



**HAL**  
open science

# A note on toric degeneration of a Bott-Samelson-Demazure-Hansen variety

Narasimha Chary Bonala

► **To cite this version:**

Narasimha Chary Bonala. A note on toric degeneration of a Bott-Samelson-Demazure-Hansen variety. 2017. hal-01617037

**HAL Id: hal-01617037**

**<https://hal.science/hal-01617037v1>**

Preprint submitted on 16 Oct 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A NOTE ON TORIC DEGENERATION OF A BOTT-SAMELSON-DEMAZURE-HANSEN VARIETY

B. NARASIMHA CHARY

ABSTRACT. In this paper we study the geometry of toric degeneration of a Bott-Samelson-Demazure-Hansen (BSDH) variety, which was algebraically constructed in [Pas10] and [PK16]. We give some applications to BSDH varieties. Precisely, we classify Fano, weak Fano and log Fano BSDH varieties and their toric limits in Kac-Moody setting. We prove some vanishing theorems for the cohomology of tangent bundle (and line bundles) on BSDH varieties. We also recover the results in [PK16], by toric methods.

**Keywords:** Bott-Samelson-Demazure-Hansen varieties, canonical line bundle, tangent bundle and toric varieties.

## 1. INTRODUCTION

Bott-Samelson-Demazure-Hansen (for short, BSDH) varieties are natural desingularizations of Schubert varieties in the flag varieties. These were algebraically constructed by M. Demazure and H.C. Hansen independently by adapting a differential geometric approach from the paper of Bott and Samelson (see [BS58], [Dem74] and [Han73]). Briefly, the BSDH varieties are iterated projective line bundles, given by factoring the Schubert variety using Bruhat decomposition. These varieties depend on the given expression of the Weyl group element corresponding to the Schubert variety (see for instance [CKP15, Page 32]). We also see in this paper some properties of these varieties which depend on the given expression.

In [GK94], M. Grossberg and Y. Karshon constructed toric degenerations of BSDH varieties by complex geometric methods. In [Pas10] B. Pasquier and in [PK16] A.J. Parameswaran and P. Karuppuchamy constructed these toric degenerations algebraically. B. Pasquier used these degenerations to study the cohomology of line bundles on BSDH varieties (see [Pas10]). In [PK16], the authors studied the limiting toric variety for a simple simply connected algebraic group by geometric methods. In this paper we study the limiting toric variety of a BSDH variety in more detail by methods of toric geometry and we prove some applications to BSDH varieties. We also recover the results in [PK16] and extend them to the Kac-Moody setting. The key idea for many results in this article is that the toric limit is a ‘Bott tower’. These are studied in [Cha] and some of their properties can be transferred to BSDH varieties by using the semi-continuity theorem.

---

The author is supported by AGIR Pole MSTIC project run by the University of Grenoble Alpes, France.

Let  $G$  be a Kac-Moody group over the field of complex numbers (for the definition see [Kum12]). Let  $B$  be a Borel subgroup containing a fixed maximal torus  $T$ . