal{A}\mathcal{Y})} \tag{5.15}$$

where

$$\begin{aligned} D(\mathcal{X}, \mathcal{Y}) &= -4(1 - \mathcal{A}\mathcal{Y})(\mathcal{X} - \mathcal{A}\mathcal{Y}) + (-1 - \mathcal{X} + \mathcal{Y} + \mathcal{A}^2\mathcal{Y})^2 \\ &= (\mathcal{A}^2 - 1)^2\mathcal{Y}^2 - 2(\mathcal{A} - 1)^2\mathcal{X}\mathcal{Y} + \mathcal{X}^2 - 2(\mathcal{A} - 1)^2\mathcal{Y} - 2\mathcal{X} + 1. \end{aligned} \tag{5.16}$$

Remark 5.3. The equation $D(\mathcal{X}, \mathcal{Y}) = 0$, the locus where S has a real double critical point, is a conic in the $(\mathcal{X}, \mathcal{Y})$ plane. Its discriminant is $-16\mathcal{A}(\mathcal{A} - 1)^2 < 0$ and so it is the equation of an ellipse. When $\mathcal{A} = 0$ (equivalently, $\mathsf{a} \to \infty$), it becomes a parabola.

Remark 5.4. The two critical points in equation (5.15) have a singularity on the line $\mathcal{X} = \mathcal{A}\mathcal{Y}$, where one of them becomes $\pm\infty$ (depending on the sign of $\mathcal{X} - \mathcal{A}\mathcal{Y}$) while the other stays finite at $\frac{\mathcal{A}\mathcal{Y} - 1}{(1 - \mathcal{A} + \mathcal{A}^2)\mathcal{Y} - 1}$ as can be easily verified by an application of l'Hôpital's rule.

Keeping χ and hence $\mathcal{X}$ fixed, we solve for $\mathcal{Y}$ (and hence for y) from $D(\mathcal{X}, \mathcal{Y}) = 0$ to obtain two solutions:

$$\mathcal{Y}_\pm = \frac{(1 + \mathcal{X})(1 - \mathcal{A})^2 \pm 2(1 - \mathcal{A})\sqrt{-\mathcal{A}\mathcal{X}^2 + \mathcal{X}(1 + \mathcal{A}^2) - \mathcal{A}}}{(1 - \mathcal{A}^2)^2}, \qquad y_\mp = -\log(\mathcal{Y}_\pm). \tag{5.17}$$

We note $y_- < y_+$ and both depend on χ; when $\chi = \mathsf{a}$ we have $y_- = y_+ = \log(1 + \mathcal{A})$; when $\chi = 0$ (hence $\mathcal{X} = 1$) we have $y_+ = \infty$ (as $\mathcal{Y}_- = 0$) and $y_- = -\log \frac{4}{(1+\mathcal{A})^2}$.

We call the curve

$$\mathcal{C} = \{(\chi, y) \in \mathbb{R}^2 : \mathsf{a} \ge \chi \ge 0, D(\mathcal{X}, \mathcal{Y}) = 0\} \tag{5.18}$$

the *arctic curve*. It is depicted in Figure 10. We denote by $\mathcal{L}$ — which we call *the liquid region*, the inside of the arctic curve; that is, the domain

$$\mathcal{L} = \{(\chi, y) \in \mathbb{R}^2 : \mathsf{a} \ge \chi \ge 0, y_- \le y \le y_+\}. \tag{5.19}$$

We now discuss what happens to the critical points at various (χ, y) positions in the plane. To do that, we first take care of the singularity on the line $\mathcal{X} = \mathcal{A}\mathcal{Y}$ or, equivalently, the line $\chi = a + y$. The curve $\mathcal{C}$ is tangent to this line, at the tangency points (χ_T, y_T) given by:

$$
\begin{aligned}
(\mathcal{X}_T, \mathcal{Y}_T) &= \left(\frac{\mathcal{A}}{1 - \mathcal{A} + \mathcal{A}^2}, \frac{1}{1 - \mathcal{A} + \mathcal{A}^2} \right), \\
(\chi_T, y_T) &= (-\log \mathcal{X}_T, -\log \mathcal{Y}_T)
\end{aligned}
\tag{5.20}
$$

where by construction $0 < \chi_T < a, y_T = \chi_T - a < 0$. We have three cases:

- when $D(\mathcal{X}, \mathcal{Y}) = 0$ and thus $(\chi, y) \in \mathcal{C}$, S has a double critical point: $z = \frac{1+\mathcal{X}-(1+\mathcal{A}^2)\mathcal{Y}}{2(\mathcal{X}-\mathcal{A}\mathcal{Y})}$. This happens twice for fixed χ: when $y = y_-$ or $y = y_+$ (the two cases coinciding at $\chi = a, y_- = y_+ = \log(1 + \mathcal{A})$). First, the case $y = y_+ > 0$. We have $z \in (1, \frac{1}{\mathcal{X}})$. When $(\chi, y) \to (0, \infty)$ along $\mathcal{C}$, $z \to 1$ from above (so $z > 1$ always). When $(\chi, y) \to (a, \log(1 + \mathcal{A}))$ from above along $\mathcal{C}$, $z \to \frac{1}{\mathcal{A}}$ from below. $\mathcal{C}$ is tangent at $(a, \log(1 + \mathcal{A}))$ to the line $\chi = a$. Second, the case $y = y_- \leq \log(1 + \mathcal{A})$. If $0 \leq \chi < \chi_T$, $z \in (-\infty, -1]$ with $z = -1$ at $\chi = 0$ and $z \to -\infty$ when $\chi \to \chi_T$ from below. When $\chi \in (\chi_T, a]$, $z \in [\frac{1}{\mathcal{A}}, \infty)$ with $z \to \infty$ when $\chi \to \chi_T$ from above and $z \to \frac{1}{\mathcal{A}}$ from below as $\chi \to a$;
- when $D(\mathcal{X}, \mathcal{Y}) < 0$, which means (χ, y) is in the interior of $\mathcal{L}$, the two distinct critical points are complex conjugate with arguments $\pm\theta(\chi, y)$ where we take $\theta(\chi, y) \in (0, \pi)$. $\theta(\chi, y)$ varies from 0 to π as y descends from y_+ to y_- through the liquid region $\mathcal{L}$ for fixed $\chi < \chi_T$. In the case $\chi > \chi_T$ both double critical points z^t corresponding to (χ, y_+) and z^b corresponding to (χ, y_-) are positive, with $0 < z^t < z^b$, so if one recenters the complex plane at any real number between z^t and z^b and considers the arguments of the two complex conjugate critical points corresponding to $y_- < y < y_+$, we have a similar situation as above. See Figure 17.
- when $D(\mathcal{X}, \mathcal{Y}) > 0$ (and thus for (χ, y) not in $\mathcal{L}$) there are two distinct real critical points. We study what happens for fixed χ. If $y > y_+$, $1 < z_- < z_+ < \frac{1}{\mathcal{X}}$ with $z_- = 1, z_+ = \frac{1}{\mathcal{X}}$ in the limit $y \to \infty$ and $z_\pm$ converging to the unique double critical point in the limit $y \to y_+$. If $y < y_-$, the situation is more complicated due to the singularity at $\mathcal{X} = \mathcal{A}\mathcal{Y}$ (equivalently on the line $\chi = a + y$). We again distinguish two cases. First, $\chi < \chi_T$ fixed. For $y \in (\chi - a, y_-)$, if $y \to y_-$, the two roots $z_\pm$ converge to the corresponding (negative) real double critical point. When $y \to \chi - a$ from above, $z_- \to -\infty$ and $z_+ \to \frac{\mathcal{A}\mathcal{Y}-1}{(1-\mathcal{A}+\mathcal{A}^2)\mathcal{Y}-1} < 0$ from below. When y passes below the line $\chi - a$ and goes to $-\infty$, z_- goes from ∞ (for y just below the line $y = \chi - a$) to $\frac{1}{\mathcal{A}}$ (at $y \to -\infty$) while z_+ goes from $\frac{\mathcal{A}\mathcal{Y}-1}{(1-\mathcal{A}+\mathcal{A}^2)\mathcal{Y}-1}$ at $y = \chi - a$ to $\mathcal{A}$ at $y \to -\infty$. Second, the case $\chi > \chi_T$. z_- goes from the double critical point $z_c > \frac{1}{\mathcal{A}}$ to $\frac{1}{\mathcal{A}}$ as y goes from y_- to $-\infty$, passing through $z_- = \frac{\mathcal{A}\mathcal{Y}-1}{(1-\mathcal{A}+\mathcal{A}^2)\mathcal{Y}-1}$ at the line $y = \chi - a$. z_+ goes from z_c to ∞ as y moves from y_- and approaches the line $y = \chi - a$ from above, and then jumps and goes from ∞ to $\mathcal{A}$ as y descends from just below the line $y = \chi - a$ to $-\infty$.

Remark 5.5. In light of the above discussion, the curve $\mathcal{C}$ is the locus in the (χ, y) plane where S has a double critical point. Given a point $(\chi, y) \in \mathcal{C}$, one computes the corresponding double critical point $z_c(\chi, y) = \frac{1+\mathcal{X}-(1+\mathcal{A}^2)\mathcal{Y}}{2(\mathcal{X}-\mathcal{A}\mathcal{Y})}$. Importantly, one can go backwards as well and for each double critical point z_c, one can compute the corresponding $(\chi, y) = (\chi(z_c), y(z_c))$ by solving for (χ, y) in the following system of two equations

$$
\left(z \frac{\partial}{\partial z} \right) S(z_c; \chi, y) = 0, \qquad \left(z \frac{\partial}{\partial z} \right)^2 S(z_c; \chi, y) = 0 \tag{5.21}
$$

and in doing so one obtains a *time/rational parametrization* of $\mathcal{C}$ as

$$
\mathcal{C} = \{(\chi(z_c), y(z_c)) : z_c \in \mathbb{R}\backslash(-1, 1]\} \tag{5.22}
$$

with the time parameter played by the double critical point of S. In brief $z_c \mapsto (\chi(z_c), y(z_c))$ takes $\mathbb{R}\backslash(-1, 1]$ to $\mathcal{C}$: as z_c increases from $1+$ to ∞ we draw the upper part of $\mathcal{C}$ from $(0, \infty)$ to (χ_T, y_T), and then as z_c increases from $-\infty$ to -1 we draw the lower part of $\mathcal{C}$ from (χ_T, y_T) to $(0, -\log \frac{4}{(1+\mathcal{A})^2})$.

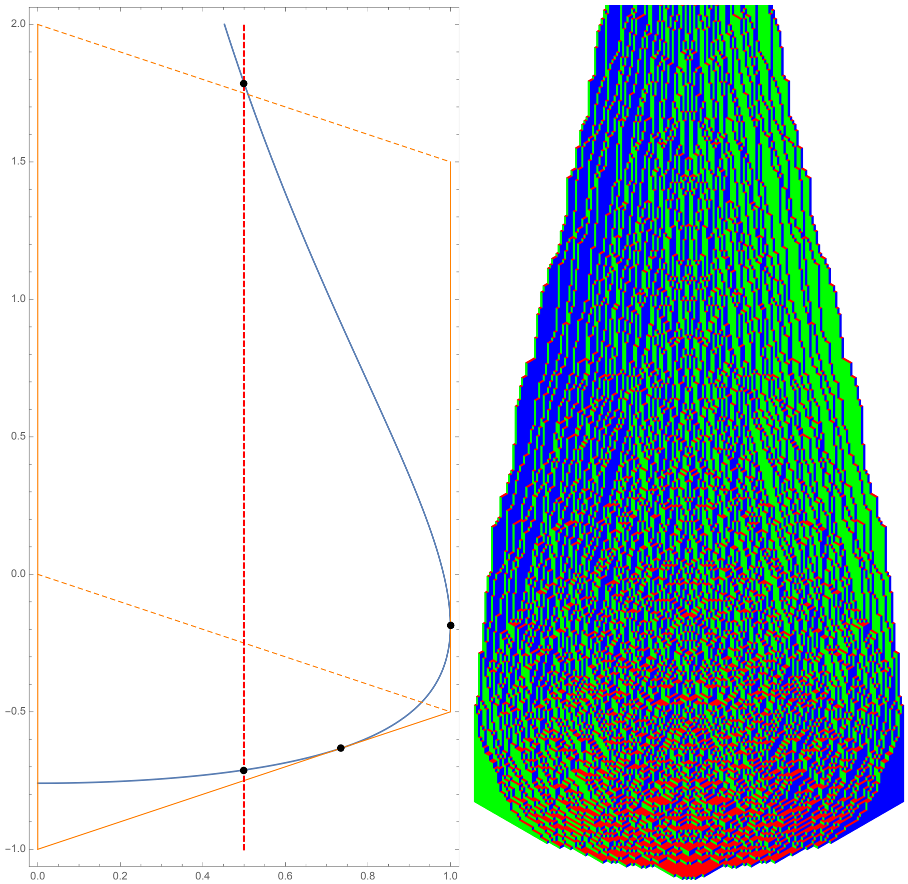

Figure 10: Left: a portion of the arctic curve for $a = 1$ in the lozenge $(\chi, h) = (\chi, y - \chi/2)$ coordinates; the vertical dashed line is of abscissa $\chi = 1/2$ and intersects the arctic curve in the two points $h_\pm = y_\pm - \chi/2$; the points of tangency of the arctic curve to the boundary of the domain on the bottom and right are also pictured. Right: a simulation (after symmetrization) of a boxed plane partition with base of size 100×100 and $q \approx 0.98$ using the algorithm from [BBB+15].

Fix a point (χ, y) in the plane, with $0 \le \chi \le a$ and $y \ge \chi - a$, and let $z_\pm$ be the corresponding critical points. In the asymptotic analysis that follows, the existence of a closed contour C_0 on which $\Re(S(z) - S(z_0)) = 0$ (where z_0 is one of the two critical points), and which passes to the right of 1 (but the left of $\exp(\chi) = \frac{1}{\chi}$) and to the left of 0 (thus encompassing the cut on the interval $[0, 1]$), will be the key ingredient in the proof. Such a contour, for (χ, y) in the critical region and thus for complex conjugate critical points ($z_0 = z_+$ in that case though the choice makes no difference as $S(\overline{z}) = \overline{S(z)}$), is depicted in Figure 18 (the circle-looking contour in the left panel), along with the directions of ascent off it. For its existence, we argue as follows. We have the following limits at 0 and $-\infty$ (along the real axis, say):

$$\lim_{z \to 0-} \Re S(z) = \infty, \qquad \lim_{z \to -\infty} \Re S(z) = -\infty \tag{5.23}$$

and so by the intermediate value theorem there will be a point z on the negative real axis with $\Re(S(z) - S(z_0)) = 0$. Likewise on the interval $(1, \frac{1}{\chi})$ $\Re(S(z) - S(z_0)) = 0$ changes sign and we thus have a point z in this interval with $\Re(S(z) - S(z_0)) = 0$. Connecting the contours in the upper and lower half-planes will yield the desired C_0. For a precise technical description of this argument see Lemma 6.4 in [Bor07] and note that the f in that statement is exactly our S without the log term. For $\chi = 0$ the preceding discussion becomes simpler. In this case, for z on the unit circle $\{|z| = 1\}$, we have, by direct computation, $2\Re(S(z)) = S(z) + S(1/z) = 0$. Moreover in the case of complex conjugate critical points $z_\pm$ we have $|z_\pm| = 1$ and so the C_0 is just the unit circle.

We finally remark that in the case $a \to \infty$ the above argument simplifies considerably and C_0 is just the circle around the origin of radius $\exp(\chi/2)$.

Before stating our first asymptotic result, we fix a few useful notations. We concentrate on the case $0 < \chi < \chi_T$, but the case $\chi_T < \chi < a$ can be treated similarly. We will study the case $\chi = 0$ separately.

For $(\chi, y) \in \mathcal{L}$, we denote by γ_+ any simple contour (path) joining the two corresponding critical points z_+ and z_- just to the right of 1. It depends of course on $\theta(\chi, y)$ which is the argument of z_+. By γ_- we denote a contour (path) joining the same two points but which passes to the left of 0. We state the result for $i' \leq i$. For $i > i'$ the only difference is one replaces γ_+ by γ_-.

Finally, one only needs to look in the half-space $y \geq \chi - a$. The reason for this is combinatorial: below the line $y = \chi - a$ we will see only particles, as the partition $\mu^{(N-i)}$ has at most i parts due to the interlacing constraints. Therefore the kernel will not be of interest around points (χ, y) in the half-space given by $y < \chi - a$.

Theorem 5.6. *Let $(\chi, y) \in (0, a) \times \mathbb{R}$ and $y > \chi - a$. Let*

$$\begin{aligned} (i, i') &= \left(\left\lfloor \frac{\chi}{r} \right\rfloor + \mathsf{i}, \left\lfloor \frac{\chi}{r} \right\rfloor + \mathsf{i}' \right), \\ (k, k') &= \left(\left\lfloor \frac{y}{r} \right\rfloor + \mathsf{k}, \left\lfloor \frac{y}{r} \right\rfloor + \mathsf{k}' \right) \end{aligned} \tag{5.24}$$

where $\mathsf{i}', \mathsf{i} \in \mathbb{N}, \mathsf{k}, \mathsf{k}' \in \mathbb{Z}'$ are fixed. Then $ri, ri' \to \chi$ and $rk, rk' \to y$ as $r \to 0+$. When $rN \to a$ we find that $\lim_{r \to 0+} K_{1,2}(i, k; i', k')$ vanishes if $y > y_+$, and is otherwise equal to

$$\lim_{r \to 0+} K_{1,2}(i, k; i', k') = \begin{cases} 0, & \text{if } y > y_+, \\ \frac{1}{2\pi \mathrm{i}} \int_C (1 - e^{-\chi} z)^{\mathsf{i} - \mathsf{i}'} z^{\mathsf{k}' - \mathsf{k} - 1} \mathrm{d}z, & \text{otherwise} \end{cases} \tag{5.25}$$

where C is

- *$\gamma_\pm$ if $y_- \leq y \leq y_+$ (γ_+ if and only if $\mathsf{i}' \leq \mathsf{i}$);*
- *a positively oriented circle of radius $1 + \epsilon$ centered at the origin for some $0 < \epsilon \ll 1$ if $y < y_-$ and $\chi < \chi_T$;*
- *a positively oriented circle containing the cut $\mathcal{P}$ but passing to the right of 1 if $y < y_-$ and $\chi > \chi_T$.*

Proof. We will give a similar argument to the one given in Section 3.1 of [OR03]. The interested reader will note the same argument has been applied before (modulo notation, conventions, and some minor technical details), in various related models (for both normal and/or strict as opposed to symmetric/free boundary plane partitions — see [OR03, OR07, Vul07, FS03, BMRT12]) and the analysis caries over almost mutatis mutandis.

Throughout the proof we restrict to the case $\mathsf{i}' \leq \mathsf{i}$, in which case the z contour is on the outside in the double contour integral formula for $K_{1,2}$. The other case follows similarly. We also write $S(z)$ in lieu of $S(z; \chi, y)$ whenever possible.

The idea is that we deform the original integration contours for $K_{1,2}$ around the complex plane and have them pass through the critical point z_+ (and in the case of complex conjugate critical points, both z_+ and z_-). We want to make $\Re(S(z) - S(z_+))$ negative everywhere except at z_+ (or $z_\pm$ in the complex conjugate case), while making $\Re(S(w) - S(z_+))$ positive everywhere except z_+ (respectively $z_\pm$). We then observe $S(z) - S(w) = (S(z) - S(z_+)) - (S(w) - S(z_+))$ and employ multiple times the following simple limit:

$$\int_\gamma \exp\left(\frac{1}{r} f(z)\right) dz \to 0, \ r \to 0+ \tag{5.26}$$

if f is smooth and $\Re f < 0$ for all but finitely many points along the simple closed contour γ.

In Figures 19, 18, 20 and 21 we illustrate, for various situations that will arise, the level lines of $\Re(S(z) - S(z_+))$. The first three figures correspond to points (χ, y) with $\chi < \chi_T$, sitting above the arctic curve, inside the liquid region $\mathcal{L}$, and below the arctic curve but above the line $\chi = y + a$ respectively. The last corresponds to (χ, y) below the arctic curve but with $\chi > \chi_T$, the only case that needs special treatment when $\chi > \chi_T$.

$K_{1,2}$ is explicitly given in Theorem 5.1. In the limit, the integrand $\frac{F(N-i,z)}{F(N-i',w)}\frac{w^{k'+1}}{z^{k+1}}$ is approximated by

$$\exp\frac{1}{r}\left(S(z;\chi,y)-S(w;\chi,y)\right) \tag{5.27}$$

and it is this function that will provide the dominant asymptotic contribution.

