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Exploring the Links Between Edge-Preserving Collaborative Filters via Gamma Convergence*

Sravan Danda[†], Aditya Challa[†], B.S.Daya Sagar[†], and Laurent Najman[‡]

Abstract. Edge-aware filtering is an important pre-processing step in many computer vision applications. In literature, there exist collaborative edge-aware filters that work well in practice but are based only on heuristics and/or principles. For instance, Tree Filter (TF) which is proposed recently based on a minimum spanning tree (MST) heuristic yields promising results. However the usage of an arbitrary MST for filtering is theoretically not justified. In this article, we introduce an edge-aware generalization of the TF, termed as *UMST filter* based on all MSTs. The major contribution of this paper is establishing theoretical links between filters based on MSTs and filters based on geodesics via the notion of Γ -convergence. More precisely, we compute the Γ -limit of Shortest Path Filters (SPFs) and show that it is the same as UMST filter. Consequently, TF can be viewed as an approximate Γ -limit of the SPFs, thereby providing a theoretical basis to its working. Further, we propose and provide a detailed analysis of two different implementations of the UMST filter based on shortest paths.

Key words. Optimization, Image Filtering, Γ -convergence, MST, Shortest Paths

AMS subject classifications. 90C27, 94A08, 94A12

1. Introduction. Image filtering has been a fundamental problem in computer vision for several years. Edge-preserving filtering is a crucial step in many low-level vision problems such as image abstraction [32], texture removal [32], texture editing [32], scene simplification [32], stereo matching [32], optical flow [32] etc. Real world images often contain noise and irrelevant information such as texture along with the object boundaries (which are the major image structures). The goal of an image filtering algorithm is thus to preserve the image structures while getting rid of the redundant information. Hence for several of the applications, it is important for any filtering algorithm to preserve object boundaries.

In the literature, there exist several edge-aware smoothing filters such as bilateral filter (BF) [33], guided filter (GF) [24], weighted least squares filter (WLS) [20], L_0 smoothing [35], propagated image filter [10], morphological amoebas or adaptive kernel based filters [25], tree filter (TF) [5] and relative total variation filter (RTV) [36] etc. Although these filters work well in practice, some of them are not extensively studied from a theoretical perspective. In this article, we study the recent state-of-art edge-aware Tree Filter (TF) which is based on a Minimum Spanning Tree (MST) heuristic. TF admits a linear time algorithm [37] and

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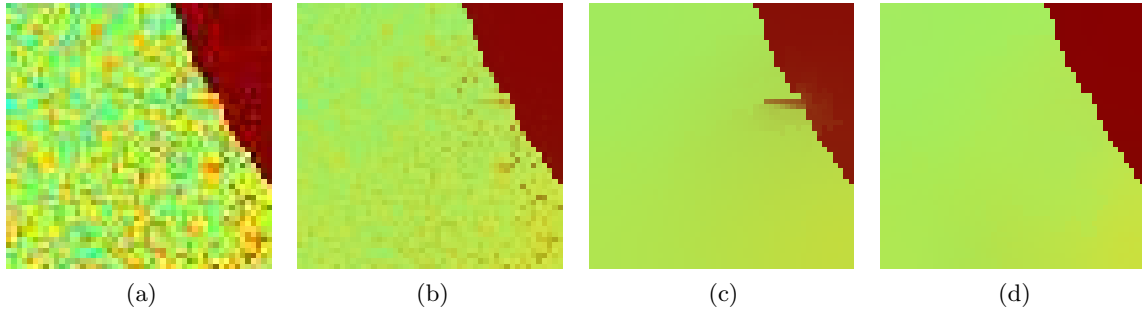


Figure 1. (a) Original image (b) Bilateral filter (c) Tree filter + Bilateral filter (d) Power Tree Filter + Bilateral filter.

34 yields promising results in applications such as texture removal, stereo matching and scene
 35 simplification. However it exhibits a leak problem at some of the object boundaries. This
 36 problem occurs due to the presence of some object boundary edges in the MST and cannot
 37 be avoided as any spanning tree connects all the nodes in a connected graph. Although, the
 38 authors in [5] tried to negate the leak effect using a bilateral filter as a post-processing step,
 39 the problem still persists (see Figure 1(c)). Also, the filtering results vary with the choice of
 40 MST which is undesirable.

41 This motivated us to explore the theoretical foundations of the TF for a deeper under-
 42 standing on how it works. Further, the links between the TF and the other edge-aware filtering
 43 methods might provide a possibility to design novel edge-aware filters. This article aims to
 44 answer this question and is an extended version of the conference paper [17], our contributions
 45 are the following:

- 46 1. We introduce an edge-aware filter based on the union of all MSTs of the image graph
 47 namely UMST filter, a generalization of the TF .
- 48 2. We compute the Γ -limit of the SPFs i.e. the Power Tree Filter (PTF) and show that
 49 it is precisely given by the UMST filter (see section 4 for details). Consequently, we
 50 provide a theoretical basis for the TF as it can be seen as an approximation of the
 51 Γ -limit of the SPFs.
- 52 3. We propose two different implementations of the Γ -limit which serve as an alternative
 53 to the TF (see section 5 for details) with a detailed analysis on how they work.

54 The rest of the paper is organized as follows: In section 2, we briefly recall the notions
 55 of TF, Γ -convergence and introduce UMST filter. In section 3, we develop SPFs as edge-
 56 aware filters starting from Gaussian-like filters and discuss their properties, links with other
 57 geodesic based methods. In section 4, we compute the Γ -limit of the SPFs and show that it
 58 is precisely the UMST filter. In section 5, we discuss approximations of UMST filter i.e. TF
 59 and propose two approximations based on shortest paths. We provide a detailed analysis of
 60 each of these implementations. In section 6, the conclusions follow and we speculate some
 61 possible directions to extend the ideas in the paper.

62 **2. Union Minimum Spanning Tree Filter.** In this section, we briefly recall the Tree Filter
 63 (TF) and provide our motivation on why one should consider using a filter based on union of

64 all the MSTs (UMST).

65 **2.1. Tree Filter.** Suppose I is a given image which possibly contains noise, we let I_i
 66 denote the color or intensity of the pixel i in the image I . Let S denote the tree filtered
 67 image. The authors in [5] construct a 4-adjacency edge-weighted graph, with the weights
 68 between adjacent pixels reflecting the color or intensity dissimilarity. More formally, if i and
 69 j are 4-adjacent pixels, they use w_{ij} defined by

$$70 \quad (1) \quad w_{ij} = \|I_i - I_j\|$$

71 One can construct a MST on this edge-weighted graph, I_{MST} . Since a spanning tree connects
 72 every pair of pixels and does not contain cycles, there exists a unique path between every pair
 73 of pixels. Let $D(i, j)$ denote the number of edges on the path between i and j on I_{MST} . For
 74 each pair i and j , the collaborative weights $t_i(j)$ are given by:

$$75 \quad (2) \quad t_i(j) = \frac{\exp\left(\frac{-D(i,j)}{\sigma}\right)}{\sum_q \exp\left(\frac{-D(i,q)}{\sigma}\right)}$$

76 where σ controls the falling rate and the summation over q is over all the pixels in the graph.
 77 The tree filtered value at pixel i is given by

$$78 \quad (3) \quad S_i = \sum_j t_i(j) I_j$$

79 Here the summation over j is over all the pixels in the graph.

80 It is reasonable to assume that the pixel color or intensities vary vastly across objects
 81 and are similar within objects. In other words, the higher weight edges mostly correspond
 82 to object boundaries and lower weight edges mostly correspond to object interiors. We shall
 83 work under this assumption in the rest of the article. The TF works on the following intuition:
 84 most of the higher weight edges do not appear while the lower weight edges mostly do appear
 85 in any MST. The collaboration across object boundaries is thus low while smoothing within
 86 objects is achieved well.

87 **2.2. Why UMST filter?.** The edges in the image graph that do not belong to object
 88 boundaries induce a disconnected subgraph. On the other hand, a MST of a graph is con-
 89 nected, hence any MST contains one or more object boundary edges. These edges cause a
 90 leak effect in the tree filtered image (see [Figure 1\(c\)](#)). Also, the filtering results vary with the
 91 MST used, making the choice of an arbitrary MST debatable. On the other hand, a filter
 92 based on UMST would ensure the following:

- 93 1. The filtering results would not depend on arbitrary MST computations.
- 94 2. There would be a significant reduction in the leak effect when compared to the TF
 95 (see [Figure 1\(d\)](#)).

