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The mean-field equation of a leaky integrate-and-fire neural network: measure solutions and steady states

Grégory Dumont ^{*} Pierre Gabriel [†]

October 16, 2017

Abstract

Neural network dynamics emerge from the interaction of spiking cells. One way to formulate the problem is through a theoretical framework inspired by ideas coming from statistical physics, the so-called mean-field theory. In this document, we investigate different issues related to the mean-field description of an excitatory network made up of leaky integrate-and-fire neurons. The description is written in the form a nonlinear partial differential equation which is known to blow up in finite time when the network is strongly connected. We prove that in a moderately connected regime the equation is globally well-posed in the space of measures, and that there exist stationary solutions. In the case of weak connectivity we also demonstrate the uniqueness of the steady state and its global exponential stability. The method relies on a contraction argument of Doeblin's type in the linear case, which corresponds to a population of non-interacting units.

Keywords: Neural network; leaky integrate-and-fire; piecewise deterministic Markov process; Doeblin condition; measure solution; relaxation to steady state

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1 Introduction

The dynamics of neural networks is extremely complex. In the brain, a population of neurons is ruled by the interaction of thousands of nervous cells that exchange information by sending and receiving action

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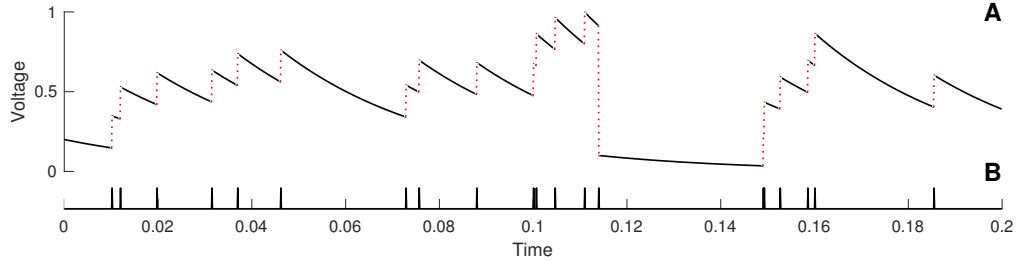


Figure 1: Simulation of the LIF model. The upper panel shows the time evolution of the membrane potential, the lower panel illustrates the arrival times of impulses, so-called Poisson spike train. The red dots correspond to discontinuities induced by the jump process. The parameters are: $h = 0.2$, $v_r = 0.1$ and Poisson rate 100.

potentials. Neuroscience needs a theory to relate key biological properties of neurons, with emerging behavior at the network scale. From a mathematical perspective, a neural network can simply be seen as a high-dimensional dynamical system of interacting elements. Unfortunately, introducing these interactions tends to lead to models that are analytically intractable. A big challenge has been to reduce the description of neural circuits.

Most attempts to establish a mathematically tractable characterization of neural networks have made use of mean-field theory (MT), see [16, 4] for a bio-physical review on the subject. Because each neuron receives input from many others, a single cell is mostly responsive to the average activity of the population - the mean-field - rather than the specific pattern of individual units. Based on theoretical concepts coming from statistical physics, MT gives rise to a so-called mean-field equation that defines the dynamic of a large (theoretically infinite) population of neurons [16, 24]. The use of MT is nowadays well accepted in neuroscience, and, over the past few decades or so, it has brought important insights into the emergent properties of neural circuits. For instance, it has played a crucial part in the understanding of neural synchronization and emerging brain rhythms [5].

Although MT is widespread among theoreticians, most of the mean-field equations are written within the language of partial differential equations (PDEs) for which there are only few mathematical studies. In this paper, our goal is precisely to fill this gap by considering a mean-field model that prevails in neuroscience. We focus our investigation on the existence and properties of the steady state measure of a PDE that arises for the description of an excitatory network of leaky integrate-and-fire (LIF) neurons.

The LIF model is a well-established neuron model within the neuroscience community [31]. It consists in an ordinary differential equation that describes the subthreshold dynamics of a neuron membrane's potential. The equation is endowed with a discontinuous reset mechanism to account for the onset of an action potential. Whenever the membrane potential reaches the firing threshold, the neuron initiates an action potential and the membrane potential is reset, see [8] for a review and [1, 7] for historical consideration. In its normalized form, the LIF model reads

$$\begin{cases} \frac{d}{dt}v(t) = -v(t) + h \sum_{j=1}^{+\infty} \delta(t - t_j) \\ \text{If } v > 1 \text{ then } v = v_r. \end{cases}$$

Here, v_r is the reset potential, δ is the Dirac measure, h is the so-called synaptic strength, and t_j are the arrival times of action potentials that originate from presynaptic cells.

Due to the presence of Dirac masses, the LIF equation describes a stochastic jump process, or piecewise deterministic Markov process [15]. Those voltage jumps result from the activation of the synapse at the reception of an action potential, and the stochastic feature is embedded in the Poisson distribution of time arrivals [32]. It is worth saying that, despite its vast simplifications, the LIF model yields amazingly accurate predictions and is known to reproduce many aspects of actual neural data [26]. Of course, there have been several variants and generalizations of the model [31]. In Fig. 1, a simulation of the LIF model is presented. It illustrates the different processes involved in the membrane equation such as the voltage jumps at the reception of an action potential, and the reset mechanism at the initiation of an action potential.

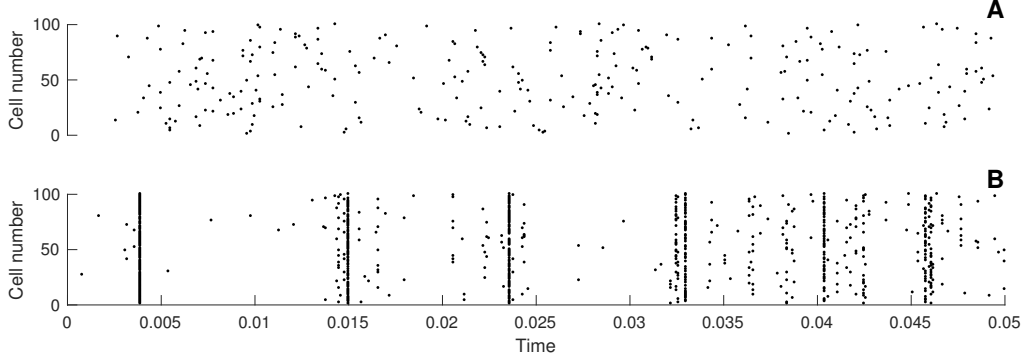


Figure 2: Simulations of the neural network. The network contains $N = 100$ neurons. In each panel is shown the spiking activity of every neuron in a raster plot (dots represent spikes). The parameters are: $h = 0.1$, $v_r = 0.1$ and Poisson rate 200. The release probability ρ is: A) $\rho = 0.01$, B) $\rho = 0.09$.

In a network, when a cell fires, the dynamic of each other neuron should be affected by the action potential. However, since synaptic transmissions are highly stochastic, the depolarization of the postsynaptic cell only occurs according to a certain probability. This probability plays the role of a coupling parameter. The dynamic of a neural network made up of LIF neurons is exposed in Fig. 2. For each simulation, we show the network raster plot where dots indicate the spiking time of individual units. The panels correspond to different values of the coupling. As we can see, for weak coupling, the network displays an asynchronous activity where each neuron fires irregularly (Fig. 2A). In contrast, when the coupling parameter is taken sufficiently large, the network enters into a synchronous state (Fig. 2B). The system seems to have a critical coupling value for which, above this value, the system is driven to a synchronous state, while below this value, it remains asynchronous [35, 36]. A great deal of attention has been devoted to the precise functional forms of these patterns, and this type of transition can be studied rigorously using MT.

As mentioned above, MT is used to simplify the description of networks and is formally derived in the limit of an infinitely large number of elements [16, 4]. In this setting, trajectories of individual units are ignored, and instead, the focus is made on the probability of finding a randomly chosen neuron in a certain state. A continuity equation describing the dynamics of the probability density function (PDF) is then derived, and the study of the PDF forms the basis of the mean-field approach. The fundamental assumption at the core of this theoretical setting is that all the neurons of the network share similar biophysical properties.

A pioneering attempt to describe neural networks within the framework of MT was made around the 1970 with the seminal work of Wilson and Cowan, followed by the Amari [42, 2]. Since then the study of neural circuits within the mean-field approach has never lost interest within the scientific community. To mention just a few, Sirovich, Omurtag and Knight [41], Nykamp and Tranchina [37], Brunel and Hakim [6, 5], and the work of Gerstner [23], were among the first to study networks of LIF neurons using MT.

Assuming that each neuron receives excitatory synaptic input with average rate $\sigma(t)$ and fires action potentials at rate $r(t)$, we denote the density function $p(t, v)$, such that $Np(t, v)dv$ gives the number of neurons with membrane potential in $[v - dv, v]$ at time t . The dynamics of the density $p(t, v)$ is prescribed by the following nonlinear partial differential equation:

$$\frac{\partial}{\partial t} p(t, v) - \underbrace{\frac{\partial}{\partial v} [vp(t, v)]}_{\text{Leak}} - \underbrace{\sigma(t) [p(t, v) - p(t, v - h)\mathbf{1}_{[h, 1]}(v)]}_{\text{Jump}} = \underbrace{\delta(v - v_r)r(t)}_{\text{Reset}}, \quad 0 < v < 1. \quad (1)$$

We show in Fig. 3 a schematic representation of the state space for the mean-field equation where the different operators take place. The firing activity of the network $r(t)$ is easily extracted from the mean-field equation. The proportion of cells crossing the threshold, see Fig. 3, is given by:

$$r(t) = \sigma(t) \int_{1-h}^1 p(t, w) dw.$$

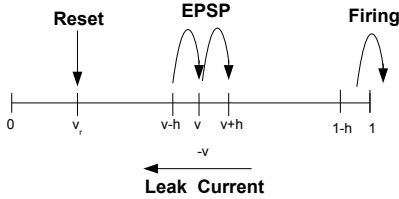


Figure 3: Schematic representation of the state space for the mean-field equation.

To account for the arrival of action potentials coming from an external source, the arrival rate $\sigma(t)$ is given by the sum of an external rate and the firing rate

$$\sigma(t) = \sigma_0 + Jr(t),$$

where J is the average number of synaptic connexion. The last equality is justified in the mean-field framework where it is assumed that single cells are only sensitive to the average population activity [41].

The equation is complemented with a zero flux boundary condition

$$p(t, 1) = 0.$$

Figure 4 portrays the dynamics of the mean-field equation where a Gaussian profile was taken as initial condition (Fig. 4A). Under the drift and the jump process, the density function gives a non zero flux at the threshold, and this flux is reinjected right away according to the reset process. This effect can be clearly seen in the third panel of the simulation presented in Fig. 4B. Asymptotically, the solution reaches a stationary profile which is shown in Fig. 4C.

Interestingly, it has been shown that the mean-field solution blows-up in finite time for any initial data in the strong connectivity regime [19]. This was attributed to the instantaneity of spikes firings and their immediate effects on the firing of other cells. This is happening when

$$J \geq 1 + \frac{1 - v_r}{h} \quad \text{and} \quad h\sigma_0 > 1.$$

Although progresses have been made, several questions remain unanswered, specially in the moderate or weak connectivity regime. For instance, as we can see from some simulation presented above, we observe that the density convergences toward a stationary state. Can we show the existence of a steady state? Can we analyze its stability properties? Answering these questions will allow us to form a deeper understanding of the asynchronous states of neural networks. Our challenge is to study the existence and properties of the mean-field equation steady state.

The paper is structured as follows. First we give a summary of the main results obtained throughout this manuscript. Section 3 is devoted to the study of the linear regime, which corresponds to a population of uncoupled neurons ($J = 0$). More precisely we prove the well-posedness of the equation in the space of measures and, via a so-called Doeblin's condition, the exponential convergence to an asynchronous state. Using the result in the linear case we show in section 4 the existence of a stationary solution in a moderate nonlinear regime

$$J < 1 + \left\lfloor \frac{1 - v_r}{h} \right\rfloor.$$

In Section 5 we prove the existence and uniqueness of global in time measure solutions to Equation (1) in the nonexplosive regime $J < 1$. Eventually, we demonstrate in section 6 the global exponential stability of the (unique) steady state in the weakly nonlinear regime $J \ll 1$. This work complements results on asynchronous state in different models [34, 38, 39].

2 Main results

In this paper we are interested in measure valued solutions to Equation (1). Considering measure solutions to structured population equations has attracted increasing interest in the last few years [9, 13, 22, 27, 28].

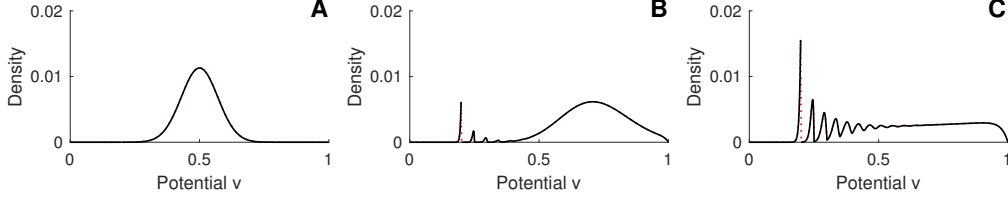


Figure 4: Simulations of the MF equation. A gaussian was taken as initial condition. The plots show in blue the evolution in time of the solution at different times. The red dots correspond to the discontinuity induced by the reset process. The parameters of the simulation are: $v_r = 0 : 3$, $h = 0.05$, $\sigma_0 = 50$, $J = 0$ A) $t = 0$, B) $t = 0.12$, C) $t = 7$.

