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► **To cite this version:**

Emmanuel Russ. H 1 – L 1 boundedness of Riesz transforms on Riemannian manifolds and on graphs. Potential Analysis, 2001, 14, pp.301-330. hal-01617991

HAL Id: hal-01617991

<https://hal.science/hal-01617991>

Submitted on 17 Oct 2017

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$H^1 - L^1$ boundedness of Riesz transforms on Riemannian manifolds and on graphs

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11/12/99

Abstract

We prove that the Riesz transforms are bounded from H^1 to L^1 on complete Riemannian manifolds and on graphs with the doubling property and the Poincaré inequality.

AMS Subject Classification (1991): 42B20, 42B30, 58G11.

Key words: Riesz transforms, heat kernel, Markov kernel, H_{at}^1 , Hörmander integral condition.

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Part I

Introduction

In [6], Coulhon and Duong showed that Riesz transforms on manifolds with the doubling property and a on-diagonal upper bound for the heat kernel are L^p -bounded for $1 < p \leq 2$. The corresponding discrete result, namely that Riesz transforms are L^p -bounded on graphs with the doubling property and a on-diagonal upper bound of the Markov kernel, is shown in [13]. In both cases, the proofs go through the L^1 case. More precisely, it is shown that the Riesz transforms are bounded from L^1 to $L^{1,\infty}$, which gives the result by interpolation between L^1 and L^2 .

In the context of those two papers, it is not clear whether the Riesz transforms are bounded from H^1 to L^1 .

Concerning the $H^1 - L^1$ boundedness, a positive result is that, on manifolds with nonnegative Ricci curvature, the Riesz transforms are $H^1 - L^1$ bounded. This is shown by Bakry in [3] with a probabilistic definition for H^1 , and by Cheng and Luo in [4], who consider the maximal H^1 space, defined by means of the heat kernel. Cheng and Luo strongly use the assumption about the Ricci curvature to get pointwise estimates about the heat kernel and show that the Riesz transforms are Calderon-Zygmund operators. They also prove that H_{at}^1 , H_{max}^1 and H_P^1 (the probabilistic space considered by Bakry) coincide.

The issue of the $H^1 - H^1$ boundedness of Riesz transforms has been investigated in \mathbb{R}^n . In [15], p.232, E. M. Stein proves that the Riesz transforms on \mathbb{R}^n are $H^1 - H^1$ bounded. In [1], P. Auscher and P. Tchamitchian show that the Riesz transform associated to an operator of the form $-\operatorname{div}(A\nabla)$ where $A : \mathbb{R}^n \rightarrow M_n(\mathbf{C})$ is a uniformly elliptic operator, is $H^1 - H^1$ bounded (Chapter 4, Proposition 10).

In the present paper, we prove the $H^1 - L^1$ boundedness in a geometric setting. Namely, we prove that Riesz transforms are $H^1 - L^1$ bounded on Riemannian manifolds and $H^1 - L^1$ bounded on graphs satisfying the doubling property and the Poincaré inequality. The result for manifolds encompasses Cheng and Luo's result, since, on complete Riemannian manifolds with non-negative Ricci curvature, the doubling property and the Poincaré inequality hold.

The basic fact used in the proof is the following result. Let (X, d, μ) be a space of homogeneous type, and T be a bounded linear operator on $L^2(X)$. Assume that there exists a measurable function $k(x, y)$ (a kernel) such that, for any function f with compact support and almost all x outside the support of f ,

$$Tf(x) = \int k(x, y)f(y)d\mu(y).$$

If there exist two constants C and $\delta > 1$ such that, for any $y, y_0 \in X$,

$$\int_{d(x,y) \geq \delta d(y_0,y)} |k(x, y) - k(x, y_0)| d\mu(x) \leq C, \quad (1)$$

then T is $H_{at}^1 - L^1$ bounded. The condition (1) is called the Hörmander integral condition.

Since the Riesz transforms are L^2 -bounded and are given by a kernel $k(x, y)$, it will be sufficient to show that k satisfies the Hörmander integral condition (1). The main result used to prove that is the Hölder regularity for solutions of the heat equation, which is a consequence of the parabolic Harnack principle, shown by L. Saloff-Coste in [14] in the case of manifolds, and by T. Delmotte in [8] in the case of graphs. Thus, we get that the Riesz transforms map continuously H^1 into L^1 .

The paper is made up as follows. First, we prove the $H^1 - L^1$ boundedness of Riesz transforms on complete Riemannian manifolds with the doubling property and the Poincaré inequality, which generalizes Bakry's and Chen and Luo's results.

Then, we give a local version of this result.

Finally, we prove the $H^1 - L^1$ boundedness of Riesz transforms on graphs with the doubling property and the Poincaré inequality. The strategy is similar but the discrete setting creates some additional technical difficulties.

I thank P. Auscher, T. Coulhon, X. T. Duong, E. M. Ouhabaz and F. Ricci for useful conversations and advice.

Part II

Preliminaries

Let X be a set equipped with a distance d and a positive measure μ . For $x \in X$ and $r > 0$, denote by $B(x, r)$ the open ball centered at x and of radius r and define $V(x, r) = \mu(B(x, r))$. Assume that each ball has finite measure. Assume also that μ has the doubling property, which means that there exists $C > 0$ such that, for any $x \in X$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

Such a space is called a space of homogeneous type.

An atom a on X is a function $a : X \rightarrow \mathbb{R}$ supported in a ball B and satisfying:

$$\begin{aligned} \int a(x) d\mu(x) &= 0, \\ \|a\|_\infty &\leq \frac{1}{V(B)}. \end{aligned} \tag{2}$$

A function $f : X \rightarrow \mathbb{R}$ is said to belong to H_{at}^1 if there exist a sequence $(\lambda_n)_{n \in \mathbb{N}} \in l^1$ and a sequence of atoms $(a_n)_{n \in \mathbb{N}}$ such that

$$f = \sum_n \lambda_n a_n$$

where the convergence is to be understood in the L^1 sense. When $f \in H_{at}^1$, set

$$\|f\|_{H_{at}^1} = \inf \sum_n |\lambda_n|,$$

the infimum being taken over all such decompositions of f .

Let T be a bounded linear operator on $L^2(X)$. Assume that this operator is given by a kernel $k(x, y)$, which means that there exists a measurable function k on $X \times X$ such that, for any function f with compact support and almost every $x \in X$ outside the support of f ,

$$Tf(x) = \int k(x, y) f(y) d\mu(y).$$

One says that k satisfies the integral Hörmander condition if there exists $C > 0$ and $\delta > 1$ such that, for any y and $y_0 \in X$,

$$\int_{d(x, y) \geq \delta d(y_0, y)} |k(x, y) - k(x, y_0)| d\mu(x) \leq C.$$

Then the following result holds:

Theorem 1 *Let (X, d, μ) be a space of homogeneous type and T a bounded linear operator on $L^2(X)$ given by a kernel k satisfying the integral Hörmander condition. Then there exists a constant $C > 0$ such that, for any atom a on X , $\|Ta\|_1 \leq C$. Hence, T can be extended to a bounded operator from $H_{at}^1(X)$ to $L^1(X)$.*

For a proof, see [5], p. 599. ■

When f is a function in $L_{loc}^1(X)$ and B is a ball of X , define

$$f_B = \frac{1}{V(B)} \int f(x) d\mu(x).$$

Say that $f \in L^1_{loc}(X)$ belongs to $BMO(X)$ if

$$\sup_B \frac{1}{V(B)} \int_B |f(x) - f_B| d\mu(x) < \infty$$

where the supremum is taken over all the balls B of X , and, if it is the case, set

$$\|f\|_{BMO} = \sup_B \frac{1}{V(B)} \int_B |f(x) - f_B| d\mu(x).$$

An important property of BMO functions, which will be used later, is the fact that there exists $C > 0$ such that, for any $f \in BMO(X)$ and any ball B of X ,

$$\frac{1}{V(B)} \int_B |f(x) - f_B|^2 d\mu(x) \leq C \|f\|_{BMO}^2. \quad (3)$$

As a consequence, there exists $C > 0$ such that, for any $f \in BMO(X)$, any ball B of X and any integer $k \geq 0$,

$$\frac{1}{V(2^k B)} \int_{2^k B} |f(x) - f_B|^2 d\mu(x) \leq C(k+1)^2 \|f\|_{BMO}^2. \quad (4)$$

Actually, (3) is equivalent to the definition of BMO . For a proof of those facts, see, for instance, [5], Theorem B. Once (3) is proved, it is easy to deduce (4). Indeed, consider the space $2^k B$ equipped with the measure $\frac{1}{V(2^k B)} \mu$ and write that

$$\begin{aligned} \|f - f_B\|_2 &\leq \sum_{i=0}^{k-1} \|f_{2^{i+1}B} - f_{2^i B}\|_2 + \|f - f_{2^k B}\| \\ &\leq Ck \|f\|_{BMO} + C \|f\|_{BMO} \\ &= C(k+1) \|f\|_{BMO}. \end{aligned}$$

In the second line, the first term follows from the very definition of BMO , the second from (3). Thus, (4) is proved. ■

Define VMO as being the closure of $C_0(X)$ (the space of all continuous functions on X with compact support) in BMO . Then, the following statement is true (see [5], Theorem 4.1, p. 638):

Theorem 2 *The dual of VMO is H^1_{at} . More precisely, if $f \in H^1_{at}(X)$, the linear functional given by*

$$v \rightarrow \int f v d\mu,$$

initially defined for $v \in C_0(X)$, has a unique bounded extension to $VMO(X)$, with a norm equivalent to $\|f\|_{H^1_{at}(X)}$. Moreover, every continuous linear functional on $VMO(X)$ has this form.