Throughout the proof, contours of integration for z and w will move around. One has to take care that the z contour never crosses the interval $[\frac{1}{X},\frac{1}{A}]$ where the poles of the function F accumulate in the limit (but the w contour can certainly cross this interval), and that the w contour never crosses the interval $[0,1]$ where the zeros of the function F (so poles in the w variable) accumulate in the limit (and again, the z contour is of course allowed to cross said interval). None of the operations described below move the z or w contours in a way that their respective forbidden intervals are crossed, as can be explicitly checked case by case.

First, the case $y>y_+,\chi<\chi_T$. We deform the contours so that the z contour, which is on the outside, passes through $z_+\in(1,\exp(\chi))$ at an angle orthogonal to the real axis — which is locally the direction of steepest descent for $\Re S$, and otherwise contains the contour C_0 where $\Re(S(z)-S(z_+))=0$ (to ensure everywhere else $\Re(S(z)-S(z_+))<0$. Similarly we deform the w contour so that it is contained in C_0 (so that $\Re(S(z)-S(z_+))>0$) and passes through z_+ parallel to the real axis. See Figure 19. We observe that the factor $1/(z-w)$ does not cause problems as it is integrable: we can bound $\int\frac{\mathrm{d}z\mathrm{d}w}{z-w}$ by the converging $\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\frac{\mathrm{d}x\mathrm{d}y}{\sqrt{x^2+y^2}}=8\delta\log(1+\sqrt{2})$ for some small positive real δ. We conclude the integral decays exponentially fast to 0 in the limit $r\to 0+$.

Second, the case $y\in(y_-,y_+),\chi<\chi_T$, which is to say $(\chi,y)\in\mathcal{L}$. We proceed as before by passing both contours through the two critical points z_+ and z_-, so that they intersect orthogonally at both. Based on the gradient of $\Re(S(z)-S(z_+))$ the final contours look like in Figure 18. We note that for this to be possible, we have to pass the w contour to the outside of the z contour on an arc passing through the left of 1 and connecting z_+ with z_- (this arc is γ_+). Doing so we pick up the residue at $z=w$ for every z between z_+ and z_- along γ_+. The total contribution is finite and equal to

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+}Res_{z=w}\left(\frac{F(N-i,z)}{F(N-i',w)}w^{k'-1/2}z^{-k-1/2}\frac{zw-1}{(z+1)(w-1)(z-w)}\right)\mathrm{d}z=\\&=\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+}\frac{F(N-i,z)}{F(N-i',z)}z^{k'-k+1}\mathrm{d}z\to\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+}(1-e^{-\chi}z)^{\mathsf{i}-\mathsf{i}'}z^{\mathsf{k}'-\mathsf{k}-1}\mathrm{d}z,\ r\to 0+\end{aligned} \tag{5.28}$$

where to take the limit we have used the estimates from Appendix B. We now argue that the integral on the remaining contours vanishes in the limit. The remaining integrals will converge to 0 exponentially again using the basic fact from (5.26). Again the denominator factor $1/(z-w)$ poses no further issues.

The situation repeats for a third time, for $y<y_-,y>\chi-a,\chi<\chi_T$. We pass both contours through $z_+<0$, but now as in the previous case the gradient of $\Re(S(z)-S(z_+))$ forces the w contour on the outside and the z on the inside, the z contour passing orthogonally to the real axis while the w contour passing parallel to it at z_+. See Figure 20. As we have to pass the whole w contour to the outside, we pick up a residue along the line $z=w$ over a whole closed contour passing through z_+ and to the left of 1 of contribution:

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{C}Res_{z=w}\left(\frac{F(N-i,z)}{F(N-i',w)}w^{k'-1/2}z^{-k-1/2}\frac{zw-1}{(z+1)(w-1)(z-w)}\right)\mathrm{d}z=\\&=\frac{1}{2\pi\mathrm{i}}\int_{C}\frac{F(N-i,z)}{F(N-i',z)}z^{k'-k+1}\mathrm{d}z\to\frac{1}{2\pi\mathrm{i}}\int_{C}(1-e^{-\chi}z)^{\mathsf{i}-\mathsf{i}'}z^{\mathsf{k}'-\mathsf{k}-1}\mathrm{d}z,\ r\to 0+\end{aligned} \tag{5.29}$$

where again we have used Appendix B. The remaining integrals, as before, converge exponentially fast to 0 in the limit.

For $\chi>\chi_T$, the two cases $y>y_+$ (above the arctic curve) and $y\in(y_-,y_+)$ (inside the liquid region) are handled using the same contours as if it were that $\chi<\chi_T$.

Finally $\chi>\chi_T$, the situation is different as now the critical points $z_\pm$ are in the interval $(\exp(a),\infty)$. We pass both contours through z_+ as before. Since the z contour cannot cross $\mathcal{P}$, we enlarge it on the left side and have it pass through infinity until it comes back on the other side and encircles $\mathcal{P}$. We can do this as there is no residue at ∞ in the z variable. Locally at z_+ the final contour C_z is parallel to the axis. We now inflate the contour C_w, passing it over C_z and picking up the stated residue over the desired contour C, so that it goes through z_+ locally perpendicular to the real axis. See Figure 21. Further note that if $\mathsf{i} = \mathsf{i}', \mathsf{k} = \mathsf{k}'$ (the diagonal of the kernel), the residue integral over C is 0, since the origin is no longer inside C.

□

Remark 5.7. We mention one qualitative difference between the case $\chi < \chi_T$ (handled in Figures 19, 18 and 20) and the case $\chi > \chi_T$. If $\chi - a < y < y_-$, asymptotically around (χ, y) (below the arctic curve), one sees only particles if $\chi < \chi_T$, while for $\chi > \chi_T$ one sees only holes. Both regions are frozen, but in different ways. See Figure 10 for a numerical visualization of this.

Remark 5.8. For a condensed version of the above argument, in the case $a = \infty$ (which simplifies calculations considerably), see [OR03]. A more detailed calculation is presented in [BMRT12]. We have opted here for the middle ground while trying to be as precise with the contours as possible. Both references consider regular (as opposed to symmetric) plane partitions, but the details end up being the same.

Remark 5.9. Up to a change of variables $z \mapsto e^{-\chi} z$ which introduces extra factors of the form $e^{\chi(\mathsf{k}'-\mathsf{k})}$ that nevertheless cancel in any pfaffian computation (as they are diagonal), the kernel

$$(\Delta \mathsf{i}, \Delta \mathsf{k}) \mapsto \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\pm} (1-z)^{\Delta \mathsf{i}} z^{-\Delta \mathsf{k} - 1} \mathrm{d}z \tag{5.30}$$

is called *the incomplete beta kernel* [OR03]; in our case $\Delta \mathsf{i} = \mathsf{i} - \mathsf{i}', \Delta \mathsf{k} = \mathsf{k} - \mathsf{k}'$. When we restrict ourselves to the same slice $\mathsf{i} = \mathsf{i}'$ a simple integration shows it becomes *the discrete sine kernel*

$$(\mathsf{k}, \mathsf{k}') \mapsto \frac{\sin(\theta(\mathsf{k} - \mathsf{k}'))}{\pi(\mathsf{k} - \mathsf{k}')} \tag{5.31}$$

where $\theta = \theta(\chi, y)$ is the argument of $z_+ = z_+(\chi, y)$.

Theorem 5.6 justifies the word *arctic curve* for $\mathcal{L}$ in the following sense: if one looks at a large system and scales appropriately, in the particle–hole description, above $\mathcal{C}$ we will see only holes in the sense that the probability of seeing a particle decays exponentially; below $\mathcal{C}$ we will see only particles — the probability of seeing particles is exponentially close to 1 that is, as can be verified from the explicit limit of the diagonal elements of the kernel $K(i, k; i, k)$. Inside $\mathcal{C}$ we will see a mixture, and this region is called the *liquid region* — denoted above by $\mathcal{L}$. We illustrate this in Figure 10.

Theorem 5.6 also gives us the limiting density of particles, given below.

Corollary 5.10. *In the limit* (5.11) *and around macroscopic point* (χ, y) *the density of particles is*

$$\begin{aligned} \rho(\chi, y) &= \frac{\theta(\chi, y)}{\pi}, \\ \theta(\chi, y) &= \arg(z_+(\chi, y)). \end{aligned} \tag{5.32}$$

Note that, for $\chi < \chi_T$, the density is 0 above $\mathcal{C}$ (as the two critical points are real so $\theta = 0$), strictly between 0 and 1 inside $\mathcal{L}$ and 1 below (as the two roots are negative below $\mathcal{C}$). For $\chi > \chi_T$, the density is 0 above $\mathcal{C}$, strictly between 0 and 1 when inside $\mathcal{L}$ where the two roots are complex conjugate, 0 again (!) for y below $\mathcal{C}$ but above the line $\chi = a + y$, and 1 below $\mathcal{C}$ and below the aforementioned line. This can immediately be translated into the density of horizontal tiles in the plane partition picture via the change of variables $y = h + \frac{\chi}{2}$ — here $h = \lim_{r \to 0+} hr$ is the macroscopic ordinate/height for lozenges.

In the case $a \to \infty$ ($\mathcal{A} \to 0$) the arctic curve $\mathcal{C}$ becomes particularly simple. It can be written as half the zero locus (the part $\chi > 0$) of

$$(1 + U + V)(1 + U - V)(1 - U + V)(1 - U - V) = 0, \qquad (U, V) = (e^{-\frac{\chi}{2}}, e^{-\frac{y}{2}}). \tag{5.33}$$

It is thus half the boundary of the amoeba of the polynomial $p(U, V) = 1 + U + V$ [KO07, KO06, KOS06] and can be recovered using the techniques in [KO07], independent of the description as a Schur process. We recall that the amoeba of a polynomial $p(U, V) \in \mathbb{C}[U, V]$ is the set

$$\{(\log |U|, \log |V|) : (U, V) \in (\mathbb{C} \setminus 0)^2, p(U, V) = 0\}. \tag{5.34}$$

We make a small digression and address the three dimensional nature of the picture. It is possible to recover the three dimensional limit surface from the above formulas. This was first obtained by Cerf and Kenyon [CK01] in the mathematical literature and by Blöte, Hilhorst and Nienhuis in the physics literature [NHB84], and while we can recover their formulas, as already noted in [OR03] (whose exposition we follow for this purpose), further analysis — in particular a concentration inequality type of result — is needed to make the computations rigorous (but see [CK01] where this is done using an alternative method). We also note that all the cited references deal with regular (non-symmetric) plane partitions, but the limit surface is the same after a mild reparametrization (so that indeed the symmetric plane partitions, as opposed to just the free boundary halves, are distributed according to the same q^{Volume} measure as the ordinary plane partitions). Here and below we consider ρ as a function of $(\chi, \hbar)$, and we extend to the symmetric case by $\rho(\chi, \hbar) = \rho(-\chi, \hbar)$. We will call X, Y, Z the three dimensional coordinates of space present in Figure 11 (right), with Z the vertical, X the south-west and Y the south-east coordinates respectively (the origin is at the hidden corner of the box). Then a parametrization of the limit surface is:

$$\begin{aligned} Z(\chi, \hbar) &= \int_{-\infty}^{\hbar} (1 - \rho(\chi, s)) ds, \\ X(\chi, \hbar) &= Z(\chi, \hbar) - \hbar - \frac{\chi}{2}, \\ Y(\chi, \hbar) &= Z(\chi, \hbar) - \hbar + \frac{\chi}{2}. \end{aligned} \tag{5.35}$$

In Figure 11 we depict two large plane partitions, coming from simulations using algorithms in [BBB$^+$15], one normal and one symmetric — and both having an a priori unbounded base in the $a \to \infty$ regime. After squaring q on one side to make them the same scale, the common limit surface, also depicted, becomes apparent.

We now address, again in the case $\chi > 0$, the limits of $K_{1,1}$ and $K_{2,2}$. It will be easier to work with the conjugated kernel $\tilde{K}(i, k; i', k')$ defined thusly. We fix (χ, y) and consider the corresponding z_+. Then (with $S(z_+) := S(z_+; \chi, y)$):

$$\begin{pmatrix} \tilde{K}_{1,1} & \tilde{K}_{1,2} \\ \tilde{K}_{2,1} & \tilde{K}_{2,2} \end{pmatrix} = \begin{pmatrix} e^{-\Re S(z_+)/r} & 0 \\ 0 & e^{\Re S(z_+)/r} \end{pmatrix} \begin{pmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \end{pmatrix} \begin{pmatrix} e^{-\Re S(z_+)/r} & 0 \\ 0 & e^{\Re S(z_+)/r} \end{pmatrix}. \tag{5.36}$$

It follows that diagonal elements of $\tilde{K}$ converge to zero.

Theorem 5.11. *With the same assumptions as in Theorem 5.6, in the asymptotic regime given by* (5.11) *we find that* $\lim_{r \to 0+} \tilde{K}_{1,1}(i, k; i', k') = \lim_{r \to 0+} \tilde{K}_{2,2}(i, k; i', k') = 0$.

Proof. It turns out the analysis was already carried out in the proof of Theorem 5.6, with one particular difference: now we can move both the z and the w contours around and deform them into one and the same contour, called $\gamma_{1,1}$ for $\tilde{K}_{1,1}$ and $\gamma_{2,2}$ for $\tilde{K}_{2,2}$ — we can do this as there is no more residue/singularity to take care of at $z = w$ as the diagonal elements of the kernel do not contain a $(z-w)$ term in the denominator of the integrand. In the frozen regions ($y > y_+$ or $y < y_-$) the contours both path through z_+, while in the liquid region ($y \in (y_-, y_+)$) both pass through z_+ and $z_- = \overline{z_+}$ respectively. In all three cases they intersect in the critical point(s) at right angles, and away from them $\gamma_{1,1}$ follows a direction of descent, while $\gamma_{2,2}$ one of ascent. In fact in all three cases, $\gamma_{1,1}$ can be taken to be the final z contour from the proof of Theorem 5.6, while $\gamma_{2,2}$ the final w contour. See Figures 19, 18 and 20: $\gamma_{1,1} = C_z$ and $\gamma_{2,2} = C_w$.

More precisely, the integrands in $\tilde{K}_{1,1}$ and $\tilde{K}_{2,2}$ can be approximated by

$$\begin{aligned} \exp \frac{1}{r} \left(\tilde{S}(z; \chi, y) + \tilde{S}(z; \chi, y) \right) &\text{ and} \\ \exp - \frac{1}{r} \left(\tilde{S}(z; \chi, y) + \tilde{S}(z; \chi, y) \right) & \end{aligned} \tag{5.37}$$

respectively as $r \to 0+$ where

$$\tilde{S}(z; \chi, y) = S(z; \chi, y) - \Re S(z_+; \chi, y) \tag{5.38}$$

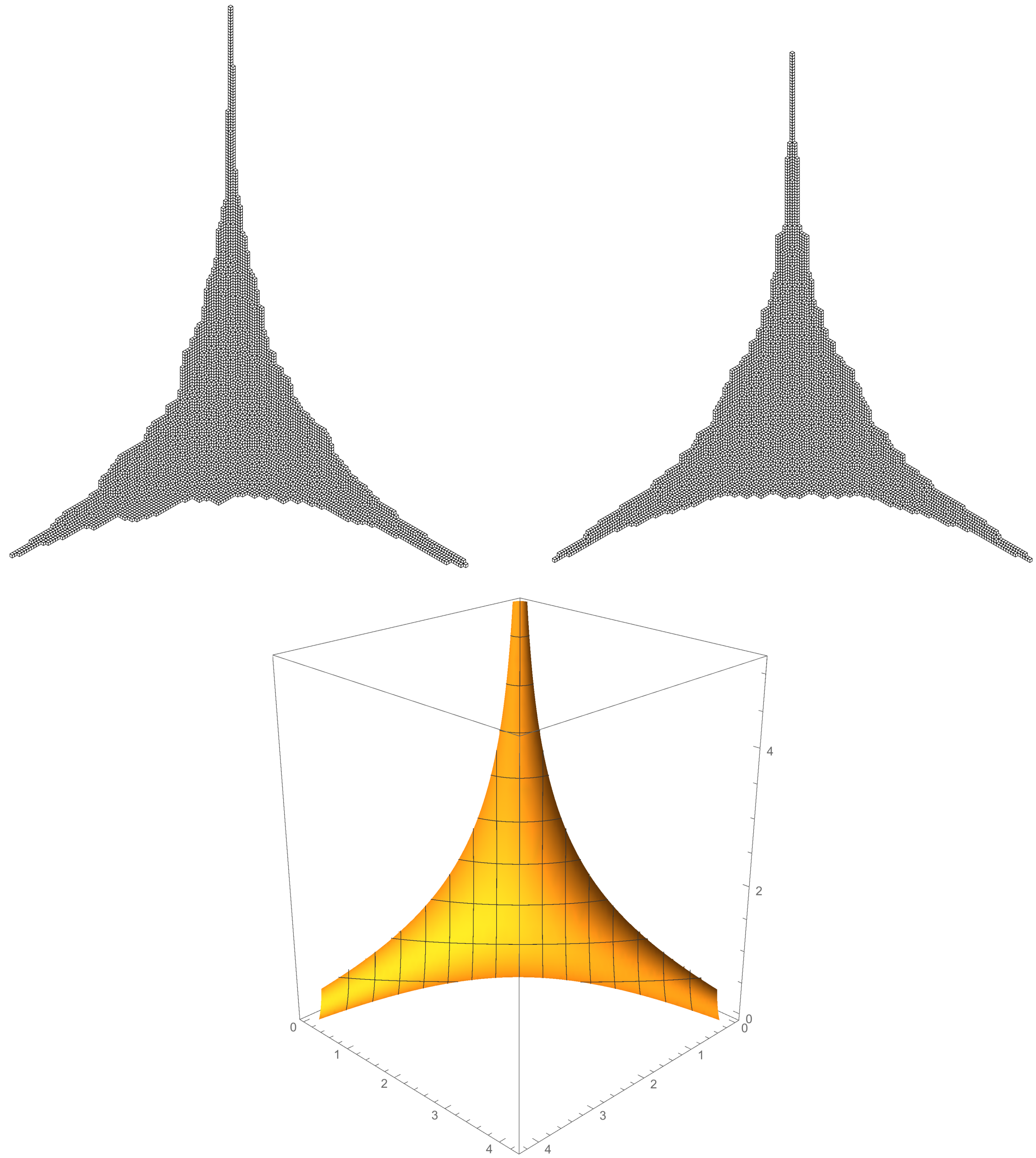

Figure 11: A random plane partition for $q = 0.953$ (top left) and a random symmetric one for q^2 (top right). The squaring of the parameter in the symmetric case is necessary to obtain the same asymptotic drawing scale. The apparent limit surface is drawn below.

with the observation that throughout the proof the dependence on χ, y will be omitted.

The important remark is that $\tilde{S}$ has real part zero at z_+ (and in the liquid region, at z_- as well), and so we deform the two contours into one and the same such that for $\tilde{K}_{1,1}$ the real part of $S(z) - S(z_+)$ decreases away from z_+ (and so will become negative), while for $\tilde{K}_{2,2}$ it increases away from z_+ thus becoming positive. Then in both cases the integrals will converge to 0 as r goes to 0 using the basic asymptotic fact (5.26). We finally remark that for $\tilde{K}_{1,1}$ deformation of the contours has to avoid the interval $(\frac{1}{\chi}, \frac{1}{\mathcal{A}})$, while for $\tilde{K}_{2,2}$ we need to avoid the interval $(0, 1)$ — both of which can be achieved.

□

Remark 5.12. $\Re S(z_+)$ can be made more explicit as both S and z_+ have explicit formulas, but we decided to avoid this to streamline the proof. In the case $a \to \infty$ the argument (originally the one from [OR03]) simplifies considerably, and it will be presented, in a slightly modified form (i.e., for a different model) in Section 6.

The discussion above, especially Theorems 5.6 and 5.11 and Remark 5.9, allows us to formulate the main as "away from the arctic curve, for $\chi > 0$, the local pfaffian correlations, in the limit, become determinantal with kernel given by the incomplete beta kernel."

Theorem 5.13. *Let $n > 0$ be a natural number, $a \in (0, \infty)$, χ, y reals with $\chi \in (0, a), y > \chi - a$, and for $1 \le s \le n$ fix n pairs $\mathsf{i}_s \in \mathbb{N}, \mathsf{k}_s \in \mathbb{Z}'$. As $r \to 0+$ assume we have $i_s \in \mathbb{N}, k_s \in \mathbb{Z}'$ depending on r and converging $ri_s \to \chi$, $rk_s \to y$ thusly:*

$$i_s = \left\lfloor \frac{\chi}{r} \right\rfloor + \mathsf{i}_s, \qquad k_s = \left\lfloor \frac{y}{r} \right\rfloor + \mathsf{k}_s, \tag{5.39}$$

and denote $U = \{(i_1, j_1), \dots, (i_n, j_n)\}$. Then in the asymptotic regime given by (5.11) we have

$$\varrho(U) \to \begin{cases} 0, & \text{if } y > y_+ \text{ or } (\chi > \chi_T \text{ and } y < y_-), \\ \det_{1 \le u,v \le n} B(\Delta \mathsf{i}_{uv}, \Delta \mathsf{k}_{uv}), & \text{if } y \in (y_-, y_+), \\ 1, & \text{if } \chi < \chi_T \text{ and } y < y_- \end{cases} \tag{5.40}$$

where $\Delta \mathsf{i}_{uv} = \mathsf{i}_u - \mathsf{i}_v, \Delta \mathsf{k}_{uv} = \mathsf{k}_u - \mathsf{k}_v$, $B(\Delta \mathsf{i}, \Delta \mathsf{k})$ is the incomplete beta kernel

$$(\Delta \mathsf{i}, \Delta \mathsf{k}) \mapsto \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\pm} (1-z)^{\Delta \mathsf{i}} z^{-\Delta \mathsf{k} - 1} \mathrm{d}z \tag{5.41}$$

with $\gamma_\pm$ connecting the two critical points $z_\pm$, γ_+ passing to the right of 0 (and γ_- passing to the left), with γ_+ being chosen in the case $\mathsf{i}_v \le \mathsf{i}_u$ (and γ_- otherwise).

In the case $\chi = 0$ ($X = 1$), the limiting regime $r \to 0+$ for the local microscopic coordinates becomes (i, k)

$$\begin{aligned} i &= \mathsf{i}, \\ k &= \left\lfloor \frac{y}{r} \right\rfloor + \mathsf{k} \end{aligned} \tag{5.42}$$

which for i is of course a relabeling we nevertheless use to be consistent with our notation so far. Because $y_+ = \infty$, the coordinate y is either in the liquid region ($y > y_- = -\log \frac{4}{(1+A)^2}$) in which case the double critical points are complex conjugate of modulus 1, or in the frozen region ($-a < y < y_+$) in which case the double critical points are real negative. Moreover, on the circle $\{|z| = 1\}$ (of interest in the liquid region) the real part of S is constant and equal to 0:

$$2\Re S(z) = S(z) + S(1/z) = 0. \tag{5.43}$$

with values increasing (positive) and decreasing (negative) outside (remark that in the notation from above $C_0 = \{|z| = 1\}$).

When one passes the contours through the critical points, the preceding arguments work almost unchanged. The few observations we make are as follows.

First, the contours for z and w exist even when $\chi = 0$ since for any finite $q = \exp(-r)$ there is enough space to the right of 1 between the largest zero of F ($= q < 1$) and its smallest pole ($= q^{-\mathsf{i}-1} > 1$). Thus, even though we have a pole at 1 in the integrand for $K_{1,2}$ and $K_{2,2}$, this will never give a residual contribution as the contours can avoid passing through 1.

Second, in both cases — y inside/outside the liquid region), we have to exchange the two contours (and if y is inside the liquid region, both have to pass through $z_+ = \overline{z_-}$), in which case we will pick up the contributions from the residues along the hyperbola $zw = 1$ in the analysis of $K_{1,1}$ and $K_{2,2}$. Their contributions are, in the limit $r \to 0+$:

$$\begin{aligned} Res_{w=\frac{1}{z}} K_{1,1} &\to \int_{\gamma_+} (1-z)^{\mathsf{i}} \left(1-\frac{1}{z}\right)^{\mathsf{i}'} z^{\mathsf{k}'-\mathsf{k}+1} \frac{z-1}{z+1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \\ Res_{w=\frac{1}{z}} K_{2,2} &\to \int_{\gamma_-} (1-z)^{-\mathsf{i}} \left(1-\frac{1}{z}\right)^{-\mathsf{i}'} z^{\mathsf{k}-\mathsf{k}'-1} \frac{1+z}{1-z} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \end{aligned} \tag{5.44}$$

where γ_+ is a closed contour passing through $z_+ < 0$ and to the right of 1 if $y < y_-$ or otherwise joins the two complex conjugate critical points (again to the right of 1) if $y > y_-$.

Third, $\tilde{K} = K$ if $y > y_-$ as the real part of S at the critical points is 0.

In view of the above, we have shown the correlations remain pfaffian when $\chi = 0$ with a kernel we have explicitly computed. We formally state the result.

Theorem 5.14. *Let $n > 0$ be a natural number, $a \in (0, \infty)$, $y > -a$ real, and for $1 \leq s \leq n$ fix n pairs $\mathsf{i}_s \in \mathbb{N}, \mathsf{k}_s \in \mathbb{Z}'$. As $r \to 0+$ assume we have $i_s \in \mathbb{N}$, as well as $k_s \in \mathbb{Z}'$ depending on r converging thusly:*

$$\begin{aligned} i_s &= \mathsf{i}_s, \\ k_s &= \left\lfloor \frac{y}{r} \right\rfloor + \mathsf{k}_s. \end{aligned} \tag{5.45}$$

Denote $U = \{(i_1, j_1), \ldots, (i_n, j_n)\}$. Then in the asymptotic regime given by (5.11) the correlations $\varrho(U)$ converge to 1 if $y < y_-$ and are otherwise pfaffian with 2×2 matrix kernel given by:

$$\begin{aligned} \mathsf{K}_{1,1}(\mathsf{i}, \mathsf{k}; \mathsf{i}', \mathsf{k}') &= \int_{\gamma_+} (1-z)^{\mathsf{i}} \left(1-\frac{1}{z}\right)^{\mathsf{i}'} z^{\mathsf{k}'-\mathsf{k}+1} \frac{z-1}{z+1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \\ \mathsf{K}_{1,2}(\mathsf{i}, \mathsf{k}; \mathsf{i}', \mathsf{k}') &= \int_{\gamma_\pm} (1-z)^{\mathsf{i}-\mathsf{i}'} z^{\mathsf{k}'-\mathsf{k}-1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \\ \mathsf{K}_{2,2}(\mathsf{i}, \mathsf{k}; \mathsf{i}', \mathsf{k}') &= \int_{\gamma_-} (1-z)^{-\mathsf{i}} \left(1-\frac{1}{z}\right)^{-\mathsf{i}'} z^{\mathsf{k}-\mathsf{k}'-1} \frac{1+z}{1-z} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \end{aligned} \tag{5.46}$$

where $\gamma_\pm$ are as in Theorem 5.13 and γ_+ is taken if and only if $\mathsf{i}' \leq \mathsf{i}$.

6 Plane overpartitions

A *plane overpartition* is a plane partition where in each row the last occurrence of an integer can be overlined or not while all the other occurrences of this integer are not overlined, and in each column the first occurrence of an integer can be overlined or not while all the other occurrences of this integer are overlined. A plane overpartition with the largest entry at most N and shape λ can be recorded as a sequence of partitions

$$\emptyset \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \cdots \prec \lambda^{(2n-1)} \prec' \lambda^{(2N)} = \lambda \tag{6.1}$$

where $\lambda^{(i)}$ is the partition whose shape is formed by all fillings greater than $N - i/2$, where the convention is that $\overline{k} = k - 1/2$. An example of a plane overpartition is given in Figure 12.

A plane partition π is called a (diagonally) strict plane partition if its diagonals $\pi^{(t)} = (\pi_{i,i+t})_{i \geq 1}$ are strict partitions, i.e. strictly decreasing sequences of integers. By deleting the overlines in a plane overpartition one obtains a strict plane partition. Conversely, a strict plane partition can be overlined to obtain a plane overpartition and there are $2^{\#\text{border components}}$ different ways to do it. A border component of a strict plane partitions is a set of rookwise-connected boxes (i.e., a connected ribbon/border strip) filled with the same number. The strict plane partition obtained by deleting the overlines in Figure 12 has five border components.

A measure that to a plane overpartition with the largest entry at most N assigns a weight $q^{|\text{sum of all entries}|}$ is an HV-ascending Schur process with $t = 1$ and $x_1 = x_2 = q^N, \ldots, x_{2N-1} = x_{2N} = q^1$.

4	$\overline{4}$	$\overline{3}$	2	2
3	3	$\overline{3}$	$\overline{2}$	
$\overline{3}$	$\overline{1}$			
1				

Figure 12: A plane overpartition $\emptyset \prec (1) \prec' (2) \prec (2,2) \prec' (3,3,1) \prec (5,3,1) \prec' (5,4,1) \prec (5,4,1,1) \prec' (5,4,2,1)$.

The asymptotics of strict plane partitions was studied in [Vul07] using point configurations. A plane partition π is represented by a finite set of points corresponding to each entry. The point configuration is a subset of $\mathbb{Z} \times \mathbb{Z}_{>0}$, where (t,x) belongs to it if x is a part of $\pi^{(t)}$. See Figure 13. The set of blue points is the point configuration corresponding to the strict plane partition obtained by deleting the overlines in Figure 12.

The above point configuration is not suitable for plane overpartitions, since there are $2^{\#\text{border components}}$ different ways to overline a strict plane partition. We need a new set of point configurations to represent these different overlinings. We explain briefly how these point configurations are obtained and refer the reader to [BCC17] and [CSV11] for details and proofs.

Starting from the point configuration of the corresponding strict plane partition we first construct a set of nonintersecting paths, where each path corresponds to a row of the plane partition. Going from $\pi_{i,j}$ to $\pi_{i,j+1}$ we take an eastbound edge followed by $\pi_{i,j} - \pi_{i,j+1}$ southbound edges if $\pi_{i,j}$ is not overlined and a southeastbound edge followed by $\pi_{i,j} - \pi_{i,j+1} - 1$ southbound edges if $\pi_{i,j}$ is overlined. See Figure 13. Once we have the nonintersecting paths we can produce a domino tiling. Dominos are placed diagonally and cover $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. It is possible to place them in such a way so that different types of dominos (divided by the direction of the position and the color of the top corner in a chessboard fashion coloring) correspond to different edges. See Figure 14. Finally, place two particles (holes) at the center of each of the two domino squares if the top corner of that domino is black (white). The point configuration is the set of all particles. See Figure 14, where particles are shown in solid colors. It corresponds to the example from Figure 12.

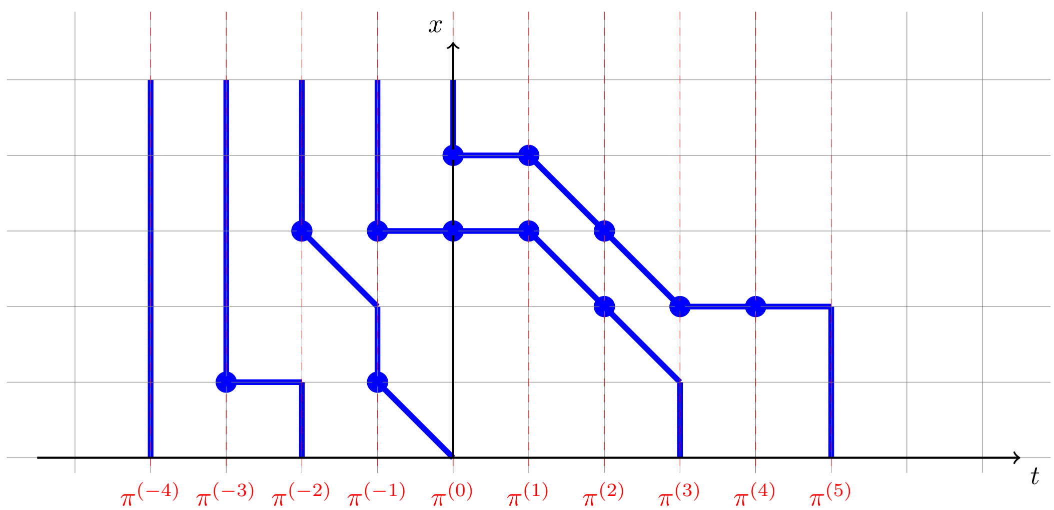

Figure 13: Strict plane partition point configuration.

One can show that a point configuration representing a plane overpartition

$$\emptyset \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \cdots \prec \lambda^{(2n-1)} \prec' \lambda^{(2N)} = \lambda \tag{6.2}$$

is a subset of $\left(\mathbb{Z}+\frac{1}{4}\right) \times \left(\mathbb{Z}+\frac{3}{4}\right) \cup \left(\mathbb{Z}+\frac{3}{4}\right) \times \left(\mathbb{Z}+\frac{1}{4}\right)$ consisting of points (t,x) such that the Maya diagram of

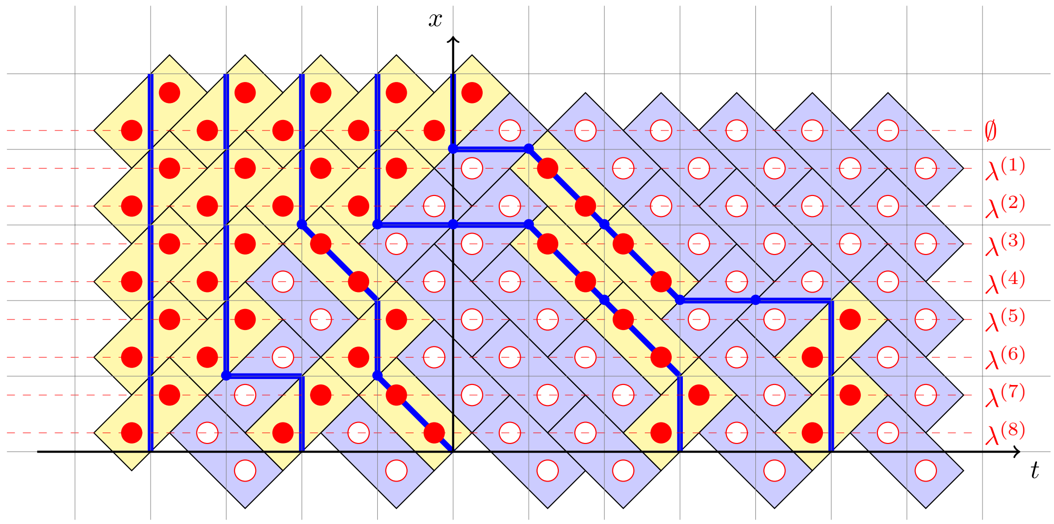

Figure 14: Plane overpartition point configuration.

$\lambda^{i(x)}$ contains a particle at the position $\bar{t}$, i.e. $\bar{t} = \lambda_j^{i(x)} - j + 1/2$, for some $j \geq 1$ where $i(x) = 2N - 2(x - 1/4)$ and

$$\bar{t} = \begin{cases} t - 3/4, & t \in \mathbb{Z} + 1/4 \\ t - 1/4, & t \in \mathbb{Z} + 3/4 \end{cases} = \lceil t - 1/2 \rceil - 1/2. \tag{6.3}$$

For the HV-ascending Schur process with $x_1 = x_2 = q^N, \ldots, x_{2N-1} = x_{2N} = q^1$:

$$\begin{aligned} F_{HV}(i,z) &= \frac{(-q^{N+2-j}z;q)_{j-1}}{(q^{N+1-j}z;q)_j} \frac{(q/z;q)_N}{(-q/z;q)_N}, i = 2j-1, j = 1,\ldots,N, \\ F_{HV}(i,z) &= \frac{(-q^{N+1-j}z;q)_j}{(q^{N+1-j}z;q)_j} \frac{(q/z;q)_N}{(-q/z;q)_N}, i = 2j, j = 1,\ldots,N. \end{aligned} \tag{6.4}$$

If we set

$$F(x,z) = F_{HV}(i(x),z) \tag{6.5}$$

then

$$\begin{aligned} F(x,z) &= \frac{(-q^{x+5/4}z;q)_{N-x-1/4}}{(q^{x+1/4}z;q)_{N-x+3/4}} \frac{(q/z;q)_N}{(-q/z;q)_N}, x \in \mathbb{Z} + \frac{3}{4}, \\ F(x,z) &= \frac{(-q^{x+3/4}z;q)_{N-x+1/4}}{(q^{x+3/4}z;q)_{N-x+1/4}} \frac{(q/z;q)_N}{(-q/z;q)_N}, x \in \mathbb{Z} + \frac{1}{4}. \end{aligned} \tag{6.6}$$

When $N \to \infty$

$$\begin{aligned} F(x,z) &= \frac{(-q^{x+5/4}z;q)_\infty}{(q^{x+1/4}z;q)_\infty} \frac{(q/z;q)_\infty}{(-q/z;q)_\infty}, x \in \mathbb{Z} + \frac{3}{4}, \\ F(x,z) &= \frac{(-q^{x+3/4}z;q)_\infty}{(q^{x+3/4}z;q)_\infty} \frac{(q/z;q)_\infty}{(-q/z;q)_\infty}, x \in \mathbb{Z} + \frac{1}{4}. \end{aligned} \tag{6.7}$$

In a compact form

$$F(x,z) = \frac{(-q^{\lceil x+1/2 \rceil}z;q)_\infty}{(q^{\lceil x \rceil}z;q)_\infty} \frac{(q/z;q)_\infty}{(-q/z;q)_\infty}. \tag{6.8}$$

Remark 6.1. For $q \in (0,1)$ and fixed x, zeros of F as a function of z are $-q^{-\lceil x+1/2 \rceil}, -q^{-\lceil x+1/2 \rceil - 1}, \ldots$, which belong to $(-\infty, -q^{-\lceil x+1/2 \rceil}]$, and $q, q^2, \ldots$, which belong to $(0, q]$. Poles are $q^{-\lceil x \rceil}, q^{-\lceil x \rceil - 1}, \ldots$, which belong to $[q^{-\lceil x \rceil}, \infty)$, and $-q, -q^2, \ldots$, which belong to $[-q, 0)$.

In our asymptotic analysis we will consider $q \in (0,1)$ and x_i such that $rx_i \to \chi$ when $r \to 0+$ where $q = e^{-r}$. We will need to avoid zeros or poles of $F(x_i, z)$ along certain contours in z for r close to $0+$. Zeros, respectively poles, could be avoided if we make sure to choose contours that do not cut $\mathcal{Z}(\chi) = (-\infty, -e^{\chi}] \cup (0,1]$, respectively $\mathcal{P}(\chi) = [-1,0) \cup [e^{\chi}, \infty)$.

Theorem 6.2. *Let $(t_1, x_1), \ldots, (t_n, x_n) \in \left(\mathbb{Z} + \frac{1}{4}\right) \times \left(\mathbb{Z} + \frac{3}{4}\right) \cup \left(\mathbb{Z} + \frac{3}{4}\right) \times \left(\mathbb{Z} + \frac{1}{4}\right)$. Pfaffian correlations are given by the following matrix kernel:*

$$K(t_i, x_i; t_j, x_j) = \begin{bmatrix} K_{1,1}(t_i, x_i; t_j, x_j) & K_{1,2}(t_i, x_i; t_j, x_j) \\ -K_{1,2}(t_j, x_j; t_i, x_i) & K_{2,2}(t_i, x_i; t_j, x_j) \end{bmatrix}_{i,j=1,\ldots,n} \tag{6.9}$$

where

$$\begin{aligned} K_{1,1}(t_i, x_i; t_j, x_j) &= [z^{\overline{t_i}} w^{\overline{t_j}}] F(x_i, z) F(x_j, w) \frac{\sqrt{zw}(z-w)}{(z+1)(w+1)(zw-1)}, \quad 1 < |w|, 1 < |z|, \\ K_{1,2}(t_i, x_i; t_j, x_j) &= \left[\frac{z^{\overline{t_i}}}{w^{\overline{t_j}}}\right] \frac{F(x_i, z)}{F(x_j, w)} \frac{\sqrt{zw}(zw-1)}{(z-w)(z+1)(w-1)}, \quad 1 < |z|, 1 < |w|, |w| \lessgtr |z|, \text{ for } i \gtrless j, \\ K_{2,2}(t_i, x_i; t_j, x_j) &= \left[\frac{1}{z^{\overline{t_i}} w^{\overline{t_j}}}\right] \frac{1}{F(x_i, z) F(x_j, w)} \frac{\sqrt{zw}(z-w)}{(z-1)(w-1)(zw-1)}, \quad 1 < |z|, 1 < |w|, \end{aligned} \tag{6.10}$$

and $\overline{t}$ is as in (6.3).