96 The first property is a direct consequence of the fact that UMST filter uses all the MSTs
 97 of the image graph. The second property can be explained intuitively as follows: The edges
 98 in the UMST is a superset of the edges of an arbitrary MST. Now, among the edges that

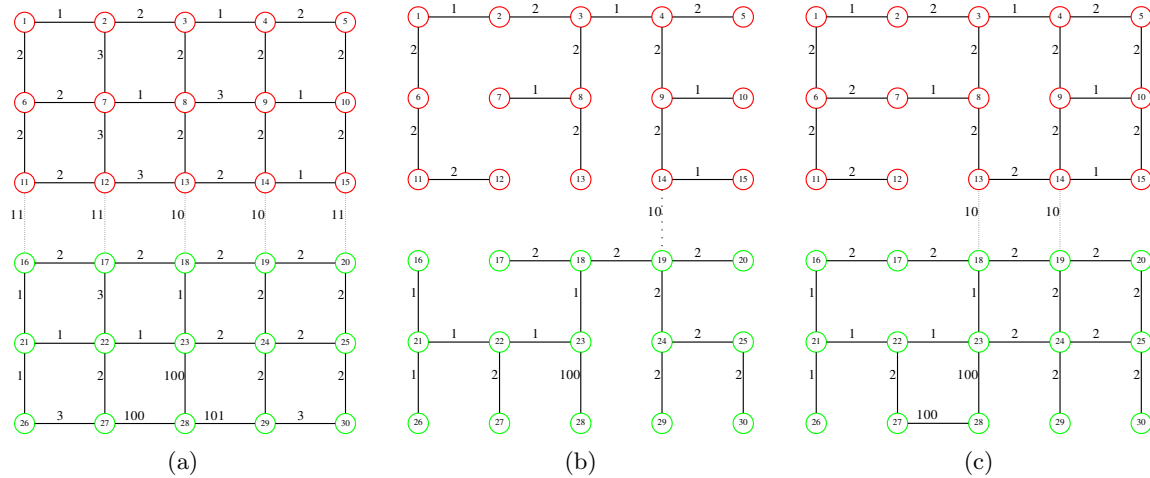


Figure 2. (a) 4-adjacency graph of a synthetic image containing two objects coloured in red and green. The pixels in this image are indexed from 1 to 30 and the weights on the edges denote the intensity dissimilarities. The edges corresponding to object boundaries are represented by dotted lines, (b) A MST obtained from (a), and (c) UMST obtained from (a). In order to illustrate that UMST filter yields better results, it should perform at least as good as TF for - removing noise at pixel numbered 28 and reducing the leak at object boundaries say at pixel numbered 13 and 14. Consider pixel numbered 28. One can see that both the edges of weights 100 incident on this pixel are present in the UMST, the noise removal is enhanced due to higher collaboration with the neighbouring pixels when compared to that of tree filter where MST had only one of the edges with weight 100. Now consider the pixel numbered 13. We see that although an extra boundary edge (edge 13 – 18) appears in the UMST, the presence of an additional interior edge incident on 13 in the UMST nullifies the effect of the boundary edge collaboration. At pixel numbered 14, the UMST filter performs better than tree filter due to the presence of the additional interior edge 13 – 14.

99 belong to UMST but not the MST are mostly object interior edges. These object interior
 100 edges dominate the collaborative effect of the object boundary edges to ensure a reduction in
 101 the leakage. **Figure 2** illustrates the above properties on a synthetic image.

102 Extending the idea of TF, we use an exponential falling weight similar to (2) for computing
 103 collaborative weights. However, we observe that there are possibly multiple paths between
 104 a given pair of pixels i and j in the UMST. In order to define the collaborative weights of
 105 the UMST filter, we need a criterion to choose a path among all the paths between i and j .
 106 We consider $\eta(i, j)$, the number of edges on a path with smallest dictionary or lexicographic
 107 order of edge weights (see **Definition 3.4** and **Definition 4.1**) replacing $D(i, j)$ in (2). This is
 108 a natural way to generalize the TF since: (a) the lesser the lexicographic order of a path, the
 109 lesser the chance of the path crossing an object boundary, (b) in the special case of the graph
 110 having a unique MST, this filter is exactly the same as TF.

111 **2.3. Lexicographic Ordering and Pass Values.** Lexicographic order of a path is related
 112 very closely to the notion of pass values used in segmentation. Pass value [15] or the minimax
 113 distance [19] between a pair of nodes is the minimum of the l^∞ norm over all the paths
 114 between them. To the best of our knowledge, this feature was first used in image filtering
 115 in [31]. Pass values between different minima of a gradient image is a measure of contrast

116 difference between objects in watershed segmentation [28]. One can observe that any image
 117 transformation on the gradient image that keeps the object boundaries intact preserves the
 118 contrast difference between objects. It is hence a desired condition for a segmentation method
 119 to preserve the contrast difference (topological watersheds [13] for instance).

120 Observe that given pixels i and j , a path with smallest lexicographic order of the edge-
 121 weights would be a special case of a path with smallest l^∞ norm. In simpler words, the smallest
 122 lexicographic order path is a critical path that determines the contrast difference between a
 123 pair of pixels. In practice, a path with smallest lexicographic order would be unique. This
 124 serves as a tie-breaker on choosing a critical path among the smallest l^∞ norm paths thus
 125 reducing the ambiguity.

126 Now, we shall recall notions of Γ -convergence before we state the main result of the paper.

127 **2.4. UMST filter and Γ -convergence.** Γ -convergence [7] is the study of asymptotic be-
 128 haviour of a sequence of minimization problems. Suppose for each $n \in \mathbb{N}$, $F_n : \mathbb{R}^l \rightarrow \mathbb{R}$ is a
 129 cost function such that $\arg \min_{x \in \mathbb{R}^l} (F_n(x)) \neq \emptyset$, what does $\lim_{n \rightarrow \infty} x_n$ minimize (assuming the
 130 limit exists) where $x_n \in \arg \min_{x \in \mathbb{R}^l} (F_n(x))$? In other words, in what sense do F_n converge
 131 to F where $F : \mathbb{R}^l \rightarrow \mathbb{R}$ so that $\lim_{n \rightarrow \infty} x_n \in \arg \min_{x \in \mathbb{R}^l} (F(x))$.

132 The usual notions of point wise and uniform convergence do not make sense when working
 133 with functionals and we need the following:

134 **Definition 2.1.** We say that $F_n \xrightarrow{\Gamma} F$ (read as F_n gamma converges to F) if: (1) for every
 135 $x \in \mathbb{R}^l$ and every sequence $(x_n)_{n \in \mathbb{N}}$, such that $x_n \rightarrow x$, we have $F(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n)$,
 136 and (2) for every $x \in \mathbb{R}^l$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$, such that $x_n \rightarrow x$, and $F(x) \geq$
 137 $\limsup_{n \rightarrow \infty} F_n(x_n)$.

138 **Theorem 2.2.** (Fundamental Theorem of Γ -Convergence) If $F_n \xrightarrow{\Gamma} F$ and x_n minimizes F_n
 139 for each $n \in \mathbb{N}$, then every limit point of the sequence $(x_n)_{n \in \mathbb{N}}$ is a minimizer of F .

140 *Proof.* Refer to [7] ■

141 In simple words, one can approximate a minimizer of F using the minimizers of F_n . **Defini-**
 142 **tion 2.1** is a simplified version of the general definition and suffices for our purposes. For a
 143 comprehensive study of Γ -convergence, we refer the interested reader to [7].

144 Γ -convergence has been proved to be very useful in many computer vision applications
 145 especially the ones based on variational formulations. The following are a few instances:
 146 In [12], the authors unified and extended a common framework of semi-supervised or seeded
 147 graph-based image segmentation methods namely graph cuts [6], random walker [22], geodesics
 148 [1, 4, 16, 19] and watershed cuts [14, 15]; In [29, 2], the elementary mathematical morphological
 149 (MM) operators have been formulated as limits of variational problems; and in [34], the authors
 150 view the local min-max filters as a limit of normalized power-weighted averaging filter; In [9],
 151 the authors propose a fast alternative to spectral clustering methods by considering their
 152 Γ -limit.

153 As edge-preserving image filtering and image segmentation are closely related problems,
 154 one can anticipate to establish links between existing filtering methods using Γ -convergence
 155 (in the similar lines as the unified seeded-segmentation framework in [12]). In fact we prove
 156 the following theorem which is the main result of the paper: *UMST filter is the Γ -limit of*

157 *shortest path edge-aware filters*. See [section 3](#) for a formal definition of shortest path filters
 158 and [section 4](#) for a proof. This result implies that one can view the UMST filter and the
 159 shortest path filters in an optimization framework.

160 **3. Shortest Path Filters and Related Methods.** In this section, we shall review the
 161 shortest path filters in detail. In particular, we develop them as a natural edge-aware extension
 162 of Gaussian-like filters. The rest of the section is dedicated to the discussions on their links
 163 with other related geodesic methods.

164 Before formally defining the SPF, we need some notions of graphs that we define below.

165 3.1. Basic Notions.

166 **Definition 3.1.** An *edge-weighted graph* $\mathcal{G} = (V, E, W)$ consists of a finite set V of nodes,
 167 and set of unordered pairs of elements of V i.e. $\{\{x, y\} \subset V : x \neq y\}$, called the edge set E ,
 168 a positive real-valued function W on the set E . We denote w_{ij} or $W(e_{ij})$ as the weight of the
 169 edge joining pixels i and j .

170 **Definition 3.2.** For $p \in \mathbb{Z}^+$, we denote by $\mathcal{G}^{(p)} = (V, E, W^{(p)})$, the graph that contains the
 171 same set of nodes and edges as of \mathcal{G} and $W^{(p)}(e_{ij}) = (W(e_{ij}))^p$ for each edge $e_{ij} \in E$ and we
 172 call $\mathcal{G}^{(p)}$ as an *exponentiated graph* of \mathcal{G} .

173 **Definition 3.3.** A *path* $P(i, j)$ between nodes i and j is a finite ordered sequence of nodes
 174 of \mathcal{G} such that there is an edge incident on every adjacent pair of nodes in the sequence. We
 175 say that a path from i to j is a *simple path* if all the nodes in the sequence are distinct.

176 **Definition 3.4.** Assume that the distinct weights in \mathcal{G} are given by $0 < w_1 < w_2 < \dots < w_k$.
 177 Given a path $P(i, j)$ in \mathcal{G} , one can assign a k -tuple (n_1, \dots, n_k) to the path, where n_r denotes
 178 the number of edges of weight w_r on the path $P(i, j)$. This k -tuple is referred to as the *edge-*
 179 *weight distribution* of the path $P(i, j)$.

180 We remark that the k -tuples associated with a path $P(i, j)$ in graph \mathcal{G} and its exponentiated
 181 graph $\mathcal{G}^{(p)}$ are identical by the virtue of it's definition. However it is important to note that:
 182 corresponding to each of the coordinates, the weights in the edge-weight distributions are
 183 different.