In our case it is a very natural framework for two reasons. First it allows to consider for instance a Dirac mass initial distribution, corresponding to a fully synchronous state of the network. Second it is very well suited for dealing with an equation which has a measure source term (the reset part in Equation (1)).

Before giving the definition of such solutions, we recall some results about measure theory (and we refer to [40] for more details). We endow the interval $[0, 1]$ with its usual topology and the associated Borel σ -algebra. We denote by $\mathcal{M}([0, 1])$ the space of signed Borel measures on $[0, 1]$, by $\mathcal{M}_+([0, 1])$ its positive cone (the set of finite positive Borel measures), and by $\mathcal{P}([0, 1])$ the set of probability measures. The Jordan decomposition theorem ensures that any $\mu \in \mathcal{M}([0, 1])$ admits a unique decomposition $\mu = \mu_+ - \mu_-$ where $\mu_+, \mu_- \in \mathcal{M}_+([0, 1])$ are mutually singular. The space $\mathcal{M}([0, 1])$ is endowed with the total variation norm defined for all $\mu \in \mathcal{M}([0, 1])$ by

$$\|\mu\|_{TV} := \mu_+([0, 1]) + \mu_-([0, 1]).$$

For any bounded Borel function f on $[0, 1]$ the supremum norm is defined by $\|f\|_\infty = \sup_{0 \leq v \leq 1} |f(v)|$ and for any $\mu \in \mathcal{M}([0, 1])$ we use the notation

$$\mu f := \int_{[0, 1]} f d\mu.$$

Endowed with the supremum norm, the space $C([0, 1])$ of continuous functions on $[0, 1]$ is a Banach space. The Riesz representation theorem ensures that $\mathcal{M}([0, 1])$ can be identified with the topological dual space of $C([0, 1])$ through the mapping

$$\begin{aligned} \mathcal{M}([0, 1]) &\rightarrow C([0, 1])' \\ \mu &\mapsto \{f \mapsto \mu f\} \end{aligned}$$

which is an isometric isomorphism:

$$\|\mu\|_{TV} = \sup_{\|f\|_\infty \leq 1} |\mu f|.$$

Recall that a sequence $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}([0, 1])$ is said to converge weak* to $\mu \in \mathcal{M}([0, 1])$ if $(\mu_n f)_{n \in \mathbb{N}}$ converges to μf for all $f \in C([0, 1])$.

Now we can give the definition of a measure solution to Equation (1). We use the notation $\mathbf{1}_\Omega$ for the indicator function of a subset $\Omega \subset [0, 1]$, and we simply denote by $\mathbf{1}$ the constant function $\mathbf{1}_{[0, 1]}$.

Definition 2.1. *Let $T \in (0, +\infty]$. We say that a family $(\mu_t)_{t \geq 0} \subset \mathcal{P}([0, 1])$ is a solution to Equation (1) on $[0, T)$ with initial datum μ_0 if*

- $t \mapsto \sigma(t) := \frac{\sigma_0}{1 - J\mu_t([1-h, 1])}$ is positive on $[0, T)$ and belongs to $L^1_{loc}([0, T))$,
- $t \mapsto \mu_t$ is weak*-continuous on $[0, T)$,
- and for all $f \in C^1([0, 1])$ and all $t \in [0, T)$

$$\mu_t f = \mu_0 f + \int_0^t \int_{[0, 1]} \left(-v f'(v) + \sigma(s) [f(v+h)\mathbf{1}_{[0, 1-h)}(v) + f(v_r)\mathbf{1}_{[1-h, 1]}(v) - f(v)] \right) d\mu_s(v) ds.$$

It will be useful to define the following operators. For any $f \in C([0, 1])$ we set

$$\mathcal{B}f(v) = f(v+h)\mathbf{1}_{[0,1-h)}(v) + f(v_r)\mathbf{1}_{[1-h,1]}(v) - f(v),$$

and, for any $f \in C^1([0, 1])$ and $\sigma > 0$,

$$\mathcal{A}_\sigma f(v) = -vf'(v) + \sigma \mathcal{B}f(v).$$

With this definition the weak formulation of Equation (1) in Definition 2.1 reads

$$\mu_t f = \mu_0 f + \int_0^t \mu_s (\mathcal{A}_{\sigma(s)} f) ds.$$

Notice that \mathcal{A} and \mathcal{B} are conservative in the sense that $\mathcal{A}\mathbf{1} = \mathcal{B}\mathbf{1} = 0$. Notice also that \mathcal{B} is a bounded operator in the sense that

$$\forall f \in C([0, 1]), \quad \|\mathcal{B}f\|_\infty \leq 2 \|f\|_\infty,$$

but in general we do not have $\mathcal{B}f \in C([0, 1])$, and thus neither $\mathcal{A}_\sigma f \in C([0, 1])$ when $f \in C^1([0, 1])$. This prevents the use of general results about the existence and uniqueness of measure solutions for structured population models (see [9, 13, 27]). For proving the well-posedness of our problem we use a duality method which is well suited for analysing the steady states.

From now on we always assume that h is small enough to satisfy

$$0 < h < v_r < 1 - h,$$

and also, to avoid pathological situations, that

$$\frac{1 - v_r}{h} \notin \mathbb{N}.$$

Now we state the main results of the paper.

Theorem 2.2. *If $J < 1 + \lfloor \frac{1-v_r}{h} \rfloor$, then there exists at least one steady state to Equation (1), i.e. there exists a probability measure $\bar{\mu}$ which satisfies*

$$\forall f \in C^1([0, 1]), \quad \bar{\mu}(\mathcal{A}_{\bar{\sigma}} f) = 0, \quad \text{where } \bar{\sigma} = \frac{\sigma_0}{1 - J\bar{\mu}([1-h, 1])}.$$

Theorem 2.3. *Assume that $J < 1$. Then for all $\mu_0 \in \mathcal{P}([0, 1])$ there exists a unique global measure solution $(\mu_t)_{t \geq 0}$ to Equation (1), in the sense of Definition 2.1.*

Theorem 2.4. *When $J < (5 - 2\sqrt{6})\left(\frac{h}{4}\right)^{\sigma_0+1}$, the steady state $\bar{\mu}$ is unique and globally exponentially stable. More precisely there exist explicit constants $t_0, a > 0$ such that for all $\mu_0 \in \mathcal{P}([0, 1])$ and all $t \geq 0$*

$$\|\mu_t - \bar{\mu}\|_{TV} \leq e^{-a(t-t_0)} \|\mu_0 - \bar{\mu}\|_{TV}.$$

3 The unconnected case ($J = 0$)

In the case $J = 0$ the equation is linear and reads

$$\frac{\partial}{\partial t} p(t, v) - \frac{\partial}{\partial v} [vp(t, v)] + \sigma_0 [p(t, v) - p(t, v-h)\mathbf{1}_{[h,1]}(v)] = \left[\sigma_0 \int_{[1-h,1]} p(t, w) dw \right] \delta_{v=v_r}. \quad (2)$$

For the sake of clarity in the current section we will denote by \mathcal{A} the operator \mathcal{A}_{σ_0} , σ_0 being a fixed positive number. Since the equation is linear, we do not need to restrict the definition of a solution to probability measures. We say that $(\mu_t)_{t \geq 0} \subset \mathcal{M}([0, 1])$ is a measure solution to Equation (2) when $t \mapsto \mu_t$ is weak*-continuous and for all $f \in C^1([0, 1])$ and all $t \geq 0$

$$\mu_t f = \mu_0 f + \int_0^t \mu_s \mathcal{A} f ds.$$

For building and studying such measure solutions we follow the method in [22], which is based on the dual equation

$$\partial_t f(t, v) + v \partial_v f(t, v) + \sigma_0 f(t, v) = \sigma_0 [f(t, v + h) \mathbf{1}_{[0, 1-h]}(v) + f(t, v_r) \mathbf{1}_{[1-h, 1]}(v)],$$

with the initial condition $f(0, \cdot) = f_0$. This equation is well-posed in the space of continuous functions, in the sense of the following lemma.

Lemma 3.1. *For any $f_0 \in C([0, 1])$, there exists a unique $f \in C(\mathbb{R}_+ \times [0, 1])$ which satisfies*

$$f(t, v) = f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} [f(t - \tau, e^{-\tau} v + h) \mathbf{1}_{[0, 1-h]}(e^{-\tau} v) + f(t - \tau, v_r) \mathbf{1}_{[1-h, 1]}(e^{-\tau} v)] d\tau.$$

Additionally if $f_0 = \mathbf{1}$ then $f = \mathbf{1}$, and if $f_0 \geq 0$ then $f \geq 0$.

Proof. The proof consists in applying the Banach fixed point theorem. Fix $T > 0$ and define on the Banach space $C([0, T] \times [0, 1])$ endowed with the supremum norm $\|\cdot\|_\infty$ the mapping Γ by

$$\Gamma f(t, v) := f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} [f(t - \tau, v e^{-\tau} + h) \mathbf{1}_{[0, 1-h]}(v e^{-\tau}) + f(t - \tau, v_r) \mathbf{1}_{[1-h, 1]}(v e^{-\tau})] d\tau.$$

It is a contraction whatever the value of T . Indeed it is an affine mapping and for $f_0 \equiv 0$ we have

$$\|\Gamma f\|_\infty \leq (1 - e^{-\sigma_0 T}) \|f\|_\infty.$$

The Banach fixed point theorem ensures the existence and uniqueness of a fixed point for Γ in $C([0, T] \times [0, 1])$, for any $T > 0$.

If $f_0 = \mathbf{1}$, it is easy to verify that $\Gamma \mathbf{1} = \mathbf{1}$ and then $f = \mathbf{1}$ by uniqueness.

For the positivity it suffices to check that when $f_0 \geq 0$, the closed positive cone of $C([0, T] \times [0, 1])$ is invariant under Γ . □

With this result we can define a family $(M_t)_{t \geq 0}$ of linear operators on $C([0, 1])$ by $M_t f_0 = f(t, \cdot)$, and we have the following properties.

Corollary 3.2. *The family of operators $(M_t)_{t \geq 0}$ is a semigroup, i.e.*

$$M_0 f = f \quad \text{and} \quad M_{t+s} f = M_t(M_s f),$$

which is conservative and positive, in the sense that for all $t \geq 0$

$$M_t \mathbf{1} = \mathbf{1} \quad \text{and} \quad f \geq 0 \implies M_t f \geq 0.$$

As a direct consequence it is a contraction for the supremum norm, meaning that for all $f \in C([0, 1])$ and all $t \geq 0$

$$\|M_t f\|_\infty \leq \|f\|_\infty.$$

Proof. For the semigroup property, it suffices to check that $(t, v) \mapsto M_{t+s}^n f(v)$ and $(t, v) \mapsto M_t^n(M_s f)(v)$ are both the unique fixed point of Γ for $f_0 = M_s^n f$, which is nothing but easy computations.

The conservativity and the positivity are immediate consequences of Lemma 3.1. □

Now we define by duality a semigroup on $\mathcal{M}([0, 1]) = C([0, 1])'$. For $\mu \in \mathcal{M}([0, 1])$ and $t \geq 0$ we define $\mu M_t \in \mathcal{M}([0, 1])$ by

$$\forall f \in C([0, 1]), \quad (\mu M_t) f = \mu(M_t f). \quad (3)$$

The properties of the right action of $(M_t)_{t \geq 0}$ are readily transferred to the left action by duality.

Corollary 3.3. *The left semigroup $(M_t^n)_{t \geq 0}$ defined on $\mathcal{M}([0, 1])$ by (3) is positive and conservative, i.e. for all $t \geq 0$ we have*

$$\begin{aligned} \mu \in \mathcal{M}_+([0, 1]) &\implies \mu M_t \in \mathcal{M}_+([0, 1]), \\ \mu \in \mathcal{P}([0, 1]) &\implies \mu M_t \in \mathcal{P}([0, 1]). \end{aligned}$$

It is also a contraction for the total variation norm, i.e. for all $\mu \in \mathcal{M}([0, 1])$ and for all $t \geq 0$

$$\|\mu M_t\|_{TV} \leq \|\mu\|_{TV}.$$

We are now ready to state the main result of the section.

Theorem 3.4. *For any $\mu_0 \in \mathcal{M}([0, 1])$ the family $(\mu_0 M_t)_{t \geq 0}$ is the unique measure solution to Equation (2).*

Additionally there exists a unique invariant probability measure $\bar{\mu}$, i.e. there exists a unique $\bar{\mu} \in \mathcal{P}([0, 1])$ such that

$$\forall t \geq 0, \quad \bar{\mu} M_t = \bar{\mu}.$$

This invariant measure is globally exponentially stable: for all $\mu \in \mathcal{M}([0, 1])$ and for all $t \geq 0$

$$\|\mu M_t - (\mu \mathbf{1}) \bar{\mu}\|_{TV} \leq e^{-a(t-t_0)} \|\mu - (\mu \mathbf{1}) \bar{\mu}\|_{TV},$$

where $t_0 = \log \frac{4}{h} > 0$ and $a = \frac{-\log\left(1 - \frac{\sigma_0}{2} \left(\frac{h}{4}\right)^{\sigma_0}\right)}{\log \frac{4}{h}} > 0$.

The discontinuity of $\mathbf{1}_{[0, 1-h)}$ and $\mathbf{1}_{[1-h, 1]}$ is an obstacle for proving directly that $(\mu_0 M_t)_{t \geq 0}$ is a measure solution to Equation (2). To work around this difficulty, we regularize these functions (see [21] for a similar approach).