Remark 6.3. Let $\chi \in \mathbb{R}$. Define

$$\tilde{K}(t_i, x_i; t_j, x_j) = \begin{pmatrix} e^{\chi \overline{t_i}/2} & 0 \\ 0 & e^{-\chi \overline{t_i}/2} \end{pmatrix} K(t_i, x_i; t_j, x_j) \begin{pmatrix} e^{\chi \overline{t_j}/2} & 0 \\ 0 & e^{-\chi \overline{t_j}/2} \end{pmatrix}. \tag{6.11}$$

Then

$$\operatorname{pf}[\tilde{K}(t_i, x_i; t_j, x_j)]_{1 \le i,j \le n} = \operatorname{pf}[K(t_i, x_i; t_j, x_j)]_{1 \le i,j \le n}. \tag{6.12}$$

Let

$$G(x, t, z) = r\left(\log F(x, z) - \overline{t}(\log z - \chi/2)\right). \tag{6.13}$$

Using Theorem 6.2

$$\tilde{K}_{1,1}(t_i, x_i; t_j, x_j) = \frac{1}{(2\pi \mathrm{i})^2} \int_{C_z} \int_{C_w} \frac{z-w}{\sqrt{zw}(z+1)(w+1)(zw-1)} \exp\left[\frac{1}{r}\left(G(\overline{t_i}, x_i, z) + G(\overline{t_j}, x_j, w)\right)\right] \mathrm{d}z \mathrm{d}w \tag{6.14}$$

where C_z and C_w are simple closed counterclockwise oriented contours such that $|z| > 1$, $z \notin \mathcal{P}(\chi)$, $|w| > 1$, and $w \notin \mathcal{P}(\chi)$ for all $z \in C_z$ and $w \in C_w$;

$$\tilde{K}_{1,2}(t_i, x_i; t_j, x_j) = \frac{1}{(2\pi \mathrm{i})^2} \int_{C_z} \int_{C_w} \frac{zw-1}{\sqrt{zw}(z-w)(z+1)(w-1)} \exp\left[\frac{1}{r}\left(G(\overline{t_i}, x_i, z) - G(\overline{t_j}, x_j, w)\right)\right] \mathrm{d}z \mathrm{d}w \tag{6.15}$$

where C_z and C_w are simple closed counterclockwise oriented contours such that $|z| > 1$, $z \notin \mathcal{P}(\chi)$, $|w| > 1$, $w \notin \mathcal{Z}(\chi)$, $|z| > |w|$ ($|z| < |w|$), for all $z \in C_z$ and $w \in C_w$ when $i \ge j$ ($i < j$);

$$\tilde{K}_{2,2}(t_i, x_i; t_j, x_j) = \frac{1}{(2\pi \mathrm{i})^2} \int_{C_z} \int_{C_w} \frac{z-w}{\sqrt{zw}(z-1)(w-1)(zw-1)} \exp\left[\frac{1}{r}\left(-G(\overline{t_i}, x_i, z) - G(\overline{t_j}, x_j, w)\right)\right] \mathrm{d}z \mathrm{d}w \tag{6.16}$$

where C_z and C_w are two simple closed counterclockwise oriented contours such that $|z| > 1$, $z \notin \mathcal{Z}(\chi)$, $|w| > 1$, $w \notin \mathcal{Z}(\chi)$ for all $z \in C_z$ and $w \in C_w$.

Before we proceed with the asymptotics, we set up some notation.

Define $C^+(R, \theta)$ ($C^-(R, \theta)$) to be the counterclockwise (clockwise) oriented arc on $|z| = R$ from $Re^{-i\theta}$ to $Re^{i\theta}$ for $R > 0$ and $0 \le \theta \le \pi$. The counterclockwise oriented circle $|z| = R$ is then $C^+(R, \theta) \cup -C^-(R, \theta)$.

For a sequence of real numbers a_i we define an (infinite) integer matrix

$$\Delta(a)_{ij} = \lceil a_i \rceil - \lceil a_j \rceil. \tag{6.17}$$

Let

$$\mathcal{D} = \{(\tau, \chi) | \tau \in \mathbb{R}, \chi \in \mathbb{R}_{\geq 0}, -1 \leq f(\tau, \chi) \leq 1\} \tag{6.18}$$

where

$$f(\tau, \chi) = \frac{(e^{\chi} + 1)(e^{\tau} - 1)}{2e^{\chi/2}(e^{\tau} + 1)}. \tag{6.19}$$

Then $(\tau, \chi) \in \mathcal{D}$ if $-\tau_c(\chi) \leq \tau \leq \tau_c(\chi)$ and $\chi \geq 0$ where

$$\tau_c(\chi) = 2 \log \frac{e^{\chi/2} + 1}{e^{\chi/2} - 1} \tag{6.20}$$

or in other words $\mathcal{D}$ is the domain bounded by three curves: $\chi = 0$ and

$$-1 \pm e^{\chi/2} - e^{\tau/2} \mp e^{\tau/2} e^{\chi/2} = 0, \quad \chi > 0. \tag{6.21}$$

This leads to the description of the domain $\mathcal{D}$ in the terms of the amoeba of the polynomial $-1+z+w+zw$. In the case when $P(z, w) = -1 + z + w + zw$ the amoeba is a domain bounded by the following four curves:

$$-1 \pm e^{\omega} - e^{\xi} \mp e^{\xi} e^{\omega} = 0, \quad \omega > 0. \tag{6.22}$$

and

$$-1 \pm e^{\omega} + e^{\xi} \pm e^{\xi} e^{\omega} = 0, \quad \omega < 0. \tag{6.23}$$

If we set $(\xi, \omega) = (\tau/2, \chi/2)$ then $\mathcal{D}$ is the half of the amoeba of $-1 + z + w + zw$ for $\omega \geq 0$.

Let

$$\theta_c(\tau, \chi) = \begin{cases} \arccos(f(\tau, \chi)), & -\tau_c(\chi) \leq \tau \leq \tau_c(\chi), \\ 0 & \tau > \tau_c(\chi), \\ \pi & \tau < -\tau_c(\chi). \end{cases} \tag{6.24}$$

Theorem 6.4. *Let $rt_i \to \tau$, $rx_i \to \chi$ when $r \to 0+$ where $q = e^{-r}$. Assume $\Delta(t - 1/2)$, $\Delta(x)$, $\Delta(x + 1/2)$ do not change with r.*

Then

$$\lim_{r \to 0+} \tilde{K}_{1,2}(t_i, x_i; t_j, x_j) = \frac{1}{2\pi \mathrm{i}} e^{-\chi \Delta(t-1/2)_{ij}} \int_{C^{\pm}(e^{-\chi/2}, \theta_c(\tau, \chi))} \frac{1}{w^{\Delta(t-1/2)_{ij}+1}} \frac{(1-w)^{\Delta(x)_{ij}}}{(1+w)^{\Delta(x+1/2)_{ij}}} \mathrm{d}w \tag{6.25}$$

where we choose $C^+(e^{-\chi/2}, \theta_c(\tau, \chi))$ if $i \geq j$ and $C^-(e^{-\chi/2}, \theta_c(\tau, \chi))$ otherwise and where $\theta_c(\tau, \chi)$ is as in (6.24)*.*

Proof. We start with (6.15). The asymptotics is determined by the limit of G when $r \to 0+$:

$$S(z; \tau, \chi) = -\mathrm{Li}_2(-e^{-\chi} z) - \mathrm{Li}_2(\frac{1}{z}) + \mathrm{Li}_2(e^{-\chi} z) + \mathrm{Li}_2(-\frac{1}{z}) - \tau(\log z - \chi/2) \tag{6.26}$$

where $\mathrm{Li}_2(z)$ is defined in Appendix B. Ignoring the log part of S, the function is analytic on $\mathbb{C} \backslash \mathcal{P}(\chi)$. We can easily compute

$$z \frac{\mathrm{d}}{\mathrm{d}z} S(z; \tau, \chi) = -\tau + \log \frac{(1 + e^{-\chi} z)(z + 1)}{(1 - e^{-\chi} z)(z - 1)} \tag{6.27}$$

where by the Cauchy–Riemann equations

$$z \frac{\mathrm{d}}{\mathrm{d}z} S(z; \tau, \chi) = x \frac{\mathrm{d}}{\mathrm{d}x} \Re S + y \frac{\mathrm{d}}{\mathrm{d}y} \Re S + \mathrm{i} \left(y \frac{\mathrm{d}}{\mathrm{d}x} \Re S - x \frac{\mathrm{d}}{\mathrm{d}y} \Re S \right). \tag{6.28}$$

The real part of $S(z;\tau,\chi)$ vanishes on the circle $z = e^{\chi/2}e^{i\theta}$. This implies that on this circle the imaginary part of $z\frac{\mathrm{d}}{\mathrm{d}z}S(z;\tau,\chi)$ vanishes too, which is the derivative of $\Re S$ in the direction of the tangent, while the real part is equal to $R\frac{\mathrm{d}}{\mathrm{d}R}\Re S(z;\tau,\chi)$. Then for $z = e^{\chi/2}e^{i\theta}$

$$R\frac{\mathrm{d}}{\mathrm{d}R}\Re S(z;\tau,\chi) = -\tau + \log\left|\frac{z+1}{z-1}\right|^2, \tag{6.29}$$

which is negative if and only if $\cos\theta < f(\tau,\chi)$.

If $(\tau,\chi) \in \mathcal{D}$ and $z = e^{\chi/2}e^{i\theta}$ then $R\frac{\mathrm{d}}{\mathrm{d}R}\Re S(z;\tau,\chi)$ changes the sign along $|z| = e^{\chi/2}$ at $\theta = \pm\theta_c(\tau,\chi)$ being positive for $|\theta| < |\theta_c(\tau,\chi)|$. We then deform contours so that the real parts of $S(z,\tau,\chi)$ and $S(w,\tau,\chi)$ are negative everywhere on the new contours excepts at the critical points. The new contours look like γ_z and γ_w in Figure 15. The integrals over these new contours vanish as $r \to 0+$, but we pick up the residue at $z = w$ on $C^+(e^\chi/2, \theta_c(\tau,\chi))$ $(C^-(e^\chi/2,\theta_c(\tau,\chi))$ when $i \geq j$ $(i < j)$. The residue is equal to

$$\frac{1}{2\pi\mathrm{i}}\int_{C^{\pm}(e^\chi/2,\theta_c(\tau,\chi))}\frac{1}{w^{\Delta(t-1/2)_{ij}+1}}\frac{(1-e^{-\chi}w)^{\Delta(x)_{ij}}}{(1+e^{-\chi}w)^{\Delta(x+1/2)_{ij}}}\mathrm{d}w. \tag{6.30}$$

The change of variables $w \mapsto e^{-\chi}w$ brings the expression from the statement of the theorem.

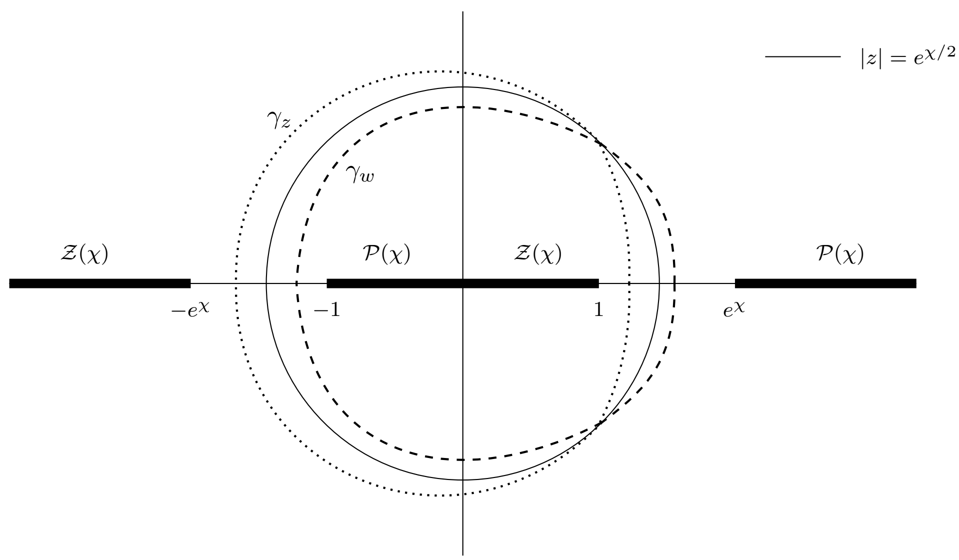

Figure 15: Deformed contours in z and w.

Some special care is needed when $\chi = 0$ because we are deforming the contours in a neighborhood of $|z| = 1$ and $|w| = 1$. In this case $(\tau,\chi) \in \mathcal{D}$ for all τ. Observe that if we deform the contours as described above, the new contour in z, respectively w, will still encompass $z = -1$, respectively $w = 1$, and we will not be picking up any residues coming from $z = -1$ or $w = 1$. Also note that while deforming the contours we remain away from $\mathcal{P}(\chi)$ for the z-contour and $\mathcal{Z}(\chi)$ for the w-contour.

Lastly, the integral $1/(z-w)$ converges at the points of intersection of the contours. We can for simplicity make them cross orthogonally; then the integral of $1/(z-w)$ along these contours is absolutely bounded by the convergent integral $\iint 1/\sqrt{x^2+y^2}dxdy$.

If (τ,χ) does not belong to $\mathcal{D}$ then $R\frac{\mathrm{d}}{\mathrm{d}R}\Re S(z;\tau,\chi)$ does not change sign. It is strictly positive for $\tau < 0$ and strictly negative for $\tau > 0$. We just need to push the contour C_z inside (for $\tau < 0$) or outside (for $\tau > 0$) of $|z| = e^{\chi/2}$ and push C_w outside (for $\tau < 0$) or inside (for $\tau > 0$) of $|w| = e^{\chi/2}$ to obtain that both $\Re S(z;\tau,\chi)$ and $\Re S(w;\tau,\chi)$ are negative along the new contours. Then when $\tau > 0$ we get that $K_{1,2}$ vanishes in the limit and when $\tau < 0$ we pick up the residue at $z = w$ along the whole circle $|z| = e^{\chi/2}$.

$\square$

Remark 6.5. The density function of a particle present at (τ, χ) is

$$\rho(\tau, \chi) = \lim_{r \to 0+} K_{1,2}(x, k; x, k), \quad \text{for } rk \to \tau \text{ and } rx \to \chi. \tag{6.31}$$

From the theorem above

$$\rho(\tau, \chi) = \frac{1}{2\pi \mathrm{i}} \int_{C^+(e^{-\chi/2}, \theta_c(\tau,\chi))} \frac{1}{z} \mathrm{d}z = \frac{\theta_c(\tau, \chi)}{\pi}. \tag{6.32}$$

A three dimensional depiction of a plane overpartition π, obtained by stacking $\pi_{i,j}$ unit cubes above the (i, j)-th position, is deciphered from the corresponding point configuration in the following way. Each particle (t, x) in the configuration is mapped to $(X, Y, Z) \in \mathbb{Z}^3_{\geq 0}$ where X is the number of particles to the right of (t, x) counting the particle itself, Y is the number of holes to the left of (t, x), or equivalently $X + \lfloor t - 1/2 \rfloor$, and Z is equal to $\lceil x + 1/2 \rceil$. The limit shape of strict plane partitions is:

$$\begin{aligned} X(\tau, \chi) &= \int_\tau^{\tau_c(\chi)} \rho(t, \chi) \mathrm{d}t, \\ Y(\tau, \chi) &= \int_{-\tau_c(\chi)}^{\tau} (1 - \rho(t, \chi)) \mathrm{d}t = \int_\tau^{\tau_c(\chi)} \rho(t, \chi) \mathrm{d}t + \tau, \\ Z(\tau, \chi) &= \chi \end{aligned} \tag{6.33}$$

where $\tau_c(\chi)$ is given by (6.20). The two formulas for Y are indeed equal from the properties of the model, but this can be confirmed directly since

$$\int_{-\tau_c(\chi)}^{0} (1 - \rho(t, \chi)) \mathrm{d}t = \int_{-\tau_c(\chi)}^{0} \frac{\pi - \arccos f(t, \chi)}{\pi} \mathrm{d}t = \int_{-\tau_c(\chi)}^{0} \frac{\arccos f(-t, \chi)}{\pi} \mathrm{d}t = \int_0^{\tau_c(\chi)} \rho(t, \chi) \mathrm{d}t. \tag{6.34}$$

In Figure 16 we depict an exactly sampled large plane overpartition for $q = 0.9$ as a domino tiling of the half plane and the corresponding strict plane partition as well. The simulations were carried out using the algorithms of [BBB$^+$15].

We now prove that under the same conditions as in Theorem 6.4 diagonal elements of the kernel $\tilde{K}$ vanish in the limit for $\chi > 0$.

Theorem 6.6. *Let $rt_i \to \tau$, $rx_i \to \chi$ when $r \to 0+$ where $q = e^{-r}$. Assume $\Delta(t - 1/2)$, $\Delta(x)$, $\Delta(x + 1/2)$ do not change with r.*

Then for $\chi > 0$

$$\lim_{r \to 0+} \tilde{K}_{1,1}(t_i, x_i; t_j, x_j) = \lim_{r \to 0+} \tilde{K}_{2,2}(t_i, x_i; t_j, x_j) = 0. \tag{6.35}$$

When $\chi = 0$ assume in addition that x_i does not change with r. Then

$$\lim_{r \to +0} \tilde{K}_{1,1}(t_i, x_i; t_j, x_j) = (-1)^{\lceil x_i \rceil} \int_{C^+(1, \theta_c(\tau, 0))} \frac{1}{w^{-\Delta(t-1/2)_{ij} + \lceil x_i \rceil - \lceil x_i + 1/2 \rceil + 1}} \frac{(1 - w)^{\lceil x_i \rceil + \lceil x_j \rceil - 1}}{(1 + w)^{\lceil x_i + 1/2 \rceil + \lceil x_j + 1/2 \rceil - 1}} \mathrm{d}w, \tag{6.36}$$

$$\lim_{r \to +0} \tilde{K}_{2,2}(t_i, x_i; t_j, x_j) = (-1)^{\lceil x_i \rceil} \int_{C^-(1, \theta_c(\tau, 0))} \frac{1}{w^{\Delta(t-1/2)_{ij} - \lceil x_i \rceil + \lceil x_i + 1/2 \rceil + 1}} \frac{(1 - w)^{-\lceil x_i \rceil - \lceil x_j \rceil + 1}}{(1 + w)^{-\lceil x_i + 1/2 \rceil + \lceil x_j + 1/2 \rceil + 1}} \mathrm{d}w. \tag{6.37}$$

Proof. We start with (6.14). The analysis was done in the proof of Theorem 6.4. We can conclude that we can deform C_z and C_w into new contours such that the real parts of $S(z, \tau, \chi)$ and $S(w, \tau, \chi)$ are negative everywhere on the new contours except at the critical points. We can choose to deform them to the same contour γ_z in Figure 15. When $\chi > 0$ there is no residue to pick up since while deforming the contours we keep $|z| > 1$, $z \notin \mathcal{P}(\chi)$, $|w| > 1$ and $z \notin \mathcal{P}(\chi)$. The proof is analogous for $\tilde{K}_{2,2}$.

In the case when when $\chi = 0$ we can proceed as above, but now for $\tilde{K}_{1,1}$ we will be picking up the residue at $z = 1/w$ along $C^+(1, \theta_c(\tau, 0))$ for $\tilde{K}_{1,1}$ and along $C^-(1, \theta_c(\tau, 0))$ for $\tilde{K}_{2,2}$. The residues are given in (6.36) and (6.37). ∎

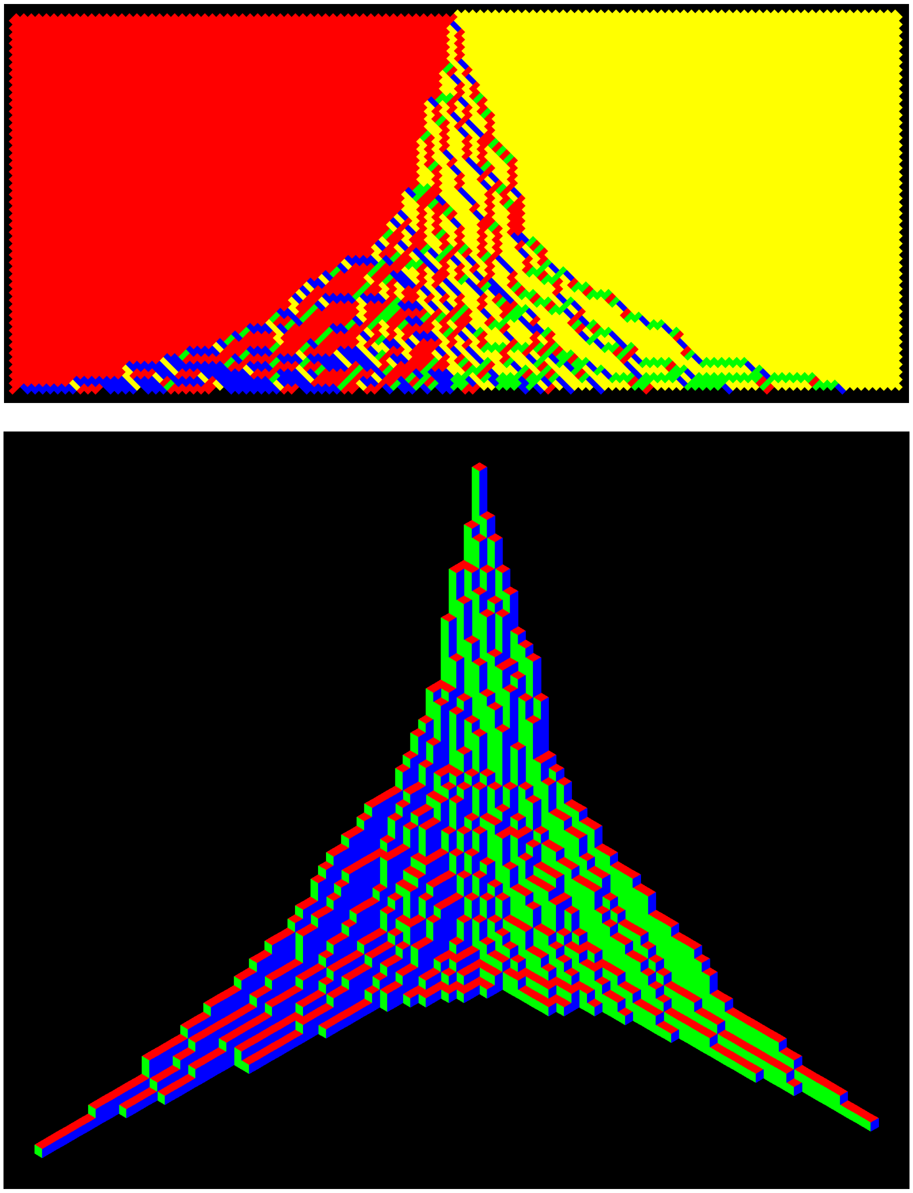

Figure 16: A large plane overpartition as a domino tiling and the corresponding strict plane partition.