184 **Definition 3.5.** Assume that the distinct weights in \mathcal{G} are given by $0 < w_1 < w_2 < \dots <$
 185 w_k . Suppose $P(i, j)$ is a path between pixels i and j in \mathcal{G} . $P(i, j)$ is said to be a *shortest*
 186 *path* between the pixels i and j in \mathcal{G} if for every path $Q(i, j)$ between i and j in \mathcal{G} , we
 187 have $\sum_{r=1}^k n_r w_r \leq \sum_{r=1}^k m_r w_r$ where (n_1, \dots, n_k) and (m_1, \dots, m_k) denote the edge-weight
 188 distributions of $P(i, j)$ and $Q(i, j)$ respectively.

189 We remark that a shortest path is always a simple path since all the weights in the edge-
 190 weighted graphs are strictly positive.

191 We shall now build an edge-aware filter from scratch: Let i and j be two pixels in $\mathcal{G}^{(p)}$.
 192 Consider the simplest weighted-average filter whose collaborative weights are given by

$$193 \quad (4) \quad g_i(j) = \frac{\exp\left(-\frac{\|i-j\|}{\sigma}\right)}{\sum_k \exp\left(-\frac{\|i-k\|}{\sigma}\right)}$$

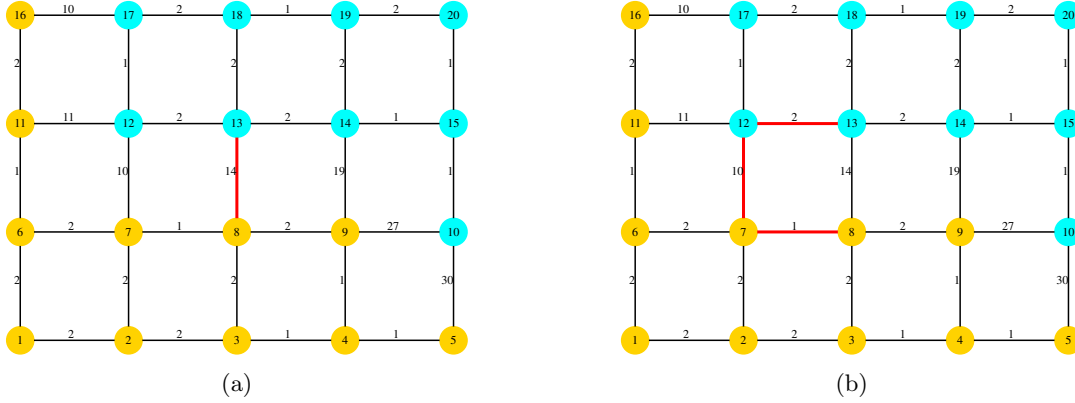


Figure 3. (a) and (b) 4-adjacency graph of a synthetic image containing two objects coloured in blue and yellow. The pixels in this image are indexed from 1 to 20 and the weights on the edges denote the intensity dissimilarities. The collaborative weight for the pair of pixels 8 and 13 (which crosses an object boundary) given by Gaussian-like filter would consider the edge highlighted in red in (a) and yields a high value. On the other hand, the SPF considers the path $\langle 8, 7, 12, 13 \rangle$ highlighted in red in (b) for computing the corresponding collaborative weight. This illustrates that SPF respects the object boundaries.

194 where $\|i - j\| : l_1$ norm between the pixels i and j and σ is the parameter controlling the
 195 level of smoothing.

196 We observe that this is a Gaussian-like filter and collaborative weights purely depend on
 197 the spatial distance between pixels i and j . More specifically, collaborative weights between
 198 pixels separated by same distance is indifferent w.r.t. existence of an object boundary between
 199 them. Hence, one has to find a way to ensure that the collaborative weights are lesser across
 200 boundaries. A natural way to extend the idea of a Gaussian-like filter is: given a pair of pixels
 201 i and j , consider the number of edges on a path with smallest sum of weights between them.
 202

Let $\Pi(P(i, j))$ denote the number of edges on a path $P(i, j)$. Define

$$203 \quad (5) \quad \Theta^{(p)}(i, j) = \inf\{\Pi(P(i, j)) \text{ where } P(i, j) \text{ is a shortest path in } \mathcal{G}^{(p)}\}$$

204 where the edge weights are given by:

$$205 \quad (6) \quad w_{ij} = \|I_i - I_j\| + 1$$

206 The SPF at pixel i is defined as:

$$207 \quad (7) \quad S_i^{(p)} = \sum_j \frac{\exp\left(-\frac{\Theta^{(p)}(i, j)}{\sigma}\right)}{\sum_k \exp\left(-\frac{\Theta^{(p)}(i, k)}{\sigma}\right)} I_j,$$

208 where σ controls the falling rate and the summations are over all pixels in the image.

209 Note that the weights in Eq. (6) are different from the ones in Eq. (1) to ensure that
 210 the weights are strictly positive. We use the edge weights as per (6) in the rest of the paper.
 211 Since, we use an increasing transformation on edge weights, shortest paths and MSTs are
 212 invariant to the modification. This assumption is needed to ensure that the converse part in
 213 Lemma 4.3 holds.

214 We remark that in the special case of all pixel values being equal in the image, SPF is
 215 exactly Gaussian-like filter. Also, in practice, the shortest path distances between a pair of
 216 pixels within the objects are close to the spatial distances and are larger than the spatial
 217 distances across object boundaries. Figure 3 illustrates on a synthetic image that the SPF is
 218 a natural edge-preserving extension of the Gaussian-like filter.

219
 220 Further, one can see that the SPF value at pixel i is a solution of the following optimization
 221 problem:

222 Consider the cost function

$$223 \quad (8) \quad Q_i^{(p)}(x) = \sum_j \exp\left(-\frac{\Theta^{(p)}(i,j)}{\sigma}\right) (x - I_j)^2$$

224 where σ controls the falling rate and the summation is over all pixels. The shortest path
 225 filtered value at pixel i is given by the minimizer of $Q_i^{(p)}(x)$ i.e.

$$226 \quad S_i^{(p)} = \arg \min_x Q_i^{(p)}(x)$$

227 SPFs are not completely new and there exist in literature, several edge-preserving filters
 228 using geodesics such as the ones discussed in [23], the adaptive kernel filter such as morpho-
 229 logical amoebas [25].

230 **3.2. Relation to Morphological Amoebas.** SPFs are also closely related to Morphological
 231 Amoebas. Morphological Amoebas are adaptive structuring elements based on shortest path
 232 distances used to build edge-aware filters. These kernels work on the assumption that the
 233 gradients are low within the objects and high across the object boundaries. In order to ensure
 234 that the kernels do not cross the object boundaries, the amoeba distance defined below is used
 235 to compute them:

$$236 \quad (9) \quad \kappa(i, j) = \min_{P(i,j)} L_\lambda(P(i, j))$$

237 where $P(i, j)$ is a path between pixels i and j , $\langle i = x_0, x_1, \dots, x_n = j \rangle$ and $\lambda \geq 0$ is a
 238 user input.

$$239 \quad (10) \quad L_\lambda(P(i, j)) = \sum_{t=0}^{n-1} (1 + \lambda \|I_{x_{t+1}} - I_{x_t}\|)$$



Figure 4. (a) and (b) synthetic images to illustrating the lexicographic ordering of paths. The lexicographic order of path in blue is lesser than that of the one in green

240 The closed ball at pixel i given by $\{j : \kappa(i, j) \leq r\}$ is the kernel used for edge-aware
 241 smoothing. The cardinality or the size of the kernel depends on r and is chosen as per the
 242 level of smoothing desired.

243 **Proposition 3.6.** $\Theta^{(1)}(i, j)$ is a constrained minima of the amoeba kernel path length given
 244 by

$$245 \quad (11) \quad \Theta^{(1)}(i, j) = \min L_0(P(i, j)) \text{ subject to } P(i, j) \in \arg \min L_1(P(i, j))$$

246 Further, the family of parameters $\Theta^{(p)}(i, j)$ are given by:

$$247 \quad (12) \quad \Theta^{(p)}(i, j) = \min L_0^p(P(i, j)) \text{ subject to } P(i, j) \in \arg \min L_1^p(P(i, j))$$

248 where

$$249 \quad (13) \quad L_\lambda^p(P(i, j)) = \sum_{t=0}^{n-1} (1 + \lambda \|I_{x_{t+1}} - I_{x_t}\|)^p$$

250

251 *Proof.* It readily follows from (5), (6), (10) and (13) ■

252 Note that the morphological amoeba lengths are a special case of the lengths given by (13).
 253 We can hence view the SPF as a generalization of the notion of morphological amoeba lengths.

254 **4. UMST Filter: Gamma Limit of Shortest Path Filters.** In this section, we shall prove
 255 that the UMST filter is the Γ -limit of SPFs. As the weights of the graphs of the shortest path
 256 filters are powers of natural numbers (see section 3), we use the term Power Tree Filter to
 257 denote the Γ -limit of Shortest Path Filters. We shall need some definitions before we prove
 258 this result.

259 4.1. Some Definitions.

260 **Definition 4.1.** Assume that graph \mathcal{G} has k distinct weights given by $w_1 < \dots < w_k$. Let
 261 (n_1, \dots, n_k) and (m_1, \dots, m_k) denote the edge-weight distributions of paths P and Q in \mathcal{G}

262 respectively. Let $l = \sup(A)$ where $A = \{r : 1 \leq r \leq k, n_r \neq m_r\}$ We define **dictionary**
 263 **ordering** or **lexicographic ordering** on the set of paths in \mathcal{G} as follows:

$$264 \quad (14) \quad P \geq Q \Leftrightarrow A = \emptyset \text{ or } n_l > m_l$$

265 See Figure 4 for an illustration on dictionary ordering. Note that dictionary ordering
 266 yields a complete ordering on the set of paths in \mathcal{G} and the ordering remains same in each of
 267 the exponentiated graphs $\mathcal{G}^{(p)}$.

268 **Definition 4.2.** Suppose $P(i, j)$ is a path between pixels i and j in \mathcal{G} . $P(i, j)$ is said to be
 269 a **smallest path w.r.t. dictionary order** between the pixels i and j in \mathcal{G} if for every path
 270 $Q(i, j)$ between i and j in \mathcal{G} , we have $Q(i, j) \geq P(i, j)$.

