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Practical stabilisation of switched affine systems with dwell-time guarantees

C. Albea Sanchez, G. Garcia, S. Hadjeras, W.P.M.H. Heemels, L. Zaccarian

Abstract—The paper deals with the problem of practical stabilisation of operating points for switched affine systems, ensuring a dwell-time associated with an LQR performance level during the transient response and an admissible chattering around the operating point. In this paper we propose a solution to this problem in which the formal guarantees are established using recent tools from hybrid dynamical systems theory. We also include insights on the quadratic performance of the proposed switching strategies.

I. INTRODUCTION

Switched systems [13] are encountered in many applications including mixing of fluids, DC-DC power conversion, event-triggered control, viral mutation in HIV treatment, mobile sensor networks, damping of vibrating structures, and several others, see [2] for references to these applications. In particular, the area of switched linear systems (SLSs) received ample attention. There exist many works about SLSs guaranteeing asymptotic stabilization of the operating point, see, e.g., [7, 15, 22], and also [13, Section 3.4] and [21, Chapter 5,6]. and a vast literature exists on the topic, see [15, 19] for recent overviews.

Interestingly, the class of switched affine systems (SASs), which we consider in this paper, received less attention. SASs are given by

$$\begin{aligned} \dot{x} &= A_\sigma x + a_\sigma & x(0) &= x_0 \\ z &= Cx, \end{aligned} \quad (1)$$

where the available input $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathcal{N} := \{1, 2, \dots, N\}$ is the switching signal, assigning a specific desired mode among N possible ones at each point in time. Moreover, in dynamics (1), $x \in \mathbb{R}^n$ is the state, $z \in \mathbb{R}^p$ is the controlled output, and A_σ and a_σ have suitable dimensions, $\sigma \in \mathcal{N}$. In [5], the authors provide an appealing switching strategy guaranteeing *asymptotic* stability of the operating point, which is an equilibrium for the average dynamics. However, that result can induce sliding modes and generate arbitrarily fast switching. Other solutions, discussed in the context of peculiar applications of SASs such as power converters, see [4, 17, 20], aim at ensuring a dwell-time associated with an admissible chattering

around the operating point. Another interesting line of work is discussed in [4], which focuses on the practical stabilisation of a family of power converters that can be modelled as SASs with all the affine dynamics in a Hamiltonian form, see also [6]. Inspired by the physical insight coming from power converters, the authors also define the set of admissible operating points having a similar form in terms of equilibria of the average dynamics, as used in [5]. Then, asymptotic stability is guaranteed without ensuring any dwell-time between consecutive switches.

In this paper we are interested in the practical stabilization of operating points for SASs described by (1). Important features of the proposed control strategy are the following ones:

- 1) stabilization of an (arbitrarily small) set around the operating point is achieved,
- 2) a positive minimal dwell-time between consecutive switches is guaranteed during the transient response, and thus infinitely fast switching (e.g., due to Zeno behaviour or sliding modes) is avoided to warrant practical implementability,
- 3) it is possible to make a design tradeoffs between the size of the asymptotically stable set around the desired operating point and a LQR performance level on the one hand and the minimal dwell-time between two switches (and thus the switching frequency) on the other one, and
- 4) a second mechanism guarantees some dwell-time in the tail of the response (some pseudo-steady-state) this obtaining a controlled chattering around the operating point.

We provide two methods to realise the latter steady-state property, based on spatial regularisation (linked to Section 4.2 in [4]) and one based on time regularisation (enforcing a positive minimal dwell-time). Rigorous proofs will be given for our statements, using recent tools from hybrid systems theory [9]. The main tool for proving stability of a compact set defined in an extended space, proceeds by an appropriate extension of Lyapunov stability theory developed in the context of hybrid dynamical systems in [9] and [16]. Due to the affine structure of the modes, a quadratic Lyapunov function (not a common Lyapunov function) can be selected satisfying a Lyapunov inequality. To the best of our knowledge, these properties have not been previously established in the literature for the general class of SASs. The works closest to the main results in this paper are [10, 11, 20]. [20] is focused on a specific converter, namely the DC-DC Boost converter and the authors provide a constrained switched differential

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inclusion derived from the specific model (including all modes of the converter, not only the steady-state modes). For this model, a stabilizing hybrid control law is derived using hybrid systems tools and, by exploiting inherent robustness properties, a spatial regularisation (not time regularisation) is applied, thereby removing Zenoness, while still preserving practical stability properties of the desired equilibrium. The work in [10] considers general SASs and based on the assumption that a control Lyapunov function (CLF) is available for the convexified dynamics, three strategies are considered that have desirable dwell-time properties in each mode. The three strategies are the well-known pulse-width modulation strategy, a steepest descent strategy based on the mode that makes the CLF decrease the fastest, and a predictive receding-horizon strategy minimising the CLF at the end of a fixed horizon (all with sample time T_s). The authors in [11] consider an LMI-based design method for practically stabilizing switching laws for equilibria of the convexified dynamics of (1) including a stability condition (see also [3, 5]). These sets of equilibria and conditions are in line with our assumptions below (see condition (2)). In principle, [11] has a sampled-data implementation with an upper-bound on the inter-sample times and using LMI-based conditions the upper-bound of the inter-sample times can be directly related to the size of the asymptotic stability set around the considered equilibrium. Practical stability is obtained using Lyapunov-Krasovskii functionals. The obtained results are useful as they allow direct tuning of the maximal sampling times. The time-regularisation proposed in this paper imposes a minimal dwell-time instead of a maximal dwell-time or a sampled-data implementation as in [10, 11], indicating that we use a different time-based switching regime not limiting the size of the dwell-time from above. Besides we present a general spatially regularised and practically stabilising switching laws not considered in [10, 11]. A preliminary version of this paper was presented at [1], but it did rely on more restrictive assumptions and did not contain the practical stability results presented here.

The paper is organized as follows. Section II is dedicated to the problem formulation. We present a hybrid dynamical control that manages a dwell-time in transient-time with an optimization procedure to tune some parameters in Section III. Then, some practical stability results that guarantee a dwell-time in the steady-state are presented in Section IV. Finally, Section V summarizes the main results and proposes future work directions.