One can state a local version of Theorem 1. Say that X satisfies the local doubling property if

$$\forall R > 0, \exists C_R > 0, \forall x \in X, \forall r \in]0, R[, V(x, 2r) \leq C_R V(x, r). \quad (5)$$

A local atom is a function $a : X \rightarrow \mathbb{R}$ supported in a ball B of radius ≤ 1 and satisfying (2).

A function f belongs to $H^{1,loc}_{at}$ if $f = \sum_n \lambda_n a_n$ where $(\lambda_n)_{n \in \mathbb{N}} \in l^1$ and the a_n 's are local atoms. The $H^{1,loc}_{at}$ norm is defined as previously.

Consider an operator T bounded on $L^2(X)$. Assume that T is given by a kernel $k(x, y)$. Say that k satisfies the local Hörmander integral condition if there exists a constant $C > 0$ such that, for any $y, y_0 \in X$ with $d(y, y_0) \leq 1$,

$$\int_{d(x,y) \geq 2d(y,y_0)} |k(x, y) - k(x, y_0)| d\mu(x) \leq C.$$

Then the following statement is true:

Theorem 3 *Let (X, d, μ) be a metric space satisfying the local doubling property. Let T be an operator bounded on $L^2(X)$, given by a kernel $k(x, y)$ satisfying the local integral Hörmander condition. Then, there exists a constant $C > 0$ such that, for any local atom a , $\|Ta\|_1 \leq C$. Therefore, T may be extended to a bounded operator from $H_{at}^{1,loc}(X)$ to $L^1(X)$.*

The proof is similar to the proof of Theorem 1. ■

Part III

The case of Riemannian manifolds

1 Statement of the main result

Let M be a complete Riemannian manifold, d the geodesic distance on M and μ the Riemannian measure on M . For $x \in M$ and $r > 0$, denote by $B(x, r)$ the geodesic ball centered at x and of radius r and by $V(x, r)$ its volume.

Say that M satisfies the doubling property if and only if there exists a constant $C > 0$ such that, for any $x \in M$ and any $r > 0$,

$$V(x, 2r) \leq CV(x, r). \quad (6)$$

This property implies that there exist two constants $C_{vol} > 0$ and $D > 0$ such that, for any $x \in M$, $r > 0$ and $\theta > 1$,

$$V(x, \theta r) \leq C_{vol} \theta^D V(x, r). \quad (7)$$

Assume also that the Poincaré inequality holds on M , which means that

$$\forall r > 0, \forall x \in M, \int_B |f - f_B|^2 d\mu \leq Cr^2 \int_{2B} |\nabla f|^2 d\mu \quad (8)$$

for all $f \in C^\infty(2B)$, where $B = B(x, r)$, $2B = B(x, 2r)$ and f_B is the mean of f over B defined by

$$f_B = \frac{1}{V(B)} \int_B f.$$

Denote by Δ the Laplace-Beltrami operator on M , by p_t the heat kernel, i. e. the kernel of $e^{-t\Delta}$ and by ∇ the Riemannian gradient.

The definitions of an atom and of $H_{at}^1(M)$ are those of the previous section (notice that M is a space of homogeneous type). We intend to show the following result:

Theorem 4 *Let M be a complete Riemannian manifold satisfying the doubling property and the Poincaré inequality. Then the Riesz transform $T = \nabla \Delta^{-\frac{1}{2}}$ is bounded from $H_{at}^1(M)$ to $L^1(M)$.*

Before proving that result, we give a few comments about it. One may define the space $H_{max}^1(M)$ in the following way. For $f \in L_{loc}^1(M)$, $t > 0$ and $x \in M$, define

$$P_t f(x) = \int p_t(x, y) f(y) d\mu(y)$$

and

$$f^+(x) = \sup_t |P_t f(x)|.$$

Say that $f \in H_{max}^1(M)$ if $f^+ \in L^1(M)$ and, if it is the case, define

$$\|f\|_{H_{max}^1} = \|f^+\|_1.$$

In [12], the following statement is proved:

Theorem 5 *Let M be a complete Riemannian manifold satisfying the doubling property and the Poincaré inequality. Then $H_{at}^1(M) = H_{max}^1(M)$.*

Thus, Theorem 4 implies that the Riesz transform is bounded from $H_{max}^1(M)$ to $L^1(M)$.

Recall that, when M has nonnegative Ricci curvature, the assumptions of Theorem 4 hold. In this setting, D. Bakry has shown in [3], Proposition 5.3, that the Riesz transforms are bounded from H_P^1 (the probabilistic Hardy space) to L^1 . Moreover, in [4], Cheng and Luo prove, again on a manifold with nonnegative Ricci curvature, that they are bounded from H_{max}^1 to L^1 , and that $H_{max}^1 = H_{at}^1 = H_P^1$. Recall also that a manifold which is roughly isometric to a manifold with non-negative Ricci curvature also satisfies the assumptions of Theorem 4 (see [7]). Finally, remember that a co-compact covering whose deck transformation group has polynomial growth also satisfies the assumptions of Theorem 4 (see [7]).

We are going to show Theorem 4. Since the Riesz transform is clearly L^2 -bounded and is given by a kernel k , it is sufficient, in view of Theorem 1, to show that k satisfies the integral Hörmander condition. To that aim, we first show estimates about some kernels which follow from the assumptions about M .

2 Kernel estimates

In what follows, we denote by p_t the heat kernel. Moreover, if y and y_0 are two fixed points in M , define, for all $x \in M$,

$$q_t(x) = p_t(x, y) - p_t(x, y_0).$$

This section is devoted to various bounds on p_t and q_t .

First, recall that, when (6) and (8) hold, one has the following estimates about p_t (see [14], Proposition 3.3):

Lemma 6 *There exist $c_1, c_2, C_1, C_2 > 0$ such that, for all $x, y \in M$ and all $t > 0$,*

$$\frac{c_1}{V(x, \sqrt{t})} e^{-\frac{c_1 d^2(x, y)}{t}} \leq p_t(x, y) \leq \frac{C_2}{V(x, \sqrt{t})} e^{-\frac{c_2 d^2(x, y)}{t}}.$$

As a consequence of the Gaussian upper bound in Lemma 6 and the doubling property (6), the following estimate is valid (see [6], Lemma 2.3 and Lemma 2.4):

Lemma 7 *For all $\gamma \in]0, 2c_2[$, there exists $C_\gamma > 0$ such that, for any $y \in M$ and any $s > 0$,*

$$\int_M |\nabla_x p_s(x, y)|^2 e^{\gamma \frac{d^2(x, y)}{s}} d\mu(x) \leq \frac{C_\gamma}{V(y, \sqrt{s})} s^{-1}.$$

Lemma 8 *There exists $\beta > 0$ such that, for all $y \in M$ and all $s, t > 0$,*

$$\int_{d(x, y) \geq t^{\frac{1}{2}}} |\nabla_x p_s(x, y)| d\mu(x) \leq C e^{-\frac{\beta}{t} s} s^{-\frac{1}{2}}.$$

One also has the following estimate about q_t :

Lemma 9 *There exist $C_3, c_3 > 0$ and $\gamma \in]0, 1[$ such that, for any $t > 0$ such that $d(y_0, y) \leq \sqrt{t}$ and any $x \in M$,*

$$|q_t(x)| \leq \frac{C_3}{V(x, \sqrt{t})} \left[\frac{d(y, y_0)}{\sqrt{t}} \right]^\gamma \exp\left(-\frac{c_3 d^2(x, y)}{t}\right). \quad (9)$$

Lemma 9 is a straightforward consequence of the Hölderian regularity of solutions of the heat equation. Indeed, one has the following proposition (see [14], Proposition 3.2):

Proposition 10 *Let $\delta \in]0, 1[$. Then, there exist $C > 0$ and $\gamma \in]0, 1[$ such that, given $x \in M$, $s \in \mathbb{R}$ and $r > 0$, any solution u of $(\frac{\partial}{\partial t} + \Delta)u = 0$ in $Q =]s, s + r^2[\times B(x, r)$ satisfies*

$$|u(t, x_1) - u(t, x_2)| \leq C \left[\frac{d(x_1, x_2)}{r} \right]^\gamma \|u\|_{\infty, Q},$$

where $(t, x_i) \in]s + \delta r^2, s + r^2[\times B(x, (1 - \delta)r)$, $i = 1, 2$.