271 Note that every smallest path w.r.t. dictionary order between pixels i and j in \mathcal{G} has the
 272 same edge-weight distribution. In particular, the number of edges on a smallest path w.r.t.
 273 dictionary order between i and j denoted by $\Pi(i, j)$ is well-defined.

274 Any MST in $\mathcal{G}^{(p)}$ is a MST in \mathcal{G} and vice-versa. This follows directly from the fact that
 275 MST is invariant to any strictly increasing transformation on the weights of a connected
 276 graph. The notions of smallest paths w.r.t. dictionary order and that of MSTs in $\mathcal{G}^{(p)}$ are
 277 hence independent of p .

278 **4.2. Gamma Limit of Shortest Path Filters.** In this subsection, we characterize the
 279 Power Tree Filter or Γ -limit of Shortest Path Filters. Firstly, we have the following result:

280 **Lemma 4.3.** Let $\mathcal{G} = (V, E, W)$. For every pair of pixels i and j in V , there exists $p_0 \geq 1$
 281 such that, a path $P(i, j)$ is a shortest path between i and j in $\mathcal{G}^{(p)}$ for all $p \geq p_0$ if and only
 282 if $P(i, j)$ is a smallest path w.r.t. dictionary order between i and j in \mathcal{G} . Further, p_0 is
 283 independent of i and j .

284 **Proof.** Let $\mathcal{G} = (V, E, W)$ and let the distinct weights in \mathcal{G} be given by $w_1 < \dots < w_k$.

285 Firstly, we shall show that for a given pair of pixels i and j , if $P(i, j)$ is a smallest path
 286 w.r.t. dictionary order between i and j in \mathcal{G} then there exists a constant p_0 such that for
 287 each $p \geq p_0$, $P(i, j)$ is a shortest path between i and j in $\mathcal{G}^{(p)}$. Let $P(i, j)$ be a small-
 288 est path w.r.t. dictionary order between i and j in \mathcal{G} . Let $Q(i, j)$ be an arbitrary simple
 289 path between i and j . Let (n_1, \dots, n_k) and (m_1, \dots, m_k) denote the edge-weight distribu-
 290 tions of paths $P(i, j)$ and $Q(i, j)$ respectively. Let $A(P, Q) = \{1 \leq r \leq k : n_r \neq m_r\}$.
 291 Suppose $A(P, Q) = \emptyset$ then $\sum_{r=1}^k n_r w_r^p \leq \sum_{r=1}^k m_r w_r^p \forall p \geq 1$. If $A(P, Q) \neq \emptyset$ then let
 292 $l = \sup(A(P, Q))$. We have $m_l > n_l$ by choice of $P(i, j)$. Also, the difference of the total
 293 weights i.e. $\sum_{r=1}^k m_r w_r^p - \sum_{r=1}^k n_r w_r^p = \Theta(w_l^p)$ with a positive leading coefficient. Hence
 294 $\exists p_{Q(i,j)} \geq 1$ such that $\sum_{r=1}^k n_r w_r^p \leq \sum_{r=1}^k m_r w_r^p \forall p \geq p_{Q(i,j)}$. Now, let \mathcal{S}_{ij} denote the set of
 295 all simple paths from i to j . Then $|\mathcal{S}_{ij}| < \infty$. Set $p_{ij} = \sup\{p_{Q(i,j)} : Q(i, j) \in \mathcal{S}_{ij}\} < \infty$. We
 296 note that given any path which is not simple, one can drop the redundant edges to construct a
 297 simple path with strictly smaller total weight. It is hence enough to show that the total weight
 298 of $P(i, j)$ is lesser than or equal to every simple path between i and j in $\mathcal{G}^{(p)}$ for sufficiently
 299 large p . As V is finite, setting $p_0 = \sup\{p_{ij} : i, j \in V\}$ completes the argument.

300 Conversely, suppose $P(i, j)$ is NOT a smallest path w.r.t. dictionary ordering between
 301 i and j , we shall construct a sequence $(p_n)_{n \geq 1}$ converging to ∞ such that $P(i, j)$ is not

302 a shortest path between i and j in $\mathcal{G}^{(p_n)}$ for each $n \geq 1$. Since $P(i, j)$ is not a smallest
 303 path w.r.t. dictionary ordering between i and j , \exists a path $T(i, j)$ between i and j such that
 304 $P(i, j) \geq T(i, j)$ holds but $T(i, j) \geq P(i, j)$ does not hold. Equivalently, if the edge weight
 305 distributions of $P(i, j)$ and $T(i, j)$ are given by (n_1, \dots, n_k) and (t_1, \dots, t_k) respectively then
 306 $A(P, T) \neq \emptyset$ and $t_l < n_l$ where $l = \sup(A(P, T))$ and $A(P, T) = \{1 \leq r \leq k : n_r \neq t_r\}$.
 307 The difference of the total weights i.e. $\sum_{r=1}^k t_r w_r^p - \sum_{r=1}^k n_r w_r^p = \Theta(w_l^p)$ and has a negative
 308 leading coefficient. Thus, \exists a constant $\rho_{ij} \geq 1$ such that for each $p \geq \rho_{ij}$, $P(i, j)$ is not a
 309 shortest path between i and j . We set $(p_n)_{n \geq 1}$ by $p_n = \rho_{ij} + n - 1$ to complete the proof. ■

310 Loosely speaking, for a large enough power p , on $\mathcal{G}^{(p)}$, between every pair of pixels, a
 311 shortest path is a smallest path w.r.t. dictionary order and vice-versa. In short, we have the
 312 following corollary:

313 **Corollary 4.4.** *Let $\Delta(i, j)$ denote the number of edges on a smallest path w.r.t. dictionary*
 314 *order between pixels i and j in \mathcal{G} . As $p \rightarrow \infty$, we have $\Theta^{(p)}(i, j) \rightarrow \Delta(i, j)$ for each pair i and*
 315 *j .*

316 **4.3. Optimization Framework for UMST Filter.** Recall the definition of UMST filter
 317 from section 2. Let \mathcal{G} denote the given image, let \mathcal{G}_{UMST} denote the UMST on the edge-
 318 weighted graph constructed from \mathcal{G} . Let $\eta(i, j)$ denote the number of edges on a smallest path
 319 w.r.t. dictionary order between pixels i and j in \mathcal{G}_{UMST} . Now consider the cost function given
 320 by:

$$\widehat{Q}_i(x) = \sum_j \exp\left(-\frac{\eta(i, j)}{\sigma}\right) (x - I_j)^2$$

322 where σ controls the falling rate and the summation is over all pixels, the UMST filtered
 323 value at pixel i is given by the minimizer of $\widehat{Q}_i(x)$.

$$U_i = \arg \min \widehat{Q}_i(x) = \sum_j \frac{\exp\left(-\frac{\eta(i, j)}{\sigma}\right)}{\sum_k \exp\left(-\frac{\eta(i, k)}{\sigma}\right)} I_j,$$

325 where U denotes the UMST filtered image.

326 Firstly, we need the following:

327 **Lemma 4.5. (Cut Property)** *For any cut C of a connected graph $\mathcal{G} = (V, E, W)$, if the*
 328 *weight of an edge e in the cut-set C is not larger than the weights of all other edges in C , then*
 329 *this edge belongs to a MST of the graph $\mathcal{G} = (V, E, W)$.*

330 **Lemma 4.6. (Cycle Property)** *For any cycle C in the graph $\mathcal{G} = (V, E, W)$, if the weight*
 331 *of an edge e of C is larger than the individual weights of all other edges of C , then this edge*
 332 *cannot belong to a MST.*

333 Before we prove the main result of the paper, we need Lemma 4.7 which is a modified
 334 version of a result stated in [27].

335 **Lemma 4.7.** *Let $\mathcal{G} = (V, E, W)$ be an edge-weighted graph. Let $\mathcal{G}_{<w}$ denote the induced*
 336 *subgraph of \mathcal{G} with the vertex set V and all the edges $e_{ij} \in E$ whose weight $w_{ij} < w$. Let*
 337 *\mathcal{G}_{UMST} denote the UMST of \mathcal{G} . Then an edge e with weight $w(e)$ belongs to the \mathcal{G}_{UMST} if and*
 338 *only if the edge e joins two connected components in $\mathcal{G}_{<w(e)}$.*

339 *Proof.* The proof directly follows from [Lemma 4.5](#) and [Lemma 4.6](#). ■

340 In simple words, an edge e is in some MST if and only if it connects two different compo-
 341 nents of the induced subgraph generated by edges of weights lower than that of e .

342 **Proposition 4.8.** *Every smallest path w.r.t. dictionary order between any two arbitrary*
 343 *nodes in $\mathcal{G} = (V, E, W)$ lies on a MST of \mathcal{G} and hence on the union of MSTs of \mathcal{G} .*

344 *Proof.* Let i and j be two arbitrary nodes in \mathcal{G} . Let $P(i, j)$ be a smallest path w.r.t.
 345 dictionary order between i and j . It is now enough to show that every edge in the path $P(i, j)$
 346 satisfies the characterization given in [Lemma 4.7](#). Suppose if possible, let $e \in P(i, j)$ be of
 347 smallest possible weight such that e is incident on nodes in a same connected component of
 348 $\mathcal{G}_{<w(e)}$. Adding e thus forms a cycle C and the other edges in C have weights strictly less
 349 than $w(e)$.