II. PROBLEM SETUP

As already mentioned in the introduction, we are interested in the practical stabilisation of a specific set of operating points, $x_e \in \mathbb{R}^n$, for SASs as in (1) satisfying the requirements 1), 2), 3) and 4) using the appropriate design of a feedback law for the switching signal σ . A necessary and sufficient condition characterizing the set of operating points that we are interested in is represented by the following standard assumption (see [5, 14]).

Assumption 1: There exists $\lambda \in \Lambda$ such that

$$\sum_{i=1}^N \lambda_i (A_i x_e + a_i) = 0, \quad (2)$$

where $\Lambda := \left\{ \lambda \in [0, 1]^N \mid \sum_{i=1}^N \lambda_i = 1 \right\}$.

Definition 1: The set of admissible operating points $\Omega_e \subset \mathbb{R}^n$ is given by

$$\Omega_e := \{x_e \in \mathbb{R}^n \mid \exists \lambda \in \Lambda \text{ s.t. (2) holds and } \sum_{i=1}^N \lambda_i A_i \text{ is Hurwitz}\}. \quad (3)$$

Hence, an operating point is admissible if it is an equilibrium point for the averaged dynamics

$$\dot{x} = \sum_{i=1}^N \lambda_i (A_i x + a_i), \lambda = [\lambda_1, \lambda_2, \dots, \lambda_N]^\top \in \Lambda$$

and a stability condition is satisfied on the corresponding convex dynamics. See also [4, 5, 11, 14] and the discussion in [1, Remark 1]. Note that the average dynamics can be perceived as the result of arbitrarily fast switching and as relaxations in the generalized sense of Krasovskii or Filippov [8].

The problem considered in this paper can be formulated as follows: *Given the SAS (1), provide a design framework such that for each $x_e \in \Omega_e$ a feedback law is constructed that determines on-line the switching signal σ that renders a (arbitrarily small) set around x_e globally asymptotically stable (i.e., practical stabilization of x_e) and satisfies the requirements 2, 3 and 4 discussed in the introduction.*

III. HYBRID MODEL AND TRADEOFF BETWEEN DWELL-TIME AND LQR PERFORMANCE LEVEL

A. Proposed architecture

Consider (1) and $x_e \in \Omega_e$ and let λ_e be a corresponding vector in Λ as in (3). We select now matrices P and Q satisfying the following property.

Property 1: The matrix $P = P^T > 0 \in \mathbb{R}^{n \times n}$ and the matrix $Q = Q^T > 0 \in \mathbb{R}^{n \times n}$ satisfy

$$\left(\sum_{i=1}^N \lambda_{e,i} A_i^T \right) P + P \left(\sum_{i=1}^N \lambda_{e,i} A_i \right) + 2Q \leq 0. \quad (4)$$

Clearly due to $\sum_{i=1}^N \lambda_{e,i} A_i$ being Hurwitz such matrices satisfying Property 1 always exist for $x_e \in \Omega_e$. Note also that Property 1 imposes less restrictive assumptions than in our preliminary work [1]. In fact, the assumptions in [1] correspond to a special case of the general construction proposed in this paper.

Using these matrices we can formulate a hybrid controller, following the formalism in [9], wherein we find the evolution (1) in the continuous-time dynamics, and the discrete-time dynamics captures the instantaneous jump

of the control signal σ from one mode to another. We represent the overall dynamics as

$$\mathcal{H} : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = f(x, \sigma), & (x, \sigma) \in \mathcal{C} \\ \begin{bmatrix} x^+ \\ \sigma^+ \end{bmatrix} \in G(x, \sigma), & (x, \sigma) \in \mathcal{D}, \end{cases} \quad (5)$$

where f represents the flow map and G is a (set-valued) map capturing the switching logic:

$$\begin{aligned} f(x, \sigma) &:= \begin{bmatrix} A_\sigma x + a_\sigma \\ 0 \end{bmatrix} \\ G(x, \sigma) &:= \left[\begin{array}{c} x \\ \operatorname{argmin}_{i \in \mathcal{N}} (x - x_e)^T P(A_i x + a_i) \end{array} \right] \end{aligned} \quad (6)$$

and where the so-called ‘‘flow’’ and ‘‘jump’’ sets \mathcal{C} and \mathcal{D} encompass, respectively, the regions in the (extended) space (x, σ) where the switching strategy continues with the current mode σ (set \mathcal{C}) or switches to a new mode (set \mathcal{D}). If switching is allowed (namely, if $(x, \sigma) \in \mathcal{D}$) then σ may switch according to G in (6). For the solution proposed in this paper, we select the flow and jump sets associated with the desired equilibrium x_e introduced in Assumption 1 as follows:

$$\mathcal{C} := \{(x, \sigma) : \tilde{x}^T P(A_\sigma x + a_\sigma) \leq -\eta \tilde{x}^T Q \tilde{x}\} \quad (7)$$

$$\mathcal{D} := \{(x, \sigma) : \tilde{x}^T P(A_\sigma x + a_\sigma) \geq -\eta \tilde{x}^T Q \tilde{x}\}, \quad (8)$$

where $\tilde{x} := x - x_e$ satisfies $\dot{\tilde{x}} = (\sum \lambda_{e,i} A_i) \tilde{x}$ and scalar $\eta \in (0, 1)$ is a design parameter that will be shown to be useful for suitably achieving a trade-off between the transient switching frequency and optimality level, as characterized later in Theorem 2.

B. Stability results

The next lemma is a fundamental step to prove Theorem 1 below.