Definition 6.7. Let $(\tau,\chi)\in\mathbb{R}\times\mathbb{R}_{\geq 0}, t,x,y\in\mathbb{Z}$. Define

$$B_{\pm}(\tau,\chi,t,x,y)=\begin{cases}\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{C^{\pm}(e^{-\chi/2},\theta_c(\tau,\chi))}\frac{1}{z^{t+1}}\frac{(1-z)^x}{(1+z)^y}\mathrm{d}z, & (\tau,\chi)\in\mathcal{D},\\ 0, & \text{otherwise}\end{cases}\tag{6.38}$$

where $\theta_c(\tau,\chi)$ is as in (6.24).

From Theorems 6.4 and 6.6 we obtain that in the limit the bulk has determinantal correlations. Precisely, after factoring out constants that cancel out in the pfaffian as in Remark 6.3, we get the following.

Theorem 6.8. *For $i=1,\dots,n$ let $rt_i\to\tau$, $rx_i\to\chi$ when $r\to 0+$ where $q=e^{-r}$. Assume $\Delta(t-1/2)$, $\Delta(x)$, $\Delta(x+1/2)$ do not change with r. Then*

$$\lim_{r\to+0}\varrho(t_1,x_1,\dots,t_n,x_n)=\det\left[B_{\pm}\left(\tau,\chi,\Delta(t-1/2)_{ij},\Delta(x)_{ij},\Delta(x+1/2)_{ij}\right)\right]_{i,j=1,\dots,n}\tag{6.39}$$

where we choose $+$ if $i\geq j$ and $-$ otherwise.

At the boundary $\chi=0$ limit correlations remain pfaffian.

Theorem 6.9. *For $i=1,\dots,n$ and $q=e^{-r}$ let t_i and x_i be such that $rt_i\to\tau$ when $r\to 0+$, and $\Delta(t-1/2)$, and x_i do not change with r. Then*

$$\lim_{r\to+0}\varrho(t_1,x_1,\dots,t_n,x_n)=\mathrm{pf}\begin{bmatrix}B^{1,1}(i,j) & B^{1,2}(i,j)\\ -B^{1,2}(j,i) & B^{2,2}(i,j)\end{bmatrix}_{1\leq i,j\leq n}\tag{6.40}$$

where

$$\begin{aligned}B^{1,1}(i,j)&=B_{+}\left(\tau,0,-\Delta(t-1/2)_{ij}+\lceil x_i\rceil-\lceil x_i+1/2\rceil,\lceil x_i\rceil+\lceil x_j\rceil-1,\lceil x_i+1/2\rceil+\lceil x_j+1/2\rceil-1\right),\\ B^{1,2}(i,j)&=B_{\pm}\left(\tau,0,\Delta(t-1/2)_{ij},\Delta(x)_{ij},\Delta(x+1/2)_{ij}\right),\\ B^{2,2}(i,j)&=B_{-}\left(\tau,0,\Delta(t-1/2)_{ij}-\lceil x_i\rceil+\lceil x_i+1/2\rceil,-\lceil x_i\rceil-\lceil x_j\rceil+1,-\lceil x_i+1/2\rceil-\lceil x_j+1/2\rceil+1\right)\end{aligned}\tag{6.41}$$

where we choose $+$ if $i\geq j$ and $-$ otherwise.

7 Conclusion

In this paper we have introduced the Schur process with two free boundary partitions. Upon charge mixing and using a new type of Wick lemma, we have seen that the process has pfaffian correlations. Killing both boundaries (forcing them to be the empty partition via the parameters), we have recovered the original process and determinantal correlations of Okounkov–Reshetikhin [OR03]. Killing just one boundary we have further recovered the pfaffian correlations of [BR05]. In this latter setting we have studied three asymptotical applications: fluctuations of symmetric last passage percolation models with i.i.d. geometric/exponential weights, limit shapes and processes for symmetric plane partitions, and for plane overpartitions — the latter via a domino tiling interpretation.

None of the aforementioned applications involved the full freedom of two free boundaries afforded by our main theorem, and we plan to pursue this freedom in future work. In particular, we plan to study asymptotics of last passage percolation, plane partitions, and plane overpartitions in this new setting. While the first is presumably related to (a possibly new version of) TASEP, the latter two no doubt lead to discretized versions of Dyson's Brownian motion in a strip with two free ends. We suspect a whole new range of processes will appear in this setting due to possible interplays between the two boundaries.

We can of course generalize the "Schur" in Schur processes to Hall–Littlewood, q-Whittaker, or even Macdonald processes. While for all these processes it is easy to compute the partition function — indeed in some sense its generic form already appears in the present work, access to correlations becomes significantly more difficult but we hope something nontrivial can still be said in the spirit of recent work by Barraquand, Borodin, Corwin and Wheeler [BBCW17].

On the other hand, there are still models we have not touched in the applications which are fully pfaffian and fall under our formalism: one can use other Littlewood identities not studied here — see [Rai00] for a flavor, or simply replace Schur functions with Schur's P and Q functions.

We finally speculate that the new type of Wick lemma and the partition function for the two free boundary process might be of interest to enumerative algebraic geometers and combinatorialists (e.g., counting maps via τ-functions) and to theoretical physicists (e.g., looking at topological vertices with two or three free boundaries or at other types of string theory partition functions).

Acknowledgments. We thank Jinho Baik, Guillaume Barraquand, Philippe Biane, Alexei Borodin, Cédric Boutillier, Ivan Corwin, Philippe Di Francesco and Patrik Ferrari for useful conversations.

Most of this work was done while the first three authors were at the Département de mathématiques et applications, École normale supérieure, Paris. We also acknowledge hospitality and support from the Galileo Galilei Institute during the 2015 program on "Statistical Mechanics, Integrability and Combinatorics". Part of this work was done while D.B. was visiting the ENS de Lyon and László Erdős at IST Austria, and remerciements are due to both institutions and to László for their hosting. J.B. recently enjoyed the hospitality from the PCMI 2017 Research Program and from the ESI programme on "Algorithmic and Enumerative Combinatorics".

We acknowledge financial support from the "Combinatoire à Paris" project funded by the City of Paris (D.B. and J.B.), from the Agence Nationale de la Recherche via the grants ANR 12-JS02-001-01 "Cartaplus" and ANR-14-CE25-0014 "GRAAL" (J.B.), from Paris Sciences & Lettres and from the ERC Advanced Grant No. 338804 (P.N.).

A Matrix kernels and Fredholm pfaffians

Here we briefly introduce the notion of matrix kernels and Fredholm pfaffians needed for the text. For more information on these, see [Rai00] or [OQR17, Appendix B]. For an antisymmetric $2n \times 2n$ matrix A, the pfaffian of A is given by

$$\operatorname{pf} A = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)} \tag{A.1}$$

and one can show that $(\operatorname{pf} A)^2 = \det A$. A matrix kernel on a space X is a matrix-valued function on $X \times X$. It is said antisymmetric if $K(x,y) = -K(y,x)^T$. We use $[K(x_i,x_j)]_{1\le i,j\le n}$ to denote the block matrix whose ijth block is $K(x_i,x_j)$. If the kernel K is antisymmetric and of even size, then so is the matrix $[K(x_i,x_j)]_{1\le i,j\le n}$ and we can compute its pfaffian. Note that this pfaffian is invariant under a permutation of $S = \{x_1, x_2, \dots, x_n\}$ and we can use $\operatorname{pf} K(S)$ to denote it.

Proposition A.1. *Let K, $\tilde{K}$ be 2×2 antisymmetric kernels on X such that there exists a complex-valued function f on X such that*

$$\tilde{K}(x,y) := \begin{pmatrix} e^{f(x)} & 0 \\ 0 & e^{-f(x)} \end{pmatrix} K(x,y) \begin{pmatrix} e^{f(y)} & 0 \\ 0 & e^{-f(y)} \end{pmatrix}. \tag{A.2}$$

Then

$$\operatorname{pf}[K(x_i,x_j)]_{1\le i,j\le n} = \operatorname{pf}[\tilde{K}(x_i,x_j)]_{1\le i,j\le n}. \tag{A.3}$$

Proof. This is straightforward since multiplication of a row and the corresponding column by a constant is equivalent to multiplication of the pfaffian by the same constant. In our case $[\tilde{K}(x_i,x_j)]_{1\le i,j\le n}$ is obtained from $[K(x_i,x_j)]_{1\le i,j\le n}$ by multiplying an odd row–column pair with $e^{f(x_i)}$ and even with $e^{-f(x_i)}$. ☐

Let (X,m) be a measure space and K a scalar kernel. The Fredholm determinant of K is defined as

$$\det(I+K)_X := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \mathrm{d}m^n(x) \det([K(x_i,x_j)]_{1\le i,j\le n}) \tag{A.4}$$

where $I(x,y)=\delta_{x,y}$. For a 2×2 antisymmetric matrix kernel K the Fredholm pfaffian is defined in analogous way by

$$\mathrm{pf}(J+K)_X=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{X^n}\mathrm{d}m^n(x)\,\mathrm{pf}[K(x_i,x_j)]_{1\le i,j\le n} \tag{A.5}$$

where J is the antisymmetric matrix kernel

$$J(x,y)=\delta_{x,y}\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{A.6}$$

Fredholm pfaffians are infinite series and for a moment we postpone the convergence issue. When X is finite and m is a counting measure then $\mathrm{pf}(J+K)_X=\sum_{S\subseteq X}\mathrm{pf}\,K(S)=\mathrm{pf}(J+K)(X)$, i.e. Fredholm and ordinary pfaffians are the same. Identities involving ordinary pfaffians and determinants can usually be generalized to Fredholm analogs, e.g. $\mathrm{pf}\,(J+K)_X^2=\det(I-JK)_X$ where $I=\delta_{x,y}\begin{pmatrix}1&0\\0&1\end{pmatrix}$, or

$$\det(I+K)_X=\mathrm{pf}\left(J+\begin{pmatrix}0&K\\-K&0\end{pmatrix}\right)_X \tag{A.7}$$

where in the last identity K is a scalar kernel.

We now discuss the convergence of Fredholm pfaffians. Typically, one is interested in absolute convergence and to bound the determinant we can use Hadamard's inequality, which says that if A is an $n\times n$ complex matrix with all entries $A_{i,j}$ satisfying $|A_{i,j}|\le D$, then $|\det(A)|\le n^{n/2}D^n$.

The Fredholm determinant (A.4) is absolutely convergent if we assume that a scalar kernel K satisfies $|K(x,y)|\le D$ for all $(x,y)\in X\times X$ and $\int_X \mathrm{d}m(x)=M<\infty$. Then by Hadamard's bound

$$|\det[K(x_i,x_j)]_{1\le i,j\le n}|\le n^{n/2}D^n, \tag{A.8}$$

which gives us that (A.4) is dominated by the absolute convergent series

$$\sum_{n=0}^{\infty}\frac{n^{n/2}(DM)^n}{n!}. \tag{A.9}$$

A little bit more generally, to check that the Fredholm pfaffian is dominated by the series (A.9) it would be enough to have that the antisymmetric kernel K satisfies

$$|\,\mathrm{pf}[K(x_i,x_j)]_{1\le i,j\le n}|\le n^{n/2}D^n\prod_{i=1}^{n}f(x_i), \tag{A.10}$$

for some measurable function f such that $\int_X \mathrm{d}m(x)f(x)=M<\infty$.

We now briefly introduce point processes. For more detail see for example [Joh06] or [BO17]. Let X be a locally compact Polish space. A configuration is any subset of X with no accumulation points. Let *Conf*(X) be the set of all configurations and let $\mathbb{P}$ be a probability measure on *Conf*(X). $\mathbb{P}$ induces a Radon measure on X^n, assigning to a bounded Borel set S the expected number of n-tuples of distinct points that fall in S. If this measure is absolutely continuous, then it has the Radon–Nikodym derivative ρ_n, known as the n-point correlation function, with respect to some reference measure m. Further, for a measurable function ϕ on X with bounded support

$$\mathbb{E}[\prod_i(1+\phi(x_i))]=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{X^n}\mathrm{d}m^n(x)\prod_{i=1}^{n}\phi(x_i)\rho_n(x_1,\dots,x_n). \tag{A.11}$$

Then for a bounded Borel set $B\subseteq X$ the gap probability p_B, which is the probability that a random configuration has no intersection with B, is given by

$$p_B=\mathbb{E}[\prod_i(1-\chi_B(x_i))]=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{B^n}\mathrm{d}m^n(x)\rho_n(x_1,\dots,x_n). \tag{A.12}$$

In the case of a finite set X, the correlation function has a simple interpretation. For an n-tuple of distinct points $S = \{x_1, \dots, x_n\}$, the n-point correlation function is equal to $\mathrm{Prob}(C_S)$ where C_S is the set of all configurations that contain S. The gap probability formula says

$$p_B = \sum_{S \subseteq B} (-1)^{\#S} \mathrm{Prob}(C_S), \tag{A.13}$$

which is simply the inclusion–exclusion formula.

A point process is called pfaffian if its n-point correlation function can be written in terms of a 2×2 antisymmetric matrix kernel K, called the correlation kernel, as

$$\rho_n(x_1, x_2, \dots, x_n) = \mathrm{pf}[K(x_i, x_j)]_{1 \le i,j \le n}. \tag{A.14}$$

Then the gap probability for a bounded Borel set $B \subseteq X$ is given by the Fredholm pfaffian

$$p_B = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \mathrm{d}m^n(x)\, \mathrm{pf}[K(x_i, x_j)]_{1 \le i,j \le n} = \mathrm{pf}(J - K)_B. \tag{A.15}$$

Remark A.2. In Section 4 we investigate the gap probabilities of $B = (s_1, \infty) \times \cdots \times (s_k, \infty) \subset (\mathbb{Z}')^k$ where m is the counting measure. B is not bounded in this case, but since it can be written as the union of the bounded sets, (A.15) holds by the dominated convergence theorem as long as we can claim the the Fredholm pfaffian is absolutely convergent. To show it is absolutely convergent we use the lemma below which gives a sufficient condition for (A.10) to be satisfied. In this case we will use $f(x) = e^{-bx}$, $b > 0$ for which $\int_B \mathrm{d}m(x) e^{-bx}$ is indeed finite.

Lemma A.3. *Let* $K = \begin{pmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \end{pmatrix}$ *be a 2×2 antisymmetric kernel and $c > d \ge 0$. Let $x, y \in \mathbb{R}$ and suppose there is a $C > 0$ such that*

$$|K_{1,1}(x,y)| \le Ce^{-cx-cy} \quad |K_{1,2}(x,y)| \le Ce^{-cx+dy} \quad |K_{2,2}(x,y)| \le Ce^{dx+dy}. \tag{A.16}$$

Then there exist $b > 0$ and $D > 0$ such that

$$|\,\mathrm{pf}[K(x_i, x_j)]_{1 \le i,j \le n}| \le n^{n/2} D^n e^{-b \sum_{i=1}^n x_i}. \tag{A.17}$$

Proof. Define

$$\tilde{K}(x, x') = \begin{pmatrix} e^{cx} & 0 \\ 0 & e^{-dx} \end{pmatrix} K(x,y) \begin{pmatrix} e^{cy} & 0 \\ 0 & e^{-dy} \end{pmatrix}. \tag{A.18}$$

Let $S = \{x_1, \dots, x_n\}$. Then $|\,\mathrm{pf}\, K(S)| = e^{(d-c)\sum_{i=1}^n x_i} |\,\mathrm{pf}\, \tilde{K}(S)| \le e^{(d-c)\sum_{i=1}^n x_i} \sqrt{(2n)^n C^{2n}}$ where the inequality comes from Hadamard's bound (since $\tilde{K}(x_i, x_j) < C$). We choose $D = C\sqrt{2}$ and $b = c - d$. ∎

B Pochhammer, theta and some elementary asymptotics

For q a parameter, the q-Pochhammer symbol of length $n \in \mathbb{N} \cup \{\infty\}$ and argument x is defined as

$$(x; q)_n := \prod_{i=0}^{n-1} (1 - xq^i). \tag{B.1}$$

We observe that, for finite n, we have $(x;q)_n = (x;q)_\infty / (q^n x; q)_\infty$ and in fact this should be taken as the definition for negative integer values of n. We also define the following theta functions (multiplicative and additive respectively):

$$\theta_q(x) := (x;q)_\infty (q/x; q)_\infty, \qquad \theta_3(z;q) := \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} z^n. \tag{B.2}$$

The Jacobi triple product identity states that

$$\begin{aligned}\theta_3(z;q) &= (q;q)_\infty \prod_{i=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots} \left(1+q^i z\right)\left(1+q^i/z\right) \\ &= (q;q)_\infty \theta_q(-z\sqrt{q}).\end{aligned} \tag{B.3}$$

If $z = e^{2\pi \mathrm{i} u}$ and $q = e^{2\pi\mathrm{i}\tau}$ for τ in the upper half plane, $\theta_3(z;q)$ is sometimes denoted in the literature as $\Theta(u;\tau)$. It is a modular form of weight $1/2$, see for example [Dol98].

Finally we notice the somewhat trivial but important relations:

$$\theta_q(qx) = -\frac{1}{x}\theta_q(x), \qquad \theta_q\left(\frac{1}{x}\right) = -\frac{1}{x}\theta_q(x), \qquad \theta_q\left(\frac{x}{q}\right) = -\frac{x}{q}\theta_q(x). \tag{B.4}$$

As a matter of convention, we will use the usual hypergeometric notation for multiple Pochhammer symbols and theta functions:

$$(a,b,\ldots;q)_n := (a;q)_n \cdot (b;q)_n \cdots, \qquad \theta_q(a,b,\ldots) := \theta_q(a)\cdot\theta_q(b)\cdots. \tag{B.5}$$

An identity we will make (albeit brief) use of is Ramanujan's ${}_1\Psi_1$ summation formula [GR04]:

$$\sum_{k\in\mathbb{Z}} z^k \frac{(a;q)_k}{(b;q)_k} = \frac{(\frac{b}{a}, q, \frac{q}{az}, az; q)_\infty}{(b, \frac{b}{az}, \frac{q}{a}, z; q)_\infty}. \tag{B.6}$$

We now turn to certain limits of Pochhammer symbols (and hence of theta functions). First we will need the dilogarithm function $\mathrm{Li}_2(z)$. It is defined by the power series representation

$$\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^2}{n^2}, \qquad |z|<1 \tag{B.7}$$

with analytic continuation given by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\mathrm{d}u, \quad z\in\mathbb{C}\backslash[1,\infty). \tag{B.8}$$

Differentiating the dilogarithm, we obtain the usual logarithm:

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\mathrm{Li}_2(z) = -\log(1-z). \tag{B.9}$$

If $q = \exp(-r)$ and $r\to 0+$, we have

$$\log(z;q)_\infty \sim -\frac{\mathrm{Li}_2(z)}{r} \tag{B.10}$$

and, furthermore,

$$\log(z;q)_{\lfloor \frac{A}{r}\rfloor} \sim \frac{1}{r}(\mathrm{Li}_2(e^{-A}z) - \mathrm{Li}_2(z)) \tag{B.11}$$

where $A>0$ is fixed. Using this, one can prove that if $u(q)$ is a function that tends to $u\in(0,1)$ as $r\to 0+$, then for fixed a and b

$$\lim_{\substack{q=e^{-r},\\ r\to 0+}} \frac{(q^a u(q);q)_\infty}{(q^b u(q);q)_\infty} = (1-u)^{b-a}. \tag{B.12}$$

C Free boundary states and pure tensors

Recall that, by Lemma 3.7, the extended free boundary state can be obtained as a sum over all admissible sets with even charge. Let us introduce the sum over *all* admissible sets

$$|\widehat{v,s}\rangle := \sum_{S\in\mathcal{S}} s^{C(S)} v^{H(S)}|S\rangle = \sum_{\lambda\in\mathcal{P},c\in\mathbb{Z}} v^{|\lambda|+c^2/2} s^c |\lambda,c\rangle \tag{C.1}$$

which has finite norm for $|v|<1$, and define the covector $\langle \widehat{u,s}|$ similarly. Lemma 3.7 implies

$$|\underline{v,t}\rangle = \frac{1}{2}\left(|\widehat{v,t^{1/2}}\rangle + |\widehat{v,-t^{1/2}}\rangle\right). \tag{C.2}$$

The basic observation of this appendix is that $|\widehat{v,s}\rangle$ and $\langle\widehat{u,s}|$ can be "factorized" in the following sense.