3.1 A regularized problem

We approximate the indicator function $\mathbf{1}_{[1-h, 1]}$ by

$$\chi_n(v) := \begin{cases} 0 & \text{if } v \leq 1 - h - \frac{h}{n}, \\ 1 + \frac{n}{h}(v - 1 + h) & \text{if } 1 - h - \frac{h}{n} \leq v \leq 1 - h, \\ 1 & \text{if } v \geq 1 - h + \frac{h}{n}, \end{cases}$$

where $n \in \mathbb{N}^*$. The sequence $(\chi_n)_{n \geq 1}$ is a decreasing sequence of continuous functions which converges pointwise to $\mathbf{1}_{[1-h, 1]}$. We define the associated regularized operators

$$\mathcal{B}^n f(v) := f(v+h)(1 - \chi_n(v)) + f(v_r)\chi_n(v) - f(v) \quad \text{and} \quad \mathcal{A}^n f(v) := -vf'(v) + \sigma_0 \mathcal{B}^n f(v).$$

As for \mathcal{A} we have the conservation property $\mathcal{A}^n \mathbf{1} = 0$. But contrary to \mathcal{A} , for $f \in C^1([0, 1])$ we have $\mathcal{A}^n f \in C([0, 1])$, and this allows to build a measure solution to the regularized equation by duality.

Consider the regularized dual equation

$$\partial_t f(t, v) + v \partial_v f(t, v) + \sigma_0 f(t, v) = \sigma_0 [f(t, v+h)(1 - \chi_n(v)) + f(t, v_r)\chi_n(v)], \quad (4)$$

with the initial condition $f(0, \cdot) = f_0$. As for the non-regularized case, this equation is well-posed on the space of continuous functions. But it is also well-posed in the space of continuously differentiable functions.

Lemma 3.5. *For $f_0 \in C([0, 1])$, there exists a unique $f \in C(\mathbb{R}_+ \times [0, 1])$ which satisfies*

$$f(t, v) = f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} [f(t - \tau, e^{-\tau} v + h)(1 - \chi_n(e^{-\tau} v)) + f(t - \tau, v_r)\chi_n(e^{-\tau} v)] d\tau.$$

Additionally

- if $f_0 = \mathbf{1}$ then $f = \mathbf{1}$,
- if $f_0 \geq 0$ then $f \geq 0$,
- if $f_0 \in C^1([0, 1])$ then $f \in C^1(\mathbb{R}_+ \times [0, 1])$ and f satisfies (4).

Proof. For the existence and uniqueness of a solution as well as the first two points we proceed as for Lemma 3.1 by applying the Banach fixed point theorem to the mapping

$$\Gamma f(t, v) := f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} [f(t - \tau, v e^{-\tau} + h)(1 - \chi_n(v e^{-\tau})) + f(t - \tau, v_r)\chi_n(v e^{-\tau})] d\tau.$$

It remains to check that when f_0 is of class C^1 then the same holds for f . To do so we prove that when $f_0 \in C^1([0, 1])$ the mapping Γ is a contraction in the Banach space $C^1([0, T] \times [0, 1])$ endowed with the norm $\|f\|_{C^1} := \|f\|_\infty + \|\partial_t f\|_\infty + \|\partial_v f\|_\infty$ when T is small enough. We have

$$\partial_t \Gamma f(t, v) = \mathcal{A}^n f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} [\partial_t f(t-\tau, v e^{-\tau} + h)(1 - \chi_n(v e^{-\tau})) + \partial_t f(t-\tau, v_r) \chi_n(v e^{-\tau})] d\tau. \quad (5)$$

and

$$\begin{aligned} \partial_v \Gamma f(t, v) &= f'_0(v e^{-t}) e^{-(1+\sigma_0)t} + \sigma_0 \int_0^t e^{-(1+\sigma_0)\tau} \partial_v f(t-\tau, v e^{-\tau} + h)(1 - \chi_n(v e^{-\tau})) d\tau \\ &\quad + \sigma_0 \int_0^t e^{-(1+\sigma_0)\tau} \frac{n}{h} \mathbf{1}_{[1-h-\frac{h}{n}, 1-h]}(v e^{-\tau}) [f(t-\tau, v_r) - f(t-\tau, v e^{-\tau} + h)] d\tau \end{aligned}$$

so when $f_0 = 0$ we have

$$\|\Gamma f\|_{C^1} \leq (1 - e^{-\sigma_0 T}) \|f\|_{C^1} + 2\sigma_0 \frac{n}{h} \log\left(1 + \frac{h}{n(1-h-\frac{h}{n})}\right) T \|f\|_\infty \leq \frac{1-2h+2\sigma_0 T}{1-2h} T \|f\|_{C^1}$$

and Γ is a contraction in $C^1([0, T] \times [0, 1])$ when $T < \frac{1-2h}{1-2h+2\sigma_0}$. This ensures that the unique fixed point f of Γ belongs to $C^1([0, T] \times [0, 1])$. To check that f satisfies (4) we can differentiate the alternative formulation of Γf

$$\Gamma f(t, v) = f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0(t-\tau)} [f(\tau, v e^{-(t-\tau)} + h)(1 - \chi_n(v e^{-(t-\tau)})) + f(\tau, v_r) \chi_n(v e^{-(t-\tau)})] d\tau$$

with respect to t and we get

$$\begin{aligned} \partial_t \Gamma f(t, v) &= -v f'_0(v e^{-t}) e^{-(1+\sigma_0)t} - \sigma_0 f_0(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 [f(t, v+h)(1 - \chi_n(v)) + f(t, v_r) \chi_n(v)] \\ &\quad - \sigma_0^2 \int_0^t e^{-\sigma_0(t-\tau)} [f(\tau, v e^{-(t-\tau)} + h)(1 - \chi_n(v e^{-(t-\tau)})) + f(\tau, v_r) \chi_n(v e^{-(t-\tau)})] d\tau \\ &\quad - \sigma_0 \int_0^t v e^{-(1+\sigma_0)\tau} \partial_v f(t-\tau, v e^{-\tau} + h)(1 - \chi_n(v e^{-\tau})) d\tau \\ &\quad - \sigma_0 \int_0^t v e^{-(1+\sigma_0)\tau} \frac{n}{h} \mathbf{1}_{[1-h-\frac{h}{n}, 1-h]}(v e^{-\tau}) [f(t-\tau, v_r) - f(t-\tau, v e^{-\tau} + h)] d\tau. \end{aligned}$$

So we have

$$\partial_t \Gamma f(t, v) + v \partial_v \Gamma f(t, v) + \sigma_0 \Gamma f(t, v) = \sigma_0 [f(t, v+h)(1 - \chi_n(v)) + f(t, v_r) \chi_n(v)]$$

and the fixed point satisfies (4). □

With this result we define a family $(M_t^n)_{t \geq 0}$ of linear operators on $C([0, 1])$ by setting $M_t^n f_0 = f(t, \cdot)$.

Lemma 3.6. *The family of operators $(M_t^n)_{t \geq 0}$ is a conservative and positive semigroup on $C([0, 1])$, hence a contraction for the supremum norm. Additionally it is strongly continuous, i.e.*

$$\forall f \in C([0, 1]), \quad \|M_t^n f - f\|_\infty \xrightarrow[t \rightarrow 0]{} 0,$$

For all $f \in C^1([0, 1])$ we have

$$\partial_t M_t^n f = \mathcal{A}^n M_t^n f = M_t^n \mathcal{A}^n f \quad (6)$$

and

$$\left\| \frac{1}{t} (M_t^n f - f) - \mathcal{A}^n f \right\|_\infty \xrightarrow[t \rightarrow 0]{} 0.$$

Proof. For the semigroup property, it suffices to check that $(t, v) \mapsto M_{t+s}^n f(v)$ and $(t, v) \mapsto M_t^n (M_s^n f)(v)$ are both the unique fixed point of Γ for $f_0 = M_s^n f$, which is nothing but easy computations.

The conservativity and the positivity are immediate consequences of the previous proposition, as well as the fact that $\partial_t M_t^n f = \mathcal{A}^n M_t^n f$. For the last equality we deduce from (5) that $\partial_t M_t^n f$ is the unique fixed point of Γ associated to $f_0 = \mathcal{A}^n f$, so $\partial_t M_t^n f = M_t^n \mathcal{A}^n f$.

For the strong continuity we use the fact that a continuous function on a compact set is uniformly continuous.

For the last point we use the strong continuity to write for $f \in C^1([0, 1])$

$$\left\| \frac{1}{t} (M_t^n f - f) - \mathcal{A}^n f \right\|_\infty \leq \frac{1}{t} \int_0^t \|M_s^n \mathcal{A}^n f - \mathcal{A}^n f\|_\infty ds \xrightarrow{t \rightarrow 0} 0$$

since $\mathcal{A}^n f \in C([0, 1])$. □

Now we can define by duality a semigroup on $\mathcal{M}([0, 1]) = C([0, 1])'$. For $\mu \in \mathcal{M}([0, 1])$ we define μM_t^n by

$$\forall f \in C([0, 1]), \quad (\mu M_t^n) f = \mu(M_t^n f).$$

The family $(M_t^n)_{t \geq 0}$ is then also a positive and conservative semigroup on $\mathcal{M}([0, 1])$ (hence a contraction for the total variation norm). Additionally the family $(\mu M_t^n)_{t \geq 0}$ is a measure solution to the regularized LIF equation.

Lemma 3.7. *For all $\mu \in \mathcal{M}([0, 1])$ the application $t \mapsto \mu M_t^n$ is weak*-continuous, and for all $f \in C^1([0, 1])$ and $t \geq 0$*

$$\mu M_t^n f = \mu f + \int_0^t \mu M_s^n \mathcal{A}^n f ds. \quad (7)$$

Proof. The continuity of $t \mapsto M_t^n f(v)$ for all $f \in C([0, 1])$ and $v \in [0, 1]$ and the dominated convergence theorem ensure the weak*-continuity of $t \mapsto \mu M_t^n$.

For the second part of the lemma it suffices to integrate the identity $\partial_s M_s^n f = M_s^n \mathcal{A}^n f$ in time on $[0, t]$ and then in space on $[0, 1]$ against the measure μ . The conclusion follows from the Fubini's theorem. □

3.2 Passing to the limit

Now we pass to the limit $n \rightarrow \infty$ to get that the family $(\mu M_t)_{t \geq 0}$ is a measure solution to Equation (2).

Lemma 3.8. *For all $T > 0$ we have*

$$\sup_{0 \leq t \leq T} \sup_{\|f\|_\infty \leq 1} \|M_t^n f - M_t f\|_\infty \xrightarrow{n \rightarrow \infty} 0.$$

Proof. From the definitions of $(M_t)_{t \geq 0}$ and $(M_t^n)_{t \geq 0}$ we get that for all $f \in C([0, 1])$ such that $\|f\|_\infty \leq 1$

$$\begin{aligned} \|M_t f - M_t^n f\|_\infty &\leq \sigma_0 \int_0^t \|M_{t-\tau} f - M_{t-\tau}^n f\|_\infty d\tau + 2\sigma_0 \sup_{0 \leq v \leq 1} \int_0^t \mathbf{1}_{[1-h-\frac{h}{n}, 1-h]}(e^{-\tau} v) d\tau \\ &\leq \sigma_0 \int_0^t \|M_{t-\tau} f - M_{t-\tau}^n f\|_\infty d\tau + 2\sigma_0 \log \left(1 + \frac{h}{n(1-2h)} \right) \end{aligned}$$

and we conclude by the Grönwall's lemma that

$$\|M_t f - M_t^n f\|_\infty \leq 2\sigma_0 \log \left(1 + \frac{h}{n(1-2h)} \right) e^{\sigma_0 t}.$$

□

Corollary 3.9. *For any $\mu_0 \in \mathcal{M}([0, 1])$ the family $(\mu_0 M_t)_{t \geq 0}$ is the unique measure solution to Equation (2).*

Proof. From Lemma 3.8 we deduce that for all $\mu \in \mathcal{M}([0, 1])$ we have $\mu M_t^n \rightarrow \mu M_t$ in the TV-norm when $n \rightarrow \infty$. This allows to pass to the limit in (7) by dominated convergence, since for all $f \in C^1([0, 1])$ we have $\mathcal{A}^n f \rightarrow \mathcal{A}f$ pointwise and $\|\mathcal{A}^n f\|_\infty \leq \|f'\|_\infty + 2\|f\|_\infty$.

The weak*-continuity of $t \mapsto \mu M_t$ follows from the weak*-continuity of $t \mapsto \mu M_t^n$, using again Lemma 3.8.