We apply Proposition 10 with $\delta = \frac{1}{8}$. Indeed, assume that $d(y_0, y) \leq \sqrt{t}$. Then, if $r = \frac{4}{3}\sqrt{t}$ and $s = \frac{4}{9}t$, one sees that $d(y_0, y) < (1 - \delta)r$ and that $s + \delta r^2 < t < s + r^2$. Fix $x \in M$ and set $u(t, z) = p_t(x, z)$. Proposition 10 shows that

$$\begin{aligned} |p_t(x, y) - p_t(x, y_0)| &\leq C \left[\frac{d(y_0, y)}{r} \right]^\gamma \sup_{(\tau, z) \in Q} p_\tau(x, z) \\ &\leq C' \left[\frac{d(x, y)}{\sqrt{t}} \right]^\gamma \sup_{(\tau, z) \in Q} p_\tau(x, z) \end{aligned}$$

where $Q =]\frac{4}{9}t, \frac{20}{9}t[\times B(y, \frac{4}{3}\sqrt{t})$. But Lemma 6 gives the following estimate: if $(\tau, z) \in Q$, then

$$p_\tau(x, z) \leq \frac{c_2}{V(x, \sqrt{\tau})} \exp(-\frac{C_2 d^2(x, z)}{\tau}).$$

When $z \in B(y, \frac{4}{3}\sqrt{t})$, $d(x, y) \leq d(x, z) + \frac{4}{3}\sqrt{t}$, so that, since $2r^2 \geq \tau \geq r^2$, one has

$$p_\tau(x, z) \leq \frac{c'_2}{V(x, \sqrt{t})} \exp(-\frac{C'_2 d^2(x, y)}{t}).$$

It follows that

$$|p_t(x, y) - p_t(x, y_0)| \leq \frac{C_3}{V(x, \sqrt{t})} \left[\frac{d(y, y_0)}{\sqrt{t}} \right]^\gamma \exp(-\frac{c'_3 d^2(x, y)}{t}).$$

Lemma 9 is proved. ■

Fix y_0 and y in M . According to the previous lemma, if $d(y, y_0) \leq \sqrt{t}$,

$$|q_t(x)| \leq \frac{C_3}{V(x, \sqrt{t})} \left[\frac{d(y, y_0)}{\sqrt{t}} \right]^\gamma e^{-c_3 \frac{d^2(x, y)}{t}}.$$

It is easy to deduce the following bound:

Lemma 11 *If $d(y_0, y) \leq \sqrt{t}$, then, for any $\alpha < 2c_3$, there exists $C_\alpha > 0$ such that*

$$\int |q_t(x)|^2 \exp \left[\alpha \frac{d^2(x, y)}{t} \right] d\mu(x) \leq \left[\frac{d(y_0, y)}{\sqrt{t}} \right]^{2\gamma} \frac{C_\alpha}{V(y, \sqrt{t})}.$$

Indeed, making use of Lemma 9 and of assumption (7), one gets that

$$\begin{aligned} \int |q_t(x)|^2 \exp \left[\alpha \frac{d^2(x, y)}{t} \right] d\mu(x) &\leq \left[\frac{d(y_0, y)}{\sqrt{t}} \right]^{2\gamma} \int \frac{C_3^2}{V^2(x, \sqrt{t})} \exp(\frac{(\alpha - 2c_3)d^2(x, y)}{t}) d\mu(x) \\ &\leq \left[\frac{d(y_0, y)}{\sqrt{t}} \right]^{2\gamma} \frac{C}{V^2(y, \sqrt{t})} \int \exp(\frac{-c_\alpha d^2(x, y)}{t}) d\mu(x) \\ &\leq \left[\frac{d(y_0, y)}{\sqrt{t}} \right]^{2\gamma} \frac{C_\alpha}{V(y, \sqrt{t})}. \blacksquare \end{aligned}$$

The estimate we will need in the following is a weighted L^2 estimate of the gradient of q_t , namely:

Lemma 12 *If $d(y_0, y) \leq \sqrt{t}$, then, for any $\alpha < 2c_3$, there exists $C'_\alpha > 0$ such that*

$$\int |\nabla_x q_t(x)|^2 \exp \left[\alpha \frac{d^2(x, y)}{t} \right] d\mu(x) \leq \frac{1}{t} \left[\frac{d(y_0, y)}{\sqrt{t}} \right]^{2\gamma} \frac{C'_\alpha}{V(y, \sqrt{t})}.$$

In the proof of this Lemma, we follow closely [10], p. 370. Let α be as in Lemma 12. Define

$$\xi(x, t) = \alpha \frac{d^2(x, y)}{t}$$

and observe that, for almost every $x \in M$,

$$\frac{\partial \xi}{\partial t} + \frac{1}{4\alpha} |\nabla \xi|^2 \leq 0. \quad (10)$$

Set

$$f(t) = \int |\nabla_x q_t(x)|^2 \exp \left[\alpha \frac{d^2(x, y)}{t} \right] d\mu(x).$$

From now on, the variables t and x are not explicitly written. On the one hand, integrating by parts, one gets that

$$f(t) = - \int q e^\xi \Delta q - \int q \nabla q \cdot \nabla (e^\xi)$$

and the Cauchy-Schwarz inequality shows that

$$f(t) \leq \left[\int q^2 e^\xi \right]^{\frac{1}{2}} \left[\left[\int e^\xi (\Delta q)^2 \right]^{\frac{1}{2}} + \left[\int e^\xi (\nabla q \cdot \nabla \xi)^2 \right]^{\frac{1}{2}} \right]. \quad (11)$$

On the other hand, computing the time derivative of f , one gets

$$\begin{aligned} f'(t) &= 2 \int e^\xi \nabla \left(\frac{\partial q}{\partial t} \right) \cdot \nabla q + \int \frac{\partial \xi}{\partial t} e^\xi |\nabla q|^2 \\ &\leq 2 \int e^\xi \nabla (\Delta q) \cdot \nabla q - \frac{1}{4\alpha} \int e^\xi |\nabla q|^2 |\nabla \xi|^2 \\ &= -2 \int e^\xi (\Delta q)^2 - 2 \int e^\xi (\Delta q) (\nabla q \cdot \nabla \xi) - \frac{1}{4\alpha} \int e^\xi |\nabla q|^2 |\nabla \xi|^2 \\ &\leq -2 \int e^\xi (\Delta q)^2 + 2 \left[\int e^\xi (\Delta q)^2 \right]^{\frac{1}{2}} \left[\int e^\xi |\nabla q|^2 |\nabla \xi|^2 \right]^{\frac{1}{2}} - \frac{1}{4\alpha} \int e^\xi |\nabla q|^2 |\nabla \xi|^2. \end{aligned} \quad (12)$$

The second line follows from (10), the third one by integration by parts. Squaring (11), dividing it by $c \int e^\xi q^2$ where c is a positive number which will be chosen later and adding to (12) leads to

$$\begin{aligned} f'(t) + c \frac{f^2(t)}{\int e^\xi q^2} &\leq (-2 + c) \int e^\xi (\Delta q)^2 \\ &+ (2 + 2c) \left[\int e^\xi (\Delta q)^2 \right]^{\frac{1}{2}} \left[\int e^\xi |\nabla q|^2 |\nabla \xi|^2 \right]^{\frac{1}{2}} \\ &+ \left(c - \frac{1}{4\alpha} \right) \left[\int e^\xi |\nabla q|^2 |\nabla \xi|^2 \right]. \end{aligned} \quad (13)$$

Choose $c = \frac{2-4\alpha}{1+16\alpha}$: one obtains

$$f'(t) + c \frac{f^2(t)}{\int e^\xi q^2} \leq 0.$$

Define

$$\phi(t) = \left[\frac{d(y, y_0)}{\sqrt{t}} \right]^{2\gamma} \frac{C_\alpha}{V(y, \sqrt{t})},$$

and, using Lemma 11, write that

$$\frac{f'(t)}{f^2(t)} \leq -\frac{c}{\phi(t)}.$$

An integration on $[0, t]$ implies that

$$f(t) \leq \frac{1}{c \int_0^t \frac{du}{\phi(u)}}.$$

Since

$$\begin{aligned} \int_0^t \frac{du}{\phi(u)} &\geq C \int_{\frac{t}{2}}^t \left(\frac{u}{d(y_0, y)^2}\right)^\gamma V(y, \sqrt{u}) du \\ &\geq C \frac{t}{2} \left(\frac{t}{d(y_0, y)^2}\right)^\gamma V(y, \sqrt{\frac{t}{2}}), \end{aligned}$$

one finally gets, using the doubling property, that

$$f(t) \leq \frac{C'_\alpha}{t} \left(\frac{d(y_0, y)^2}{t}\right)^\gamma \frac{1}{V(y, \sqrt{t})}.$$

Lemma 12 is therefore proved. ■

As an immediate consequence of this lemma, we get:

Lemma 13 *There exists $C' > 0$ and $\alpha > 0$ such that, if $d(y_0, y) \leq \sqrt{t}$, then, for any $r_0 > 0$,*

$$\int_{d(x, y) \geq r_0} |\nabla_x q_t(x)| d\mu(x) \leq \frac{C'}{\sqrt{t}} \left[\frac{d(y_0, y)}{\sqrt{t}}\right]^\gamma e^{-\alpha \frac{r_0^2}{t}}.$$

Choose $\alpha < 2c_3$. Then, using Lemma 12, write that

$$\begin{aligned} \int_{d(x, y) \geq r_0} |\nabla_x q_t(x)| d\mu(x) &\leq \left[\int |\nabla_x q_t(x)|^2 \exp\left[\alpha \frac{d^2(x, y)}{t}\right] d\mu(x) \right]^{\frac{1}{2}} \\ &\quad \left[\int_{d(x, y) \geq r_0} \exp\left[-\alpha \frac{d^2(x, y)}{t}\right] d\mu(x) \right]^{\frac{1}{2}} \\ &\leq \frac{C'}{\sqrt{t}} \left(\frac{d(y_0, y)}{\sqrt{t}}\right)^\gamma \frac{1}{\sqrt{V(y, \sqrt{t})}} \sqrt{V(y, \sqrt{t})} e^{-\frac{\alpha}{2} \frac{r_0^2}{t}} \\ &\leq \frac{C'}{\sqrt{t}} \left(\frac{d(y_0, y)}{\sqrt{t}}\right)^\gamma e^{-\frac{\alpha}{2} \frac{r_0^2}{t}} \end{aligned}$$

which proves Lemma 13. ■

3 $H^1 - L^1$ boundedness

We prove that the Riesz transform maps continuously H^1 into L^1 . As in [6], write that

$$\nabla \Delta^{-\frac{1}{2}} = \int_0^{+\infty} \nabla e^{-t\Delta} \frac{dt}{\sqrt{t}}.$$