For $k \in \mathbb{Z}'$, let $\mathcal{H}_k$ be a two-dimensional Hilbert space with orthonormal basis $\{|\circ_k\rangle, |\bullet_k\rangle\}$. and set $|0_k\rangle = |\circ_k\rangle$ for $k>0$, $|\bullet_k\rangle$ otherwise. The fermionic Fock space $\mathcal{F}$ may be identified with the infinite tensor product $\prod^{\otimes}_{k\in\mathbb{Z}'}\{\mathcal{H}_k, |0_k\rangle\}$, see e.g. [Tak03, pp. 85-87] for background on infinite tensor products of Hilbert spaces. For S an admissible set, the basis element $|S\rangle$ is identified as

$$|S\rangle = \prod^{\otimes}_{k\in\mathbb{Z}'}\left(\mathbb{1}_{k\notin S}|\circ_k\rangle + \mathbb{1}_{k\in S}|\bullet_k\rangle\right) \tag{C.3}$$

(i.e. $|\bullet_k\rangle$ represents having a "particle" at position k and $|\circ_k\rangle$ a "hole"). More generally, we may consider a pure tensor of the form

$$|T\rangle = \prod^{\otimes}_{k\in\mathbb{Z}'}\left(a_k|\circ_k\rangle + b_k|\bullet_k\rangle\right) \tag{C.4}$$

which is a well-defined element of $\mathcal{F}$ under certain conditions on the sequences a_k and b_k [Tak03, p.86, Lemma 1.7]. By (3.5), (C.3) and multilinearity, the actions of ψ_ℓ and ψ^*_ℓ on $|T\rangle$ read explicitly

$$\begin{aligned}\psi_\ell|T\rangle &= \prod^{\otimes}_{k<\ell}\left(a_k|\circ_k\rangle + b_k|\bullet_k\rangle\right)\otimes a_\ell|\bullet_\ell\rangle \otimes \prod^{\otimes}_{k>\ell}\left(a_k|\circ_k\rangle - b_k|\bullet_k\rangle\right),\\ \psi^*_\ell|T\rangle &= \prod^{\otimes}_{k<\ell}\left(a_k|\circ_k\rangle + b_k|\bullet_k\rangle\right)\otimes b_\ell|\circ_\ell\rangle \otimes \prod^{\otimes}_{k>\ell}\left(a_k|\circ_k\rangle - b_k|\bullet_k\rangle\right).\end{aligned} \tag{C.5}$$

Now, let us observe that

$$|\widehat{v,s}\rangle = \prod^{\otimes}_{k\in\mathbb{Z}'_-}\left(s^{-1}v^{-k}|\circ_k\rangle + |\bullet_k\rangle\right)\prod^{\otimes}_{k\in\mathbb{Z}'_+}\left(|\circ_k\rangle + sv^k|\bullet_k\rangle\right) \tag{C.6}$$

as seen by expanding the tensor product in the basis $|S\rangle$ and recalling the definition (3.1) of charge and energy. A similar expression for $\langle\widehat{u,s}|$ is found by duality. In particular, we readily deduce that

$$\langle\widehat{u,s}|\widehat{v,s}\rangle = \prod_{k\in\mathbb{Z}'_-}\left(1+s^{-2}(uv)^{-k}\right)\prod_{k\in\mathbb{Z}'_+}\left(1+s^2(uv)^k\right) \tag{C.7}$$

and remark in passing that, together with the other evaluation of $\langle\widehat{u,s}|\widehat{v,s}\rangle = \theta_3(s^2;uv)/(uv;uv)_\infty$ coming from (C.1), we recover the Jacobi triple product identity (B.3) with $q=uv$ and $z=s^2$. More generally, using (C.5), we obtain infinite product formulas for expressions of the form $\langle\widehat{u,s}|X|\widehat{v,s}\rangle$ with X an arbitrary product of the ψ_ℓ's and ψ^*_ℓ's. For instance, we have

$$\frac{\langle\widehat{u,s}|\psi_\ell|\widehat{v,s}\rangle}{\langle\widehat{u,s}|\widehat{v,s}\rangle} = \frac{su^\ell}{1+s^2(uv)^k}\prod_{\substack{k\in\mathbb{Z}'\\k>\ell}}\frac{1-s^2(uv)^k}{1+s^2(uv)^k} \tag{C.8}$$

and

$$\frac{\langle\widehat{u,s}|\psi_\ell\psi_{\ell'}|\widehat{v,s}\rangle}{\langle\widehat{u,s}|\widehat{v,s}\rangle} = \mathrm{sgn}(\ell-\ell')s^2u^{\ell+\ell'}\frac{\prod_{\ell'<k<\ell}\left(1-s^2(uv)^k\right)}{\prod_{\ell'\leq k\leq \ell}\left(1+s^2(uv)^k\right)}. \tag{C.9}$$

Another application of the "pure tensor formalism" is that it makes it easy to check the fermionic reflection relations (3.42). By (C.2) and duality, they follow from:

Proposition C.1. *We have*

$$s^2(v+z)\psi(z)|\widehat{v,s}\rangle = (v-z)\psi^*(v^2/z)|\widehat{v,s}\rangle \tag{C.10}$$

where both sides are well-defined (finite norm) elements of $\mathcal{F}$ for $|v|^2<|z|<1$.

Proof. Let us first consider z as a formal variable, in which case the relation amounts to having

$$s^2(v\psi_k + \psi_{k-1})|\widehat{v,s}\rangle = \left(\frac{\psi_k^*}{v^{2k-1}} + \frac{\psi_{k-1}^*}{v^{2k-2}}\right)|\widehat{v,s}\rangle \tag{C.11}$$

for all $k \in \mathbb{Z}'$. This is straightforward to check using the representation of $|\widehat{v,s}\rangle$ as a pure tensor: we may write

$$|\widehat{v,s}\rangle = C \prod_{i\in\mathbb{Z}'}^{\otimes} \frac{|\circ_i\rangle + sv^i|\bullet_i\rangle}{1+sv^i} \tag{C.12}$$

where $C := \theta_3(s;v)/(v;v)_\infty$. Using (C.5), we find that

$$s^2(v\psi_k + \psi_{k-1})|\widehat{v,s}\rangle = C \prod_{i<k-1}^{\otimes} \frac{|\circ_i\rangle + sv^i|\bullet_i\rangle}{1+sv^i} \otimes \frac{v|\circ_{k-1}\rangle \otimes |\bullet_k\rangle + |\bullet_{k-1}\rangle \otimes |\circ_k\rangle}{(1+sv^{k-1})(1+sv^k)} \otimes \prod_{i>k} \frac{|\circ_i\rangle + sv^i|\bullet_i\rangle}{1+sv^i} \tag{C.13}$$

and that the rhs of (C.11) admits the same expression.

In the case where z is a complex number, we observe that the norm of the rhs of (C.13) tends to 1 for $k \to \infty$ and decays as $|v|^{-2k}$ for $k \to -\infty$. Therefore, multiplying by z^k and summing over all $k \in \mathbb{Z}'$, we get an absolutely convergent sum for $|v|^2 < |z| < 1$, so that both sides of (C.10) are well-defined elements of $\mathcal{F}$. Note that, in contrast, $\psi(z)|\widehat{v,s}\rangle$ and $\psi^*(z)|\widehat{v,s}\rangle$ are well-defined only for $|v| < |z| < 1$, hence the extra factors $(v \pm z)$ in (C.10) are crucial to make both sides well-defined on a common annulus. $\square$

D Fermionic expectations and some hypergeometric identities

Fermionic expectations. We list some of the desired fermionic expectation values needed in the text.

Proposition D.1. *Using the notation of* (3.18) *and* (3.68)*, we have:*

$$
\begin{aligned}
\langle 0|\psi(z)\psi(w)|\underline{v}\rangle &= \frac{v^2\sqrt{zw}(z-w)}{(z+v)(w+v)(zw-v^2)}, \ \textit{for} \ \left|\frac{v}{w}\right|<1, \left|\frac{v}{z}\right|<1,\\
\langle 0|\psi(z)\psi^*(w)|\underline{v}\rangle &= \frac{\sqrt{zw}(zw-v^2)}{(z-w)(w-v)(z+v)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1, \left|\frac{w}{z}\right|<1,\\
\langle 0|\psi^*(z)\psi(w)|\underline{v}\rangle &= \frac{\sqrt{zw}(zw-v^2)}{(z-w)(w+v)(z-v)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1, \left|\frac{w}{z}\right|<1,\\
\langle 0|\psi^*(z)\psi^*(w)|\underline{v}\rangle &= \frac{v^2\sqrt{zw}(z-w)}{(z-v)(w-v)(zw-v^2)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi(z)\psi(w)|\underline{v^{er}}\rangle &= \frac{v^2(z-w)}{\sqrt{zw}(zw-v^2)}, \ \textit{for} \ \left|\frac{v^2}{zw}\right|<1,\\
\langle 0|\psi(z)\psi^*(w)|\underline{v^{er}}\rangle &= \frac{w\sqrt{w}(zw-v^2)}{\sqrt{z}(z-w)(w^2-v^2)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi^*(z)\psi(w)|\underline{v^{er}}\rangle &= \frac{z\sqrt{z}(zw-v^2)}{\sqrt{w}(z-w)(z^2-v^2)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{z}\right|<1,\\
\langle 0|\psi^*(z)\psi^*(w)|\underline{v^{er}}\rangle &= \frac{v^2zw\sqrt{zw}(z-w)}{(z^2-v^2)(w^2-v^2)(zw-v^2)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi(z)\psi(w)|\underline{v^{ec}}\rangle &= \frac{v^2zw\sqrt{zw}(z-w)}{(z^2-v^2)(w^2-v^2)(zw-v^2)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi(z)\psi^*(w)|\underline{v^{ec}}\rangle &= \frac{z\sqrt{z}(zw-v^2)}{\sqrt{w}(z-w)(z^2-v^2)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{z}\right|<1,\\
\langle 0|\psi^*(z)\psi(w)|\underline{v^{ec}}\rangle &= \frac{w\sqrt{w}(zw-v^2)}{\sqrt{z}(z-w)(w^2-v^2)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi^*(z)\psi^*(w)|\underline{v^{ec}}\rangle &= \frac{v^2(z-w)}{\sqrt{zw}(zw-v^2)}, \ \textit{for} \ \left|\frac{v^2}{zw}\right|<1,\\
\langle 0|\psi(z)\psi(w)|\underline{v^{\beta,\ or}}\rangle &= \frac{v^2\sqrt{zw}(z-w)}{(zw-v^2)(z+\beta)(w+\beta)}, \ \textit{for} \ \left|\frac{v^2}{zw}\right|<1, \left|\frac{\beta}{z}\right|<1, \left|\frac{\beta}{w}\right|<1,\\
\langle 0|\psi(z)\psi^*(w)|\underline{v^{\beta,\ or}}\rangle &= \frac{\sqrt{zw}(zw-v^2)(w+\beta)}{(z-w)(w^2-v^2)(z+\beta)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{w}\right|<1, \left|\frac{\beta}{z}\right|<1,\\
\langle 0|\psi^*(z)\psi(w)|\underline{v^{\beta,\ or}}\rangle &= \frac{\sqrt{zw}(zw-v^2)(z+\beta)}{(z-w)(z^2-v^2)(w+\beta)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{z}\right|<1, \left|\frac{\beta}{w}\right|<1,\\
\langle 0|\psi^*(z)\psi^*(w)|\underline{v^{\beta,\ or}}\rangle &= \frac{v^2\sqrt{zw}(z-w)(z+\beta)(w+\beta)}{(z^2-v^2)(w^2-v^2)(zw-v^2)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi(z)\psi(w)|\underline{v^{\alpha,\ oc}}\rangle &= \frac{v^2\sqrt{zw}(z-w)(z-\alpha)(w-\alpha)}{(z^2-v^2)(w^2-v^2)(zw-v^2)}, \ \textit{for} \ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1,\\
\langle 0|\psi(z)\psi^*(w)|\underline{v^{\alpha,\ oc}}\rangle &= \frac{\sqrt{zw}(zw-v^2)(z-\alpha)}{(z-w)(z^2-v^2)(w-\alpha)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{z}\right|<1, \left|\frac{\alpha}{w}\right|<1,\\
\langle 0|\psi^*(z)\psi(w)|\underline{v^{\alpha,\ oc}}\rangle &= \frac{\sqrt{zw}(zw-v^2)(w-\alpha)}{(z-w)(w^2-v^2)(z-\alpha)}, \ \textit{for} \ \left|\frac{w}{z}\right|<1, \left|\frac{v}{w}\right|<1, \left|\frac{\alpha}{z}\right|<1,\\
\langle 0|\psi^*(z)\psi^*(w)|\underline{v^{\alpha,\ oc}}\rangle &= \frac{v^2\sqrt{zw}(z-w)}{(zw-v^2)(z-\alpha)(w-\alpha)}, \ \textit{for} \ \left|\frac{v^2}{zw}\right|<1, \left|\frac{\alpha}{z}\right|<1, \left|\frac{\alpha}{w}\right|<1.
\end{aligned}
\tag{D.1}
$$

Proof. All of these follow easily via the boson–fermion correspondence (3.17). We write the fermionic generating series in terms of the Γ operators, we commute all the Γ_-, Γ'_- operators to the left where they act trivially on $\langle 0|$, then we reflect all the Γ_+, Γ'_+ in the free boundary states at the right using the reflection equations, and commute the result all the way to the left again until we are left with the corresponding rational factors times $\langle 0|0\rangle = 1$.

Note that we can use the first four equations and equation (3.70) to easily derive the remaining equations, e.g.

$$\langle 0|\psi(z)\psi(w)|\underline{v^{er}}\rangle = \frac{z+v}{z}\frac{w+v}{w}\langle 0|\psi(z)\psi(w)|\underline{v}\rangle. \tag{D.2}$$

We can also prove all directly by brute force (using nothing but the canonical anti-commutation relations and the definition of the basis vectors inside Fock space). As examples we derive the first and fifth equation. In the first equation the vectors in $|\underline{v}\rangle$ that give nonvanishing scalar product are $|S\rangle$ where $S = \mathbb{Z}'_{<0}\backslash\{k,l\}$ for $k,l \in \mathbb{Z}'_{<0}$ and $l \neq k$. Then

$$\begin{aligned}\langle 0|\psi(z)\psi(w)|\underline{v}\rangle &= \sum_{\substack{k,l\in\mathbb{Z}_{\geq 0}\\ l<k}} (z^{-k-1/2}w^{-l-1/2}(-1)^{k+l}v^{k+l+1} + z^{-l-1/2}w^{-k-1/2}(-1)^{k+l+1}v^{k+l+1})\\ &= \frac{v^2\sqrt{zw}}{zw-v^2}\left(-\frac{1}{z+v}+\frac{1}{w+v}\right) = \frac{v^2\sqrt{zw}(z-w)}{(z+v)(w+v)(zw-v^2)},\end{aligned} \tag{D.3}$$

assuming $|v|<|z|, |v|<|z|$.

Similarly, in the fifth equation the vectors in $|\underline{v^{er}}\rangle$ that give non vanishing scalar product are $|S\rangle$, where $S = \mathbb{Z}'_{<0}\backslash\{k,k-1\}$ for $k \in \mathbb{Z}'_{<0}$. Then

$$\begin{aligned}\langle 0|\psi(z)\psi(w)|\underline{v^{er}}\rangle &= \sum_{k\in\mathbb{Z}_{\geq 0}} (-z^{-k-3/2}w^{-k-1/2}v^{2k+2} + z^{-k-1/2}w^{-k-3/2}v^{2k+2})\\ &= \frac{v^2\sqrt{zw}}{zw-v^2}\left(-\frac{1}{z}+\frac{1}{w}\right) = \frac{v^2(z-w)}{\sqrt{zw}(zw-v^2)},\end{aligned} \tag{D.4}$$

assuming $|v^2|<|zw|$. □

We now turn to two free boundaries. Recall the notation

$$\begin{aligned}\langle \underline{u,t}| &= \sum_{\lambda\in\mathcal{P},\ c\in 2\mathbb{Z}} u^{|\lambda|+\frac{c^2}{2}}t^{c/2}\langle\lambda,c| = \sum_{c\in 2\mathbb{Z}} u^{\frac{c^2}{2}}t^{c/2}\langle\underline{u}|R^{-c},\\ |\underline{v,t}\rangle &= \sum_{\lambda\in\mathcal{P},\ c\in 2\mathbb{Z}} v^{|\lambda|+\frac{c^2}{2}}t^{c/2}|\lambda,c\rangle = \sum_{c\in 2\mathbb{Z}} v^{\frac{c^2}{2}}t^{c/2}R^c|\underline{v}\rangle.\end{aligned} \tag{D.5}$$

The following two free boundary analogue of Proposition D.1 holds.