350 Now, consider the subgraph generated by edges in $P(i, j) \cup C \setminus \{e\}$. This subgraph is
 351 connected and hence there exists a path $Q(i, j)$ (say) between i and j . It is easy to see that
 352 $Q(i, j)$ has smaller dictionary order compared to $P(i, j)$: number of edges of weight greater
 353 than $w(e)$ in $Q(i, j)$ cannot exceed to that in $P(i, j)$ since C has edges of weight strictly less
 354 than $w(e)$; number of edges of weight $w(e)$ in $Q(i, j)$ is at least one less than that of $P(i, j)$.
 355 This contradicts the fact that $P(i, j)$ is a smallest path w.r.t. dictionary order between i and
 356 j . ■

357 **Corollary 4.9.** *For every pair i and j in \mathcal{G} , we have $\eta(i, j) = \Delta(i, j)$*

358 The main result is formally stated as follows:

359 **Theorem 4.10.** *As $p \rightarrow \infty$, we have the following:*

$$360 \quad (17) \quad Q_i^{(p)}(x) \xrightarrow{\Gamma} \widehat{Q}_i(x)$$

361 *In other words, the shortest path filters converge to the UMST filter as $p \rightarrow \infty$ i.e.*

$$362 \quad (18) \quad S_i^{(p)} \longrightarrow U_i$$

363

364 *Proof.* The proof follows readily from [Proposition 4.8](#) and [Lemma 4.3](#). ■

365 The fact that UMST filter is a Γ -limit of shortest path filters is useful: The state-of-art
 366 algorithms in shortest paths and MSTs can be jointly exploited to obtain novel algorithms to
 367 compute UMST filter.

368 **5. Implementation.** In this section, we discuss several possible implementations of UMST
 369 filter. We utilize ideas from shortest paths and spanning trees to obtain two novel approxima-
 370 tion algorithms to compute UMST filter. We provide detailed analyses of our implementations
 371 along with the TF which is yet another approximation of the UMST filter.

372 **5.1. Exact Algorithms.** Thanks to UMST characterization given by Lemma 4.7, we can
 373 compute the UMST of a graph in $\mathcal{O}(|E|)$ which in the case of 4-adjacency graphs would be
 374 $\mathcal{O}(|V|)$. Also one can hope to reduce at least significant number of edges when compared to
 375 \mathcal{G} in practice.

376 A naive approach is to adapt the Floyd-Warshall [21] algorithm to calculate $\eta(i, j)$ for each
 377 pair and the computation of UMST filter takes $\mathcal{O}(|V|^3)$ time and $\mathcal{O}(|V|^2)$ space. This is very
 378 expensive in terms of both memory and space and is hence not practical. In order to reduce
 379 the complexity, one needs to exploit the properties of lexicographic ordering and that of the
 380 UMST graph structure. One approach is by borrowing ideas from Image Foresting Transform
 381 (IFT) [19].

382 Image Foresting Transform is a unified framework for several image processing operators
 383 that are based on shortest paths. Some of these operators include fuzzy-connected segmen-
 384 tation [11, 30] and distance transforms [26, 18]. In simple words, IFT is a generalization of
 385 Dijkstra’s algorithm where an image (with a specified adjacency relation), a set of seeds and
 386 a path cost function are specified and one needs to assign to every non-seeded pixel, a seed
 387 label to which it admits a path with smallest cost. The path costs are usually application-
 388 specific and are not necessarily given by sum of the weights of the edges on the path. Hence,
 389 a modified Dijkstra’s algorithm is used to handle a general class of path cost functions that
 390 arise in computer vision applications. We briefly describe the IFT framework below as in [19]
 391 and then discuss the relations with the SPF’s:

392 The IFT takes as an input, an image I , an adjacency relation \mathcal{A} (usually given by 4-
 393 adjacency in case of 2D images), a cost function \mathcal{C} for all paths and outputs an optimum
 394 spanning forest. Note that the seeds can be specified implicitly by the cost function by
 395 assigning a fixed cost for every path that starts at a certain pixel (finite for seed pixels and
 396 infinite for non-seed pixels). Although there are no restrictions on the dimension of the
 397 image and the adjacency relation, the path costs are restricted and the following are sufficient
 398 conditions for the optimal IFT algorithm [19] to be applicable:

399 For any pixel t , there is an optimum path π ending at t which is either trivial or is of the
 400 form $\tau \cdot \langle s, t \rangle$, where

- 401 • $C(\tau) \leq C(\pi)$,
- 402 • τ is an optimum path ending at s ,
- 403 • $C(\tau' \cdot \langle s, t \rangle) = C(\pi)$ if τ' is an optimum path ending at s

404 Using a single seed, the optimum-path forest obtained (which is a tree rooted at the seed)
 405 can be used to compute the shortest path filtering collaborative weights (see (5)). By varying
 406 the seed s , the SPF can be computed for the whole image. However, such an implementation
 407 would still take $\mathcal{O}(|V|^2)$ time. To the best of our knowledge, we do not have any linear-time
 408 exact algorithms for computing the UMST filter. Thus, it calls a need to develop at least a
 409 quasi-linear algorithm approximation algorithm.

410 **5.2. Approximation Algorithms.** *Single tree-based approximation:* As we have developed
 411 the UMST filter by generalizing the notion of TF in section 2, we can view TF as a heuristic
 412 approximation of the UMST filter. Note that TF uses only one MST and hence can be
 413 computed dynamically in linear time by doing an upward aggregation followed by a downward
 414 aggregation on the tree (see [37] for details). However, the usage of an arbitrary MST makes

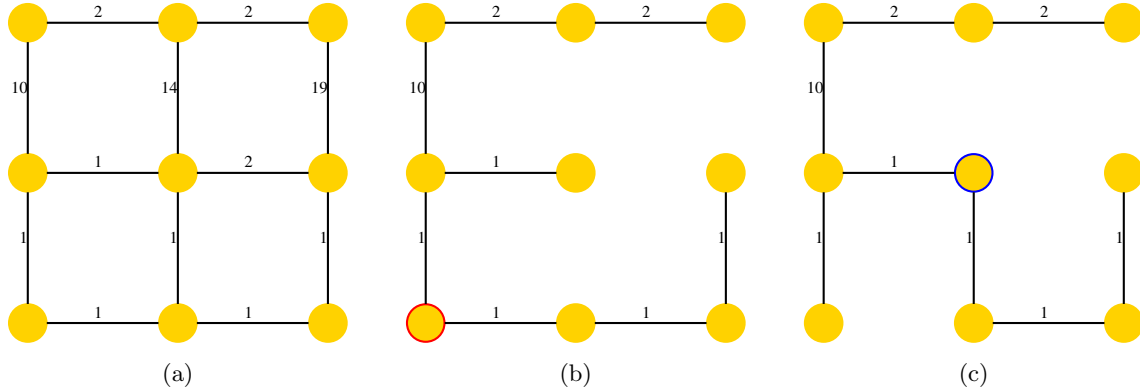


Figure 5. (a) A synthetic image with the edge-weights reflecting the intensity dissimilarities, (b) and (c) Adaptive spanning trees of the pixels circled in red and blue respectively.

Algorithm 1 Generic Algorithm to Compute UMST Filter

Input: A 4-adjacency graph \mathcal{G} of an image I , Adaptive Spanning Trees T_i for each $i \in V$

Output: Filtered image S .

- 1: **for all** pixels $i \in V$ **do**
 - 2: Starting from i on T_i , use $S_p = I_p + \sum_{q \in \text{children of } p} \exp(\frac{-1}{\sigma}) S_q$ recursively to compute S_i
 - 3: **end for**
-

415 it difficult to analyse the degree of approximation quantitatively.

416 *Multiple tree-based approximations:* One can also uses multiple spanning trees adaptively
 417 for filtering different pixels. In fact, one can find upper bounds on the approximation factors
 418 as a consequence of [Proposition 5.1](#).

419 **Proposition 5.1.** *For every pixel i in the image I , there exists a spanning tree T_i (termed
 420 as adaptive spanning tree), such that T_i contains a smallest path with respect to dictionary
 421 ordering between pixels i and any other pixel j in I .*

422 *Proof.* Let i be an arbitrary pixel in I . We shall construct an adaptive spanning tree T_i ,
 423 such that T_i contains a smallest path with respect to dictionary ordering between pixels i and
 424 any other pixel j in I . The construction is a special case of IFT (see [19]) with the following
 425 as the inputs: I is the image with 4-adjacency. The path costs are given by: $f(\pi) = \infty$ for
 426 every path π that does not start at i . All paths π_l that start at i are completely ordered in
 427 the order of decreasing lexicographic ordering using their edge-weight distributions. The path
 428 costs are determined by the order statistics of the path i.e. if $\pi_1 \geq \pi_2 \geq \dots > \pi_l$ are all the
 429 paths starting from i ordered w.r.t. lexicographic ordering then the path cost f is given by
 430 $f(\pi_t) = l - t + 1$ where $1 \leq t \leq l$. We remark that the this path cost is *monotonic incremental*
 431 (see [19]). The IFT algorithm applied thus yields T_i , an adaptive spanning tree of i with the
 432 required properties. ■

433 [Proposition 5.1](#) essentially implies that one can decompose the UMST into possibly differ-

Algorithm 2 To compute a depth-truncated adaptive spanning tree

Input: UMST of the graph, depth d and pixel i
Output: Depth-Truncated Adaptive Spanning Tree $T_{i,d}$

- 1: Set $X = \{i\}$ and $T_{i,d} = (i, \emptyset)$
- 2: **while** True **do**
- 3: break = **true**
- 4: **for** e in shortest edges from X to X^c **do**
- 5: **if** $\text{dist}(e, i, T_{i,d}) < d$ **then**
- 6: $T_{i,d} \cup e$
- 7: break = **false**
- 8: **end if**
- 9: **end for**
- 10: **if** break = **true** **then**
- 11: return $T_{i,d}$.
- 12: **end if**
- 13: **end while**

434 ent spanning trees T_i for each $i \in V$. Using each of the trees independently (see [Figure 5](#) for
435 an illustration on a synthetic image), one can obtain the exact UMST filter (see [Algorithm 1](#)).
436 The exact computation takes $\mathcal{O}(|V|^2)$ and $\mathcal{O}(|V|)$ time and space complexities respectively.
437 However, by truncating each of these trees in [Algorithm 1](#), one can obtain fast approximate
438 solutions. We present two ways to truncate the adaptive trees to obtain approximate UMST
439 filter.

440

441 *Depth-based truncation:* For each $i \in V$, we truncate the adaptive spanning tree T_i to $T_{i,d}$
442 such that it contains only the pixels j that are at most d (user-defined parameter) edges away
443 from i on T_i (see [Figure 6\(b\)](#) for an illustration and [Algorithm 2](#) for computing it)

444 We rewrite (16) as:

445 (19)
$$U_i = \frac{1}{C} \sum_l \exp\left(-\frac{l}{\sigma}\right) \sum_{j:\eta(i,j)=l} I_j,$$

446 where C is the normalizing constant. In (19), we observe that the exponential term rapidly
447 converges to 0 and hence one can approximate the above expression by

448 (20)
$$U_i \approx U_{i,d} = \frac{1}{C} \sum_{l=1}^d \exp\left(-\frac{l}{\sigma}\right) \sum_{j:\eta(i,j)=l} I_j,$$

449 where d is a parameter indicating a fixed depth. This simplification reduces the calculation
450 for each pixel drastically and hence [Algorithm 2](#) is practically $\mathcal{O}(|V|)$.

451 We shall now analyse (20) in more detail. Also as a consequence of [Proposition 5.1](#), any
452 two pixels are separated by at most $|V| - 1$ edges on a spanning tree T_i i.e. $\eta(i, j) \leq |V| - 1$.
453 Also, if $\eta(i, j) = l > 0$ then for each $0 \leq l' \leq l$, there exists at least one pixel j' such that
454 $\eta(i, j') = l'$. Assume that the intensities satisfy $1 \leq I_j \leq 255$ then we have the following:

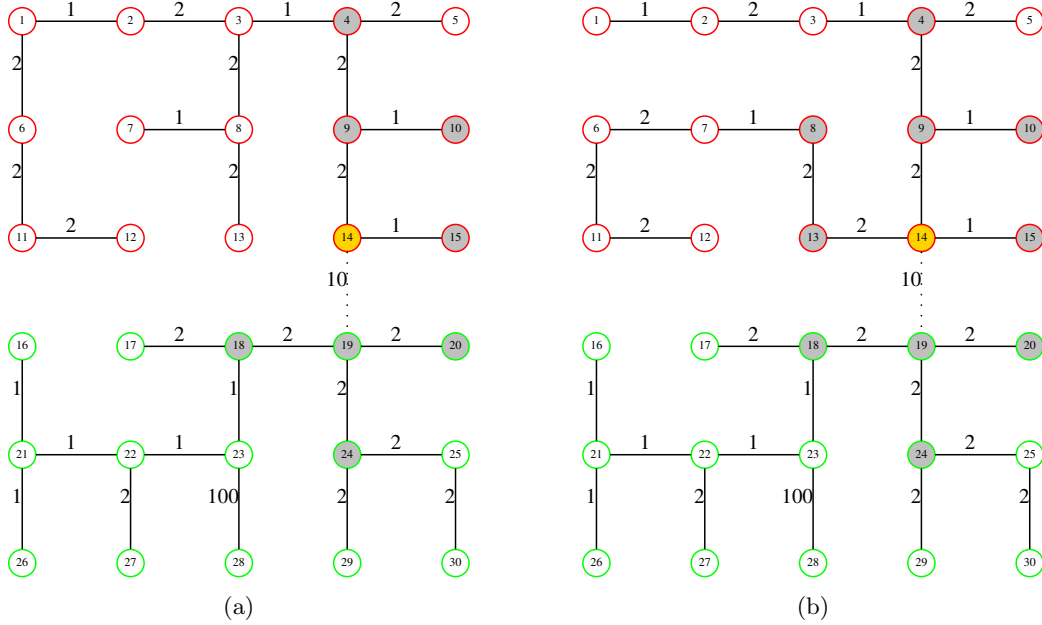


Figure 6. (a) TF of pixel numbered 14 in Figure 2, here the pixels with significant collaboration (distance ≤ 2) are highlighted in grey (b) Depth-based truncation of UMST filter at pixel numbered 14 in Figure 2, here the pixels with significant collaboration (depth ≤ 2) are highlighted in grey. Observe that the collaboration within the object is higher due to the usage of adaptive spanning tree instead of an arbitrary MST.

$$455 \quad (21) \quad \frac{U_i - U_{i,d}}{U_i} = \frac{\sum_{l=d+1}^{|V|-1} \exp(-\frac{l}{\sigma}) \sum_{j:\eta(i,j)=l} I_j}{\sum_{l=0}^{|V|-1} \exp(-\frac{l}{\sigma}) \sum_{j:\eta(i,j)=l} I_j}$$

$$456 \quad (22) \quad \leq \frac{\exp(-\frac{d+1}{\sigma}) \sum_{j:\eta(i,j) \geq d+1} I_j}{I_i + \sum_{l=1}^d \exp(-\frac{l}{\sigma}) \sum_{j:\eta(i,j)=l} I_j}$$

$$457 \quad (23) \quad \leq \frac{\exp(-\frac{d+1}{\sigma}) \sum_{j:\eta(i,j) \geq d+1} I_j}{1 + \sum_{l=1}^d \exp(-\frac{l}{\sigma})}$$

$$458 \quad (24) \quad = \frac{\exp(-\frac{d+1}{\sigma})}{1 + \sum_{l=1}^d \exp(-\frac{l}{\sigma})} \sum_{j:\eta(i,j) \geq d+1} I_j$$

$$459 \quad (25) \quad = \frac{\exp(-\frac{d+1}{\sigma})}{1 - \exp(-\frac{d+1}{\sigma})} (1 - \exp(-\frac{1}{\sigma})) \sum_{j:\eta(i,j) \geq d+1} I_j$$

460 For an image with 10^6 pixels, setting $\sigma = 0.1$, the expression in (25) is bounded above by
 461 $\frac{1}{100}$ whenever $d \geq 220$. However, the empirical results illustrated by Figure 7 indicate that
 462 the filtered value of the pixel as a function of depth, d , stabilizes beyond a depth of 15 for
 463 $\sigma = 0.1$.

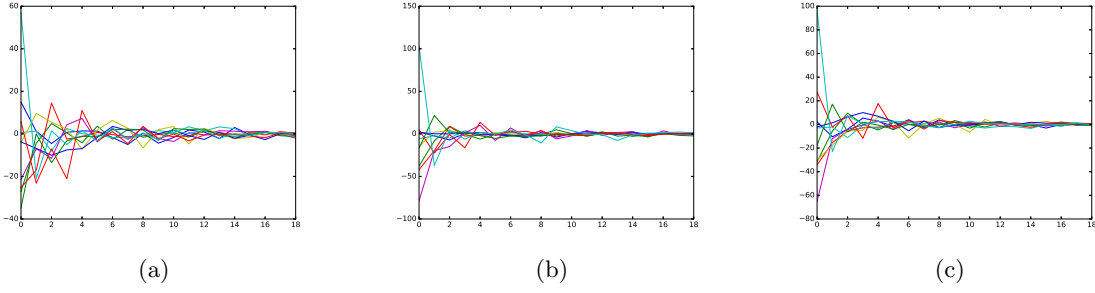


Figure 7. From a color image, many pixels have been chosen randomly and each curve in a sub figure represents a pixel. The RGB bands are separately processed and plotted in three sub figures. In each of the sub figures, a curve denotes the first difference of the depth-truncated approximate UMST filtered values as a function of depth. Note that the differences stabilize to 0 at a depth of 15 indicating that (20) yields good approximation to UMST filter.

Algorithm 3 To compute an order-truncated adaptive spanning tree

Input: UMST of the graph with vertex set I and edge set E , kernel size N and pixel i , path cost function f as defined in the proof of [Proposition 5.1](#)

Output: Order-Truncated Adaptive Spanning Tree $\hat{T}_{i,N}$

- 1: Set $\hat{T}_{i,N} = \emptyset$, $\mathcal{Q} = I$, $Parent(j) = null$ for each $j \in I$ and $count = 0$
 - 2: **while** $\mathcal{Q} \neq \emptyset$ and $count < N$ **do**
 - 3: Remove from \mathcal{Q} a pixel j such that $f(P^*(j))$ is minimum and add it to $\hat{T}_{i,N}$
 - 4: $count+ = 1$
 - 5: **for** each pixel k such that $(j, k) \in E$ **do**
 - 6: **if** $f(P^*(j) \cdot \langle j, k \rangle) < f(P^*(k))$ **then**
 - 7: set $Parent(k) = j$
 - 8: **end if**
 - 9: **end for**
 - 10: **end while**
 - 11: Return $\hat{T}_{i,N}$
-

464 In what follows, we view a rooted spanning tree T as a directed spanning tree: for every
 465 pixel j , a path $P^*(j)$ recursively as $\langle j \rangle$ if parent of j i.e. $Parent(j) = nil$, and $P^*(j) =$
 466 $P^*(s) \cdot \langle s, j \rangle$ if $Parent(j) = s \neq nil$ (notations are borrowed from [19]).