This integral representation shows that T is given by the following kernel k :

$$k(x, y) = \int_0^{+\infty} \nabla_x p_t(x, y) \frac{dt}{\sqrt{t}}.$$

As was explained before, to get Theorem 4, we just have to check that k satisfies the integral Hörmander condition (1). Let y and y_0 be two fixed points in M . Then

$$\begin{aligned}
\int_{d(x,y) \geq 2d(y_0,y)} |k(x,y) - k(x,y_0)| d\mu(x) &\leq \int_0^{+\infty} \int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x [p_t(x,y) - p_t(x,y_0)]| d\mu(x) \frac{dt}{\sqrt{t}} \\
&= \int_0^{d^2(y,y_0)} \int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x [p_t(x,y) - p_t(x,y_0)]| d\mu(x) \frac{dt}{\sqrt{t}} \\
&\quad + \int_{d^2(y,y_0)}^{+\infty} \int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x [p_t(x,y) - p_t(x,y_0)]| d\mu(x) \frac{dt}{\sqrt{t}} \\
&= I_1 + I_2.
\end{aligned} \tag{14}$$

As for I_1 , write that

$$I_1 \leq \int_0^{d^2(y,y_0)} \int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x p_t(x,y)| d\mu(x) \frac{dt}{\sqrt{t}} + \int_0^{d^2(y,y_0)} \int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x p_t(x,y_0)| d\mu(x) \frac{dt}{\sqrt{t}}.$$

Lemma 8 ensures that there exists β such that

$$\int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x p_t(x,y)| d\mu(x) \leq C e^{-4\beta \frac{d^2(y_0,y)}{t}} t^{-\frac{1}{2}}.$$

Moreover, if $d(x,y) \geq 2d(y,y_0)$, then $d(x,y_0) \geq \frac{1}{2}d(y,y_0)$, so that, using Lemma 8 again,

$$\begin{aligned}
\int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x p_t(x,y_0)| d\mu(x) &\leq \int_{d(x,y_0) \geq \frac{1}{2}d(y_0,y)} |\nabla_x p_t(x,y_0)| d\mu(x) \\
&\leq C e^{-\beta \frac{d^2(y_0,y)}{4t}} t^{-\frac{1}{2}}.
\end{aligned}$$

Finally, there exists α such that

$$\begin{aligned}
I_1 &\leq C \int_0^{d^2(y,y_0)} e^{-\alpha \frac{d^2(y_0,y)}{t}} \frac{dt}{t} \\
&= C \int_0^1 e^{-\frac{\alpha}{t}} \frac{dt}{t},
\end{aligned}$$

which is finite and does not depend on $d(y_0,y)$.

We now turn to I_2 . Using Lemma 13 and the fact that $d(y,y_0) \leq \sqrt{t}$, one has, for some $\beta > 0$,

$$\begin{aligned}
I_2 &= \int_{d^2(y,y_0)}^{+\infty} \int_{d(x,y) \geq 2d(y_0,y)} |\nabla_x q_t(x)| d\mu(x) \frac{dt}{\sqrt{t}} \\
&\leq C \int_{d^2(y,y_0)}^{+\infty} \left[\frac{d(y,y_0)}{\sqrt{t}} \right]^\gamma e^{-\beta \frac{d^2(y,y_0)}{t}} \frac{dt}{t} \\
&\leq C' \int_1^{+\infty} \frac{dt}{t^{1+\frac{\gamma}{2}}},
\end{aligned}$$

which is also finite and independent of $d(y_0,y)$. Thus, $k(x,y)$ satisfies the integral Hörmander condition and the Riesz transform maps continuously H^1 into L^1 . Theorem 4 is proved. ■

It should be noticed that the exponential term in Lemma 13 is not used to check that I_2 is finite, whereas the exponential term in Lemma 8 is essential to ensure that I_1 is finite.

Part IV

The case of Riemannian manifolds: a local theorem

In this part, we state and prove a result analogous to Theorem 4 with local assumptions about the manifold, instead of global ones.

1 Statement of the main result

Let M be a complete Riemannian manifold. Say that M satisfies the local doubling property if

$$\forall R > 0, \exists C_R > 0, \forall x \in M, \forall r \in]0, R[, V(x, 2r) \leq C_R V(x, r). \quad (15)$$

Say that M has at most exponential volume growth at infinity when

$$\exists C, c > 0, \forall x \in M, \forall r \leq 1, \forall \theta > 1, V(x, \theta r) \leq C e^{c\theta} V(x, r). \quad (16)$$

Finally, say that M satisfies the local Poincaré inequality if

$$\forall R > 0, \exists C_R > 0, \forall r \in]0, R[, \forall x \in M, \forall f \in C^\infty(B(x, 2r)), \quad (17)$$

$$\int_B |f - f_B|^2 d\mu \leq C_R r^2 \int_{2B} |\nabla f|^2 d\mu$$

where $B = B(x, r)$, $2B = B(x, 2r)$.

We intend to show the following statement:

Theorem 14 *Let M be a Riemannian manifold satisfying the local doubling property (15), the local Poincaré inequality (17) and having at most exponential volume growth at infinity (which means (16)). Then there exists a constant $C > 0$ such that, for any $f \in H_{at}^{1,loc}$,*

$$\|\|\nabla f\|\|_1 \leq C \left[\|\|\Delta^{\frac{1}{2}} f\|\|_{H_{at}^{1,loc}} + \|f\|_{H_{at}^{1,loc}} \right].$$

It is important to notice the differences between the assumptions of Theorem 4 and Theorem 14. In Theorem 4, the doubling property and the Poincaré inequality are global, whereas they are local in Theorem 14. Since the doubling property is only local, we only consider local atoms, i. e. atoms supported in balls with small radii.

When M has Ricci curvature bounded below, the assumptions of Theorem 14 hold. In [2], Bakry shows that, when M has Ricci curvature bounded below, then, for any $1 < p < \infty$, there exists $C_p > 0$ such that, for any $f \in C^\infty(M)$ with compact support,

$$\|\|\nabla f\|\|_p \leq C_p \left[\|\|\Delta^{\frac{1}{2}} f\|\|_p + \|f\|_p \right].$$

But this paper does not give any result about the $H^1 - L^1$ continuity. Note that this result is generalized for $1 < p \leq 2$ by Coulhon and Duong in [6], Theorem 1.2, when M only has the local doubling property and a on-diagonal upper bound of the heat kernel for small time.

We shall work with the operator $\tilde{T} = \nabla(\Delta + b)^{-\frac{1}{2}}$ where $b > 0$ will be chosen later. This operator, which is called the corrected Riesz transform, is used by Coulhon and Duong for the proof of Theorem 1.2 in [6]. The local norm in Theorem 14, as well as in Theorem 1.2 in [6], appears thanks to the parameter b (see [2], Lemme 4.2). It is clear that \tilde{T} is L^2 -bounded and is given by a kernel $k(x, y)$. In view of Theorem 3, it is sufficient to prove that k satisfies the local Hörmander integral condition.

2 Kernel estimates

As before, for any fixed points $y_0, y \in M$ and any $t > 0$, define

$$q_t(x) = p_t(x, y) - p_t(x, y_0).$$

Under the assumptions of Theorem 14, one has estimates about the heat kernel which correspond to those of the previous section. Namely, (15) and (17) imply the two following statement:

Lemma 15 *There exist $c_1, C_1, c_2, C_2 > 0$ such that, for all $t < 1$, all $x \in M$ and every $x_1, x_2 \in B(x, 1)$,*

$$\frac{c_1}{V(x_1, \sqrt{t})} e^{-C_1 \frac{d^2(x_1, x_2)}{t}} \leq p_t(x_1, x_2) \leq \frac{C_2}{V(x_1, \sqrt{t})} e^{-c_2 \frac{d^2(x_1, x_2)}{t}}.$$

This result is given in [14], Proposition 3.3.

Similarly to Lemma 9, one has the following result:

Lemma 16 *There exist C_3, c_3 and $\gamma \in]0, 1[$ such that, for any $t \in]0, 1[$ and any $x, y, y_0 \in M$ satisfying $d(y_0, y) \leq \sqrt{t}$,*

$$|q_t(x)| \leq \frac{C_3}{V(x, \sqrt{t})} \left[\frac{d(y_0, y)}{\sqrt{t}} \right]^\gamma e^{-c_3 \frac{d^2(x, y)}{t}}.$$

This result is a consequence of Proposition 10 applied with $R = 1$.

