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Modal expansion in elastic open waveguides with perfectly matched layers

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**Abstract**

The modal basis of elastic open waveguides contains two continua of radiation modes and a discrete set of trapped modes. A third set also exists: the leaky modes. However, they do not belong to the modal basis as they spatially grow to infinity. Herein, the excitation of elastic open waveguides is investigated. For numerical purpose, the infinite transverse direction is modelled with a truncated perfectly matched layer (PML). Indeed, the PML offers a natural way to reveal the contribution of leaky modes. The PML gives access to the improper Riemann sheets by redefining the branch cuts, yielding two rotated continua of radiation modes (PML modes). The cases of an infinite medium and an open waveguide are considered. It is shown that all sets are necessary for achieving convergence of the modal expansion.

**Keywords:** modal analysis, open waveguide, leaky modes, perfectly matched layer, Riemann surface

**1 Introduction**

Numerically, elastic waveguides can be modelled with a cross-section finite element discretization thanks to the so-called Semi-Analytical Finite Element method (SAFE) [1]. For open waveguides, the SAFE can be combined with a PML [2], among various other methods. The PML eigenvalue problem has been studied by way of the convergence of trapped and leaky modes [2, 3]. Yet, the use of these modes and of PML modes in a modal expansion may remain unclear. Besides, a biorthogonality relation must be derived. This talk aims to bring insights on the full modal solution computed with a PML, through numerical and theoretical studies of elastic open waveguides.

**2 Numerical modelling**

For simplicity, the waveguide has two layers and is axisymmetric. The first layer has a radius  $r = a$ , and the second layer is infinite. The latter is truncated by a PML of thickness  $h$  and

starting from the radial position  $d$ . The PML is introduced using an analytic continuation in the radial direction:

$$\tilde{r}(r) = \int_0^r \gamma(\xi) d\xi \tag{1}$$

$\gamma(r)$  is a complex attenuation function ( $\text{Im}\gamma > 0$ ) through the PML ( $r > d$ ). At the end of the PML ( $r = d + h$ ), a Dirichlet condition is applied. Assuming an axial wavenumber and a time-harmonic dependance  $e^{j(kz - \omega t)}$ , the elastodynamics variational formulation is written on the cross-section. The finite element discretization yields the quadratic eigenvalue problem [2]:

$$(\mathbf{K}_1 + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3 - \omega^2\mathbf{M})\mathbf{U} = \mathbf{0} \tag{2}$$

$\mathbf{K}_1$ ,  $\mathbf{K}_3$  and  $\mathbf{M}$  are symmetric. Because of the PML, all the matrices are complex valued. It has been shown in [1] that the so-called Auld’s real orthogonality relationship must be used for modal expansion in waveguides with complex material properties (viscoelastic case). It can be checked that this relation actually still applies with a PML. Then, the total displacement field can be expanded on  $M$  positive-going modes as follows:

$$\mathbf{U}(z, \omega) = \sum_{m=1}^M \mathbf{E}_m \mathbf{F}(k_m) e^{ik_m z}. \tag{3}$$

$\mathbf{E}_m$  is the excitability matrix defined in [1], and  $\mathbf{F}$  stands for external forces.

**3 Numerical results**

A homogeneous elastic medium, excited by a unit point load applied at the centre in  $z$  direction, is modelled. Figure 1 shows that a good accuracy with respect to analytical results can be achieved within a limited distance range. For a given number of modes, this range increases with the complex thickness  $h\hat{\gamma} = \int_d^{d+h} \gamma(\xi) d\xi$ . These results are consistent with Olyslager’s [5], obtained for Green functions in scalar acoustic waveguides.

In a bilayer open waveguide, leaky modes dominate at small distances. Numerical results

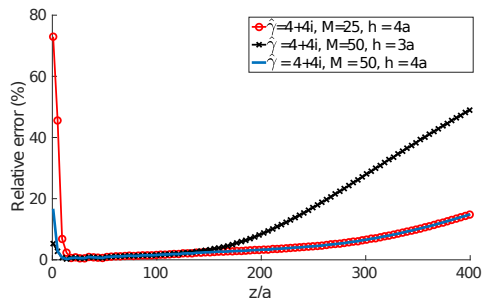


Figure 1: Relative error in an infinite medium on longitudinal displacement at  $r = 0$  and at a fixed frequency between numerical and analytical results.

(not presented here for conciseness) show that at great distances, long term diffraction occurs and cannot be properly described by leaky modes but by PML modes that stand for body waves (see section 4). It is noteworthy that a truncated PML enables to compute this phenomenon, while other types of absorbing layers have been proved to be inefficient [4].

#### 4 On the modal decomposition

So far, the  $M$  modes included in Eq. (3) have not been precisely defined. First, let us go back to the initial elastic waveguide problem, unbounded and without PML. The key points of the analysis are the definitions of the Riemann surface and of the two branch cuts of the problem, such that outgoing waves in the transverse direction decrease at infinity. The proper Riemann sheet yields two sets of modes. The first one is made of discrete proper poles, including trapped modes. The second one is the branch cuts contribution, yielding two continua of radiation modes. Leaky modes are also poles of the problem, but they lie on the improper Riemann sheets (they grow to infinity in the transverse direction).

With an infinite PML, the proper Riemann sheet is modified. It is shown that both branch cuts are rotated by an angle of  $-\arg\gamma$ . This gives access to parts of the initial improper Riemann sheets (numbered **2,3,4** - see Figure 2). Hence, the modal basis changes as well. The first set including trapped modes is unchanged. The continua of radiation modes are now defined by the rotated branch cuts. Besides, another discrete set made of revealed leaky modes is now included in the modal decomposition.

For numerical purpose, the PML needs to

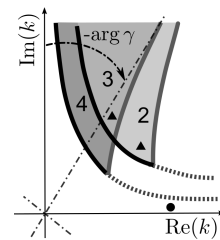


Figure 2: Proper Riemann sheet of a viscoelastic open waveguide with PML: initial branch cuts (black lines) and rotated branch cuts (grey line), trapped modes (circle) and leaky modes (triangle).

be truncated. With a finite PML, the continua of radiation modes become discrete [5]. These modes are called PML modes. Although non intrinsic to the physics (PML modes mainly depend on user-defined PML parameters), they have to be included in the modal expansion to achieve convergence, as mentioned in section 3.

#### 5 Acknowledgements

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#### References

- [1] F. Treyssède and L. Laguerre, Numerical and analytical calculation of modal excitability for elastic wave generation in lossy waveguides, *Journal of the Acoustical Society of America* **133** (2013), pp. 3287–3837.
- [2] F. Treyssède, K.L. Nguyen, A.-S. Bonnet-BenDhia and C. Hazard, Finite element computation of trapped and leaky elastic waves in open stratified waveguides, *Wave motion* **51** (2014), pp. 1093–1107.
- [3] S. Kim and J. E. Pasciak, The computation of resonances in open systems using a perfectly match layer, *Mathematical of Computation* **78** (2009), pp. 1375–1398.
- [4] J. Hu and C. R. Menyuk, Understanding leaky modes: slab waveguide revisited, *Adv. Opt. Photon.* **1** (2009), pp. 58–106.
- [5] F. Olyslager, Discretization of Continuous Spectra Based on Perfectly Matched Layers, *SIAM Journal on Applied Mathematics* **64** (2004), pp. 1408–1433.