467 *Order-based truncation:* For each $i \in V$, we truncate the adaptive spanning tree T_i to $\hat{T}_{i,N}$
 468 such that it contains only the pixels j among the closest N (user-defined parameter) pixels
 469 w.r.t. the lexicographic ordering from i on T_i . (see [Figure 8\(b\)](#) for an illustration [Algorithm 3](#)
 470 for an computing it)

471 The order-based truncation of the adaptive spanning tree would precisely compute the
 472 Γ -limit of the morphological amoeba filters with $\lambda = 1$ in (9). More formally, we have the
 473 following result:

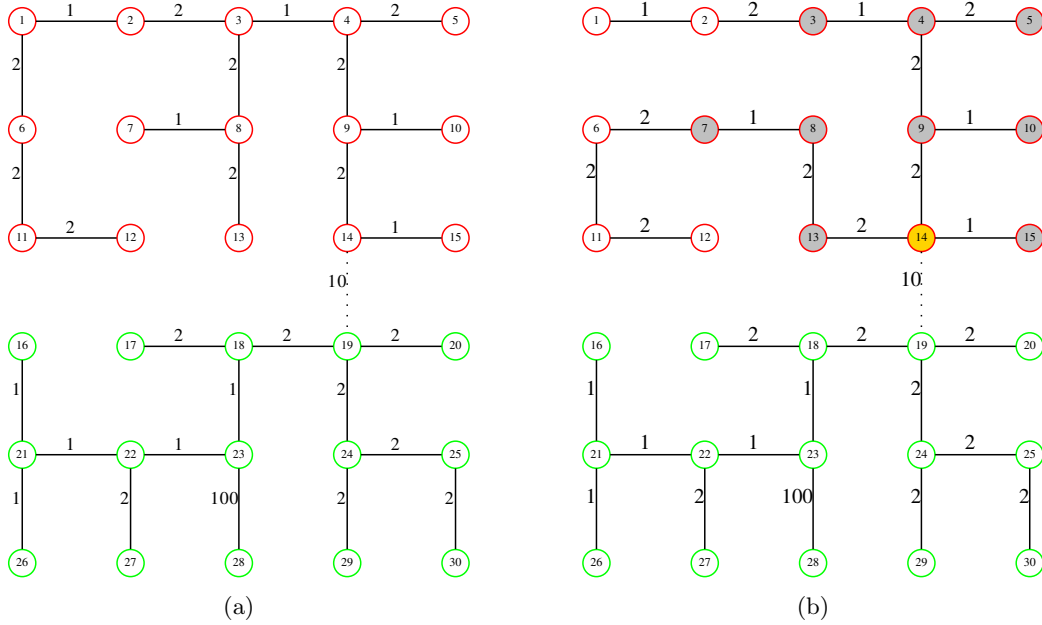


Figure 8. (a) TF of pixel numbered 14 in Figure 2 (b) Order-based truncation of PTF at pixel numbered 14 in Figure 2, here the pixels that are closest w.r.t. lexicographic order from 14 (top 10 including itself) are highlighted in grey. Observe that the collaboration in the PTF within the object is very high due to the usage of power spanning tree instead of an arbitrary MST.

474 **Proposition 5.2.** As $p \rightarrow \infty$, we have

$$475 \quad (26) \quad S_{i,N}^{(p)} \longrightarrow \hat{U}_{i,N}$$

476 where $S_{i,N}^{(p)}$ is the morphological amoeba filter at pixel i (with $\lambda = 1$ and kernel size N) and
 477 $\hat{U}_{i,N}$ is the collaborative filter using the closest N pixels to i w.r.t. the lexicographic ordering
 478 on the adaptive spanning tree T_i .

479 *Proof.* The proof follows directly from Lemma 4.3. ■

480 We remark that the choice of the kernel size N determines the trade-off between the level of
 481 smoothing and the computational cost. In practice, using $N \approx 100$, one can obtain a good
 482 edge-aware filter. As the kernel size is fixed and small, Algorithm 3 runs practically in $\mathcal{O}(|V|)$
 483 time. We shall see the comparison of the performance of our approximations with that of tree
 484 filter in the experiments section.

485 **5.3. Experiments.** In this section, we demonstrate that our approximations of the Γ -
 486 limit perform similar to the TF (in fact marginally better) at an additional computational
 487 cost. To process 1000 pixels, it takes about 0.29 seconds, 37 seconds and 738 seconds for
 488 TF, order-based and depth-based approximations respectively. Consequently, we provide an
 489 empirical evidence that TF is a fast approximation to the Γ -limit of the SPFs. For all our
 490 experiments, we have used identical σ ($= 10$) parameter for computing the TF, depth-based

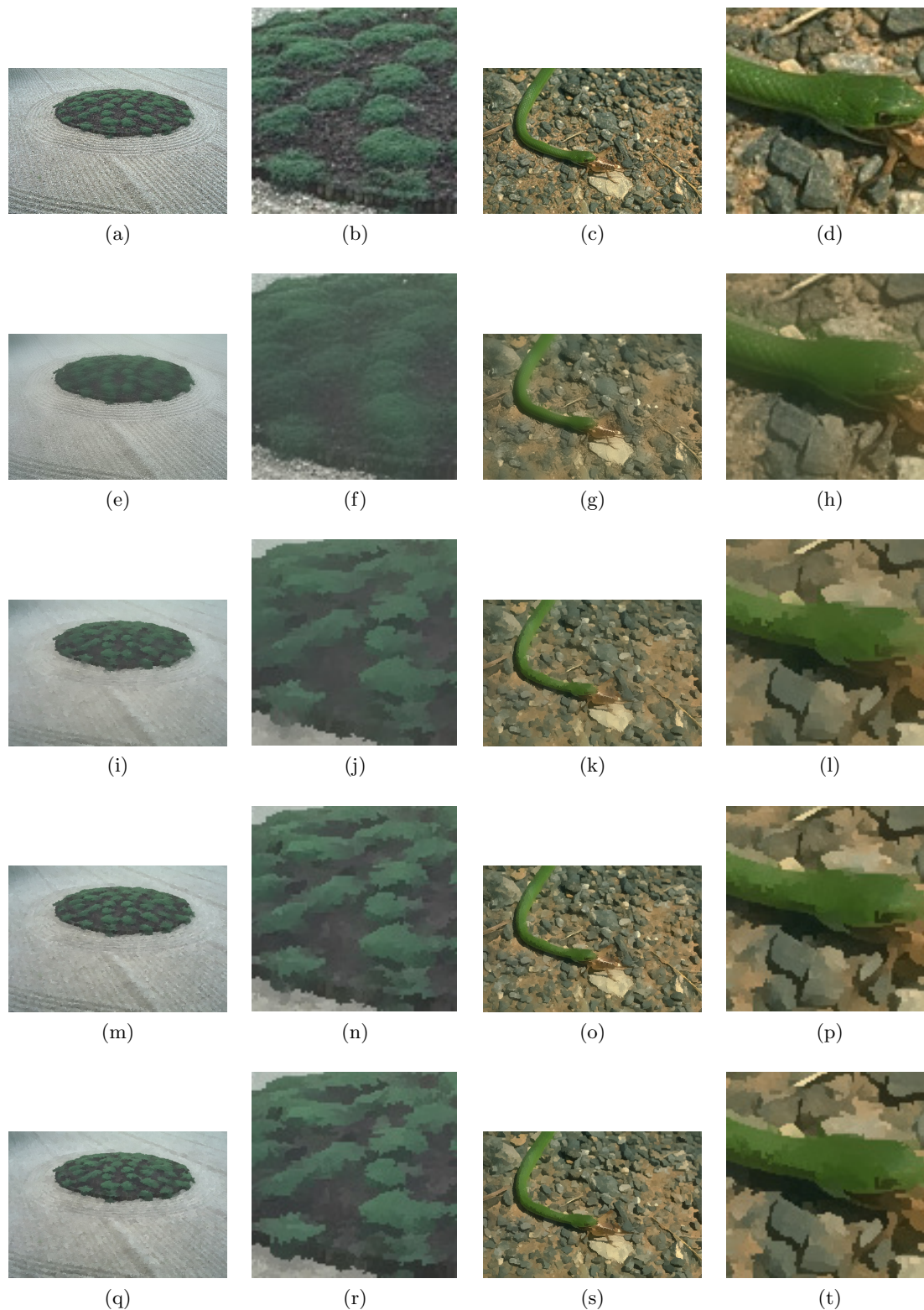


Figure 9. Subtle differences of the performance of TF vs our multi-tree approximations of UMSTF on BSDS500 [3] images are illustrated. (a), (b), (c), (d) original images (e), (f), (g), (h) Bilateral Filter (σ color = 100, σ space = 10) (i), (j), (k), (l) Tree Filter ($\sigma = 10$) (m), (n), (o), (p) Depth-based multi-tree truncation ($\sigma = 10$, depth = 15) (q), (r), (s), (t) Order-based multi-tree truncation ($\sigma = 10$, $N = 100$). In the second and fourth columns, observe the green patches and the snake's eye respectively. The leaks are more prominent in TF when compared to our approximations.

491 (with $depth = 15$) and order-based (with $N = 100$) multi-tree approximations of UMSTF.
 492 The experiments are performed on Intel (R) Xeon(R) CPU W3565 at 3.20GHz with RAM
 493 size of 6 GigaBytes.

494 Firstly, [Figure 9](#) shows a qualitative comparison of BF, TF, depth-based and order-based
 495 multi-tree approximations on some natural images. We observe that BF yields blurry images
 496 and erases some object boundaries as expected. On the other hand, TF and our approxima-
 497 tions yield similar results. However, on a closer look, one can observe that the boundaries are
 498 marginally better preserved in our approximations (compare green patches in [Figure 9](#) second
 499 column, snake’s eye in [Figure 9](#) fourth column).

500 For a quantitative comparison of these filters, we have computed the PSNR and structural
 501 similarity index (SSIM) [8] on images corrupted with synthetic noises. In general, higher PSNR
 502 values and higher SSIM (SSIM equal to 1 implies identical structures) indicate that the image
 503 structures are better preserved. However, SSIM is a superior measure when compared to
 504 PSNR as the latter estimates absolute errors while the former takes structural information
 505 into consideration. To see this, observe that the mean PSNR values of BF in [Table 1](#) is
 506 higher than that of other two filters over three iterations of random salt and pepper noise.
 507 However, a visual comparison of these filters (see [Figure 10](#)) suggests that noise is better
 508 eliminated by TF and our approximation. The mean SSIM values however (see [Table 2](#)) are
 509 in-line with the visual results and indicate that order-based approximation of UMST filter
 510 performs better than TF in presence of salt and pepper noise. Further, the scatter plot (see
 511 [Figure 11](#)) of the SSIM values of TF and order-based approximation of UMST filter on three
 512 iterations of random Poisson, salt and pepper, Gaussian and speckle noises on these images
 513 (House, Barbara, Lena and Pepper) indicate that our approximation is slightly better than
 514 TF irrespective of the type of noise.