Proposition D.2. *We have:*

$$\begin{aligned}
\langle\underline{u,t}|\psi(z)\psi(w)|\underline{v,t}\rangle &= \frac{((uv)^2;(uv)^2)^2_\infty\theta_{(uv)^2}(\frac{w}{z})}{(-\frac{v}{z},-\frac{v}{w},uz,uw;uv)_\infty\theta_{(uv)^2}(u^2zw)}\cdot\frac{\theta_3\left((\frac{tzw}{v^2})^2;(uv)^4\right)}{(uv;uv)_\infty}\cdot\frac{v^2}{tw\sqrt{zw}},\\
&\quad for\ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1, |uz|<1, |uw|<1, |uv|<1,\\
\langle\underline{u,t}|\psi(z)\psi^*(w)|\underline{v,t}\rangle &= \frac{((uv)^2;(uv)^2)^2_\infty\theta_{(uv)^2}(u^2zw)}{(-\frac{v}{z},\frac{v}{w},uz,-uw;uv)_\infty\theta_{(uv)^2}(\frac{w}{z})}\cdot\frac{\theta_3\left((\frac{tz}{w})^2;(uv)^4\right)}{(uv;uv)_\infty}\cdot\sqrt{\frac{w}{z}},\\
&\quad for\ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1, |uz|<1, |uw|<1, \left|\frac{w}{z}\right|<1, |uv|<1,\\
\langle\underline{u,t}|\psi^*(z)\psi(w)|\underline{v,t}\rangle &= \frac{((uv)^2;(uv)^2)^2_\infty\theta_{(uv)^2}(u^2zw)}{(\frac{v}{z},-\frac{v}{w},-uz,uw;uv)_\infty\theta_{(uv)^2}(\frac{w}{z})}\cdot\frac{\theta_3\left((\frac{tw}{z})^2;(uv)^4\right)}{(uv;uv)_\infty}\cdot\sqrt{\frac{z}{w}},\\
&\quad for\ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1, |uz|<1, |uw|<1, \left|\frac{w}{z}\right|<1, |uv|<1,\\
\langle\underline{u,t}|\psi^*(z)\psi^*(w)|\underline{v,t}\rangle &= \frac{((uv)^2;(uv)^2)^2_\infty\theta_{(uv)^2}(\frac{w}{z})}{(\frac{v}{z},\frac{v}{w},-uz,-uw;uv)_\infty\theta_{(uv)^2}(u^2zw)}\cdot\frac{\theta_3\left((\frac{tv^2}{zw})^2;(uv)^4\right)}{(uv;uv)_\infty}\cdot\frac{v^2t}{w\sqrt{zw}},\\
&\quad for\ \left|\frac{v}{z}\right|<1, \left|\frac{v}{w}\right|<1, |uz|<1, |uw|<1, |uv|<1.
\end{aligned} \tag{D.6}$$

Proof. Once we use the boson–fermion correspondence (3.17), the computations are identical to the ones in the proof of Proposition 2.2. We use the same ping-pong argument to eliminate the Γ operators (bouncing Γ_+/Γ'_+ towards the left and reflecting, bouncing Γ_-/Γ'_- towards the right and reflecting, infinitely often) and then we are left with simple expectation values that evaluate to theta functions. We outline the computation of the $\psi\psi$ correlator:

$$\begin{aligned}\langle \underline{u,t}|\psi(z)\psi(w)|\underline{v,t}\rangle &= \langle \underline{u,t}|z^{C-\frac{1}{2}}R\Gamma_-(z)\Gamma'_+\left(-\frac{1}{z}\right)w^{C-\frac{1}{2}}R\Gamma_-(w)\Gamma'_+\left(-\frac{1}{w}\right)|\underline{v,t}\rangle \\ &= \frac{((uv)^2;(uv)^2)^2_\infty \theta_{(uv)^2}(\frac{w}{z})}{(-\frac{v}{z},-\frac{v}{w},uz,uw;uv)_\infty \theta_{(uv)^2}(u^2zw)} \cdot \langle \underline{u,t}|z^{C-\frac{1}{2}}Rw^{C-\frac{1}{2}}R|\underline{v,t}\rangle\end{aligned} \tag{D.7}$$

where the theta and Pochhammer factors come from eliminating the Γ operators via the ping-pong argument, and the remaining inner product can be evaluated as

$$\begin{aligned}\langle \underline{u,t}|z^{C-\frac{1}{2}}Rw^{C-\frac{1}{2}}R|\underline{v,t}\rangle &= \frac{1}{\sqrt{zw}}\sum_{\substack{\lambda,\mu\in\mathcal{P}\\ c,d\in 2\mathbb{Z}}} u^{|\lambda|+c^2/2}t^{c/2}v^{|\mu|+d^2/2}t^{d/2}\langle\lambda,c|z^C Rw^C R|\mu,d\rangle \\ &= \frac{1}{\sqrt{zw}}\sum_{\substack{\lambda,\mu\in\mathcal{P}\\ c,d\in 2\mathbb{Z}}} u^{|\lambda|+c^2/2}t^{c/2}v^{|\mu|+d^2/2}t^{d/2}w^{d+1}z^{d+2}\langle\lambda,c|\mu,d+2\rangle \\ &= \frac{v^2}{tw\sqrt{zw}}\sum_{\lambda\in\mathcal{P}}(uv)^{|\lambda|}\sum_{c\in 2\mathbb{Z}}(uv)^{c^2/2}\left(\frac{tzw}{v^2}\right)^c \\ &= \frac{v^2}{tw\sqrt{zw}}\cdot\frac{\theta_3((\frac{tzw}{v^2})^2;(uv)^4)}{(uv;uv)_\infty}\end{aligned} \tag{D.8}$$

where we have used the orthogonality relation $\langle\lambda,c|\mu,d+2\rangle = \delta_{\lambda,\mu}\delta_{c,d+2}$. The other three correlators are computed along similar lines. □

Remark D.3. We can also compute other two free boundary correlators: using the notation from Section 3.3.2, we can compute for instance

$$\langle \underline{u,t}|\psi(z)\psi^*(w)|\underline{v^{\alpha,\ oc},t}\rangle \tag{D.9}$$

by first using (3.72) in conjunction with the arguments above. We omit such computations for brevity.

Hypergeometric identities. The pure tensor formalism of Appendix C provides another approach to computing the correlators in Proposition D.2. It is instructive as the two approaches combined lead to interesting multivariate (in the present case, bivariate) bilateral basic hypergeometric summation formulae.

Note first that, by (C.2), we may consider expectation values between $\langle\widehat{u,s}|$ and $|\widehat{v,s}\rangle$, then deduce those between $\langle\underline{u,t}|$ and $|\underline{v,t}\rangle$ by bilinearity. For the remainder of this section, let us denote, for A an operator acting on $\mathcal{F}$,

$$\langle A\rangle := \frac{\langle\widehat{u,s}|A|\widehat{v,s}\rangle}{\langle\widehat{u,s}|\widehat{v,s}\rangle}. \tag{D.10}$$

Let us compute $\langle\psi(z)\psi(w)\rangle$ in two different ways. On the one hand, we may write

$$\langle\psi(z)\psi(w)\rangle = \sum_{\ell,\ell'\in\mathbb{Z}'} z^\ell w^{\ell'}\langle\psi_\ell\psi_{\ell'}\rangle \tag{D.11}$$

and rewrite (C.9) in the form

$$\begin{aligned}\langle\psi_\ell\psi_{\ell'}\rangle &= \operatorname{sgn}(\ell-\ell')s^2u^{\ell+\ell'}\frac{(s^2(uv)^{\ell'+1},-s^2(uv)^{\ell+1};uv)_\infty}{(s^2(uv)^\ell,-s^2(uv)^{\ell'};uv)_\infty} \\ &= \operatorname{sgn}(L-L')s^2u^{L+L'+1}\frac{(s^2\sqrt{uv};uv)_L(-s^2\sqrt{uv};uv)_{L'}}{(s^2uv\sqrt{uv};uv)_{L'}(-s^2uv\sqrt{uv};uv)_L}\end{aligned} \tag{D.12}$$

where we set $\ell = L + 1/2$, $\ell' = L' + 1/2$ so that the Pochhammer symbols in the third line have integer length. On the other hand, we may proceed as in the proof of Proposition D.2, writing

$$|\widehat{v,s}\rangle = \sum_{c\in\mathbb{Z}} s^c v^{c^2/2} R^c |\underline{v}\rangle, \qquad \langle\widehat{u,s}| = \sum_{c\in\mathbb{Z}} s^c v^{c^2/2} R^c \langle\underline{u}| \tag{D.13}$$

and using again the ping-pong method. Doing this for both $\langle\psi(z)\psi(w)\rangle$ and $\langle\psi(z)\psi^*(w)\rangle$ (for brevity we omit the computation of $\langle\psi_\ell\psi^*_{\ell'}\rangle$), we arrive at the following bivariate bilateral basic hypergeometric summation formulae.

Theorem D.4. *We have:*

$$\begin{aligned} &s^2 u\sqrt{zw} \sum_{-\infty<L'<L<\infty} (z^L w^{L'} - z^{L'} w^L) u^{L+L'} \frac{(s^2\sqrt{uv};uv)_L(-s^2\sqrt{uv};uv)_{L'}}{(s^2uv\sqrt{uv};uv)_{L'}(-s^2uv\sqrt{uv};uv)_L} = \\ &\quad \frac{((uv)^2;(uv)^2)^2_\infty \theta_{(uv)^2}(\frac{w}{z})}{(uz,uw,-\frac{v}{z},-\frac{v}{w};uv)_\infty} \cdot \frac{v^2}{s^2 w\sqrt{zw}} \cdot \frac{\theta_3(\frac{s^2zw}{v^2};uv)}{\theta_3(s^2;uv)} \end{aligned} \tag{D.14}$$

and

$$\begin{aligned} &s^2\sqrt{zuv/w} \sum_{-\infty<L'<L<\infty} ((zu)^L(v/w)^{L'} + (zu)^{L'}(v/w)^L) \frac{(s^2\sqrt{uv};uv)_L(-s^2\sqrt{uv};uv)_{L'}}{(s^2uv\sqrt{uv};uv)_{L'}(-s^2uv\sqrt{uv};uv)_L} \\ &+ \frac{s^2\sqrt{zuv/w}}{1+s^2\sqrt{uv}} \sum_{L\in\mathbb{Z}} (zuv/w)^L \frac{(-s^2\sqrt{uv};uv)_L}{(-s^2uv\sqrt{uv};uv)_L} = \\ &\quad \frac{((uv)^2;(uv)^2)^2_\infty \theta_{(uv)^2}(u^2zw)}{(-\frac{v}{z},\frac{v}{w},uz,-uw;uv)_\infty \theta_{(uv)^2}(\frac{w}{z})} \cdot \sqrt{\frac{w}{z}} \cdot \frac{\theta_3(\frac{s^2z}{w};uv)}{\theta_3(s^2;uv)}. \end{aligned} \tag{D.15}$$

Using (C.8), we can also compute $\langle\psi(z)\rangle$ in two different ways to obtain the following bilateral basic hypergeometric summation formula.

Proposition D.5. *We have:*

$$\sum_{k\in\mathbb{Z}'} \frac{s(zu)^k}{1+s^2(uv)^k} \cdot \frac{(s^2(uv)^{k+1};uv)_\infty}{(-s^2(uv)^{k+1};uv)_\infty} = \frac{((uv)^2;(uv)^2)_\infty}{(uz,-\frac{v}{z};uv)_\infty} \cdot \frac{1}{s}\sqrt{\frac{v}{z}} \cdot \frac{\theta_3(\frac{s^2z}{v};uv)}{\theta_3(s^2;uv)}. \tag{D.16}$$

Remark D.6. Using a bit of q-Pochhammer symbol gymnastics with $q = uv$ throughout, one can rearrange equation (D.16) to look like

$$\sum_{k\in\mathbb{Z}} (zu)^k \frac{(-s^2/\sqrt{q};q)_k}{(s^2\sqrt{q};q)_k} = \frac{(q,-q,-\frac{s^2uz}{\sqrt{q}},-\frac{q\sqrt{q}}{s^2uz};q)_\infty}{(uz,-\frac{q}{uz},s^2\sqrt{2},-\frac{q\sqrt{q}}{s^2};q)_\infty} \tag{D.17}$$

thus becoming a special case of Ramanujan's ${}_1\Psi_1$ summation formula (B.6).

Remark D.7. If instead of computing $\langle\psi(z)\rangle$ two different ways, we compute $\langle\psi^*(w)\rangle$, we arrive at the same formula as above up to switching u and v and changing z into $1/w$. Indeed one could have furthermore looked at the other two 2-point correlators $\langle\psi^*(z)\psi(w)\rangle$ and $\langle\psi^*(z)\psi^*(w)\rangle$ but this would have lead to nothing new.

Remark D.8. The proofs of the identities obtained so far are combinatorial. Once one applies the boson–fermion correspondence (itself lending to a combinatorial interpretation — see [BCC17]), the product side of the identities is obtained by applying the commutation between Γ_+ and Γ_- and the reflection equations the Γ's satisfy at the free boundaries (all are applied "infinitely often"). Both of these relations have interpretations in terms of bijective Fomin growth diagrams and the RSK correspondence — see [BCC17], [BBB+15] and references therein. In some sense the hypergeometric identities thus obtained already come with combinatorial proofs. We plan to address this in more detail in a future note, but see [Cor03] for related bijections proving the univariate identities mentioned so far.

Remark D.9. Instead of univariate/bivariate hypergeometric sums, one can obtain truly multivariate ones by considering correlators of the sort $\langle\psi(z_1)\dots\psi(z_n)\rangle$ or $\langle\psi(z_1)\psi^*(w_1)\dots\psi(z_n)\psi^*(w_n)\rangle$ and computing them two different ways. We have not explored this avenue, but plan to do so in the future.

We conclude this section with a combinatorial proof of a pfaffian evaluation, a version of Okada's [Oka06] pfaffian version of the Frobenius determinant ([Fro82]; see also Lemma 4.3 in [Rai10]).

Proposition D.10. *We have:*

$$\begin{aligned}\operatorname*{pf}_{1\leq i<j\leq 2n} & \frac{\theta_{(uv)^2}(z_j/z_i)\theta_3((tu^2z_iz_j)^2;(uv)^4)}{z_j\theta_{(uv)^2}(u^2z_iz_j)\theta_3(t^2;(uv)^4)} = \\ & = u^{2n(n-1)}\frac{\theta_3((t\prod_{1\leq i\leq 2n}z_i)^2u^{4n};(uv)^4)}{\theta_3(t^2;(uv)^4)}\prod_{i=1}^{2n}z_i^{2n-i-1}\prod_{1\leq i<j\leq 2n}\frac{\theta_{(uv)^2}(z_j/z_i)}{\theta_{(uv)^2}(u^2z_iz_j)}.\end{aligned} \tag{D.18}$$

Proof. Wick's lemma for two free boundaries 3.10 yields

$$\operatorname*{pf}_{1\leq i<j\leq 2n}\frac{\langle\underline{u,t}|\psi(z_i)\psi(z_j)|\underline{v,t}\rangle}{\langle\underline{u,t}|\underline{v,t}\rangle} = \frac{\langle\underline{u,t}|\psi(z_1)\dots\psi(z_{2n})|\underline{v,t}\rangle}{\langle\underline{u,t}|\underline{v,t}\rangle} \tag{D.19}$$

and we can compute both sides using the ping-pong method. The RHS evaluates to a product, while the entries in the pfaffian on the LHS can also be evaluated explicitly. Cancelling out univariate factors on both sides yields the result. □

Remark D.11. To make the connection with Okada's pfaffian explicit (Theorem 1.1 in [Oka06]), write $u = \exp(2\pi\mathrm{i}\Upsilon), v = \exp(2\pi\mathrm{i}\Xi), z_k = \exp(2\pi\mathrm{i}\zeta_k), t = \exp(2\pi\mathrm{i}\tau)$. We then have

$$x_k^{\mathrm{Okada}} = \Upsilon+\zeta_k, \qquad z^{\mathrm{Okada}} = \frac{\pi}{4}+\tau, \qquad w^{\mathrm{Okada}} = \frac{3\pi}{4}+\tau \tag{D.20}$$

while the lattice (elliptic curve) Λ used in op. cit. is generated, in our notation, by 1 and $2(\Upsilon+\Xi)$. All follows from writing the ratio of the two θ_3 functions in the pfaffian as $\theta_{(uv)^2}(\mathrm{i}tu^2z_iz_j, -\mathrm{i}tu^2z_iz_j)/\theta_{(uv)^2}(\mathrm{i}t, -\mathrm{i}t)$.

Remark D.12. When $v = 0$ (and we can put $u = 1$ without loss of generality) this becomes a pfaffian evaluation due to Laksov–Lascoux–Thorup and Stembridge [LLT89, Ste90] (notice that the present paper can be seen as a generalization of results in [Ste90] to the case of two free boundaries):

$$\operatorname*{pf}_{1\leq i<j\leq 2n}\frac{z_i-z_j}{1-z_iz_j} = \prod_{1\leq i<j\leq 2n}\frac{z_i-z_j}{1-z_iz_j}. \tag{D.21}$$

E Extra figures used in Section 5

In this Section we draw the contours omitted from Section 5 due to text flow considerations.

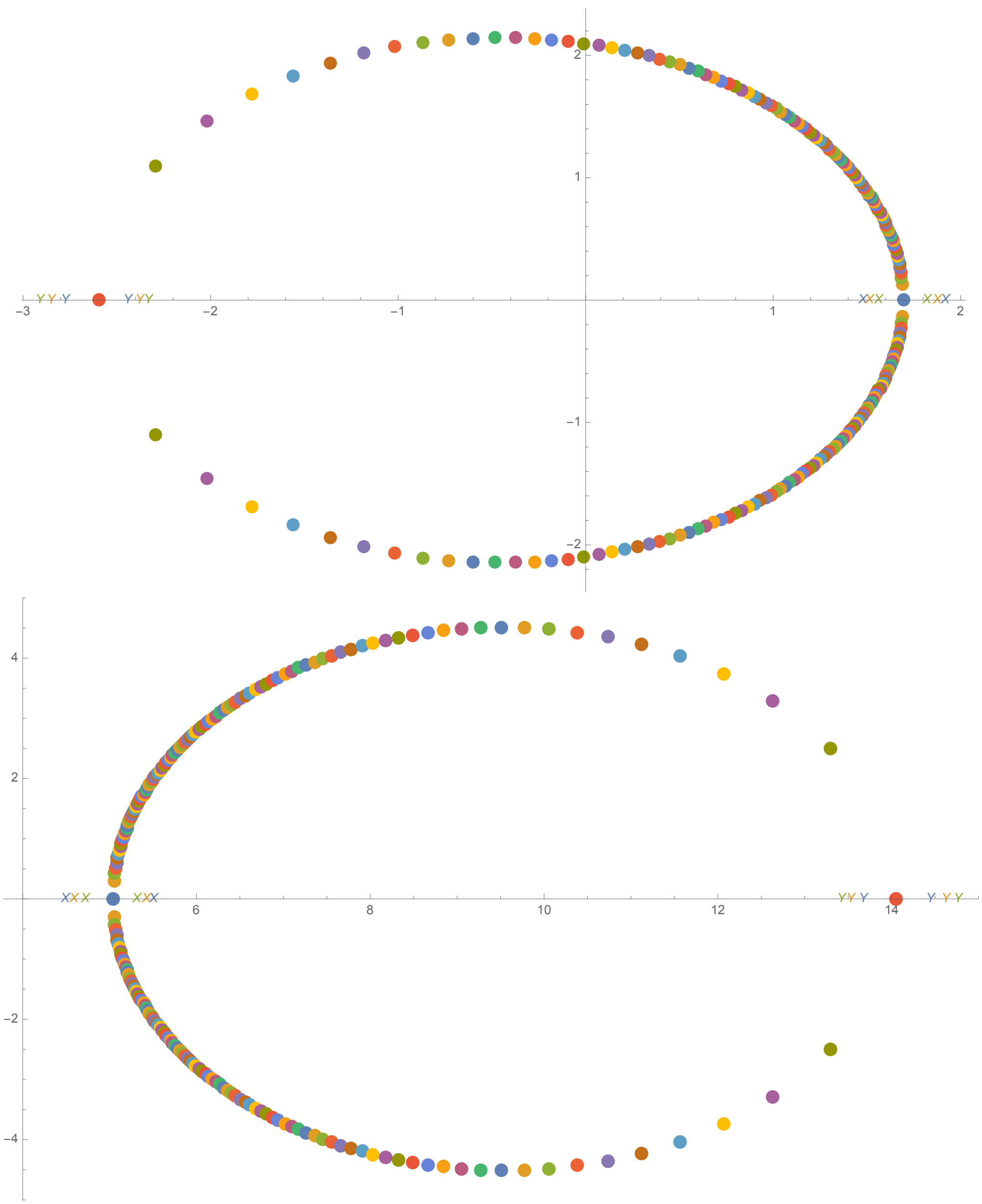

Figure 17: The critical points of S, for $a = 2$ and for $\chi = 1 < \chi_T$ (top), $\chi = 1.97 > \chi_T$ (bottom) fixed, as y varies from just above y_+ to just below y_-; they are distinct real for $y > y_+$ (denoted by an X) approaching the double critical point at $y = y_+$; they are complex conjugate for $y_+ > y > y_-$ approaching the double critical point at $y = y_-$ and they become distinct real again (denoted by Y) diverging from said double critical point as y decreases below y_-.