For the uniqueness we use that if $(\mu_t)_{t \geq 0}$ is a solution to Equation (2) then for all $n \in \mathbb{N}^*$, all $t > 0$, all $\mu \in \mathcal{M}([0, 1])$, and all $f \in C^1([0, 1])$ we have

$$\begin{aligned} \frac{d}{ds} \left(\int_0^s \mu_\tau M_{t-s}^n f d\tau \right) &= \mu_s M_{t-s}^n f - \int_0^s \mu_\tau \mathcal{A}^n M_{t-s}^n f d\tau \\ &= \mu M_{t-s}^n f + \int_0^s \mu_\tau (\mathcal{A} - \mathcal{A}^n) M_{t-s}^n f d\tau \\ &= \mu M_{t-s}^n f + \int_0^s \mu_\tau (\mathcal{B} - \mathcal{B}^n) M_{t-s}^n f d\tau. \end{aligned} \quad (8)$$

For proving the validity of these computations we write for all $h > 0$

$$\begin{aligned} \frac{1}{h} \left[\int_0^{s+h} \mu_\tau M_{t-s-h}^n f d\tau - \int_0^s \mu_\tau M_{t-s}^n f d\tau \right] &= \\ \frac{1}{h} \int_s^{s+h} \mu_\tau M_{t-s}^n f d\tau + \int_s^{s+h} \mu_\tau \frac{M_{t-s-h}^n f - M_{t-s}^n f}{h} d\tau + \int_0^s \mu_\tau \frac{M_{t-s-h}^n f - M_{t-s}^n f}{h} d\tau. \end{aligned}$$

For the convergence of the first term above we use the weak*-continuity of $\tau \rightarrow \mu_\tau$ to get

$$\frac{1}{h} \int_s^{s+h} \mu_\tau M_{t-s}^n f d\tau \xrightarrow{h \rightarrow 0} \mu_s M_{t-s}^n f.$$

For the second term we use that $t \mapsto \mu_t$ is locally bounded for the TV-norm due to the uniform boundedness principle, because it is weak*-continuous. Using (6) we deduce

$$\left| \int_s^{s+h} \mu_\tau \frac{M_{t-s-h}^n f - M_{t-s}^n f}{h} d\tau \right| \leq h \sup_{s \leq \tau \leq s+h} \|\mu_\tau\|_{TV} \|\mathcal{A}^n f\|_\infty \xrightarrow{h \rightarrow 0} 0.$$

For the last term we also use (6) to get by dominated convergence

$$\int_0^s \mu_\tau \frac{M_{t-s-h}^n f - M_{t-s}^n f}{h} d\tau \xrightarrow{h \rightarrow 0} - \int_0^s \mu_\tau \mathcal{A}^n M_{t-s}^n f d\tau.$$

Now that (8) is proved, we integrate on $[0, t]$ to obtain

$$\int_0^t \mu_\tau f d\tau = \int_0^t \mu M_{t-s}^n f ds + \int_0^t \int_0^s \mu_\tau (\mathcal{B} - \mathcal{B}^n) M_{t-s}^n f d\tau ds$$

and by dominated convergence, when $n \rightarrow \infty$,

$$\int_0^t \mu_\tau f d\tau = \int_0^t \mu M_s f ds.$$

Differentiating it with respect to t we get that $\mu_t f = \mu M_t f$ and then $\mu_t = \mu M_t$ because $C^1([0, 1])$ is dense in $C([0, 1])$. \square

3.3 Exponential contraction and invariant measure

We first prove the exponential contraction of any couple of probability solutions.

Proposition 3.10. *For all $\mu, \tilde{\mu} \in \mathcal{P}([0, 1])$ we have*

$$\forall t \geq 0, \quad \|\mu M_t - \tilde{\mu} M_t\|_{TV} \leq e^{-a(t-t_0)} \|\mu - \tilde{\mu}\|_{TV}$$

with

$$t_0 = \log \frac{4}{h} > 0 \quad \text{and} \quad a = \frac{-\log \left(1 - \frac{\sigma_0}{2} \left(\frac{h}{4} \right)^{\sigma_0} \right)}{\log \frac{4}{h}} > 0.$$

Notice that the values of t_0 and a are explicit (in terms of the coefficients of the model) but not optimal. The optimization of these constants is also an interesting issue that could be addressed in a future work. The proof of Proposition 3.10 relies on a so-called Doeblin's condition (see for instance [29, 33]).

Lemma 3.11. *The semigroup $(M_t)_{t \geq 0}$ satisfies the Doeblin's condition*

$$\forall f \geq 0, \forall v \in [0, 1], \quad M_{t_0} f(v) \geq c(\nu f),$$

with $t_0 = \log \frac{4}{h} > 0$, $c = \frac{\sigma_0}{2} \left(\frac{h}{4}\right)^{\sigma_0} \in (0, 1)$, and $\nu = \frac{2}{h} \mathbf{1}_{[\frac{h}{2}, h]}$ the uniform probability measure on $[\frac{h}{2}, h]$.

Proof. We start with the definition of $(M_t)_{t \geq 0}$ which gives for $f \geq 0$

$$\begin{aligned} M_t f(v) &= f(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} [M_{t-\tau} f(e^{-\tau} v + h) \mathbf{1}_{[0, 1-h]}(e^{-\tau} v) + M_{t-\tau} f(v_r) \mathbf{1}_{[1-h, 1]}(e^{-\tau} v)] d\tau \\ &\geq f(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} M_{t-\tau} f(e^{-\tau} v + h) \mathbf{1}_{[0, 1-h]}(e^{-\tau} v) d\tau. \end{aligned}$$

Iterating this inequality we deduce

$$\begin{aligned} M_t f(v) &\geq f(v e^{-t}) e^{-\sigma_0 t} + \sigma_0 \int_0^t e^{-\sigma_0 \tau} f((e^{-\tau} v + h) e^{-(t-\tau)}) \mathbf{1}_{[0, 1-h]}(e^{-\tau} v) d\tau \\ &\geq \sigma_0 e^{-\sigma_0 t} \int_0^t f((e^{-\tau} v + h) e^{-(t-\tau)}) \mathbf{1}_{[0, 1-h]}(e^{-\tau} v) d\tau. \end{aligned}$$

Let $t_1 = -\log h$ the time after which all the neurons which did not undergo potential jumps have a voltage between 0 and h (i.e. $\forall \tau \geq t_1, \forall v \in [0, 1], v e^{-\tau} \in [0, h]$), and let $t_2 > 0$ to be chosen later. For $t = t_0 := t_1 + t_2$ we have

$$\begin{aligned} M_{t_0} f(v) &\geq \sigma_0 e^{-\sigma_0 t_0} \int_{t_1}^{t_0} f(e^{-t_0} v + h e^{-(t_0-\tau)}) d\tau \\ &\geq \sigma_0 e^{-\sigma_0 t_0} \int_{t_1}^{t_0} f(e^{-t_0} v + h e^{-(t_0-\tau)}) e^{-(t_0-\tau)} d\tau \\ &= \frac{\sigma_0}{h} e^{-\sigma_0 t_0} \int_{v e^{-t_0} + h e^{-t_2}}^{v e^{-t_0} + h} f(w) dw \quad (w = e^{-t_0} v + h e^{-(t_0-\tau)}) \\ &\geq \frac{\sigma_0}{h} e^{-\sigma_0 t_0} \int_{2h e^{-t_2}}^h f(w) dw \end{aligned}$$

For the last inequality we have used that $v e^{-t_0} \leq e^{-t_1 - t_2} = h e^{-t_2}$. So if we choose $t_2 = \log 4$ we get

$$M_{t_0} f(v) \geq \frac{\sigma_0}{2} e^{-\sigma_0 t_0} \frac{2}{h} \int_{\frac{h}{2}}^h f(w) dw = \frac{\sigma_0}{2} \left(\frac{h}{4}\right)^{\sigma_0} \nu(f).$$

□

Proof of Proposition 3.10. We divide the proof into three steps.

Step 1: We first prove that for all $v, v' \in [0, 1]$ we have

$$\|\delta_v M_{t_0} - \delta_{v'} M_{t_0}\|_{TV} \leq (1-c) \|\delta_v - \delta_{v'}\|_{TV}.$$

Define the linear operator U on $C([0, 1])$ by

$$\forall f \in C([0, 1]), \forall v \in [0, 1], \quad Uf(v) := \frac{1}{1-c} (M_{t_0} f(v) - c(\nu f)).$$

This operator is conservative ($U\mathbf{1} = \mathbf{1}$) and Lemma 3.11 ensures that it is positive. As a consequence it is a contraction ($\|Uf\|_\infty \leq \|f\|_\infty$) and we deduce that for all $v, v' \in [0, 1]$

$$\sup_{\|f\|_\infty \leq 1} |M_{t_0} f(v) - M_{t_0} f(v')| = (1-c) \sup_{\|f\|_\infty \leq 1} |Uf(v) - Uf(v')| \leq (1-c) \sup_{\|f\|_\infty \leq 1} |f(v) - f(v')|.$$

Step 2: We extend the result to general measures. For $\mu, \tilde{\mu} \in \mathcal{P}([0, 1])$ we have $(\mu - \tilde{\mu})_+([0, 1]) = (\tilde{\mu} - \mu)_+([0, 1]) = \frac{1}{2}\|\mu - \tilde{\mu}\|_{TV}$. We deduce from the step 1 that

$$\begin{aligned}
\|\mu M_{t_0} - \tilde{\mu} M_{t_0}\|_{TV} &= \|(\mu - \tilde{\mu})_+ M_{t_0} + (\mu - \tilde{\mu})_- M_{t_0}\|_{TV} \\
&= \|(\mu - \tilde{\mu})_+ M_{t_0} - (\tilde{\mu} - \mu)_+ M_{t_0}\|_{TV} \\
&= \sup_{\|f\|_\infty \leq 1} \left| \int_v M_{t_0} f(v) d(\mu - \tilde{\mu})_+(v) - \int_{v'} M_{t_0} f(v') d(\tilde{\mu} - \mu)_+(v') \right| \\
&= \frac{1}{(\mu - \tilde{\mu})_+([0, 1])} \sup_{\|f\|_\infty \leq 1} \left| \iint_{vv'} [M_{t_0} f(v) - M_{t_0} f(v')] d(\mu - \tilde{\mu})_+(v) d(\tilde{\mu} - \mu)_+(v') \right| \\
&\leq \sup_{v, v'} \|\delta_v M_{t_0} - \delta_{v'} M_{t_0}\|_{TV} (\mu - \tilde{\mu})_+([0, 1]) \\
&\leq (1 - c) \sup_{v, v'} \|\delta_v - \delta_{v'}\|_{TV} \frac{1}{2} \|\mu - \tilde{\mu}\|_{TV} = (1 - c) \|\mu - \tilde{\mu}\|_{TV}.
\end{aligned}$$

Step 3: Conclusion. For $t \geq 0$ we define $n = \lfloor \frac{t}{t_0} \rfloor$ and we get by induction

$$\|\mu M_t - \tilde{\mu} M_t\|_{TV} \leq (1 - c)^n \|\mu M_{t - nt_0} - \tilde{\mu} M_{t - nt_0}\|_{TV} \leq e^{n \log(1 - c)} \|\mu - \tilde{\mu}\|_{TV}.$$

This ends the proof since

$$n \log(1 - c) \leq \left(\frac{t}{t_0} - 1\right) \log(1 - c) = -a(t - t_0).$$

□

Corollary 3.12. *There exists a unique invariant measure $\bar{\mu} \in \mathcal{P}([0, 1])$ for the semigroup $(M_t)_{t \geq 0}$ and for all $\mu \in \mathcal{M}([0, 1])$ we have*

$$\forall t \geq 0, \quad \|\mu M_t - (\mu \mathbf{1}) \bar{\mu}\|_{TV} \leq e^{-a(t - t_0)} \|\mu - (\mu \mathbf{1}) \bar{\mu}\|_{TV}.$$

Proof. We proved in the step 2 of the proof of Proposition 3.10 that the mapping $\mu \mapsto \mu M_{t_0}$ is a contraction in the Cauchy space $(\mathcal{P}([0, 1]), \|\cdot\|_{TV})$. We deduce that it admits a unique fixed point $\bar{\mu} \in \mathcal{P}([0, 1])$. The semigroup property ensures that for all $t \geq 0$, $\bar{\mu} M_t$ is also a fixed point of M_{t_0} . By uniqueness we get that $\bar{\mu} M_t = \bar{\mu}$, meaning that $\bar{\mu}$ is invariant under $(M_t)_{t \geq 0}$. This concludes the proof since the exponential convergence is an immediate consequence of Proposition 3.10. □

4 Existence of a steady state when $J < 1 + \lfloor \frac{1 - v_r}{h} \rfloor$

This section is devoted to the proof of Theorem 2.2. The result in the previous section ensures that for all $\sigma > 0$ there exists a unique $\mu^\sigma \in \mathcal{P}([0, 1])$ such that $\mu^\sigma \mathcal{A}_\sigma = 0$. The question we address here is the existence of $\bar{\sigma} > 0$ such that

$$\bar{\sigma} = \frac{\sigma_0}{1 - J \mu^{\bar{\sigma}}([1 - h, 1])}. \quad (9)$$

In this case the measure $\bar{\mu} := \mu^{\bar{\sigma}}$ is a steady state for the nonlinear equation. We will prove the existence of such a steady state under the condition $J < 1 + \lfloor \frac{1 - v_r}{h} \rfloor$. This condition is quite optimal because we know that if $J \geq 1 + \frac{1 - v_r}{h}$ then there does not exist global solutions to the nonlinear equation (blow-up in finite time for any initial data [19]).

For finding $\bar{\sigma}$ which satisfies (9), we define the two functions

$$F : \begin{cases} (0, +\infty) & \rightarrow [0, +\infty) \\ \sigma & \mapsto \mu^\sigma([1 - h, 1]) \end{cases} \quad \text{and} \quad G : \begin{cases} (0, +\infty) & \rightarrow \mathbb{R} \\ \sigma & \mapsto \frac{1}{J} \left(1 - \frac{\sigma_0}{\sigma}\right) \end{cases}.$$

and we prove the existence of $\bar{\sigma}$ such that $F(\bar{\sigma}) = G(\bar{\sigma})$. To do so we need informations on the function F and it requires some regularity results on the regularity of the invariant measure μ^σ .