Lemma 1: Consider a point $x_e \in \Omega_e$ and matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$, satisfying Property 1. Then, for each $x \in \mathbb{R}^n$, denoting $\tilde{x} := x - x_e$,

$$\min_{i \in \mathcal{N}} \tilde{x}^T P(A_i x + a_i) \leq -\tilde{x}^T Q \tilde{x}. \quad (9)$$

□

Proof. First notice that the left hand side of (2) is linear in the components of λ_e . Moreover, λ_e belongs to the compact set $\Lambda = \left\{ \lambda_n \in [0, 1]^N \mid \sum_{i=1}^N \lambda_{n,i} = 1 \right\}$. Then, the following minimum is obtained at the extreme points:

$$\begin{aligned} & \min_{i \in \mathcal{N}} \tilde{x}^T P(A_i x + a_i) \\ &= \min_{\lambda_n \in \Lambda} \tilde{x}^T P \left(\sum_{i=1}^N \lambda_{n,i} A_i x + \sum_{i=1}^N \lambda_{n,i} a_i \right) \\ &= \min_{\lambda_n \in \Lambda} \tilde{x}^T P \sum_{i=1}^N \lambda_{n,i} A_i \tilde{x} + \tilde{x}^T P \left(\sum_{i=1}^N \lambda_{n,i} A_i x_e + \sum_{i=1}^N \lambda_{n,i} a_i \right) \\ &\leq \tilde{x}^T P \sum_{i=1}^N \lambda_{e,i} A_i \tilde{x} + \tilde{x}^T P \left(\sum_{i=1}^N \lambda_{e,i} A_i x_e + \sum_{i=1}^N \lambda_{e,i} a_i \right) \\ &\leq -\tilde{x}^T Q \tilde{x}, \end{aligned}$$

where in the last step we used relations (2) and (4). □

A similar sufficient condition was presented in [7] for a class of SLSs with $N = 2$, which is quadratically stabilized via output feedback if there exists an asymptotically stable convex combination of the different modes.

The proof of Lemma 1 is relevant in terms of the nature of the switching signals generated by our solution. In particular, property (9) combined with (7) shows that unless $\tilde{x} = \tilde{x}^+ = 0$ (which means that we are at the equilibrium $x = x_e$), the solution always jumps to the interior of the flow set \mathcal{C} . Indeed, $\tilde{x} \neq 0$ implies

$$-\tilde{x}^T Q \tilde{x} < -\eta \tilde{x}^T Q \tilde{x},$$

because $\eta < 1$. This fact, together with stability (ensuring boundedness of solutions) and the sector growth condition

$$|\dot{\tilde{x}}| = |\tilde{x}| \leq |A_\sigma(x - x_e)| + |A_\sigma x_e + a_\sigma| \leq \kappa_1 |\tilde{x}| + \kappa_2 \quad (10)$$

with κ_1 and κ_2 positive constants, coming from the linearity of the flow dynamics (1), implies that there is a uniform lower bound on the dwell time between each pair of consecutive resets before solutions approach $x = x_e$. Clearly, this lower bound shrinks to zero as solutions approach $x = x_e$ because only arbitrarily fast switching can make x_e an equilibrium, in general. As compared to existing approaches (see e.g., [5]) that essentially rely on a ‘‘sliding-mode’’ type of paradigm, thereby typically leading to very fast switching and chattering along a sliding surface, our solution is instead characterized by relatively slow switching, where the transient switching frequency can be adjusted, to a certain extent, using parameter η .

Following up on standard stability theory for hybrid systems [9], we will establish suitable stability properties of the point x_e in terms of uniform global attractivity of a bounded (and closed) set in the higher-dimensional space spanned by (x, σ) . In particular, we will establish properties of the compact attractor

$$\mathcal{A} := \{(x, \sigma) : x = x_e, \sigma \in \mathcal{N}\}, \quad (11)$$

encompassing the fact that we are interested in uniform stability and convergence to a set where $x = x_e$ and σ assumes some unspecified value or pattern within the desired limit set of solutions.

Theorem 1: Consider a point x_e and a vector λ_e satisfying Assumption 1 and matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ satisfying Property 1. Then attractor (11) is uniformly globally asymptotically stable (UGAS) for hybrid system (5)–(8).

Proof. Let us take the candidate Lyapunov function

$$V(\tilde{x}) := V(x - x_e) := \frac{1}{2} \tilde{x}^T P \tilde{x} \quad (12)$$

being $\tilde{x} := x - x_e$. In the flow set, \mathcal{C} , using its definition in (7), we get

$$\langle \nabla V(\tilde{x}), f(\tilde{x}, \sigma) \rangle = \tilde{x}^T P(A_\sigma(\tilde{x} + x_e) + a_\sigma) \leq -\eta \tilde{x}^T Q \tilde{x}. \quad (13)$$

Across jumps we trivially get:

$$V(\tilde{x}^+) - V(\tilde{x}) = \frac{1}{2} \{ \tilde{x}^T P \tilde{x} - \tilde{x}^T P \tilde{x} \} = 0. \quad (14)$$

because $\tilde{x}^+ = x^+ - x_e = x - x_e = \tilde{x}$.

Uniform global asymptotic stability is then shown by [16, Theorem 1]. In particular, since the distance of x to the attractor (11) is defined by $|x|_{\mathcal{A}} = |\tilde{x}|$, we have that [16, eq. (6)] holds from the structure of V and from (13) and (14). [16, Theorem 1] also requires building the restricted hybrid system $\mathcal{H}_{\delta,\Delta}$ by intersecting \mathcal{C} and \mathcal{D} with the set

$$S_{\delta,\Delta} := \{(\tilde{x}, \sigma) : |\tilde{x}| \geq \delta \quad \text{and} \quad |\tilde{x}| \leq \Delta\} \quad (15)$$

and then proving (semi-global) practical persistence flow for $\mathcal{H}_{\delta,\Delta}$, for each fixed values of (δ, Δ) . In particular, practical persistent flow amounts to showing that there exists $\gamma \in \mathcal{K}_\infty$ and $M \geq 0$, such that, all solutions to $\mathcal{H}_{\delta,\Delta}$ satisfy

$$t \geq \gamma(j) - M, \quad \forall t \in \bigcup_{j \in \text{dom}_j \xi} I^j \times \{j\} \quad (16)$$

(see [16] for details). To establish (16), notice that after each jump, from the definition of G in (6) and from Property (9) (in Lemma 1), we have

$$\tilde{x}^T(A_{\sigma^+}x + B_{\sigma^+}) \leq -\tilde{x}^T Q \tilde{x} < -\eta \tilde{x}^T Q \tilde{x}, \quad (17)$$

where we used the fact that $\eta < 1$ and that $(0, \sigma) \notin S_{\delta,\Delta}$. Therefore, if any solution to $\mathcal{H}_{\delta,\Delta}$ performs a jump from $S_{\delta,\Delta}$, it will remain in $S_{\delta,\Delta}$ (because \tilde{x} remains unchanged) and then, from (8), it jumps to the interior of the flow set $\mathcal{C} \cap S_{\delta,\Delta}$. Moreover, from the strict inequality in (17), then all non-terminating solutions must flow for some time and since $\mathcal{C} \cap S_{\delta,\Delta}$ is bounded, there is a uniform dwell-time $\rho(\delta, \Delta)$ between each pair of consecutive jumps. This dwell-time (δ, Δ) clearly implies [16, eq. (4)] with the class \mathcal{K}_∞ function $\gamma(j) = \rho(\delta, \Delta)j$ and $M = 1$. Then, all the assumptions of [16, Theorem 1] hold and UGAS of \mathcal{A} is concluded. \square

Note that according to Theorem 1 system (5)–(8) may exhibit a Zeno behaviour when $x \rightarrow x_e$, and consequently an infinitely fast switching may be expected, which is not acceptable in practice. This is why we propose in Section IV an additional dwell-time logic to obtain a temporal-regularisation of the dynamics, thereby weakening asymptotic convergence into practical convergence.