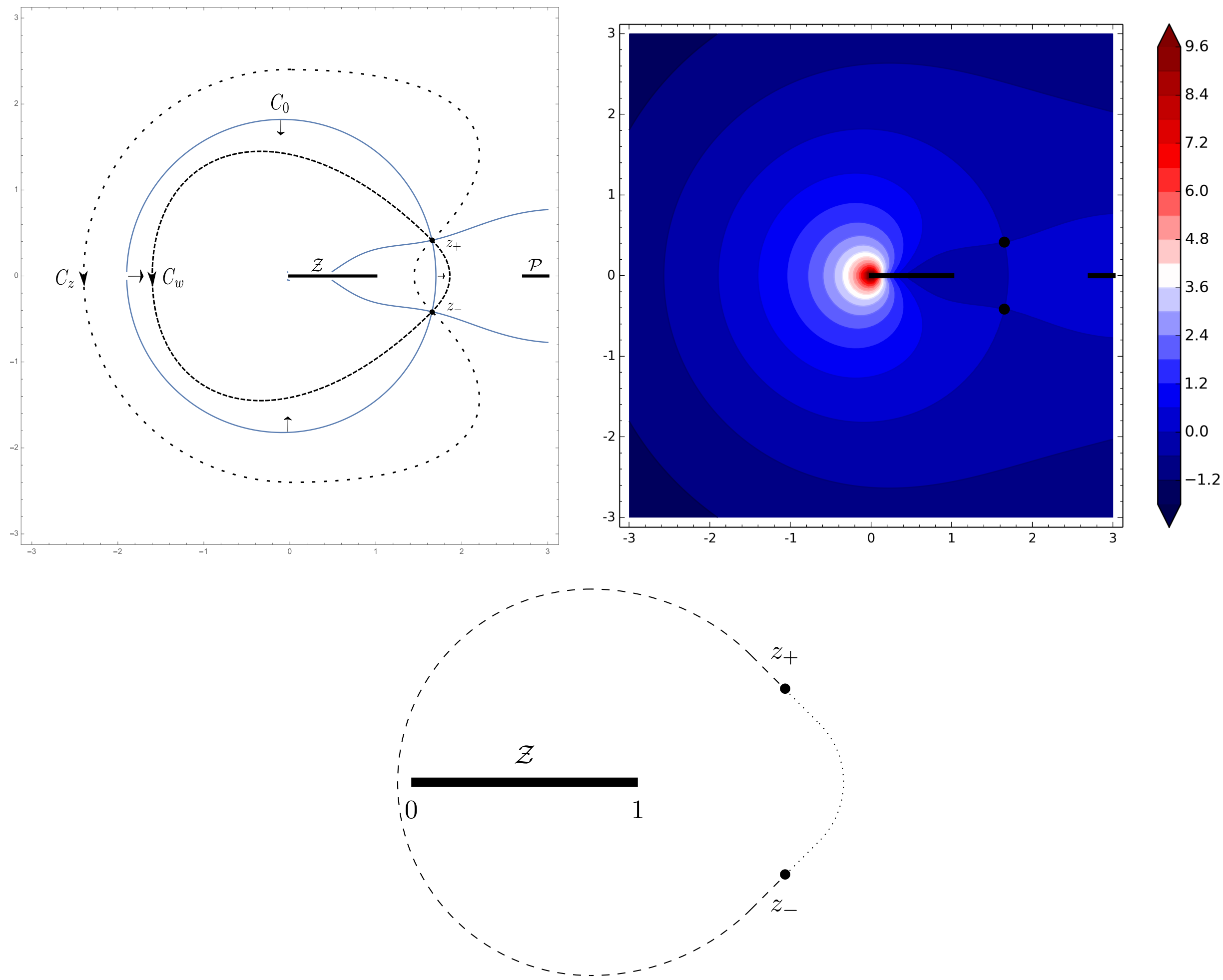

Figure 18: Top: the curve $\Re(S(z)-S(z_+))=0$ (left) with arrows pointing towards the direction of ascent and the final z and w contours from the proof of Theorem 5.6; some level lines for $\Re(S(z)-S(z_+))$ (right). Here $z_+=\overline{z_-}\approx 1.65+0.41\mathrm{i}$ is one of the complex conjugate critical points of S, for $a=2, \chi=1<\chi_T, y=y_+-0.2$. We remark (χ, y) is in the liquid region $\mathcal{L}$. The two original contours (z contour on the outside) are also drawn, along with the complex conjugate critical points z_+, z_- (the dots). Bottom: the contours γ_+ (dotted) and γ_- (dashed) from Theorem 5.6.

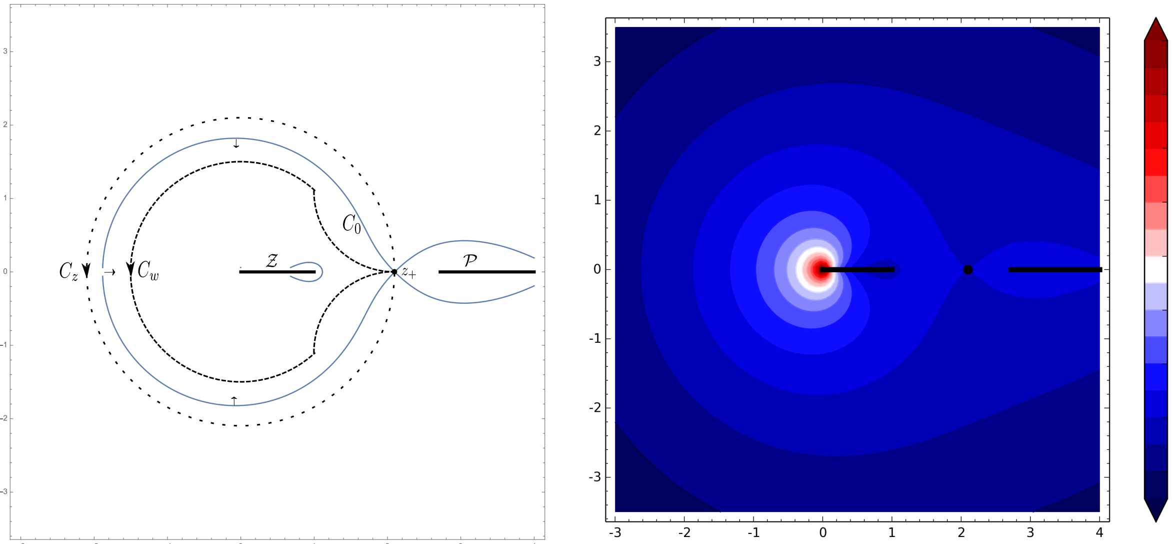

Figure 19: The curve $\Re(S(z) - S(z_+)) = 0$ (left) with arrows pointing towards the direction of ascent and the final z and w contours from the proof of Theorem 5.6; some level lines for $\Re(S(z) - S(z_+))$ (right). Here $z_+ \approx 2.09$ is one of the distinct real critical points of S, for $a = 2, \chi = 1 < \chi_T, y = y_+ + 0.2$. We remark (χ, y) is above the arctic curve.

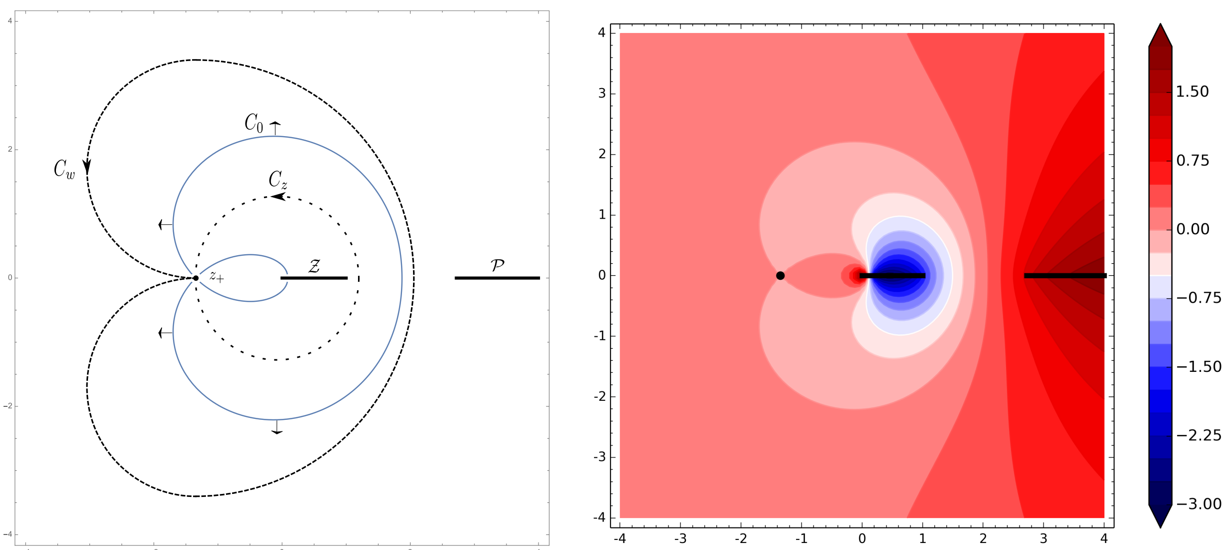

Figure 20: The curve $\Re(S(z) - S(z_+)) = 0$ (left) with arrows pointing towards the direction of ascent and the final z and w contours from the proof of Theorem 5.6; some level lines for $\Re(S(z) - S(z_+))$ (right). Here $z_+ \approx -1.34$ is one of the distinct real critical points of S, for $a = 2, \chi = 1 < \chi_T, y = y_+ - 0.05$. We remark (χ, y) is below the arctic curve but above the line $\chi = a + y$.

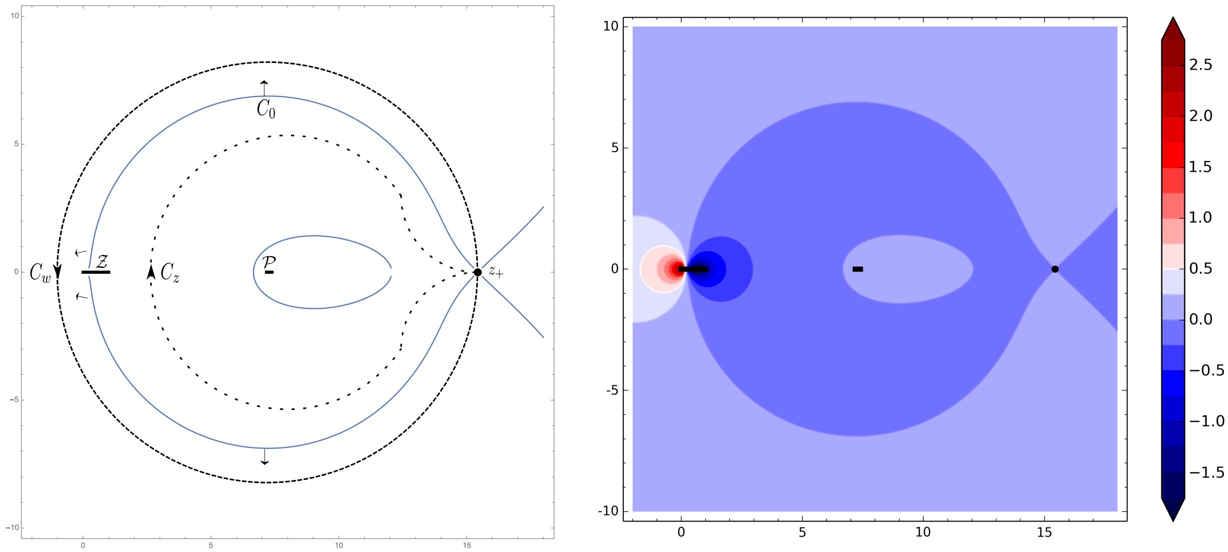

Figure 21: The curve $\Re(S(z) - S(z_+)) = 0$ (left) with arrows pointing towards the direction of ascent and the final z and w contours from the proof of Theorem 5.6; some level lines for $\Re(S(z) - S(z_+))$ (right). Here $z_+ \approx 15.43$ is one of the distinct real critical points of S, for $\mathsf{a} = 2, \chi = 1.97 > \chi_T, \mathsf{y} = \mathsf{y}_- - 0.0005$. We remark (χ, y) is below the arctic curve but above the line $\chi = \mathsf{a} + \mathsf{y}$.

References

[AESW51] M. Aissen, A. Edrei, I. J. Schoenberg, and A. Whitney. On the generating functions of totally positive sequences. *Proc. Nat. Acad. Sci. U. S. A.*, 37:303–307, 1951.

[AZ13] A. Alexandrov and A. Zabrodin. Free fermions and tau-functions. *J. Geom. Phys.*, 67:37–80, 2013, arXiv:1212.6049 [math-ph].

[BBB+15] D. Betea, C. Boutillier, J. Bouttier, G. Chapuy, S. Corteel, and M. Vuletić. Perfect sampling algorithm for Schur processes, 2015, arXiv:1407.3764 [math.PR].

[BBC+16] C. Boutillier, J. Bouttier, G. Chapuy, S. Corteel, and S. Ramassamy. Dimers on rail yard graphs. *Ann. Inst. Henri Poincaré D*, to appear, 2016, arXiv:1504.05176 [math-ph].

[BBCS17a] J. Baik, G. Barraquand, I. Corwin, and T. Suidan. Facilitated exclusion process, 2017, arXiv:1707.01923 [math.PR].

[BBCS17b] J. Baik, G. Barraquand, I. Corwin, and T. Suidan. Pfaffian Schur processes and last passage percolation in a half-quadrant. *Ann. Probab.*, to appear, 2017, arXiv:1606.00525v3 [math.PR].

[BBCW17] G. Barraquand, A. Borodin, I. Corwin, and M. Wheeler. Stochastic six-vertex model in a half-quadrant and half-line open ASEP, 2017, arXiv:1704.04309 [math.PR].

[BCC17] J. Bouttier, G. Chapuy, and S. Corteel. From Aztec diamonds to pyramids: steep tilings. *Trans. Amer. Math. Soc.*, 369(8):5921–5959, 2017, arXiv:1407.0665 [math.CO].

[BDJ99] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999, arXiv:math/9810105 [math.CO].

[BG16] A. Borodin and V. Gorin. Lectures on integrable probability. In *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 155–214. Amer. Math. Soc., Providence, RI, 2016, arXiv:1212.3351 [math.PR].

[BMRT12] C. Boutillier, S. Mkrtchyan, N. Reshetikhin, and P. Tingley. Random skew plane partitions with a piecewise periodic back wall. *Ann. Henri Poincaré*, 13(2):271–296, 2012, arXiv:0912.3968 [math-ph].

[BO17] A. Borodin and G. Olshanski. The ASEP and determinantal point processes. *Comm. Math. Phys.*, 353(2):853–903, 2017, arXiv:1608.01564 [math-ph].

[Bor07] A. Borodin. Periodic Schur process and cylindric partitions. *Duke Math. J.*, 140(3):391–468, 2007, arXiv:math/0601019 [math.CO].

[BP08] A. Borodin and S. Péché. Airy kernel with two sets of parameters in directed percolation and random matrix theory. *J. Stat. Phys.*, 132(2):275–290, 2008, arXiv:0712.1086 [math-ph].

[BR99] J. Baik and E. M. Rains. Symmetrized random permutations, 1999, arXiv:math/9910019 [math.CO].

[BR01a] J. Baik and E. M. Rains. Algebraic aspects of increasing subsequences. *Duke Math. J.*, 109(1):1–65, 2001, arXiv:math/9905083 [math.CO].

[BR01b] J. Baik and E. M. Rains. The asymptotics of monotone subsequences of involutions. *Duke Math. J.*, 109(2):205–281, 2001, arXiv:math/9905084 [math.CO].

[BR05] A. Borodin and E. M. Rains. Eynard-Mehta theorem, Schur process, and their Pfaffian analogs. *J. Stat. Phys.*, 121(3-4):291–317, 2005, arXiv:math-ph/0409059.

[CFP10] I. Corwin, P. L. Ferrari, and S. Péché. Limit processes for TASEP with shocks and rarefaction fans. *J. Stat. Phys.*, 140(2):232–267, 2010, arXiv:1002.3476 [math-ph].

[CFP12] I. Corwin, P. L. Ferrari, and S. Péché. Universality of slow decorrelation in KPZ growth. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(1):134–150, 2012, arXiv:1001.5345 [math.PR].

[CK01] R. Cerf and R. Kenyon. The low-temperature expansion of the Wulff crystal in the 3D Ising model. *Comm. Math. Phys.*, 222(1):147–179, 2001.

[CK11] M. Ciucu and C. Krattenthaler. The interaction of a gap with a free boundary in a two dimensional dimer system. *Comm. Math. Phys.*, 302(1):253–289, 2011, arXiv:0912.2023 [math.CO].

[Cor03] S. Corteel. Particle seas and basic hypergeometric series. *Adv. in Appl. Math.*, 31(1):199–214, 2003.

[CSV11] S. Corteel, C. Savelief, and M. Vuletić. Plane overpartitions and cylindric partitions. *J. Combin. Theory Ser. A*, 118(4):1239–1269, 2011, arXiv:0903.2039 [math.CO].

[DFR12] P. Di Francesco and N. Reshetikhin. Asymptotic shapes with free boundaries. *Comm. Math. Phys.*, 309(1):87–121, 2012, arXiv:0908.1630 [math-ph].

[Dol98] I. Dolgachev. Lectures on modular forms. Available online at `http://www.math.lsa.umich.edu/~idolga/ModularBook.pdf`, 1998.

[FN17] P. L. Ferrari and P. Nejjar. Fluctuations of the competition interface in presence of shocks. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(1):299–325, 2017, arXiv:1603.07498 [math-ph].

[Fro82] G. Frobenius. Ueber die elliptischen Functionen zweiter Art. *J. Reine Angew. Math.*, 93:53–68, 1882.

[FS03] P. L. Ferrari and H. Spohn. Step fluctuations for a faceted crystal. *J. Statist. Phys.*, 113(1-2):1–46, 2003, arXiv:cond-mat/0212456 [cond-mat.stat-mech].

[Gho17] P. Ghosal. Correlation functions of the Pfaffian Schur process using Macdonald difference operators, 2017, arXiv:1705.05859 [math.PR].

[GR04] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.

[JM83] M. Jimbo and T. Miwa. Solitons and infinite-dimensional Lie algebras. *Publ. Res. Inst. Math. Sci.*, 19(3):943–1001, 1983.

[Joh00] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000, arXiv:math/9903134 [math.CO].

[Joh06] K. Johansson. Random matrices and determinantal processes. In *Mathematical statistical physics*, pages 1–55. Elsevier B. V., Amsterdam, 2006, arXiv:math-ph/0510038.

[Kac90] V. G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[KO06] R. Kenyon and A. Okounkov. Planar dimers and Harnack curves. *Duke Math. J.*, 131(3):499–524, 2006, arXiv:math/0311062 [math.AG].

[KO07] R. Kenyon and A. Okounkov. Limit shapes and the complex Burgers equation. *Acta Math.*, 199(2):263–302, 2007, arXiv:math-ph/0507007.

[KOS06] R. Kenyon, A. Okounkov, and S. Sheffield. Dimers and amoebae. *Ann. of Math. (2)*, 163(3):1019–1056, 2006, arXiv:math-ph/0311005.

[LLT89] D. Laksov, A. Lascoux, and A. Thorup. On Giambelli's theorem on complete correlations. *Acta Math.*, 162(3-4):143–199, 1989.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[MJD00] T. Miwa, M. Jimbo, and E. Date. *Solitons*, volume 135 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000. Differential equations, symmetries and infinite-dimensional algebras, Translated from the 1993 Japanese original by Miles Reid.

[Nej17] P. Nejjar. Transition to shocks in TASEP and Decoupling of Last Passage Times, 2017, arXiv:1705.08836 [math.PR].

[NHB84] B. Nienhuis, H. J. Hilhorst, and H. W. J. Blöte. Triangular SOS models and cubic-crystal shapes. *J. Phys. A*, 17(18):3559–3581, 1984.

[Oka06] S. Okada. An elliptic generalization of Schur's Pfaffian identity. *Adv. Math.*, 204(2):530–538, 2006, arXiv:math/0412038 [math.CA].

[Oko01] A. Okounkov. Infinite wedge and random partitions. *Selecta Math. (N.S.)*, 7(1):57–81, 2001, arXiv:math/9907127 [math.RT].

[OQR17] J. Ortmann, J. Quastel, and D. Remenik. A Pfaffian representation for flat ASEP. *Comm. Pure Appl. Math.*, 70(1):3–89, 2017, arXiv:1501.05626 [math.PR].

[OR03] A. Okounkov and N. Reshetikhin. Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. *J. Amer. Math. Soc.*, 16(3):581–603 (electronic), 2003, arXiv:math/0107056 [math.CO].

[OR07] A. Okounkov and N. Reshetikhin. Random skew plane partitions and the Pearcey process. *Comm. Math. Phys.*, 269(3):571–609, 2007, arXiv:math/0503508 [math.CO].

[Pan15] G. Panova. Lozenge tilings with free boundaries. *Lett. Math. Phys.*, 105(11):1551–1586, 2015, arXiv:1408.0417 [math.PR].

[Rai00] E. M. Rains. Correlation functions for symmetrized increasing subsequences, 2000, arXiv:math/0006097 [math.CO].

[Rai10] E. M. Rains. Transformations of elliptic hypergeometric integrals. *Ann. of Math. (2)*, 171(1):169–243, 2010, arXiv:math/0309252 [math.QA].

[SI04] T. Sasamoto and T. Imamura. Fluctuations of the one-dimensional polynuclear growth model in half-space. *J. Statist. Phys.*, 115(3-4):749–803, 2004, arXiv:cond-mat/0307011 [cond-mat.stat-mech].

[Sta99] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ste90] J. R. Stembridge. Nonintersecting paths, Pfaffians, and plane partitions. *Adv. Math.*, 83(1):96–131, 1990.

[Tak03] M. Takesaki. *Theory of operator algebras. III*, volume 127 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 8.

[Tho64] E. Thoma. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Math. Z.*, 85:40–61, 1964.

[TW94] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159(1):151–174, 1994, arXiv:hep-th/9211141.

[Vul07] M. Vuletić. The shifted Schur process and asymptotics of large random strict plane partitions. *Int. Math. Res. Not. IMRN*, (14):Art. ID rnm043, 53, 2007, arXiv:math-ph/0702068.