Lemma 4.1. For any $\sigma > 0$, the invariant measure μ^σ is absolutely continuous with respect to the Lebesgue measure. Its density p_σ satisfies $vp_\sigma(v) \in W^{1,\infty}([0, v_r] \cup (v_r, 1])$ and

$$\forall v \in (0, 1), \quad 0 < p_\sigma(v) \leq \min \left\{ \frac{\sigma v^{\sigma-1}}{h^\sigma}, \frac{\sigma}{v} \right\}.$$

Proof. Recall that μ^σ satisfies

$$\int_{[0,1]} \left[\frac{v}{\sigma} f'(v) + f(v) - f(v+h) \mathbf{1}_{[0,1-h)}(v) \right] d\mu^\sigma(v) = f(v_r) \mu^\sigma([1-h, 1]), \quad \forall f \in C^1([0, 1]). \quad (10)$$

Thus the derivative of $v\mu^\sigma(dv)$ in the distributional sense is a finite measure and then $v\mu^\sigma(dv)$ is a function with bounded variation. We deduce that there exist $\alpha \geq 0$ and $p_\sigma \in L^1_+(0, 1)$ such that $\mu^\sigma = \alpha \delta_0 + p_\sigma(v) dv$. More precisely $vp_\sigma(v)$ is a $W^{1,1}$ function on the intervals $(0, h)$, (h, v_r) and $(v_r, 1)$ (so that it has a left and a right trace at $v = h$ and $v = v_r$) with a jump at $v = v_r$ given by

$$v_r p_\sigma(v_r^-) - v_r p_\sigma(v_r^+) = \sigma \int_{1-h}^1 p_\sigma(w) dw$$

and also *a priori* at $v = h$ given by

$$h p_\sigma(h^-) - h p_\sigma(h^+) = \mu^\sigma(\{0\}) = \alpha.$$

We will actually prove that μ^σ does not charge 0, *i.e.* $\alpha = 0$, so that there is no jump at $v = h$. Consider $f \in C^1_c([0, 1])$ which satisfies $f(0) = 1$, and define $f_n(v) := f(nv)$. For all $n > [1/h]$ Equation (10) written with f_n gives

$$\int_{[0,1]} \left[\frac{v}{\sigma} f'_n(v) + f_n(v) \right] d\mu^\sigma(v) = 0.$$

By dominated convergence we have

$$\int_{[0,1]} f_n(v) d\mu^\sigma(v) \xrightarrow{n \rightarrow \infty} \mu^\sigma(\{0\}) = \alpha$$

and

$$\int_{[0,1]} v f'_n(v) d\mu^\sigma(v) = \int_{[0,1]} n v f'(nv) d\mu^\sigma(v) \xrightarrow{n \rightarrow \infty} 0.$$

We conclude that $\alpha = 0$, so that $\mu^\sigma = p_\sigma(v) dv$. The function p_σ has no jump at $v = h$ and it satisfies the following equation on $(0, v_r) \cup (v_r, 1)$

$$\frac{-1}{\sigma} (vp_\sigma(v))' + p_\sigma(v) = p_\sigma(v-h) \mathbf{1}_{[0,1-h)}(v).$$

For the bound on p_σ we start by studying p_σ on the interval $(0, h)$. On this interval the equation satisfied by p_σ is

$$(vp_\sigma(v))' = p_\sigma(v)$$

which gives after integration

$$p_\sigma(v) = \left(\frac{v}{h} \right)^{\sigma-1} p_\sigma(h^-). \quad (11)$$

By positivity of p_σ we have on both intervals $(0, v_r)$ and $(v_r, 1)$ the differential inequality

$$(vp_\sigma(v))' \leq \sigma p_\sigma(v)$$

from which we get

$$\forall 0 < w < v \leq 1, \quad p_\sigma(v) \leq \left(\frac{v}{w} \right)^{\sigma-1} p_\sigma(w). \quad (12)$$

Integrating from $w = 0$ to $w = 1$ we deduce

$$1 \geq \int_0^v p_\sigma(w) dw \geq \frac{p_\sigma(v)}{v^{\sigma-1}} \int_0^v w^{\sigma-1} dw = \frac{vp_\sigma(v)}{\sigma}$$

and

$$p_\sigma(v) \leq \frac{\sigma}{v} \quad \forall 0 < v \leq 1.$$

Combining with (12) we get that

$$\forall 0 < w < v \leq 1, \quad p_\sigma(v) \leq \frac{\sigma}{w^\sigma} v^{\sigma-1}$$

and with (11) we obtain

$$p_\sigma(v) \leq \frac{\sigma v^{\sigma-1}}{h^\sigma} \quad \forall 0 < v \leq 1.$$

From (12) we deduce that the support of p_σ is necessarily of the form $[0, v_m]$ with $0 < v_m \leq 1$ and p_σ is strictly positive in the interior of its support. Assume that $v_m < 1$. Then integrating the equation satisfied by p_σ between v_m and $\tilde{v} := \min\{v_m + h, 1\} > v_m$ we get the contradiction

$$0 = \int_{v_m-h}^{\tilde{v}-h} p_\sigma(v) dv > 0.$$

So $v_m = 1$ and $p_\sigma(v) > 0$ for all $v \in (0, 1)$. □

We are now ready to prove Theorem 2.2, which is a direct consequence of the following lemma about the asymptotic behavior of the function F .

Lemma 4.2. *The function F is continuous and satisfies*

$$\lim_{\sigma \rightarrow 0} F(\sigma) = 0 \quad \text{and} \quad \lim_{\sigma \rightarrow +\infty} F(\sigma) = \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor\right)^{-1}.$$

Proof of Theorem 2.2. If $J < 1 + \lfloor \frac{1-v_r}{h} \rfloor$ then Lemma 4.2 ensures the existence of $\bar{\sigma} > 0$ such that $F(\bar{\sigma}) = G(\bar{\sigma})$. Then $\bar{\sigma}$ satisfies (9) and $\mu^{\bar{\sigma}}$ is a steady state of the nonlinear equation. □

Proof of Lemma 4.2. Fix $\sigma > 0$ and consider a sequence $(\sigma_n)_{n \in \mathbb{N}} \subset [\sigma/2, 2\sigma]$ which converges to σ . Since the associated sequence $(p_{\sigma_n})_{n \in \mathbb{N}}$ satisfies

$$\forall n \in \mathbb{N}, \forall v \in [0, 1], \quad 0 \leq p_{\sigma_n}(v) \leq \frac{2\sigma v^{\sigma/2-1}}{h^{2\sigma}}$$

we deduce from the Dunford-Pettis theorem that there exists a subsequence, still denoted by (p_{σ_n}) , which converges L^1 -weak to a limit $q \in \mathcal{P}([0, 1]) \cap L^1(0, 1)$, *i.e.*

$$\int_0^1 p_{\sigma_n}(v) \varphi(v) dv \rightarrow \int_0^1 q(v) \varphi(v) dv$$

for all $\varphi \in L^\infty(0, 1)$. Passing to the limit in the weak formulation we get that q is solution to (10). By uniqueness we get that $q = p_\sigma$ and the whole sequence converges to p_σ . This gives the continuity of the application

$$\sigma \mapsto \int_0^1 p_\sigma(v) \varphi(v) dv$$

for any $\varphi \in L^\infty(0, 1)$, and as a consequence the continuity of F since $\mathbf{1}_{[1-h, 1]} \in L^\infty(0, 1)$.

For the limit at 0 we readily deduce from Lemma 4.1 that

$$0 \leq F(\sigma) \leq \frac{\sigma}{1-h} \xrightarrow{\sigma \rightarrow 0} 0.$$

If we want to be more precise we can prove that p_σ converges weak* to δ_0 when $\sigma \rightarrow 0$. Indeed for any sequence $(\sigma_n)_{n \in \mathbb{N}}$ which tends to 0 we can extract from $(p_{\sigma_n})_{n \geq 0}$ a subsequence, still denoted $(p_{\sigma_n})_{n \geq 0}$, which converges weak* to a probability measure μ . Lemma 4.1 ensures that

$$\forall 0 < \epsilon \leq v \leq 1, \forall \sigma > 0, \quad 0 \leq p_\sigma(v) \leq \frac{\sigma}{\epsilon}$$

so we deduce that for all $f \in C_c([\varepsilon, 1])$ we have

$$\mu f = \lim_{n \rightarrow \infty} \int_0^1 p_{\sigma_n}(v) f(v) dv = \lim_{n \rightarrow \infty} \int_\varepsilon^1 p_{\sigma_n}(v) f(v) dv = 0.$$

As a consequence $\text{supp } \mu = 0$ and since $\mu \in \mathcal{P}([0, 1])$ we deduce that $\mu = \delta_0$, and then $p_\sigma \xrightarrow{*} \delta_0$ when $\sigma \rightarrow 0$ since the sequence (σ_n) is arbitrary.

We finish with the limit at $+\infty$. Let $(\sigma_n)_{n \in \mathbb{N}}$ a sequence which tends to $+\infty$. We can extract from $(p_{\sigma_n})_{n \geq 0}$ a subsequence, still denoted $(p_{\sigma_n})_{n \geq 0}$, which weakly converges to a probability measure μ . We want to identify the limit μ . We have that

$$\int_0^1 p_{\sigma_n}(v) f(v) dv \rightarrow \mu f \quad (n \rightarrow \infty)$$

for all $f \in C([0, 1])$. We define

$$D := \{f \in C^1([0, 1]), f(v_r) = f(1)\}$$

which satisfies the property that

$$\forall f \in D, \quad \mathcal{A}f \in C([0, 1])$$

and as a consequence

$$\forall f \in D, \quad 0 = \frac{1}{\sigma_n} \mu^{\sigma_n} \mathcal{A}_{\sigma_n} f = \frac{1}{\sigma_n} \mu^{\sigma_n} (v f'(v)) + \mu^{\sigma_n} \mathcal{B}f \xrightarrow[n \rightarrow \infty]{} \mu \mathcal{B}f.$$

This property that $\mu \mathcal{B}f = 0$ for all $f \in D$, *i.e.*

$$\int_{[0, 1]} \left[f(v) - f(v+h) \mathbf{1}_{[0, 1-h]}(v) \right] d\mu(v) = f(v_r) \mu([1-h, 1]), \quad \forall f \in D.$$

allows to prove that

$$\mu = \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor \right)^{-1} \sum_{k=0}^{\lfloor \frac{1-v_r}{h} \rfloor} \delta_{v_r+kh}.$$

We prove this step by step. First for $f \in C_c^1([0, h])$ we get $\int_{[0, h]} f(v) d\mu(v) = 0$ and so $\text{supp } \mu \cap [0, h) = \emptyset$. We easily deduce by induction, choosing $f \in C_c^1(kh, (k+1)h)$, that $\text{supp } \mu \cap [0, kh) = \emptyset$ for any $k \in \mathbb{N}$ such that $kh \leq v_r$ and then with one more step, with $f \in C_c^1(kh, v_r)$, that $\text{supp } \mu \cap [0, v_r) = \emptyset$. Keeping going we get by choosing $f \in C_c^1(v_r+kh, v_r+(k+1)h)$ that $\text{supp } \mu \cap (v_r+kh, v_r+(k+1)h) = \emptyset$ for any $k \in \mathbb{N}$. Finally we have proved that

$$\text{supp } \mu \subset \left\{ \{v_r+kh\}, 0 \leq k \leq \left\lfloor \frac{1-v_r}{h} \right\rfloor \right\}$$

so there exists a finite family of nonnegative real numbers α_k such that

$$\mu = \sum_{k=0}^{\lfloor \frac{1-v_r}{h} \rfloor} \alpha_k \delta_{v_r+kh}.$$

For $1 \leq k \leq \left\lfloor \frac{1-v_r}{h} \right\rfloor$, considering $f \in C_c^1(v_r+(k-1/2)h, \min\{v_r+(k+1/2)h, 1\})$ such that $f(v_r+kh) = 1$ as a test function, we get that $\alpha_k = \alpha_{k-1}$. Now let $f \in C^1([0, 1])$ such that $f(v_r) = f(1) = 1$ and $\text{supp } f \subset [v_r-h/2, v_r+h/2] \cup [v_r + \lfloor \frac{1-v_r}{h} \rfloor h, 1]$ (recall that $1-h < v_r + \lfloor \frac{1-v_r}{h} \rfloor h < 1$ since $\frac{1-v_r}{h} \notin \mathbb{N}$). Using this function as a test function we get that $\alpha_0 = \alpha_{\lfloor \frac{1-v_r}{h} \rfloor}$. Finally all the α_k are equal and

$$\mu = \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor \right)^{-1} \sum_{k=0}^{\lfloor \frac{1-v_r}{h} \rfloor} \delta_{v_r+kh}.$$

Since this limit does not depend on the subsequence, we deduce that for any $f \in C([0, 1])$ we have

$$\lim_{\sigma \rightarrow +\infty} \int_0^1 f(v) p_\sigma(v) dv = \int_0^1 f(v) d\mu(v) = \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor\right)^{-1} \sum_{k=0}^{\left\lfloor \frac{1-v_r}{h} \right\rfloor} f(v_r + kh).$$

Since $\mathbf{1}_{[1-h, 1]}$ is not continuous we cannot conclude directly for the limit of F . For all $n \geq 1$ we define $\tilde{\chi}_n(v) := \chi_n(v - \frac{h}{n})$. For all $v \in [0, 1]$ we have $\tilde{\chi}_n(v) \leq \mathbf{1}_{[1-h, 1]}(v) \leq \chi_n(v)$ and since $v_r + \lfloor \frac{1-v_r}{h} \rfloor h \in [1-h, 1]$

$$\begin{aligned} \lim_{\sigma \rightarrow +\infty} \int_0^1 \chi_n(v) p_\sigma(v) dv &= \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor\right)^{-1} \left[\chi_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h - h\right) + \chi_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h\right) \right], \\ \lim_{\sigma \rightarrow +\infty} \int_0^1 \tilde{\chi}_n(v) p_\sigma(v) dv &= \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor\right)^{-1} \tilde{\chi}_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h\right). \end{aligned}$$

We deduce that for all $n \geq 1$

$$\tilde{\chi}_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h\right) \leq \left(1 + \left\lfloor \frac{1-v_r}{h} \right\rfloor\right) \lim_{\sigma \rightarrow +\infty} F(\sigma) \leq \chi_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h - h\right) + \chi_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h\right).$$

The fact that $v_r + \lfloor \frac{1-v_r}{h} \rfloor h \in (1-h, 1)$ (since $\frac{1-v_r}{h} \notin \mathbb{N}$) ensures that

$$\lim_{n \rightarrow \infty} \chi_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h\right) = \lim_{n \rightarrow \infty} \tilde{\chi}_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h\right) = 1$$

and

$$\lim_{n \rightarrow \infty} \chi_n\left(v_r + \left\lfloor \frac{1-v_r}{h} \right\rfloor h - h\right) = 0,$$

and the conclusion follows. \square

5 Global well-posedness for $J < 1$

For a connectivity $J < 1$, we know from [18] that Equation (1) is globally well-posed in $L^1([0, 1])$. We prove here Theorem 2.3 which ensures that it is still the case in the larger space $\mathcal{M}([0, 1])$.