C. Optimality and parameters tuning

Theorem 1 establishes UGAS of the attractor, which results in desirable uniform stability and convergence properties. However, we are interested in further providing a suitable performance guarantee for our solution, which follows the same paradigm as the one discussed, in a continuous-time setting, in [5]. This performance guarantee, may, for example, refer to desirable levels of dissipated energy, current peak, response time among others.

Within the considered hybrid context, we first recall that solutions are parametrized by ordinary time t (measuring amount of flow) and discrete-time j (measuring the number of switches) so that the domain of a solution ξ

(see [9, Ch. 2]) corresponds to a finite or infinite union of intervals of the following form:

$$\text{dom } \xi = \bigcup_{j \in \text{dom}_j \xi} I^j \times \{j\}, \quad (18)$$

with $I^j = [t_j, t_{j+1}]$ being a bounded time interval having the so-called ‘‘jump times’’ t_k as extremes, or possibly being a last unbounded interval open to the right and of the form $I^j = [t_j, +\infty)$. In (18), we use the notation $\text{dom}_j \xi := \{j \in \mathbb{Z} : (t, j) \in \text{dom } \xi, \text{ for some } t \in \mathbb{R}\}$, namely $\text{dom}_j \xi$ includes all $j \in \mathbb{Z}$ such that I^j is non-empty. Within this context, we represent a LQ performance metric focusing on flowing characteristics of the plant state, using the expression

$$J(\xi) := \sum_{k \in \text{dom}_j \xi} \int_{t_k}^{t_{k+1}} |\tilde{z}(\tau, k)|^2 d\tau, \quad (19)$$

where $\xi = (x, \sigma) : \text{dom } \xi \rightarrow \mathbb{R}^n \times \mathcal{N}$ is a solution to hybrid system (5)–(8), and $\tilde{z}(t, j) := C\tilde{x}(t, j)$ for all $(t, j) \in \text{dom } \xi$ is a suitable performance output.

With our hybrid switching solution, we may then give the following guarantee on the performance cost (19).

Theorem 2: Consider hybrid system (5)–(8) satisfying Assumption 1 and Property 1. If

$$C^T C \leq Q, \quad (20)$$

then the following bound holds along any solution $\xi = (x, \sigma)$ of (5)–(8):

$$J(\xi) \leq \frac{1}{2\eta} \tilde{x}(0, 0)^T P \tilde{x}(0, 0), \quad (21)$$

where $\tilde{x}(t, j) = x(t, j) - x_e$, for all $(t, j) \in \text{dom}(\xi)$.

Proof. To prove the optimality property in (21), consider any solution $\xi = (x, \sigma)$ to \mathcal{H} . Then for each $(t, j) \in \text{dom } \xi$ and denoting $t = t_{j+1}$ to simplify notation, we have from (13)

$$\begin{aligned} V(\tilde{x}(t, j)) - V(\tilde{x}(0, 0)) &= \sum_{k=0}^j V(\tilde{x}(t_{k+1}, k)) - V(\tilde{x}(t_k, k)) \\ &= \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \langle \nabla V(\tilde{x}(\tau, k)), f(x(\tau, k), u(\tau, k)) \rangle d\tau \\ &\leq \sum_{k=0}^j \int_{t_k}^{t_{k+1}} -\eta \tilde{x}^T(\tau, k) Q \tilde{x}(\tau, k) d\tau \\ &\leq -\eta \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \tilde{x}^T(\tau, k) C^T C \tilde{x}(\tau, k) d\tau, \end{aligned} \quad (22)$$

where the last inequality comes from applying (20). Now, considering $\tilde{z}(\tau, k) = C\tilde{x}(\tau, k)$, taking the limit as $t + j \rightarrow +\infty$ and using the fact that UGAS established in Theorem 1 implies $\lim_{t+j \rightarrow +\infty} V(\tilde{x}(t, j)) = 0$, we get from (22)

$$\eta J(\xi) \leq V(\tilde{x}(0, 0)) = \frac{1}{2} \tilde{x}(0, 0)^T P \tilde{x}(0, 0),$$

as to be proven. \square

Remark 1: It should be emphasized that once matrices P and Q have been fixed compliantly with requirement (20), the guaranteed performance level for our scheme (in terms of size of the upper bound for index J in (19) along solutions) is proportional to the inverse of $\eta \in (0, 1)$ (see (21)). To this end, large values of η (as close as possible to 1) are expected to lead to improved LQ performance along solutions.

On the other hand, one may appreciate by looking at the flow and jump sets in (7) and (8), that smaller values of η correspond to strictly smaller jump sets (and larger flow sets), which reveals that solutions are expected to flow longer before switches of control input σ are experienced. Therefore we anticipate that solutions with smaller values of η exhibit a smaller switching frequency during the transient response. In other words, one may use parameter η to find a tradeoff between suitable transient switching frequency and transient performance along solutions. This operation clearly affects the level of guaranteed optimality, according to (21). \dashv

D. Computation of P and Q

The problem addressed next is the computation of parameters P , Q , following an optimization capturing the goal of reducing as much as possible the right hand side in bound (21). To this end, we make the following natural selection of matrix Q :

$$Q = C^T C + \nu I, \quad (23)$$

where $\nu > 0$ is a (typically small) positive constant, which may be selected equal to zero if $C^T C > 0$. Then it is clear that selection (23) ensures $Q > 0$, as required, in addition to ensuring bound (20).