From the previous estimates, one gets first the following result:

Lemma 17 *For any $\beta < 2c_3$, there exists $C_\beta > 0$ such that, for any $s > 0$ and any $y \in M$ satisfying $d(y_0, y) \leq \sqrt{s}$,*

$$\int |q_s(x)|^2 e^{\beta \frac{d^2(x, y)}{s}} d\mu(x) \leq \phi(s)$$

where

$$\phi(s) = \begin{cases} \frac{C_\gamma}{V(y, \sqrt{s})} \left[\frac{d(y_0, y)}{\sqrt{s}} \right]^{2\gamma} & \text{if } s \leq 1, \\ \frac{C_\gamma}{V(y, 1)} d(y_0, y)^{2\gamma} & \text{if } s \geq 1. \end{cases}$$

When $s \leq 1$, just use the upper bound given by Lemma 16. Then, observe that the map

$$(s, x) \mapsto q_s(x) = p_s(x, y) - p_s(x, y_0)$$

is a solution of $\frac{\partial}{\partial s} u + \Delta u = 0$, which implies that

$$\int |q_s(x)|^2 e^{\beta \frac{d^2(x, y)}{s}} d\mu(x)$$

is non-increasing in s . This fact is a consequence of the following integral maximum principle (see [9], Theorem 1.1):

Theorem 18 *Suppose that u is a solution of $\frac{\partial}{\partial s} u + \Delta u = 0$. Then the function*

$$I(t) = \int_M u^2(x, t) e^{\xi(x, t)} d\mu(x)$$

is non-increasing, provided that the function $\xi(x, t)$ is locally Lipschitz and satisfies the relation

$$\frac{\partial \xi}{\partial t} + \frac{1}{2} |\nabla \xi|^2 \leq 0.$$

Therefore, one has the claimed estimate for $s \geq 1$. ■

We are now going to show the following upper bound:

Lemma 19 *For any $\beta < c_3$, there exists $C_\beta > 0$ such that, for any $s > 0$ satisfying $d(y_0, y) \leq \sqrt{s}$,*

$$\int |\nabla_x q_s(x)|^2 e^{\beta \frac{d^2(x,y)}{s}} d\mu(x) \leq \psi(s)$$

where

$$\psi(s) = \begin{cases} \frac{C_\beta}{sV(y, \sqrt{s})} \left[\frac{d(y_0, y)}{\sqrt{s}} \right]^{2\gamma} & \text{if } s \leq 1, \\ \frac{C_\beta}{sV(y, 1)} d(y_0, y)^{2\gamma} & \text{if } s \geq 1. \end{cases}$$

Doing the same computations as in the proof of Lemma 12, one gets that, for any $s > 0$,

$$\int |\nabla_x q_s(x)|^2 e^{\beta \frac{d^2(x,y)}{s}} d\mu(x) \leq \frac{C}{\int_0^s \frac{du}{\phi(u)}}.$$

But, when $s \leq 1$, one has

$$\begin{aligned} \int_0^s \frac{du}{\phi(u)} &\geq \int_{\frac{s}{2}}^s \frac{du}{\phi(u)} \\ &\geq C \frac{s}{2} \left[\frac{\sqrt{s}}{d(y, y_0)} \right]^{2\gamma}, \end{aligned}$$

and when $s \geq 1$,

$$\begin{aligned} \int_0^s \frac{du}{\phi(u)} &\geq \int_1^s \frac{du}{\phi(u)} \\ &\geq C(s-1)V(y, 1) \left[\frac{1}{d(y, y_0)} \right]^{2\gamma}. \end{aligned}$$

Lemma 19 follows. ■

Finally, this lemma implies the following:

Lemma 20 *There exists $\beta > 0$ such that, for any $s > 0$ satisfying $d(y, y_0) \leq \sqrt{s}$,*

$$\int_{d(x,y) > r_0} |\nabla_x q_s(x)| d\mu(x) \leq \frac{\chi(s)}{\sqrt{s}} e^{-\beta \frac{r_0^2}{s}} ds$$

where the function χ is defined by

$$\chi(s) = \begin{cases} C \left[\frac{d(y_0, y)}{\sqrt{s}} \right]^\gamma & \text{if } s \leq 1, \\ C d(y_0, y)^\gamma e^{cs} & \text{if } s \geq 1. \end{cases}$$

One uses the Cauchy-Schwarz inequality like in the proof of Lemma 13, in conjunction with the assumption (16). ■

3 $H^{1,loc} - L^1$ boundedness

As said before, following [6], we consider, for some $b > 0$, the operator $\tilde{T} = \nabla(\Delta + b)^{-\frac{1}{2}}$ and intend to prove that it is $H_{at}^{1,loc} - L^1$ bounded. It is given by the kernel

$$k(x, y) = \int_0^{+\infty} \frac{e^{-tb}}{\sqrt{t}} \nabla_x p_t(x, y) dt.$$

We want to show that this kernel satisfies the local integral Hörmander condition. Let y and y_0 be two fixed points in M satisfying $d(y_0, y) \leq 1$. Write that

$$\begin{aligned}
I &= \int_{d(x,y) \geq 2d(y,y_0)} |k(x,y) - k(x,y_0)| d\mu(x) \\
&\leq \int_0^{+\infty} \frac{e^{-tb}}{\sqrt{t}} \int_{d(x,y) \geq 2d(y,y_0)} |\nabla_x p_t(x,y) - \nabla_x p_t(x,y_0)| d\mu(x) dt \\
&= \int_0^{d^2(y,y_0)} \frac{e^{-tb}}{\sqrt{t}} \int_{d(x,y) \geq 2d(y,y_0)} |\nabla_x p_t(x,y) - \nabla_x p_t(x,y_0)| d\mu(x) dt \\
&+ \int_{d^2(y,y_0)}^1 \frac{e^{-tb}}{\sqrt{t}} \int_{d(x,y) \geq 2d(y,y_0)} |\nabla_x p_t(x,y) - \nabla_x p_t(x,y_0)| d\mu(x) dt \\
&+ \int_1^{+\infty} \frac{e^{-tb}}{\sqrt{t}} \int_{d(x,y) \geq 2d(y,y_0)} |\nabla_x p_t(x,y) - \nabla_x p_t(x,y_0)| d\mu(x) dt \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Remember that $d(y, y_0) \leq 1$. As for I_1 , the triangular inequality shows that

$$\begin{aligned}
I_1 &\leq C \int_0^{d^2(y,y_0)} \frac{e^{-tb}}{\sqrt{t}} \int_{d(x,y) \geq 2d(y,y_0)} |\nabla_x p_t(x,y)| d\mu(x) dt \\
&+ \int_0^{d^2(y,y_0)} \frac{e^{-tb}}{\sqrt{t}} \int_{d(x,y) \geq 2d(y,y_0)} |\nabla_x p_t(x,y_0)| d\mu(x) dt \\
&\leq C \int_0^{d^2(y,y_0)} \frac{e^{-tb}}{\sqrt{t}} \frac{dt}{\sqrt{t}} e^{-\beta \frac{d^2(y,y_0)}{t}} \\
&\leq C \int_0^{d^2(y,y_0)} \frac{dt}{t} e^{-\beta \frac{d^2(y,y_0)}{t}} \\
&= C \int_0^1 \frac{dt}{t} e^{-\frac{\beta}{t}}
\end{aligned}$$

which is finite and independent of y and y_0 .

Then,

$$\begin{aligned}
I_2 &\leq C \int_{d^2(y,y_0)}^1 \frac{e^{-tb}}{\sqrt{t}} \frac{dt}{t} e^{-\beta \frac{d^2(y,y_0)}{t}} \left[\frac{d(y,y_0)}{\sqrt{t}} \right]^\gamma \\
&= C \int_1^{\frac{1}{d^2(y,y_0)}} \frac{e^{-tb d^2(y,y_0)}}{t^{1+\frac{\gamma}{2}}} e^{-\frac{\beta}{t}} dt \\
&\leq \int_0^{+\infty} e^{-\frac{\beta}{t}} \frac{dt}{t^{1+\frac{\gamma}{2}}}
\end{aligned}$$

which is also finite and does not depend on the points y, y_0 .

Finally,

$$\begin{aligned} I_3 &\leq \int_1^{+\infty} \frac{e^{-tb}}{\sqrt{t}} \frac{dt}{\sqrt{t}} d(y_0, y)^\gamma e^{-\beta \frac{4d^2(y, y_0)}{t}} e^{ct} \\ &\leq \int_1^{+\infty} \frac{e^{-tb'}}{t} dt \end{aligned}$$

which is finite and independent of y and y_0 . In the last line, one has to choose b larger than $2c$ and to use the fact that $d(y, y_0) \leq 1$.

Thus, the kernel k satisfies the local Hörmander integral condition, which proves that \tilde{T} is $H_{at}^{1,loc} - L^1$ bounded. ■

Part V

The case of graphs

1 Statement of the main result

Let Γ be an infinite connected graph, endowed with its natural metric and a symmetric weight $\mu_{xy} = \mu_{yx}$ on $\Gamma \times \Gamma$. Assume that x and y are neighbours if and only if $\mu_{xy} \neq 0$. Define, for every $x \in \Gamma$,

$$m(x) = \sum_{y \sim x} \mu_{xy}.$$

For every real $r \geq 0$, the ball $B(x, r)$ is defined as follows:

$$B(x, r) = \{y \in \Gamma; d(y, x) \leq r\},$$

and, if A is a subset of Γ , its volume is

$$V(A) = \sum_{x \in A} m(x).$$

When A is a ball $B(x, r)$, $V(A)$ will be denoted by $V(x, r)$.