Our method of proof relies on duality arguments and divides into several steps. First we remark that if $(\mu_t)_{t \geq 0}$ is a measure solution to Equation (1), then for all $t \geq 0$ there exists a function ϕ_t , obtained by solving a nonlinear dual equation which involves only μ_0 , such that $\mu_t([1-h, 1]) = \mu_0 \phi_t$. Once ϕ_t is known for all t , we deduce the value of $\sigma(t)$ and we can see Equation (1) as a time-inhomogeneous but linear equation. We solve this equation in a similar way as for the linear case $J = 0$. This method of construction allows us to prove a Duhamel formula for Equation (1) which is then used to prove uniqueness. The Duhamel formula is also the corner stone to prove the exponential stability of the steady state in the weakly connected regime (Section 6).

In what follows, μ is a **fixed probability measure**. Due to the discontinuity of the indicator functions, we need to apply the method described above to a regularized problem before passing to the limit, as in Section 3. The first step consists in building, for $t \geq 0$, a regularized version of the function ϕ_t . For $T > 0$ we denote $X^T := C(\{(s, t, v), 0 \leq s \leq t < T, 0 \leq v \leq 1\})$ and for $n \in \mathbb{N}^*$ and T small enough we define $\psi_n \in X^T$ as the unique solution to the nonlinear equation

$$\partial_s \psi_n(s, t, v) = v \partial_v \psi_n(s, t, v) + \frac{\sigma_0}{1 - J \mu \psi_n(0, s, \cdot)} [\psi_n(s, t, v) - \psi_n(s, t, v+h)(1 - \chi_n(v)) - \psi_n(s, t, v_r) \chi_n(v)],$$

with the terminal condition $\psi_n(t, t, v) = \chi_n(v)$. More precisely ψ_n is defined in the following lemma, where we have set $T^* = \frac{(1-J)^2}{2\sigma_0}$.

Lemma 5.1. *There exists a unique function $\psi_n \in \{f \in X^{T^*}, 0 \leq f \leq 1\}$ which satisfies*

$$\begin{aligned} \psi_n(s, t, v) &= \chi_n(ve^{s-t}) e^{\frac{\sigma_0}{1-J}(s-t)} + \int_s^t \left(\frac{\sigma_0}{1-J} - \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} \right) e^{\frac{\sigma_0}{1-J}(s-\tau)} \psi_n(\tau, t, ve^{s-\tau}) d\tau \\ &\quad + \int_s^t \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} [\psi_n(\tau, t, ve^{s-\tau} + h)(1 - \chi_n(ve^{s-\tau})) + \psi_n(\tau, t, v_r)\chi_n(ve^{s-\tau})] d\tau. \end{aligned}$$

Proof. Let $T \in (0, T^*)$. We use the Banach fixed point theorem for the mapping

$$\begin{aligned} \Gamma f(s, t, v) &= \chi_n(ve^{s-t}) e^{\frac{\sigma_0}{1-J}(s-t)} + \int_s^t \left(\frac{\sigma_0}{1-J} - \frac{\sigma_0}{1-J\mu f(0, \tau, \cdot)} \right) e^{\frac{\sigma_0}{1-J}(s-\tau)} f(\tau, t, ve^{s-\tau}) d\tau \\ &\quad + \int_s^t \frac{\sigma_0}{1-J\mu f(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} [f(\tau, t, ve^{s-\tau} + h)(1 - \chi_n(ve^{s-\tau})) + f(\tau, t, v_r)\chi_n(ve^{s-\tau})] d\tau \end{aligned}$$

on the invariant complete metric space $\{f \in X^T, 0 \leq f \leq 1\}$. This mapping is a contraction when $T < \frac{(1-J)^2}{2\sigma_0}$ since

$$\begin{aligned} \|\Gamma f_1 - \Gamma f_2\|_\infty &\leq \left[\frac{\sigma_0}{1-J} \|f_1 - f_2\|_\infty + 2 \left\| \frac{\sigma_0}{1-J\mu f_1} - \frac{\sigma_0}{1-J\mu f_2} \right\|_\infty + \left\| \frac{\sigma_0}{1-J\mu f_2} \right\|_\infty \|f_1 - f_2\|_\infty \right] T \\ &\leq \frac{2\sigma_0}{(1-J)^2} T \|f_1 - f_2\|_\infty. \end{aligned}$$

□

In a second step we define for any $f_0 \in C([0, 1])$ the function $f \in X^{T^*}$ as the unique solution to the linear equation

$$\partial_s f(s, t, v) = v\partial_v f(s, t, v) + \frac{\sigma_0}{1-J\mu\psi_n(0, s, \cdot)} [f(s, t, v) - f(s, t, v+h)(1 - \chi_n(v)) - f(s, t, v_r)\chi_n(v)]$$

with the terminal condition $f(t, t, v) = f_0(v)$. This definition is made more precise in the following lemma.

Lemma 5.2. *For all $f_0 \in C([0, 1])$ there exists a unique $f \in X^{T^*}$ which verifies*

$$\begin{aligned} f(s, t, v) &= f_0(ve^{s-t}) e^{\frac{\sigma_0}{1-J}(s-t)} + \int_s^t \left(\frac{\sigma_0}{1-J} - \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} \right) e^{\frac{\sigma_0}{1-J}(s-\tau)} f(\tau, t, ve^{s-\tau}) d\tau \\ &\quad + \int_s^t \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} [f(\tau, t, ve^{s-\tau} + h)(1 - \chi_n(ve^{s-\tau})) + f(\tau, t, v_r)\chi_n(ve^{s-\tau})] d\tau. \end{aligned}$$

Additionally if $f_0 \geq 0$, then $f \geq 0$.

This allows to define a positive semigroup $(N_{s,t}^n)_{0 \leq s \leq t < T^*}$ on $C([0, 1])$ by $N_{s,t}^n f_0(v) := f(s, t, v)$. The semigroup property means that

$$N_{t,t}^n f = f \quad \text{and} \quad \forall \tau \in [s, t], \quad N_{s,t}^n f = N_{s,\tau}^n (N_{\tau,t}^n f).$$

Additionally we easily check that $N_{s,t}^n \mathbf{1} = \mathbf{1}$ and, together with the positivity property, it ensures the contraction property

$$\|N_{s,t}^n f\|_\infty \leq \|f\|_\infty.$$

A fundamental remark here is that the uniqueness in Lemmas 5.1 and 5.2 ensures that

$$\psi_n(s, t, v) = N_{s,t}^n \chi_n(v).$$

For all $t \in [0, T^*)$ we define

$$\sigma_n(t) := \frac{\sigma_0}{1-J\mu\psi_n(0, t, \cdot)} = \frac{\sigma_0}{1-J\mu N_{0,t}^n \chi_n}$$

and for the sake of conciseness we denote by \mathcal{A}_t^n the operator $\mathcal{A}_{\sigma_n(t)}^n$, i.e.

$$\forall f \in C^1([0, 1]), \quad \mathcal{A}_t^n f(v) := \mathcal{A}_{\sigma_n(t)}^n f(v) = -vf'(v) + \sigma_n(t) \mathcal{B}^n f(v).$$

This operator is the infinitesimal generator of the semigroup $(N_{s,t}^n)$ in the sense of the following lemma, where we have set $T^{**} = \frac{(1-J)^2(1-2h)}{3\sigma_0}$.

Lemma 5.3. *If $f \in C^1([0, 1])$, then the function $(s, t, v) \mapsto N_{s,t}^n f(v)$ is continuously differentiable on the set $\{0 \leq s \leq t \leq T^{**}, 0 \leq v \leq 1\}$ and we have*

$$\forall 0 \leq s \leq t \leq T^{**}, \quad \partial_s N_{s,t}^n f = -\mathcal{A}_s^n N_{s,t}^n f, \quad \text{and} \quad \partial_t N_{s,t}^n f = N_{s,t}^n \mathcal{A}_t^n f.$$

Proof of Lemmas 5.2 and 5.3. First let $f_0 \in C([0, 1])$, $0 < T < T^*$, and define on X^T the mapping

$$\begin{aligned} \Gamma f(s, t, v) &= f_0(v e^{s-t}) e^{\frac{\sigma_0}{1-J}(s-t)} + \int_s^t \left(\frac{\sigma_0}{1-J} - \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} \right) e^{\frac{\sigma_0}{1-J}(s-\tau)} f(\tau, t, v e^{s-\tau}) d\tau \\ &+ \int_s^t \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} [f(\tau, t, v e^{s-\tau} + h)(1 - \chi_n(v e^{s-\tau})) + f(\tau, t, v_r)\chi_n(v e^{s-\tau})] d\tau. \end{aligned}$$

For any $f_1, f_2 \in X^T$ we have

$$\|\Gamma f_1 - \Gamma f_2\|_\infty \leq \min \left\{ \frac{2\sigma_0}{1-J}, \frac{\sigma_0}{(1-J)^2} \right\} \|f_1 - f_2\|_\infty T$$

and this ensures that Γ is a contraction on X^T endowed with the supremum norm. We deduce the existence and uniqueness of a fixed point for Γ from the Banach fixed point theorem.

If $f_0 \geq 0$ the positive cone of X^T is invariant under Γ so the fixed point belongs to this cone.

Now assume that $T < T^{**}$ and $f_0 \in C^1([0, 1])$. In this case we can apply the Banach fixed point theorem in the space $\{f \in X^T : \partial_v f, \partial_t f \in X^T\}$ with the norm $\|f\|_{C^1} := \|f\|_\infty + \|\partial_v f\|_\infty + \|\partial_t f\|_\infty$. Indeed, computing

$$\begin{aligned} \partial_t \Gamma f(s, t, v) &= \mathcal{A}_t^n f_0(v e^{s-t}) e^{\frac{\sigma_0}{1-J}(s-t)} + \int_s^t \left(\frac{\sigma_0}{1-J} - \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} \right) e^{\frac{\sigma_0}{1-J}(s-\tau)} \partial_t f(\tau, t, v e^{s-\tau}) d\tau \\ &+ \int_s^t \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} [\partial_t f(\tau, t, v e^{s-\tau} + h)(1 - \chi_n(v e^{s-\tau})) + \partial_t f(\tau, t, v_r)\chi_n(v e^{s-\tau})] d\tau \end{aligned}$$

$$\begin{aligned} \partial_v \Gamma f(s, t, v) &= e^{s-t} f_0'(v e^{s-t}) e^{\frac{\sigma_0}{1-J}(s-t)} + \int_s^t \left(\frac{\sigma_0}{1-J} - \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} \right) e^{\frac{\sigma_0}{1-J}(s-\tau)} e^{s-\tau} \partial_v f(\tau, t, v e^{s-\tau}) d\tau \\ &+ \int_s^t \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} e^{s-\tau} \partial_v f(\tau, t, v e^{s-\tau} + h)(1 - \chi_n(v e^{s-\tau})) \\ &+ \int_s^t \frac{\sigma_0}{1-J\mu\psi_n(0, \tau, \cdot)} e^{\frac{\sigma_0}{1-J}(s-\tau)} e^{s-\tau} \frac{n}{h} \mathbf{1}_{[1-h-\frac{h}{n}, 1-h]}(v e^{s-\tau}) [f(\tau, t, v_r) - f(\tau, t, v e^{s-\tau} + h)] d\tau \end{aligned}$$

we get for $f_1, f_2 \in \{f \in X^T : \partial_v f, \partial_t f \in X^T\}$

$$\|\partial_t \Gamma f_1 - \partial_t \Gamma f_2\|_\infty \leq \min \left\{ \frac{2\sigma_0}{1-J}, \frac{\sigma_0}{(1-J)^2} \right\} \|\partial_t(f_1 - f_2)\|_\infty T$$

$$\|\partial_v \Gamma f_1 - \partial_v \Gamma f_2\|_\infty \leq \left[\min \left\{ \frac{2\sigma_0}{1-J}, \frac{\sigma_0}{(1-J)^2} \right\} \|\partial_v(f_1 - f_2)\|_\infty + \frac{2\sigma_0}{1-J} \frac{1}{1-2h} \|f_1 - f_2\|_\infty \right] T$$

and finally

$$\|\Gamma f_1 - \Gamma f_2\|_{C^1} \leq \min \left\{ \frac{4\sigma_0}{(1-J)(1-2h)}, \frac{3\sigma_0}{(1-J)^2(1-2h)} \right\} T \|f_1 - f_2\|_{C^1}.$$

We deduce that the unique fixed point of Γ satisfies $\partial_v f, \partial_t f \in X^T$. We can also compute

$$\begin{aligned} \partial_s \Gamma f(s, t, v) &= v \partial_v \Gamma f(s, t, v) + \frac{\sigma_0}{1-J} [\Gamma f(s, t, v) - f(s, t, v)] \\ &+ \frac{\sigma_0}{1-J\mu\psi_n(0, s, \cdot)} [f(s, t, v) - f(s, t, v+h)(1 - \chi_n(v)) - f(s, t, v_r)\chi_n(v)] \end{aligned}$$

and this ensures that the fixed point also satisfies $\partial_s f \in X^T$ and $\partial_s f = -\mathcal{A}_s^n f$.