Once parameter Q is selected, under the assumption that the convex combination $\sum_{i=1}^N \lambda_{e,i} A_i$ is Hurwitz, the following convex optimization expressed by linear matrix inequalities always leads to a feasible solution:

$$\begin{aligned} \min_{P=P^T > 0} \text{Trace}(P), \text{ subject to:} \quad (24) \\ \sum_{i=1}^N \lambda_{e,i} A_i^T P + P \sum_{i=1}^N \lambda_{e,i} A_i^T \leq -2Q, \end{aligned}$$

and this optimal solution clearly satisfies Property 1.

IV. PRACTICAL GLOBAL RESULTS USING SPACE- OR TIME-REGULARIZATION

The hybrid control law proposed above can provide arbitrarily fast switching as the solution approaches x_e , because given an initial condition in \mathcal{A} , one sees that the hybrid dynamics (5)–(8) has at least one solution that keeps jumping onto \mathcal{A} . Infinitely fast switching is not desirable in terms of energy efficiency and reliability, because every switch dissipates energy and reduces the switch lifespan. For this reason, we propose a few modifications of the hybrid law, aiming at reducing the number of switches when $\tilde{x} = x - x_e$ is close to zero. This goal is reasonable for the proposed law, because it is possible to show that away

from \mathcal{A} , during *transients*, our control law already joins a desirable property of dwell time between switches, as long as Assumption 1 and Property 1 hold. The dwell time induced by the practical modification introduced here can be denoted, with a slight abuse of notation, *steady-state* dwell time property, as it regulates the dwell time during the controlled chattering enforced (on the “steady-state” tail of the solution) around x_e .

To best formalize this *steady-state* dwell-time property, given any x_e and P, Q satisfying the above assumptions, consider system (5)–(8) and denote it by the shortcut notation $\mathcal{H} := (f, G, \mathcal{C}, \mathcal{D})$ and then, for a non-negative scalar ε , define the following restricted jump set:

$$\mathcal{D}_\varepsilon := \mathcal{D} \setminus \{(x, \sigma) : V(x - x_e) < \varepsilon\}, \quad (25)$$

where V has been introduced in (12). A useful practical dwell-time property for \mathcal{H} is then established below.

Lemma 2: There exists a positive scalar T^* such that for each $T \leq T^*$, there exist a scalar ε_T such that all solutions to \mathcal{H} jumping from set $\mathcal{D}_{\varepsilon_T}$ flow for at least T ordinary time after the jump before reaching set \mathcal{D} . Moreover, as T tends to zero, we have that ε_T tends to zero as well.

Proof. To prove the lemma, it is enough to fix any scalar $\varepsilon = \varepsilon^*$ in (25) and show that there exists T^* such that all solutions starting from $\mathcal{D}_{\varepsilon^*}$ flow for at least T^* ordinary time after the jump before reaching set \mathcal{D} . The rest of the lemma follows trivially from the fact that smaller values of $\varepsilon < \varepsilon^*$ are associated with the solutions starting in $\mathcal{D}_{\varepsilon^*}$ (already characterized by T^*) plus additional solutions starting in the compact set $\overline{\mathcal{D}_\varepsilon} \setminus \mathcal{D}_{\varepsilon^*}$, that enjoy a dwell time property because any jump from this set maps to the interior of the flow set (and then one can consider the minimum flowing time over this compact set of initial conditions). Clearly, the dwell-time T is expected to converge to zero as ε converges to zero, thereby defining the function ε_T discussed in the lemma.

Let us then fix a scalar $\varepsilon = \varepsilon^*$ in (25) and first notice that any solution jumping from $\mathcal{D}_{\varepsilon^*}$ at time $(t_j, j - 1)$ satisfies, before and after the jump:

$$|\tilde{x}(t_j, j)|_Q^2 := \tilde{x}^T Q \tilde{x} \geq q_m |\tilde{x}|^2 \geq \frac{q_m}{p_M} V(\tilde{x}) \geq \frac{q_m \varepsilon^*}{p_M} =: 2\varepsilon_Q, \quad (26)$$

where the dependence on (t, j) has been omitted at the right hand side, and where we denoted by q_m and q_M the minimum and maximum eigenvalues of Q , respectively, and by p_M the maximum eigenvalue of P . Define now the function $\chi(\tau) := 2\varepsilon_Q - |\tilde{x}(t_j + \tau, j)|_Q^2$ and notice that (26) implies $\chi(0) \leq 0$. Consider now the flow dynamics in (6) and introduce scalars $b_\sigma = A_\sigma x_e + a_\sigma$ to get

$$\dot{\tilde{x}} = A_\sigma x + a_\sigma = A_\sigma \tilde{x} + b_\sigma, \quad (27)$$

so that we may characterize the variation of χ as:

$$\dot{\chi} = -2\tilde{x}^T Q (A_\sigma \tilde{x} + b_\sigma) \leq \kappa_1 |\tilde{x}|_Q^2 + \kappa_2 |\tilde{x}|_Q \quad (28)$$

where $\kappa_1 := 2 \frac{q_M}{q_m} \max_{\sigma \in N} |A_\sigma|$ and $\kappa_2 := 2 \frac{q_M}{\sqrt{q_m}} \max_{\sigma \in N} |b_\sigma|$. Using now $|\tilde{x}|_Q^2 \leq |\chi| + 2\varepsilon_Q$, which also gives $|\tilde{x}|_Q \leq \sqrt{|\chi|} +$

$\sqrt{2\varepsilon_Q}$, because $|\chi|$ and ε_Q are both non-negative, we get the bound:

$$\begin{aligned}\dot{\chi}(\tau) &\leq \kappa_1(|\chi(\tau)| + 2\varepsilon_Q) + \kappa_2(\sqrt{|\chi(\tau)|} + \sqrt{2\varepsilon_Q}) \quad (29) \\ &= \kappa_1|\chi(\tau)| + \kappa_2\sqrt{|\chi(\tau)|} + \kappa_3, \quad \forall \tau \leq t_{j+1} - t_j,\end{aligned}$$

where $\kappa_3 = 2\kappa_1\varepsilon_Q + \kappa_2\sqrt{2\varepsilon_Q} > 0$. Denote by ϕ the solution to the differential equation induced by (29) starting at zero. This solution is continuous and strictly increasing because $\kappa_i > 0$ for all $i = 1, 2, 3$. Then there exists T_1 such that $\phi(T_1) = \varepsilon_Q$ and from standard comparison theory, and recalling that $\chi(0) \leq 0$ (by (26)), we have $\chi(\tau) \leq \varepsilon_Q$ for all $\tau \leq T_1$, which implies

$$|\tilde{x}(t_j + \tau, j)|_Q^2 \geq 2\varepsilon_Q - \chi(\tau) \geq \varepsilon_Q, \quad \forall \tau \leq T_1. \quad (30)$$