The graph Γ is said to satisfy the doubling property if there exists a constant $C > 0$ such that, for every $x \in \Gamma$ and $r > 0$

$$V(x, 2r) \leq CV(x, r). \quad (18)$$

Note that this property implies that there exist two constants $C_{vol} > 0$ and $D > 0$ such that, for any $x \in \Gamma, \theta > 1$ and $r > 0$,

$$V(x, \theta r) \leq C_{vol} \theta^D V(x, r). \quad (19)$$

The graph Γ is said to satisfy the Poincaré inequality if there exists a constant $C > 0$ such that, for every function f from Γ to \mathbb{R} , every $x_0 \in \Gamma$ and $r > 0$, one has

$$\sum_{x \in B(x_0, r)} m(x) |f(x) - f_B|^2 \leq Cr^2 \sum_{x, y \in B(x_0, 2r)} \mu_{xy} |f(x) - f(y)|^2, \quad (20)$$

where

$$f_B = \frac{1}{V(x_0, r)} \sum_{x \in B(x_0, r)} m(x) f(x).$$

Finally, one says that Γ satisfies $\Delta(\alpha)$ for $\alpha > 0$ if the two following conditions hold:

$$\begin{aligned} x \sim y &\Rightarrow \mu_{xy} \geq \alpha m(x), \\ \forall x \in \Gamma, &\quad \mu_{xx} \geq \alpha m(x). \end{aligned}$$

Denote by $\|f\|_p$ the L^p norm of a function f , that is to say

$$\|f\|_p = \left[\sum_x |f(x)|^p m(x) \right]^{\frac{1}{p}}.$$

One may then consider on Γ a discrete-time Markov kernel. Set

$$p(x, y) = \frac{\mu_{xy}}{m(x)}$$

and define the iterated kernel p_k as follows:

$$\begin{aligned} p_0(x, y) &= \delta(x, y), \\ p_k(x, y) &= \sum_z p(x, z)p_{k-1}(z, y). \end{aligned}$$

The definition of an atom and of the space H_{at}^1 is given in the first section (note that Γ is a space of homogeneous type).

The gradient of a function f is defined by

$$\nabla f(x) = \sum_{y \sim x} |f(y) - f(x)|$$

where $x \sim y$ means that y is a neighbour of x .

Define the linear operator P by

$$Pf(x) = \sum_y p(x, y)f(y)$$

and notice that

$$P^k f(x) = \sum_y p_k(x, y)f(y).$$

One easily checks that P is self-adjoint on $L^2(\Gamma, m(x))$. The Riesz transform T is defined as

$$T = \nabla(I - P)^{-\frac{1}{2}},$$

where the unbounded linear operator $(I - P)^{-\frac{1}{2}}$ is defined by means of spectral theory.

We intend to show the following result:

Theorem 21 *Let Γ be an infinite graph satisfying the doubling property and the Poincaré inequality. Then, the Riesz transform is bounded from $H_{at}^1(\Gamma)$ to $L^1(\Gamma)$.*

One can define the space H_{max}^1 in the following way. If $f \in L^1(\Gamma)$, set

$$f^+(x) = \sup_k P^k f(x)$$

and say that $f \in H_{max}^1$ if and only if $f^+ \in L^1$. If it is the case, define

$$\|f\|_{H_{max}^1} = \|f^+\|_1.$$

The following statement is shown in [11]:

Theorem 22 *Let Γ be a graph satisfying the doubling property, the Poincaré inequality and the condition $\Delta(\alpha)$. Then $H_{at}^1(\Gamma) = H_{max}^1(\Gamma)$.*

Therefore, Theorem 21 shows that the Riesz transform is bounded from $H_{max}^1(\Gamma)$ to $L^1(\Gamma)$.

To prove Theorem 21, since the Riesz transform is L^2 -bounded and is given by a kernel $k(x, y)$, we only have to check that this kernel satisfies the Hörmander integral condition, which requires some estimates about kernels.

2 Kernel estimates

Fix y_0 and y in Γ . For any $k \in \mathbb{N}$ and $x \in \Gamma$, define

$$q_k(x) = \frac{p_k(y, x) - p_k(y_0, x)}{m(x)}.$$

This section is devoted to various bounds about p_k and q_k .

Recall that, when (18) and (20) hold, one has the following estimate about p_k :

Theorem 23 *Let Γ satisfy the doubling property, the Poincaré inequality and $\Delta(\alpha)$ for $\alpha > 0$. Then, there exist $c_1, C_1, c_2, C_2 > 0$ such that $d(x, y) \leq k \Rightarrow \frac{c_1 m(y)}{V(x, \sqrt{k})} e^{-\frac{c_1 d(x, y)^2}{k}} \leq p_k(x, y) \leq \frac{C_2 m(y)}{V(x, \sqrt{k})} e^{-\frac{c_2 d(x, y)^2}{k}}$.*

This theorem is shown by T. Delmotte in [8], Theorem 1.7.

As a consequence of the Gaussian upper bound in Lemma 23 and of the doubling property (18), one gets the following upper bounds (see Lemma 7 and Lemma 2 in [13]):

Lemma 24 *For all $\gamma \in]0, c_1[$, there exists $C_\gamma > 0$ such that*

$$\sum_y |\nabla_y p_k(y, x)|^2 e^{\frac{\gamma d^2(x, y)}{k}} m(y) \leq \frac{C_\gamma m^2(x)}{k V(x, \sqrt{k})}.$$

Lemma 25 *There exists $\beta > 0$ such that, for any $x \in \Gamma$, $l \in \mathbb{N}$, $k \in \mathbb{N}^*$,*

$$\sum_{y \notin B(x, \sqrt{l})} |\nabla_y p_k(y, x)| m(y) \leq C m(x) e^{-\frac{\beta l}{k}} k^{-\frac{1}{2}}.$$

T. Delmotte also shows that the solutions of $m(x)u(n+1, x) = \sum_y \mu_{xy} u(n, y)$ satisfy a Hölder regularity property (see Proposition 4.1 in [8]):

Proposition 26 *There exists $h > 0$ and $C > 0$ such that, for all $x_0 \in \Gamma$, $n_0 \in \mathbb{Z}$ and $R \in \mathbb{N}$, if u is a solution of $m(x)u(n+1, x) = \sum_y \mu_{xy} u(n, y)$ on $Q = (\mathbb{Z} \cap [n_0 - 2R^2, n_0]) \times B(x_0, 2R)$, $x_1, x_2 \in B(x_0, R)$ and $n \in \mathbb{Z} \cap [n_0 - R^2, n_0]$, then*

$$|u(n, x_1) - u(n, x_2)| \leq C \left[\frac{d(x_1, x_2)}{R} \right]^h \sup_Q |u|.$$

As a consequence of Theorem 23 and of Proposition 26, one has the following estimate about q_k :

Lemma 27 *There exists $C_3, c_3 > 0$ and $h \in]0, 1[$ such that, for any $k \in \mathbb{N}$, $x, y, y_0 \in \Gamma$ such that $d(y_0, y) \leq \sqrt{k}$,*

$$|q_k(x)| \leq \frac{C_3}{V(x, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^h e^{-c_3 \frac{d^2(y_0, x)}{k}}.$$

Assume first that $d(y, y_0) \leq \frac{1}{2}\sqrt{k}$ and that $x \in \Gamma$. Proposition 26 may be applied to $u(k, z) = p_k(z, x)$, with $R \sim \frac{1}{2}\sqrt{k}$ and $n_0 \sim \frac{5}{4}k$. Since $y \in B(y_0, R)$ and $k \in \mathbb{Z} \cap [n_0 - R^2, n_0]$, one gets:

$$|p_k(y, x) - p_k(y_0, x)| \leq C \left[\frac{d(y, y_0)}{R} \right]^h \sup_Q p_l(z, x)$$

where

$$Q = (\mathbb{Z} \cap [n_0 - 2R^2, n_0]) \times B(y_0, 2R).$$

But, thanks to Theorem 23, when $n_0 - 2R^2 \leq l \leq n_0$ and $z \in B(y_0, 2R)$,

$$\begin{aligned} p_l(z, x) &\leq \frac{C_2 m(x)}{V(x, \sqrt{l})} e^{-\frac{c_2 d^2(x, z)}{l}} \\ &\leq \frac{C_2 m(x)}{V(x, \sqrt{n_0 - 2R^2})} e^{-\frac{c_2 d^2(x, z)}{n_0}}. \end{aligned}$$

One has

$$\begin{aligned} -\frac{d^2(x, z)}{n_0} &\leq -\frac{d^2(x, y_0)}{2n_0} + \frac{d^2(y_0, z)}{n_0} \\ &\leq -\frac{d^2(x, y_0)}{2n_0} + 4, \end{aligned}$$

so that

$$p_l(z, x) \leq \frac{C_3 m(x)}{V(x, \sqrt{k})} e^{-\frac{c_2 d^2(y_0, x)}{k}}.$$

It follows that

$$|p_k(y, x) - p_k(y_0, x)| \leq C_4 \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^h \frac{m(x)}{V(x, \sqrt{k})} e^{-\frac{c_2 d^2(y_0, x)}{k}}. \quad (21)$$

Finally, if $d(y, y_0) \leq \sqrt{k}$, consider a point y_1 such that $d(y, y_1) \leq \frac{1}{2}\sqrt{k}$ and $d(y_1, y_0) \leq \frac{1}{2}\sqrt{k}$, and apply (21) to $|p_k(y, x) - p_k(y_1, x)|$ and to $|p_k(y_1, x) - p_k(y_0, x)|$. Lemma 27 is shown. ■