From the computation of $\partial_t \Gamma f$ we see that if $N_{s,t}^n f_0$ is the fixed point of Γ with terminal condition f_0 then $\partial_t N_{s,t}^n f_0$ is the fixed of Γ with terminal condition $\mathcal{A}_t^n f_0$. By uniqueness we deduce that $\partial_t N_{s,t}^n f_0 = N_{s,t}^n \mathcal{A}_t^n f_0$. □

The third step consists in defining the measure $\mu N_{s,t}^n$ by duality

$$\forall f \in C([0, 1]), \quad (\mu N_{s,t}^n) f := \mu(N_{s,t}^n f).$$

The following lemma ensures that $t \mapsto \mu N_{0,t}^n$ is a solution to a regularized version of Equation (1) on the interval $[0, T^{**})$.

Lemma 5.4. *The mapping $t \mapsto \mu N_{0,t}^n$, which is defined on $[0, T^*)$, takes its values in $\mathcal{P}([0, 1])$ and is weak*-continuous. Additionally for all $t \in [0, T^{**})$ we have*

$$\forall f \in C^1([0, 1]), \quad \mu N_{0,t}^n f = \mu f + \int_0^t \mu N_{0,s}^n \mathcal{A}_s^n f ds. \quad (13)$$

Proof. The positivity property in Lemma 5.2 ensures that $\mu N_{0,t}^n \in \mathcal{M}_+([0, 1])$. Additionally we easily check that $N_{0,t}^n \mathbf{1} = \mathbf{1}$. Together with the positivity this implies that $\mu N_{0,t}^n \in \mathcal{P}([0, 1])$, and also that $\|\mu N_{0,t}^n f\|_\infty \leq \|f\|_\infty$ for all $f \in C([0, 1])$. The weak*-continuity of $t \mapsto \mu N_{0,t}^n$ follows from the continuity of $t \mapsto N_{0,t}^n f(v)$ for all $f \in C([0, 1])$, $v \in [0, 1]$, and from the dominated convergence theorem.

For (13) we prove a little bit more, namely that for all $f \in C^1([0, 1])$ the mapping $t \mapsto \mu N_{0,t}^n f$ is continuously differentiable and that

$$\frac{d}{dt}(\mu N_{0,t}^n f) = \mu N_{0,t}^n \mathcal{A}_t^n f.$$

Indeed, from Lemma 5.3 and by dominated convergence we have

$$\frac{1}{h}(\mu N_{0,t+h}^n f - \mu N_{0,t}^n f) = \mu \left[\frac{1}{h}(N_{0,t+h}^n f - N_{0,t}^n f) \right] \xrightarrow{h \rightarrow 0} \mu(\partial_t N_{0,t}^n f) = \mu N_{0,t}^n \mathcal{A}_t^n f.$$

□

In the fourth step we pass to the limit $n \rightarrow \infty$.

Lemma 5.5. *For all $t \in [0, T^*)$, the sequence $(\mu N_{0,t}^n)_{n \in \mathbb{N}^*}$ is convergent for the total variation norm. Denoting $(\mu_t)_{0 \leq t < T^*} \subset \mathcal{P}([0, 1])$ the limit family, we have for all $t \in [0, T^{**})$*

$$\forall f \in C^1([0, 1]), \quad \mu_t f = \mu f + \int_0^t \mu_s \mathcal{A}_{\sigma(s)} f ds.$$

Proof. We check that $(\mu N_{0,t}^n)_{n \in \mathbb{N}^*}$ is a Cauchy sequence. Let $n, p \in \mathbb{N}^*$, $0 \leq s \leq t < T^*$, and $f \in C([0, 1])$ such that $\|f\|_\infty \leq 1$. We have, using that $\mu \in \mathcal{P}([0, 1])$ and the Fubini's theorem,

$$\begin{aligned} \|N_{s,t}^n f - N_{s,t}^{n+p} f\|_\infty &\leq 2 \int_s^t \left| \frac{\sigma_0}{1 - J \mu N_{0,\tau}^n \chi_n} - \frac{\sigma_0}{1 - J \mu N_{0,\tau}^{n+p} \chi_{n+p}} \right| d\tau + \frac{2\sigma_0}{1-J} \int_s^t \|N_{\tau,t}^n f - N_{\tau,t}^{n+p} f\|_\infty d\tau \\ &\leq \frac{2\sigma_0 J}{(1-J)^2} \int_s^t |\mu N_{0,\tau}^n \chi_n - \mu N_{0,\tau}^{n+p} \chi_{n+p}| d\tau + \frac{2\sigma_0}{1-J} \int_s^t \|N_{\tau,t}^n f - N_{\tau,t}^{n+p} f\|_\infty d\tau \\ &\leq \frac{2\sigma_0 J}{(1-J)^2} \int_s^t |\mu N_{0,\tau}^n \chi_n - \mu N_{0,\tau}^n \chi_{n+p}| d\tau \\ &\quad + \frac{2\sigma_0 J}{(1-J)^2} \int_s^t |\mu N_{0,\tau}^n \chi_{n+p} - \mu N_{0,\tau}^{n+p} \chi_{n+p}| d\tau + \frac{2\sigma_0}{1-J} \int_s^t \|N_{\tau,t}^n f - N_{\tau,t}^{n+p} f\|_\infty d\tau \\ &\leq \frac{2\sigma_0 J}{(1-J)^2} \int_s^t \mu |N_{0,\tau}^n (\chi_n - \chi_{n+p})| d\tau \\ &\quad + \frac{2\sigma_0}{(1-J)^2} \int_s^t \|\mu N_{\tau,t}^n - \mu N_{\tau,t}^{n+p}\|_{TV} d\tau + \frac{2\sigma_0}{1-J} \int_s^t \|N_{\tau,t}^n f - N_{\tau,t}^{n+p} f\|_\infty d\tau \\ &\leq \frac{2\sigma_0 J}{(1-J)^2} \sup_{v \in [0,1]} \int_0^t |N_{0,\tau}^n (\chi_n - \chi_{n+p})(v)| d\tau \\ &\quad + \frac{2\sigma_0}{(1-J)^2} \int_s^t \|\mu N_{\tau,t}^n - \mu N_{\tau,t}^{n+p}\|_{TV} d\tau + \frac{2\sigma_0}{1-J} \int_s^t \|N_{\tau,t}^n f - N_{\tau,t}^{n+p} f\|_\infty d\tau. \end{aligned}$$

We bound the quantity

$$\Omega_n(s, t) := \sup_{p \in \mathbb{N}} \sup_{v \in [0, 1]} \int_s^t |N_{s, \tau}^n(\chi_n - \chi_{n+p})(v)| d\tau$$

for $n \in \mathbb{N}^*$, $0 \leq s \leq t < T^*$. From the definition of the semigroup $(N_{s, t}^n)$ we have for all $v \in [0, 1]$, $n \in \mathbb{N}^*$, and $0 \leq s \leq t < T^*$

$$\begin{aligned} \int_s^t |N_{s, \tau}^n(\chi_n - \chi_{n+p})(v)| d\tau &\leq \int_s^t |(\chi_n - \chi_{n+p})(ve^{s-\tau})| d\tau + \frac{\sigma_0}{1-J} \int_s^t \int_s^\tau |N_{\tau', \tau}^n(\chi_n - \chi_{n+p})(ve^{s-\tau'})| d\tau' d\tau \\ &+ \frac{\sigma_0}{1-J} \int_s^t \int_s^\tau |N_{\tau', \tau}^n(\chi_n - \chi_{n+p})(ve^{s-\tau'} + h)(1 - \chi_n(ve^{s-\tau'})) + N_{\tau', \tau}^n(\chi_n - \chi_{n+p})(v_r)\chi_n(ve^{s-\tau'})| d\tau' d\tau \\ &\leq \int_s^t \mathbf{1}_{[1-h-\frac{h}{n}, 1-h]}(ve^{s-\tau}) d\tau + \frac{\sigma_0}{1-J} \int_s^t \int_{\tau'}^t |N_{\tau', \tau}^n(\chi_n - \chi_{n+p})(ve^{s-\tau'})| d\tau d\tau' \\ &+ \frac{\sigma_0}{1-J} \int_s^t \int_{\tau'}^t |N_{\tau', \tau}^n(\chi_n - \chi_{n+p})(ve^{s-\tau'} + h)(1 - \chi_n(ve^{s-\tau'})) + N_{\tau', \tau}^n(\chi_n - \chi_{n+p})(v_r)\chi_n(ve^{s-\tau'})| d\tau d\tau' \\ &\leq \log \left(1 + \frac{h}{n(1-2h)} \right) + \frac{3\sigma_0}{1-J} \int_s^t \Omega_n(\tau', t) d\tau'. \end{aligned}$$

Taking the supremum in the left hand side we get the inequality

$$\Omega_n(s, t) \leq \log \left(1 + \frac{h}{n(1-2h)} \right) + \frac{3\sigma_0}{1-J} \int_s^t \Omega_n(\tau, t) d\tau$$

which gives by Grönwall's lemma

$$\Omega_n(s, t) \leq \log \left(1 + \frac{h}{n(1-2h)} \right) e^{\frac{3\sigma_0}{1-J}(t-s)}.$$

Coming back to the first computations of the proof we get

$$\begin{aligned} \|\mu N_{s, t}^n f - \mu N_{s, t}^{n+p} f\|_\infty &\leq \frac{2\sigma_0 J}{(1-J)^2} \log \left(1 + \frac{h}{n(1-2h)} \right) e^{\frac{3\sigma_0}{1-J}t} + \frac{2\sigma_0}{(1-J)^2} \int_s^t \|\mu N_{\tau, t}^n - \mu N_{\tau, t}^{n+p}\|_{TV} d\tau \\ &+ \frac{2\sigma_0}{1-J} \int_s^t \|N_{\tau, t}^n f - N_{\tau, t}^{n+p} f\|_\infty d\tau \end{aligned}$$

which gives by the Grönwall's lemma

$$\|\mu N_{s, t}^n f - \mu N_{s, t}^{n+p} f\|_\infty \leq \frac{2\sigma_0}{(1-J)^2} \left[J \log \left(1 + \frac{h}{n(1-2h)} \right) e^{\frac{3\sigma_0}{1-J}t} + \int_s^t \|\mu N_{\tau, t}^n - \mu N_{\tau, t}^{n+p}\|_{TV} d\tau \right] e^{\frac{2\sigma_0}{1-J}(t-s)}.$$

Finally

$$\begin{aligned} \|\mu N_{s, t}^n - \mu N_{s, t}^{n+p}\|_{TV} &\leq \sup_{\|f\|_\infty \leq 1} \|\mu N_{s, t}^n f - \mu N_{s, t}^{n+p} f\|_\infty \\ &\leq \frac{2\sigma_0 e^{\frac{2\sigma_0}{1-J}t}}{(1-J)^2} \left[J \log \left(1 + \frac{h}{n(1-2h)} \right) e^{\frac{3\sigma_0}{1-J}t} + \int_s^t \|\mu N_{\tau, t}^n - \mu N_{\tau, t}^{n+p}\|_{TV} d\tau \right] \end{aligned}$$

and by the Grönwall's lemma

$$\|\mu N_{s, t}^n - \mu N_{s, t}^{n+p}\|_{TV} \leq \frac{2\sigma_0 J}{(1-J)^2} \log \left(1 + \frac{h}{n(1-2h)} \right) \exp \left(\frac{5\sigma_0}{1-J}t + \frac{2\sigma_0 e^{\frac{2\sigma_0}{1-J}t}}{(1-J)^2}(t-s) \right).$$

We deduce that for all $t \in [0, T^*)$ the sequence $(\mu N_{0, t}^n)_{n \in \mathbb{N}^*} \subset \mathcal{P}([0, 1])$ is a Cauchy sequence, hence convergent to a limit $\mu_t \in \mathcal{P}([0, 1])$, and additionally

$$\sup_{0 \leq t < T^*} \|\mu N_{0, t}^n - \mu_t\|_{TV} \leq \frac{2\sigma_0 J}{(1-J)^2} \log \left(1 + \frac{h}{n(1-2h)} \right) \exp \left(\frac{5\sigma_0}{1-J}T^* + \frac{2\sigma_0 e^{\frac{2\sigma_0}{1-J}T^*}}{(1-J)^2}T^* \right).$$

This allows to pass to the limit in Lemma 5.4, and the proof is complete. \square

The last step consists in proving that any solution to Equation (1) satisfies a Duhamel formula.