Consider now equation (17) and define the function ¹

$$\varsigma(\tilde{x}) := \frac{\tilde{x}^T(A_\sigma\tilde{x} + b_\sigma)}{|\tilde{x}|_Q^2} + 1,$$

which, from (17) clearly satisfies $\varsigma(\tilde{x}(t_j, j)) \leq 0$ after the jump from $\mathcal{D}_{\varepsilon^*}$. We prove below the existence of T^* such that

$$\varsigma(\tilde{x}(t_j + \tau, j)) \leq 1 - \eta, \quad \forall \tau < T^*, \quad (31)$$

which trivially proves $\tilde{x}(t_j + \tau, j)^T(A_\sigma\tilde{x}(t_j + \tau, j) + b_\sigma) \leq -\eta|\tilde{x}(t_j + \tau, j)|_Q^2$, which in turn implies that the solution does not belong to \mathcal{D} , thus completing the proof of the lemma.

To prove (31), we proceed again with bounding the derivative of ς . Straightforward derivations provide, along flowing solutions according to (6):

$$\begin{aligned}\dot{\varsigma} &= -\frac{2\tilde{x}^T P(A_\sigma\tilde{x} + b_\sigma)\tilde{x}^T Q(A_\sigma\tilde{x} + b_\sigma)}{|\tilde{x}|_Q^4} \\ &+ \frac{\tilde{x}^T(PA_\sigma + A_\sigma^T P)A_\sigma\tilde{x} + \tilde{x}^T(2A_\sigma^T P + PA_\sigma)b_\sigma + b_\sigma^T P b_\sigma}{|\tilde{x}|_Q^2} \\ &\leq \varsigma_1 + \varsigma_2 \frac{1}{|\tilde{x}|_Q} + \varsigma_3 \frac{1}{|\tilde{x}|_Q^2},\end{aligned}$$

where $\varsigma_1, \varsigma_2, \varsigma_3$ are sufficiently large positive scalars (and where we used $|\tilde{x}| \leq \frac{1}{\sqrt{q_m}}|\tilde{x}|_Q$ in several places). Consider now any time $\tau \leq T_1$, and use bound (30) to obtain $\dot{\varsigma} \leq \varsigma_1 + \varsigma_2\varepsilon_Q^{-1/2} + \varsigma_3\varepsilon_Q^{-1}$, which, together with $\varsigma(\tilde{x}(t_j, j)) \leq 0$, and integrating $\dot{\varsigma}$, immediately gives (31) for $T^* := \min\{T_1, T_2\}$, where $T_2 := \frac{1-\eta}{\varsigma_1 + \varsigma_2\varepsilon_Q^{-1/2} + \varsigma_3\varepsilon_Q^{-1}}$. \square

Lemma 2 ensures that some dwell-time is achievable if solutions remain sufficiently far from \mathcal{A} . Then we have two possibilities to modify our control law to ensure that dwell-time is enjoyed by solutions. One of them corresponds to replacing the jump set \mathcal{D} by the restricted version in \mathcal{D}_ε and forcing solutions to flow in $\mathcal{D} \setminus \mathcal{D}_\varepsilon$ (this is called space regularization and is addressed in Section IV-A), and the other one corresponds to forcing solutions not to jump unless some dwell time has expired (this is called time

regularization and is addressed in Section IV-B). Both solutions are associated to desirable properties, that arise from the result of Lemma 2.

A. Space regularization

Based on Lemma 2, for any value of a positive scalar ε , let us consider introducing the following space-regularized version of $\mathcal{H} = (f, G, \mathcal{C}, \mathcal{D})$ in (5)–(8):

$$\mathcal{H}_\varepsilon := (f, G, \mathcal{C}_\varepsilon, \mathcal{D}_\varepsilon) \quad (32a)$$

$$\mathcal{C}_\varepsilon := \mathcal{C} \cup \{(x, \sigma) : V(x) \leq \varepsilon\}, \quad (32b)$$

where \mathcal{D}_ε has been introduced in (25). The regularized dynamics (32), (25) is clearly motivated by the fact that jumps are forbidden when solutions are ε -close to the attractor because they are forced to flow. Then it makes sense to introduce the following ε -inflated version of attractor \mathcal{A} :

$$\mathcal{A}_\varepsilon := \{(\tilde{x}, \sigma) : V(\tilde{x}) \leq \varepsilon, \sigma \in \bar{N}\}, \quad (33)$$

which evidently reduces to \mathcal{A} as ε tends to zero.

Mainly using Lemma 2 the following desirable results are enjoyed by hybrid system \mathcal{H}_ε .

Theorem 3: Consider point x_e and a vector λ_e satisfying Assumption 1 and matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ satisfying Property 1. The following holds:

- 1) for any positive scalar ε , set \mathcal{A}_ε in (33) is UGAS for dynamics \mathcal{H}_ε in (32);
- 2) set \mathcal{A} is globally practically asymptotically stable for (32), with respect to parameter ε ;
- 3) \exists a scalar $T > 0$ such that all solutions to \mathcal{H}_ε enjoy a dwell-time property corresponding to T , namely given any solution φ to \mathcal{H}_ε , all $(t, j) \in \text{dom } \varphi$ satisfy $t \geq \frac{j}{T} - 1$;
- 4) as long as ε is sufficiently large, the scalar T of the previous item can be made arbitrarily close to T^* .

Proof. First notice that sets \mathcal{C}_ε and \mathcal{D}_ε are both closed. Indeed, \mathcal{C}_ε is the union of two closed sets and \mathcal{D}_ε is the intersection of two closed sets (\mathcal{D} and the complement of the open set where $V(\tilde{x}) < \varepsilon$). Then, due to the properties of f and G , system \mathcal{H}_ε satisfies the hybrid basic conditions of [9, As. 6.5] and we may apply several useful results pertaining well-posed hybrid systems.