Therefore, the following bound holds:

Lemma 28 *If $d(y_0, y) \leq \sqrt{k}$, then, for any $\alpha < 2c_3$, there exists $C_\alpha > 0$ such that*

$$\sum_x |q_k(x)|^2 \exp \left[\alpha \frac{d^2(x, y)}{k} \right] m(x) \leq \left[\frac{d(y_0, y)}{\sqrt{k}} \right]^{2h} \frac{C_\alpha}{V(y, \sqrt{k})}.$$

The proof is similar to Lemma 11. ■

We intend to show the following estimate, which is the discrete equivalent of Lemma 12 (see also the analogous estimate in [13], Lemma 7):

Lemma 29 *For any $k \in \mathbb{N}^*$ satisfying $d(y, y_0) \leq \sqrt{k}$ and any $\gamma < c_3$,*

$$\sum_x |\nabla_x q_k(x)|^2 \exp \left[\gamma \frac{d^2(y, x)}{k} \right] m(x) \leq \frac{C'_\gamma}{kV(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{2}}.$$

The main step in the proof of this lemma is the following temporal estimate:

Lemma 30 *For any $x \in \Gamma$, any $k \in \mathbb{N}^*$ satisfying $d(y, y_0) \leq \sqrt{k}$,*

$$|q_{k+1}(x) - q_k(x)| \leq \frac{C}{kV(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{h}{2}}.$$

To begin the proof, notice that, for any $k \in \mathbb{N}^*$, one has

$$q_{k+1} = Pq_k.$$

Indeed, if $x \in \Gamma$,

$$\begin{aligned}
[Pq_k](x) &= \sum_z p(x, z) q_k(z) \\
&= \sum_z p(x, z) \frac{p_k(y, z) - p_k(y_0, z)}{m(z)} \\
&= \frac{1}{m(x)} \sum_z p(z, x) [p_k(y, z) - p_k(y_0, z)] \\
&= \frac{1}{m(x)} [p_{k+1}(y, x) - p_{k+1}(y_0, x)] \\
&= q_{k+1}(x).
\end{aligned}$$

Let l be an integer satisfying $l \sim \frac{k}{2}$. One has

$$\begin{aligned}
|q_{k+1}(x) - q_k(x)| &= \left| \sum_z p_l(x, z) [q_{l+1}(z) - q_l(z)] \right| \\
&= \left| \sum_z \frac{p_l(x, z)}{m(z)} [q_{l+1}(z) - q_l(z)] m(z) \right| \\
&\leq \left\| \frac{p_l(x, \cdot)}{m} \right\|_2 \|q_{l+1} - q_l\|_2.
\end{aligned}$$

But

$$\begin{aligned}
\left\| \frac{p_l(x, \cdot)}{m} \right\|_2^2 &= \sum_z \frac{p_l(x, z) p_l(x, z)}{m(z)} \\
&= \sum_z p_l(x, z) \frac{p_l(z, x)}{m(x)} \\
&= \frac{p_k(x, x)}{m(x)} \\
&\leq \frac{C}{V(x, \sqrt{k})}.
\end{aligned}$$

Moreover, if $m \sim \frac{l}{2}$, then

$$\begin{aligned}
\|q_{l+1} - q_l\|_2 &= \|(I - P)P^m q_m\|_2 \\
&\leq \|(I - P)P^m\|_{2 \rightarrow 2} \|q_m\|_2 \\
&\leq \frac{C}{l} \|q_m(y, \cdot)\|_2.
\end{aligned} \tag{22}$$

The last line follows from Lemma 6 in [13], which we recall here:

Lemma 31 *One has* $\left\| (I - P) \left[\frac{p_k(x, \cdot)}{m} \right] \right\|_2 \leq \frac{C}{kV^{\frac{1}{2}}(x, \sqrt{k})}$.

Recall that Lemma 31 is a straightforward consequence of the condition $\Delta(\alpha)$, which implies that $-1 \notin Sp(P)$.

From the fact that

$$\begin{aligned}
\|q_m\|_1 &= \sum_x |q_m(x)| m(x) \\
&\leq \sum_x p_m(y, x) + \sum_x p_m(y_0, x) \\
&= 2
\end{aligned}$$

and that

$$\begin{aligned}
\|q_m\|_\infty &= \sup_x \frac{|p_m(y, x) - p_m(y_0, x)|}{m(x)} \\
&\leq \frac{C}{V(y, \sqrt{m})} \left[\frac{d(y, y_0)}{\sqrt{m}} \right]^h,
\end{aligned}$$

we may conclude that

$$\begin{aligned}
\|q_m\|_2 &\leq \|q_m\|_1^{\frac{1}{2}} \|q_m\|_\infty^{\frac{1}{2}} \\
&\leq \frac{C}{V^{\frac{1}{2}}(y, \sqrt{m})} \left[\frac{d(y, y_0)}{\sqrt{m}} \right]^{\frac{h}{2}}.
\end{aligned}$$

It follows from (22) that

$$\|q_{l+1} - q_l\|_2 \leq \frac{C}{lV^{\frac{1}{2}}(y, \sqrt{l})} \left[\frac{d(y, y_0)}{\sqrt{l}} \right]^{\frac{h}{2}}$$

and finally that

$$\begin{aligned}
|q_{k+1}(x) - q_k(x)| &\leq \frac{C}{kV^{\frac{1}{2}}(y_0, \sqrt{k})V^{\frac{1}{2}}(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{h}{2}} \\
&\leq \frac{C}{V(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{h}{2}}.
\end{aligned}$$

In the last line, we use the fact that $d(y, y_0) \leq \sqrt{k}$ and the doubling property. Thus, Lemma 30 is proved. ■

Let us deduce Lemma 29 from Lemma 30. The following computations are inspired from those which give Lemma 7 in [13]. Write that

$$\begin{aligned}
\sum_{x \sim z} |q_k(x) - q_k(z)|^2 e^{\gamma \frac{d^2(y, x)}{k}} m(x) &\leq C \sum_{d(x, z) \leq 1} |q_k(x) - q_k(z)|^2 m(x) p(x, z) e^{\gamma \frac{d^2(y, x)}{k}} \\
&= CI_k.
\end{aligned} \tag{23}$$

From now on, we will work with I_k .

$$\begin{aligned}
I_k &= \sum_{d(x, z) \leq 1} q_k(x) [q_k(x) - q_k(z)] m(x) p(x, z) e^{\gamma \frac{d^2(y, x)}{k}} \\
&\quad - \sum_{d(x, z) \leq 1} q_k(z) [q_k(x) - q_k(z)] m(x) p(x, z) e^{\gamma \frac{d^2(y, x)}{k}} \\
&= \sum_{d(x, z) \leq 1} q_k(x) [q_k(x) - q_k(z)] m(x) p(x, z) e^{\gamma \frac{d^2(y, x)}{k}} \\
&\quad + \sum_{d(x, z) \leq 1} q_k(x) [q_k(x) - q_k(z)] m(x) p(x, z) e^{\gamma \frac{d^2(y, z)}{k}}.
\end{aligned}$$

In the last line, we inverted x and z and we used the reversibility of p with respect to m . Hence, we get that

$$\begin{aligned}
I_k &\leq 2 \sum_{x \sim z} q_k(x) [q_k(x) - q_k(z)] m(x) p(x, z) e^{\gamma \frac{d^2(y, x)}{k}} \\
&+ \sum_{x \sim z} q_k(x) [q_k(x) - q_k(z)] m(x) p(x, z) \left[e^{\gamma \frac{d^2(y, z)}{k}} - e^{\gamma \frac{d^2(y, x)}{k}} \right] \\
&= 2I_k^{(1)} + I_k^{(2)}.
\end{aligned}$$

The previous upper bounds allow to estimate $I_k^{(1)}$ and $I_k^{(2)}$. Indeed,

$$\begin{aligned}
I_k^{(1)} &= \sum_x q_k(x) e^{\gamma \frac{d^2(y, x)}{k}} m(x) \sum_z p(x, z) [q_k(x) - q_k(z)] \\
&= \sum_x q_k(x) e^{\gamma \frac{d^2(y, x)}{k}} m(x) [q_k(x) - q_{k+1}(x)]
\end{aligned}$$

hence, thanks to Lemma 30,

$$\begin{aligned}
|I_k^{(1)}| &\leq \frac{C}{kV(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{h}{2}} \sum_x |q_k(x)| e^{\gamma \frac{d^2(x, y)}{k}} m(x) \\
&\leq \frac{C}{kV(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{2}} \sum_x \frac{1}{V(x, \sqrt{k})} e^{(\gamma - c_3) \frac{d^2(y, x)}{k}} m(x) \\
&\leq \frac{C_\gamma}{kV^2(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{2}} \sum_y e^{(\gamma' - \alpha) \frac{d^2(x, y)}{k}} m(x) \\
&\leq \frac{C_\gamma}{kV(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{2}}.
\end{aligned}$$

Note that, in this computation, it is possible to choose $\gamma' \in]\gamma, c_3[$ because $\gamma < c_3$.