Lemma 5.6. *Let $(\mu_t)_{0 \leq t < T}$ be a solution to Equation (1). Then for any $\sigma > 0$ the following Duhamel formula is verified*

$$\forall f \in C([0, 1]), \forall t \geq 0, \quad \mu_t f = \mu_0 M_t f + \int_0^t (\sigma(s) - \sigma) \mu_s \mathcal{B} M_{t-s} f ds \quad (14)$$

where $(M_t)_{t \geq 0}$ is the semigroup generated by \mathcal{A}_σ (see Section 3).

Proof. Using the semigroup $(M_t^n)_{t \geq 0}$ defined in Section 3.1 we have

$$\begin{aligned} \frac{d}{ds} \left(\int_0^s \mu_\tau M_{t-s}^n f d\tau \right) &= \mu_s M_{t-s}^n f - \int_0^s \mu_\tau \mathcal{A}^n M_{t-s}^n f d\tau \\ &= \mu_0 M_{t-s}^n f + \int_0^s \mu_\tau (\mathcal{A}_{\sigma(\tau)} - \mathcal{A}^n) M_{t-s}^n f d\tau \\ &= \mu_0 M_{t-s}^n f + \int_0^s \mu_\tau (\sigma(\tau) \mathcal{B} - \sigma \mathcal{B}^n) M_{t-s}^n f d\tau. \end{aligned}$$

Integrating between 0 and t we get

$$\int_0^t \mu_\tau f d\tau = \int_0^t \mu_0 M_s^n f ds + \int_0^t \int_0^s \mu_\tau (\sigma(\tau) \mathcal{B} - \sigma \mathcal{B}^n) M_{t-s}^n f d\tau ds.$$

Differentiating with respect to t we obtain (by using dominated convergence, Fubini's theorem, and a change of variable)

$$\begin{aligned} \mu_t f &= \mu_0 M_t^n f + \int_0^t \mu_r (\sigma(r) \mathcal{B} - \sigma \mathcal{B}^n) f dr + \int_0^t \int_0^s \mu_r (\sigma(r) \mathcal{B} - \sigma \mathcal{B}^n) \mathcal{A}^n M_{t-s}^n f dr ds \\ &= \mu_0 M_t^n f + \int_0^t \mu_r (\sigma(r) \mathcal{B} - \sigma \mathcal{B}^n) f dr + \int_0^t \mu_r (\sigma(r) \mathcal{B} - \sigma \mathcal{B}^n) \left(\int_r^t \mathcal{A}^n M_{t-s}^n f ds \right) dr \\ &= \mu_0 M_t^n f + \int_0^t \mu_r (\sigma(r) \mathcal{B} - \sigma \mathcal{B}^n) \left(f + \int_0^{t-r} \mathcal{A}^n M_s^n f ds \right) dr \\ &= \mu_0 M_t^n f + \int_0^t \mu_r (\sigma(r) \mathcal{B} - \sigma \mathcal{B}^n) M_{t-r}^n f dr \end{aligned}$$

and then passing to the limit $n \rightarrow \infty$

$$\mu_t f = \mu_0 M_t f + \int_0^t (\sigma(r) - \sigma) \mu_r \mathcal{B} M_{t-r} f dr.$$

□

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. With Lemma 5.5 we have proved for any $\mu_0 \in \mathcal{P}([0, 1])$ the existence of a local solution, on $[0, T^{**})$. But since T^{**} is independent of μ_0 , we can iterate the procedure to get a global solution. The uniqueness is a consequence of the Duhamel formula (14). Let $(\mu_t)_{0 \leq t < T}$ and $(\tilde{\mu}_t)_{0 \leq t < \tilde{T}}$ two solutions. Then for any $t \in [0, \min\{T, \tilde{T}\})$ we get from the Duhamel formula with $\sigma = \frac{\sigma_0}{1-J}$

$$\begin{aligned} \|\mu_t - \tilde{\mu}_t\|_{TV} &\leq \|\mu_0 - \tilde{\mu}_0\|_{TV} + 2 \int_0^t |\sigma(s) - \tilde{\sigma}(s)| ds + \frac{2\sigma_0}{1-J} \int_0^t \|\mu_s - \tilde{\mu}_s\|_{TV} ds \\ &\leq \|\mu_0 - \tilde{\mu}_0\|_{TV} + \frac{2\sigma_0}{(1-J)^2} \int_0^t \|\mu_s - \tilde{\mu}_s\|_{TV} ds \end{aligned}$$

and by the Grönwall's lemma

$$\|\mu_t - \tilde{\mu}_t\|_{TV} \leq \|\mu_0 - \tilde{\mu}_0\|_{TV} e^{\frac{2\sigma_0}{(1-J)^2} t}.$$

□

6 Global exponential stability when $J \ll 1$

The Duhamel formula (14) combined with the exponential contraction of $(M_t)_{t \geq 0}$ allows to get the uniqueness of a steady state and its global exponential stability for the nonlinear equation for J small enough. It is a consequence of the following proposition.

Proposition 6.1. *Let $(\mu_t)_{t \geq 0}$ be a measure solution to Equation (1). Then for all $t \geq 0$ we have*

$$\|\mu_t - \bar{\mu}\|_{TV} \leq \frac{e^{\omega t}}{1-c} \|\mu_0 - \bar{\mu}\|_{TV}$$

where $c = \frac{\sigma_0}{2} \left(\frac{h}{4}\right)^{\sigma_0}$ and $\omega = \frac{2\sigma_0 J}{(1-c)(1-J)^2} + \frac{\log(1-c)}{\log \frac{4}{h}}$.

Proof. The proof relies on the Duhamel formula (14). First for $\mu \in \mathcal{M}([0, 1])$ define the measure $\mu\mathcal{B}$ by

$$\forall f \in C([0, 1]), \quad (\mu\mathcal{B})f := \mu(\mathcal{B}f).$$

The conservation property $\mathcal{B}\mathbf{1} = 0$ ensures that for any $\mu \in \mathcal{M}([0, 1])$ we have $(\mu\mathcal{B})([0, 1]) = 0$. This allows to deduce from Proposition 3.10, using also that \mathcal{B} is bounded by 2, that for all $\mu \in \mathcal{M}([0, 1])$ and all $t \geq 0$

$$\|\mu\mathcal{B}M_t\|_{TV} \leq e^{-a(t-t_0)} \|\mu\mathcal{B}\|_{TV} \leq 2e^{-a(t-t_0)} \|\mu\|_{TV},$$

where $t_0 = \log \frac{4}{h}$ and $a = \frac{-\log(1-c)}{t_0}$. Using this inequality in the Duhamel formula (14) with $\sigma = \bar{\sigma}$ we get

$$\begin{aligned} \|\mu_t - \bar{\mu}\|_{TV} &\leq \|(\mu_0 - \bar{\mu})M_t\|_{TV} + \sigma_0 J \int_0^t \left| \frac{(\mu_s - \bar{\mu})([1-h, 1])}{(1-J\mu_s([1-h, 1]))(1-J\bar{\mu}([1-h, 1]))} \right| \|\mu_s \mathcal{B}M_{t-s}\|_{TV} ds \\ &\leq \|\mu_0 - \bar{\mu}\|_{TV} e^{-a(t-t_0)} + \frac{2\sigma_0 J}{(1-J)^2} \int_0^t \|\mu_s - \bar{\mu}\|_{TV} e^{-a(t-s-t_0)} ds. \end{aligned}$$

Denoting $\theta(t) = \|\mu_t - \bar{\mu}\|_{TV} e^{at}$ this also reads

$$\theta(t) \leq \|\mu_0 - \bar{\mu}\|_{TV} e^{at_0} + \frac{2\sigma_0 J e^{at_0}}{(1-J)^2} \int_0^t \theta(s) ds = \frac{1}{1-c} \|\mu_0 - \bar{\mu}\|_{TV} + \frac{2\sigma_0 J}{(1-c)(1-J)^2} \int_0^t \theta(s) ds$$

and the Grönwall's lemma ensures that

$$\theta(t) \leq \frac{e^{\frac{2\sigma_0 J}{(1-c)(1-J)^2} t}}{1-c} \|\mu_0 - \bar{\mu}\|_{TV}.$$

□

Proof of Theorem 2.4. From Proposition 6.1 we only have to check that if $J < (5 - 2\sqrt{6})\left(\frac{h}{4}\right)^{\sigma_0+1}$ then $\frac{2\sigma_0 J}{(1-c)(1-J)^2} < \frac{-\log(1-c)}{\log \frac{4}{h}}$.

Using that $\log(x) \leq x - 1$ we have

$$\frac{-(1-c)\log(1-c)}{2\sigma_0 \log \frac{4}{h}} \geq \frac{c(1-c)}{2\sigma_0 \log \frac{4}{h}} = \frac{1-c}{4 \log \frac{4}{h}} \left(\frac{h}{4}\right)^{\sigma_0} \geq \frac{1}{4} \left(\frac{h}{4}\right)^{\sigma_0+1} (1-c).$$

Since $J_0 := 5 - 2\sqrt{6} \in (0, 1)$ satisfies $\frac{J_0}{(1-J_0)^2} = \frac{1}{8}$ we have for all $J < J_0 \left(\frac{h}{4}\right)^{\sigma_0+1}$

$$\frac{J}{(1-J)^2} < \frac{1}{8} \left(\frac{h}{4}\right)^{\sigma_0+1}.$$

The conclusion follows from the bound

$$c \leq \frac{1}{2e \log \left(\frac{4}{h}\right)} \leq \frac{1}{6e \log 2} < \frac{1}{2}.$$

□

Conclusion

The mean-field model considered along this paper is a standard equation capturing the spiking population rate of a local neural circuit [16]. While not specifically a model of any particular brain region, it describes a population of self recurrent excitatory LIF neurons receiving stochastic Poisson spike trains. Although the mean-field equation (1) is widespread among physicists, it has received only little attention by mathematicians, and there is nowadays, no identified mathematical framework to study its solution properties, see [18, 19] for a first step in that direction. It has thus become necessary to investigate systematically the conditions under which the solution to the mean-field equation exists and to understand its stability properties.

In the mean-field limit, the level of recurrent excitation is control by a parameter J which reflects the average number of connexion per cell. Interestingly, this parameter plays a critical part in the emergence of a finite time blow-up of the solution [19]. This effect was first noticed in [17] for the perfect integrate-and-fire, observed numerically with leaky integrate-and-fire neurons [35, 36], and soon theoretically explained in [19] using similar ideas to [11]. There is extensive numerical evidence that the blow-up of the mean-field equation is nothing but the emergent synchrony patterns of firing across neurons.

An important result that has been proved in [17] is the existence and stability of a unique stationary state for a moderate coupling scenario, i.e. for moderate values of J , the connectivity parameter. When the average number of connexions is not too big ($J < 1$), the asynchronous state of a network of perfect integrate-and-fire neurons is stable. Our paper extends the stability property to networks with cells having a leaky membrane potential. Unfortunately, if we have been able to extend the existence of a steady state, the uniqueness and stability only hold for weak coupling ($J \ll 1$).

Mean-field equations have gain intensive visibility over the past decades, however, most of the work has been done with the diffusion approximation equation. Assuming h small enough, formal computations give:

$$p(t, v) - p(t, v - h) = h \frac{\partial}{\partial v} p(t, v) - \frac{h^2}{2} \frac{\partial^2}{\partial v^2} p(t, v) + o(h^2).$$

Plugging this second order approximation into the mean-field equation (1) leads to the diffusive PDE presented in [6, 5] and studied mathematically in a sequel of papers [11, 14, 12, 10]. Although the diffusion equation is more common in the literature - several textbooks dedicate a chapter to it [24, 3, 20, 25] - and has the advantage to offer a clear expression of the steady state, recent modeling discussions suggested that it is not an appropriate description for most neural networks [30]. In any case, it seems crucial to us to relate our theoretical findings to the mathematical results established for the diffusion approximation.

We first note that we get the same type of results for the stability of the steady state equation in the weakly coupled case ($J \ll 1$). A difference should nonetheless be noted, for the diffusion equation, the exponential stability is only local [14], while it is global in our case. Furthermore, with the diffusion equation, the global stability can not arise since it may blow up for a certain class of initial condition [11]. On the other hand, similar open issues hold for moderate coupling, where no precise conclusion can be formulated. Nonetheless, in the diffusion case, it can be shown that for some particular connectivity regimes, there exist at least two steady states. Numerically, both in the diffusion or non-diffusion scenario, the same steady state is always observed, suggesting that there is only one stable fixed point. Note that for strong coupling, for both mean-field equations, the steady-state does not exist, and obviously its stability property is not an issue [11, 19].

Probably, the most straightforward discussion that we should be having is about the stability and uniqueness of the steady state for moderate coupling. While the existence of a unique stable steady state has been addressed for an excitatory network of non leaky cells [17], it is still an open issue for the LIF. Another important discussion should address the discontinuous mechanism proposed by [17] to restart the flow of the solution after the blow-up. While there is no intuitive difficulties in proposing a similar discontinuous mapping for the PDE considered along this paper, defining a solution at the blow-up time and extending it beyond the blow-up is not a trivial task and it will be the subject of a new research.

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