Proof of item 3). This item follows in a straightforward way from Lemma 2. Indeed, solutions to \mathcal{H}_ε can only jump from \mathcal{D}_ε . Since $\mathcal{D}_\varepsilon \subset \mathcal{D}$, any such solution φ flows for at least T time after each jump, before reaching again \mathcal{D}_ε and possibly jumping again, which clearly implies $t + 1 \geq \frac{j}{T}$ (where the “1” takes care of the initial condition), as to be proven.

Proof of item 4). Again using Lemma 2 it is possible to obtain a dwell time T arbitrarily close to (but smaller than) T^* , because of the existence of a large enough value of ε enforcing that type of dwell-time property after all jumps from the corresponding set \mathcal{D}_ε .

Proof of item 1). Consider the following Lyapunov function candidate:

$$V_\varepsilon(\tilde{x}) = \max\{V(\tilde{x}) - \varepsilon, 0\}, \quad (34)$$

¹To avoid overloading notation, the hybrid time is only specified on the \tilde{x} component, but the state variable σ should be evaluated at the same hybrid time in the derivations at the end of the proof of Lemma 2.

which is clearly positive definite with respect to \mathcal{A}_ε and radially unbounded. Since outside set \mathcal{A}_ε the hybrid dynamics \mathcal{H}_ε coincides with the one of \mathcal{H} , then equations (13) and (14) hold for any (\tilde{x}, σ) not in \mathcal{A}_ε , which implies that

$$\langle \nabla V_\varepsilon(\tilde{x}), f(x, \sigma) \rangle < 0 \quad \forall \tilde{x} \in \mathcal{C}_\varepsilon \setminus \mathcal{A}_\varepsilon \quad (35)$$

$$V_\varepsilon(\tilde{x}^+) - V_\varepsilon(\tilde{x}) = 0, \quad \forall \tilde{x} \in \mathcal{D}_\varepsilon \setminus \mathcal{A}_\varepsilon \quad (36)$$

Moreover, from the property established in item 3), all complete solutions to \mathcal{H}_ε must flow for some time, and therefore from (35), we have that no solution can keep V_ε constant and non-zero. UGAS of \mathcal{A}_ε by applying the nonsmooth invariance principle in [18], also using the well posedness result established at the beginning of the proof.

Proof of item 2). The proof follows in a straightforward way from the previous item, after noticing that given any neighborhood \mathcal{I} of \mathcal{A} , there exists a small enough $\varepsilon > 0$ such that $\mathcal{A}_\varepsilon \subset \mathcal{I}$. \square

B. Time regularization

Based on Lemma 2, for any value of $T < T^*$, we may introduce the following additional state variable τ to dynamics (5):

$$\mathcal{H}_T : \begin{cases} \begin{cases} \dot{\tilde{x}} = f(x, \sigma), \\ \dot{\tau} = 1 - \text{dz}\left(\frac{\tau}{T}\right), \end{cases} & (x, \sigma) \in \mathcal{C}_T \\ \begin{cases} \begin{bmatrix} x^+ \\ \sigma^+ \end{bmatrix} \in G(x, \sigma), \\ \tau^+ = 0, \end{cases} & (x, \sigma) \in \mathcal{D}_T, \end{cases} \quad (37a)$$

where the dz denotes the non-negative side of the unit deadzone function defined as $\text{dz}(s) := \max\{0, s - 1\}$, for all $s \geq 0$ and the jump and flow sets are the following time-regularized versions of \mathcal{C} and \mathcal{D} in (5)–(8):

$$\begin{aligned} \mathcal{C}_T &:= \mathcal{C} \times [0, 2T] \cup \{(x, u, \tau) : \tau \in [0, T]\} \\ \mathcal{D}_T &:= \mathcal{D} \times [T, 2T]. \end{aligned} \quad (37b)$$

The above regularization is clearly motivated by the fact that jumps are forbidden when the timer τ is too small, namely not enough time has elapsed since the last jump. Then all solutions are forced to flow for at least T ordinary time after each jump. Note also that the deadzone trick at the right hand side of equation (37a) allows a solution to flow forever while ensuring that timer τ remains in a compact set.

Before proceeding any further, we emphasize that forcing a solution to flow regardless of whether it belongs to \mathcal{D} or not, may lead to an increase of function V . It is useful to quantify how much increase V can experience from the set where $V(\tilde{x}) \leq \varepsilon_T$ (where ε_T is introduced in Lemma 2). To this end, we exploit the affine nature of the dynamics and observe that along solutions of (37) we have $\dot{V}(\tilde{x}) \leq |\tilde{x}| |P| |\dot{\tilde{x}}| = |\tilde{x}| |P| |\dot{\tilde{x}}| \leq |\tilde{x}| |P| (\kappa_1 |\tilde{x}| + \kappa_2) \leq 2\alpha V(\tilde{x}) + 2\beta \sqrt{V(\tilde{x})}$, where α and β are large enough positive scalars and where we used (10) and the positive definiteness of P in the expression of V . Proceeding as in

[12, page 203], we obtain along any solution ϕ satisfying $(t, j) \in \text{dom } \phi$ and $(t + T, j) \in \text{dom } \phi$,

$$\begin{aligned} \sqrt{V(\phi(t + \tau, j))} &\leq e^{\alpha\tau} \sqrt{V(\phi(t, j))} + \beta \int_0^\tau e^{\alpha s} ds \\ &= e^{\alpha\tau} \sqrt{V(\phi(t, j))} + \frac{\beta}{\alpha} (e^{\alpha\tau} - 1), \quad \forall \tau \in [0, T]. \end{aligned}$$

Therefore, assuming that $V(\phi(t, j)) \leq \varepsilon_T$, we obtain for all $\tau \in [0, T]$,

$$V(\phi(t + \tau, j)) \leq \varepsilon_V(T) := e^{\alpha T} \varepsilon_T + \frac{\beta}{\alpha} (e^{\alpha T} - 1). \quad (38)$$

This bound motivates introducing the following set:

$$\mathcal{E}_T := \{(\tilde{x}, u, \tau) : V(\tilde{x}) \leq \varepsilon_V(T), \sigma \in \bar{N}, \tau \in [0, 2T]\}, \quad (39)$$

which enjoys the nice property of shrinking to $\mathcal{A} \times \{0\}$, as T converges to zero.