As for I_2 , if we denote by f the map defined by

$$f(x) = e^{\frac{\gamma}{k} x^2}$$

we may write, according to the mean-value theorem, that

$$|f(b) - f(a)| \leq \frac{2\gamma}{k} |b - a| (\sup a, b) e^{\frac{\gamma}{k} (\sup a, b)^2}.$$

Applying this inequality with $a = d(y, x)$, $b = d(y, z)$ when $d(x, z) \leq 1$, so that $|d(y, x) - d(y, z)| \leq 1$, we get, if

we notice that $\sup(a, b) \leq a + 1$, that

$$\begin{aligned}
|I_k^{(2)}| &\leq \frac{2\gamma}{k} \sum_{d(x,z) \leq 1} m(x)p(x,z) |q_k(x)| |q_k(x) - q_k(z)| [d(x,y) + 1] e^{\frac{\gamma}{k}[d(y,x)+1]^2} \\
&= 2\sqrt{\frac{\gamma}{k}} \sum_{d(x,z) \leq 1} m(x)p(x,z) |q_k(x)| |q_k(x) - q_k(z)| \sqrt{\frac{\gamma}{k} [d(y,x) + 1]^2} e^{\frac{\gamma}{k}[d(y,x)+1]^2} \\
&\leq 2\sqrt{\frac{\gamma}{k}} \sum_{d(x,z) \leq 1} m(x)p(x,z) |q_k(x)| |q_k(x) - q_k(z)| e^{\frac{\gamma}{k}[d(y,x)+1]^2} \\
&\leq 2\sqrt{\frac{\gamma}{k}} \left[\sum_{d(x,z) \leq 1} m(x)p(x,z) [q_k(x)]^2 e^{\frac{\gamma}{k}[d(y,x)+1]^2} \right]^{\frac{1}{2}} \\
&\quad \left[\sum_{d(x,z) \leq 1} m(x)p(x,z) |q_k(x) - q_k(z)|^2 e^{\frac{\gamma}{k}[d(y,x)+1]^2} \right]^{\frac{1}{2}}.
\end{aligned}$$

But

$$\frac{\gamma}{k} [d(y,x) + 1]^2 = \frac{\gamma}{k} [d^2(y,x) + 2d(y,x) + 1]$$

and, in the sums which define I_k , we may assume that $d(y,x) \leq 2k + 1$ and that $d(y,z) \leq 2k + 1$, otherwise, since $d(x,z) \leq 1$, we obtain $d(y,x) > 2k + 1$ and $d(y,z) > 2k + 1$. Then, $p_k(y,x) = p_k(y,z) = 0$. Moreover, since $d(y_0, y) < \kappa\sqrt{k}$, one has $d(y_0, x) > k$ and $d(y_0, z) > k$, and $p_k(y_0, x) = p_k(y_0, z) = 0$. All that implies $q_k(x) = q_k(z) = 0$, and the corresponding terms do not appear in the sum which defines I_k .

Finally,

$$\frac{\gamma}{k} [d(x,y) + 1]^2 \leq \frac{\gamma}{k} d^2(x,y) + C$$

and we can write that

$$\begin{aligned}
|I_k^{(2)}| &\leq C\sqrt{\frac{\gamma}{k}} \left[\sum_y [q_k(x)]^2 e^{\gamma'' \frac{d^2(x,y)}{k}} m(x) \right]^{\frac{1}{2}} \sqrt{I_k} \\
&\leq \frac{C_\gamma}{\sqrt{k}\sqrt{V(y, \sqrt{k})}} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^h \sqrt{I_k} \\
&\leq \frac{C_\gamma}{\sqrt{k}\sqrt{V(y, \sqrt{k})}} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{4}} \sqrt{I_k}
\end{aligned}$$

since $d(y, y_0) < \kappa\sqrt{k}$.

If we use simultaneously the estimates about $I_k^{(1)}$ and $I_k^{(2)}$, we find that

$$I_k \leq \frac{C_\gamma}{kV(y, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{2}} + \frac{C_\gamma}{\sqrt{k}V(x, \sqrt{k})} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{4}} \sqrt{I_k}$$

from which we get the right estimate for I_k , hence Lemma 29 by (23). ■

Thanks to Lemma 29, we can finally show the following Lemma:

Lemma 32 *There exist $C > 0$ and $\alpha > 0$ such that, for any $x, y_0, y \in \Gamma$, any $k \in \mathbb{N}^*$ satisfying $d(y_0, y) < \kappa\sqrt{k}$ and any $l \in \mathbb{N}^*$,*

$$\sum_{d(x,y) \geq l} |\nabla_x q_k(x)| m(x) \leq \frac{C}{\sqrt{k}} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{4}} e^{-\alpha \frac{l^2}{k}}.$$

Indeed, we just have to write that, if $\alpha < \frac{c_3}{2}$,

$$\begin{aligned} \sum_{x \sim z, d(y,x) \geq l} |q_k(x) - q_k(z)| m(x) &\leq \left[\sum_{x \sim z, d(y,x) \geq l} |q_k(x) - q_k(z)|^2 e^{2\alpha \frac{d^2(y,x)}{k}} m(x) \right]^{\frac{1}{2}} \\ &\leq \left[\sum_{d(y,x) \geq l} e^{-2\alpha \frac{d^2(x,y)}{k}} m(x) \right]^{\frac{1}{2}} \\ &\leq \frac{C}{\sqrt{kV(y, \sqrt{k})}} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{4}} \sqrt{V(y, \sqrt{k})} e^{-\frac{\alpha l^2}{2k}} \end{aligned}$$

which is Lemma 32 (one may apply Lemma 29 because $2\alpha \in]0, c_3[$). ■

3 $H^1 - L^1$ boundedness

We want to prove Theorem 21. Like in [13], write that

$$\nabla(I - P)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} a_k \nabla P^k$$

where the a_k 's are defined by

$$(1 - x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} a_k x^k.$$

Thus, T is given by the kernel

$$r(x, y) = \sum_{k=0}^{\infty} a_k \frac{\nabla p_k(x, y)}{m(y)} \quad (24)$$

and we are going to prove that this kernel satisfies the integral Hörmander condition. Let y and y_0 be two fixed points in Γ . One has

$$\begin{aligned} \sum_{d(x,y) \geq 2d(y_0,y)} |r(x, y) - r(x, y_0)| m(x) &\leq \sum_{k=0}^{\infty} a_k \sum_{d(x,y) \geq 2d(y_0,y)} \left| \frac{\nabla_x p_k(x, y)}{m(y)} - \frac{\nabla_x p_k(x, y_0)}{m(y_0)} \right| m(x) \\ &= \sum_{k \leq \frac{d^2(y, y_0)}{\kappa^2}} a_k \sum_{d(x,y) \geq 2d(y_0,y)} \left| \frac{\nabla_x p_k(x, y)}{m(y)} - \frac{\nabla_x p_k(x, y_0)}{m(y_0)} \right| m(x) \\ &+ \sum_{k > \frac{d^2(y, y_0)}{\kappa^2}} a_k \sum_{d(x,y) \geq 2d(y_0,y)} \left| \frac{\nabla_x p_k(x, y)}{m(y)} - \frac{\nabla_x p_k(x, y_0)}{m(y_0)} \right| m(x) \\ &= S_1 + S_2. \end{aligned}$$

One estimates S_1 in the same way as I_1 in (14), by means of the triangle inequality and Lemma 25 instead of Lemma 8. Also using the fact that $a_k \sim \frac{1}{\sqrt{\pi k}}$, one obtains that

$$\begin{aligned} S_1 &\leq C \sum_{k \leq \frac{d^2(y, y_0)}{\kappa^2}} a_k e^{-\alpha \frac{d^2(y, y_0)}{k}} k^{-\frac{1}{2}} \\ &\leq C \int_0^{\frac{d^2(y, y_0)}{\kappa^2}} e^{-\alpha \frac{d^2(y, y_0)}{t}} \frac{dt}{t} \\ &= C \int_0^1 e^{-\frac{\alpha \kappa^2}{u}} \frac{du}{u}, \end{aligned}$$

which is finite and independent of the points y and y_0 .

The treatment of S_2 is completely similar to the one of I_2 in (14). Just use Lemma 32, applied with

$$q_k(x) = \frac{p_k(x, y)}{m(y)} - \frac{p_k(x, y_0)}{m(y_0)},$$

instead of Lemma 13. Using the fact that $a_k \sim \frac{1}{\sqrt{\pi k}}$ again, one finds that

$$\begin{aligned} S_2 &\leq C \sum_{k > \frac{d^2(y, y_0)}{\kappa^2}} \frac{a_k}{k} \left[\frac{d(y, y_0)}{\sqrt{k}} \right]^{\frac{3h}{4}} e^{-\alpha \frac{d^2(y, y_0)}{k}} \\ &\leq C \int_{\frac{d^2(y, y_0)}{\kappa^2}}^{\infty} \frac{dt}{t} \left[\frac{d(y, y_0)}{\sqrt{t}} \right]^{\frac{3h}{4}} e^{-\alpha \frac{d^2(y, y_0)}{t}} \\ &\leq C \int_1^{\infty} \frac{dt}{t} \frac{1}{t^{\frac{3h}{4}}} \end{aligned}$$

which is finite and independent of the points y, y_0 . Thus, the kernel r satisfies the integral Hörmander condition and we have proved that the Riesz transform is $H^1 - L^1$ bounded. Theorem 21 is proved. ■

Like in the setting of manifolds, notice that the exponential term in Lemma 32 is not really used, but that getting it is easy.

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