Table 1

Peak Signal to Noise Ratios (PSNR) measured in db on filtered images corrupted by salt and pepper noise. A higher value indicates a better filter

	Mean PSNR values obtained on Salt Pepper Noise		
	Bilateral Filter	Tree Filter	PTF Order-based
House	24.71	25.04	24.67
Barbara	24.08	22.19	22.47
Lena	22.58	22.59	22.64
Pepper	21.70	21.96	20.59
Mean	23.27	22.95	22.59

515 Although our approximations yield marginally better results than that of TF, they are
 516 computationally expensive. However, it is important to note that our approximations can be
 517 implemented in parallel as each pixel is processed independently of the other. Also we note
 518 that one has to choose an appropriate filter depending on the type of noise. For instance, BF
 519 outperforms the tree-based filters in presence of Gaussian noise (see [Table 3](#) and [Table 4](#)). To
 520 summarize, we have demonstrated that TF is a fast approximation of the Γ -limit of SPFs.

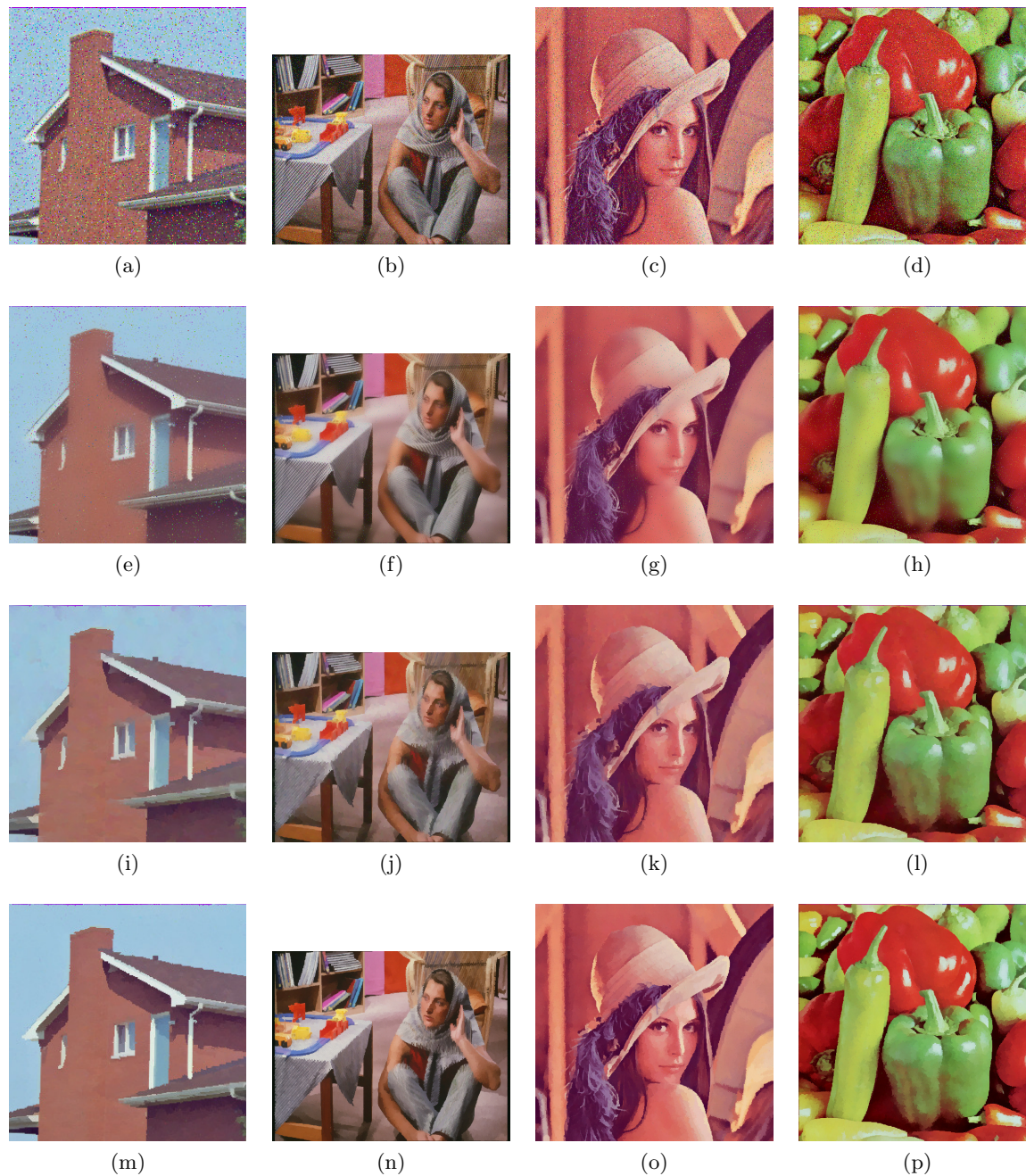


Figure 10. Visual illustration of edge-preserving filters on images contaminated with salt and pepper noise. BF erases some object boundaries and does not eliminate the noise completely while TF and order-based approximation eliminate noise and yield similar results. (a), (b), (c), (d) Salt and pepper noisy images (e), (f), (g), (h) Bilateral Filter (σ color = 100, σ space = 10) (i), (j), (k), (l) Tree Filter ($\sigma = 10$) (m), (n), (o), (p) Order-based multi-tree truncation ($\sigma = 10$, $N = 100$)

Table 2

Structural Similarity Indices (SSIM) on filtered images corrupted by salt and pepper noise. A higher value indicates a better filter and a value close to 1 indicates an ideal filter

	Mean SSIM values obtained on Salt Pepper Noise		
	Bilateral Filter	Tree Filter	PTF Order-based
House	0.69	0.80	0.83
Barbara	0.72	0.66	0.72
Lena	0.69	0.75	0.79
Pepper	0.62	0.74	0.74
Mean	0.68	0.74	0.77

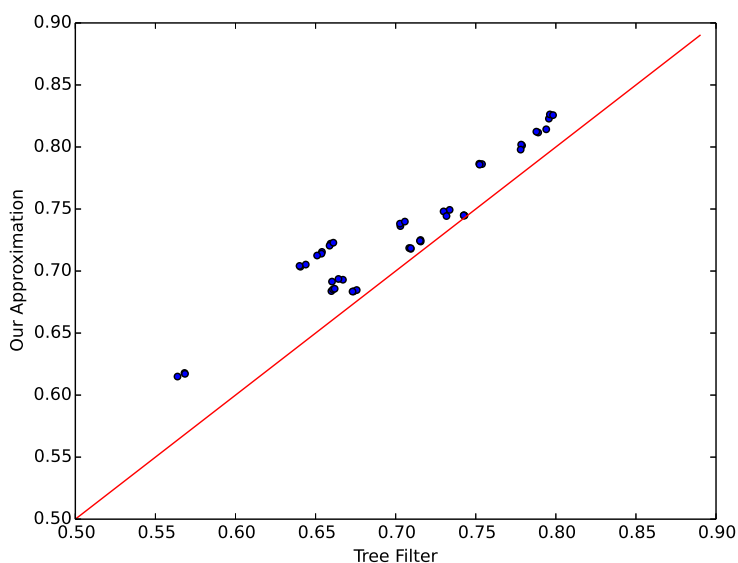


Figure 11. Scatter plot of SSIM values of TF versus Order-based approximation of UMST filter on three iterations of random Poisson, Salt and Pepper, Gaussian and Speckle noises on House, Barbara, Lena and Pepper images.

Table 3

Peak Signal to Noise Ratios (PSNR) measured in db on filtered images corrupted by Gaussian noise. A higher value indicates a better filter

	Mean PSNR values obtained on Gaussian Noise		
	Bilateral Filter	Tree Filter	PTF Order-based
House	25.12	23.76	23.97
Barbara	24.14	21.18	21.58
Lena	22.67	21.47	21.78
Pepper	22.15	21.09	20.93
Mean	23.52	21.88	22.07

Table 4

Structural Similarity Indices (SSIM) on filtered images corrupted by Gaussian noise. A higher value indicates a better filter and a value close to 1 indicates an ideal filter

	Mean SSIM values obtained on Gaussian Noise		
	Bilateral Filter	Tree Filter	PTF Order-based
House	0.79	0.73	0.75
Barbara	0.77	0.57	0.62
Lena	0.76	0.66	0.69
Pepper	0.75	0.67	0.68
Mean	0.77	0.66	0.69

521 **6. Conclusions.** In this paper, we have analysed the edge-aware filters from scratch by de-
 522 veloping shortest path filters as a natural extension of Gaussian-like filters. We have provided
 523 a common optimization framework for the filters based on shortest paths and the ones based
 524 on minimum spanning trees using the notion of Γ -convergence. We have thus established a
 525 theoretical justification of the MST heuristic based tree filter by proving that the tree filter
 526 is an approximate Γ -limit of shortest path filters. Further, we have proposed two different
 527 approximation algorithms of the Γ -limit by leveraging ideas from shortest paths and minimum
 528 spanning trees.

529 Establishing methods based on principles and/or heuristics as limits of solutions of opti-
 530 mization problems would enable us to design efficient algorithms. We believe that extending
 531 the ideas from our paper, one can obtain efficient parallel algorithms for practical applica-
 532 tions. Further, we believe that one could design novel edge-aware image filters based on these
 533 theoretical foundations. In this line of research, ultimately we aim to show that Γ -convergence
 534 serves as a powerful tool in applications beyond image segmentation and filtering.

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