Mainly using Lemma 2 the following desirable results are enjoyed by hybrid system \mathcal{H}_T in (37).

Theorem 4: Consider point x_e and a vector λ_e satisfying Assumption 1 and matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ satisfying Property 1. The following holds:

- 1) all solutions to \mathcal{H}_T enjoy a dwell-time property corresponding to T ;
- 2) for any positive scalar $T < T^*$, there exists a compact set $\mathcal{A}_T \subset \mathcal{E}_T$ which is UGAS for dynamics \mathcal{H}_T in (37);
- 3) set $\mathcal{A} \times \{0\}$ is globally practically asymptotically stable for (37), with respect to parameter T (namely as long as T is sufficiently small, the UGAS set \mathcal{A}_T characterized in the previous item can be made arbitrarily close to $\mathcal{A} \times \{0\}$).

Proof. Similar to the proof of Theorem 3 we start by noticing that hybrid system (37) enjoys the hybrid basic conditions of [9, As. 6.5], because sets \mathcal{C}_T and \mathcal{D}_T are both closed and f and G enjoy desirable properties. Then we may apply several useful results pertaining well-posed hybrid systems (specifically, in the proof of item 2 below).

Proof of item 1. The dwell-time property of solutions follows in a straightforward way from the fact that solutions are forced to not jump until the timer variable τ has reached the value τ . Since $\dot{\tau} = 1$ for all $\tau \leq T$, then all solutions flow for at least T ordinary time after each jump (because $\tau^+ = 0$ across jumps).

Proof of item 2. Consider the two hybrid systems \mathcal{H}_ε and \mathcal{H}_T in (32) and (37), respectively. For any positive value of $T < T^*$, we have shown in the proof of item 4 of Theorem 3 that it suffices to pick $\varepsilon = \varepsilon_T$ (coming from Lemma 2) to obtain UGAS of the attractor $\mathcal{A}_{\varepsilon_T}$ in (33) and a dwell time of T for all solutions to $\mathcal{H}_{\varepsilon_T}$.

Since the (x, σ) dynamics of $\mathcal{H}_{\varepsilon_T}$ and \mathcal{H}_T coincide, except for the dwell time restriction on \mathcal{H}_T , the above mentioned dwell-time property of solutions to $\mathcal{H}_{\varepsilon_T}$ ensures that (possibly after an initial flow of at most T ordinary time) the (x, σ) component of each solution to \mathcal{H}_T remaining outside $\mathcal{A}_{\varepsilon_T} \times [0, 2T]$, coincides with a solution to $\mathcal{H}_{\varepsilon_T}$, therefore any such solution to \mathcal{H}_T must approach

$\mathcal{A}_{\epsilon_T} \times [0, 2T]$, which is a strict subset of \mathcal{E}_T in (39). Two things may happen then. Either the solution approaches $\mathcal{A}_{\epsilon_T} \times [0, 2T]$ without ever reaching it, so it eventually remains in \mathcal{E}_T , or it reaches $\mathcal{A}_{\epsilon_T} \times [0, 2T]$ and may then be forced to flow by the dwell-time logic of \mathcal{H}_T . However in this last case we get from bound (38) that such solution cannot flow outside \mathcal{E}_T . As a consequence, \mathcal{E}_T is uniformly attractive and reached in finite time by all solutions, in addition to being strongly forward invariant for \mathcal{H}_T .

We now use the well-posedness property established at the beginning of the proof to exploit a number of regularity results from [9, Ch. 6 & 7]. Denote by $\Omega(\mathcal{E}_T)$ the ω -limit set of \mathcal{E}_T (see [9, Def. 6.23]) and note that it cannot be empty, and must satisfy $\Omega(\mathcal{E}_T) \subset \mathcal{E}_T$, because \mathcal{E}_T is bounded and strongly forward invariant. Then using again strong forward invariance of \mathcal{E}_T we get boundedness of all solutions starting from \mathcal{E}_T and we may apply [9, Prop. 6.26] to obtain that $\Omega(\mathcal{E}_T)$ is compact, nonempty, uniformly attractive from \mathcal{E}_T , and strongly forward invariant. Since also \mathcal{E}_T is uniformly attractive, we may then apply a global version ² of [9, Prop. 7.5] applied to the compact attractor $\Omega(\mathcal{E}_T)$, to conclude global asymptotic stability of $\Omega(\mathcal{E}_T)$, which is equivalent to UGAS from [9, Thm 3.40 & Thm 7.12].

Proof of item 3. Item 3 follows in a straightforward way by recalling from Lemma 2 that ε_T converges to zero as T goes to zero, and then that also $\varepsilon_V(T)$ in (38) enjoys the same property. As a consequence, set \mathcal{E}_T in (39) shrinks to $\mathcal{A} \times \{0\}$ as T goes to zero, and since we established in item 2 that $\mathcal{A}_T \subset \mathcal{E}_T$ for all $T > 0$, we can make \mathcal{A}_T arbitrarily close to $\mathcal{A} \times \{0\}$ by selecting T sufficiently small. \square

V. CONCLUSIONS AND FUTURE WORK

This article deals with the practical stabilization of switched affine systems controlled by using a hybrid dynamical systems paradigm that allows managing the switching pseudo-frequency during transients and at the steady state. We showed practical asymptotic stability results using a space- and time-regularization that allows: 1) stabilising the set where $V(x - x_e) \leq \varepsilon$ being ε a sufficiently small tuning parameter enabling the adjustment of the steady-state switching pseudo-frequency, 2) inducing a positive minimal dwell-time in each mode, T during transient operation, whose size can be adjusted by tuning the parameter η , 3) obtaining a tradeoff between transient dwell-time on the one hand and an LQR performance level, tuning the parameter η that, in addition to affecting the transient dwell time, also affects the LQ optimality level during the transient responses. Simulations and experimental results were not included here due to space constraints and are regarded as future work that will be published in application-related journals.

²A global version of [9, Prop. 7.5] is trivially obtained by establishing its hypotheses for any arbitrary positive value of μ .

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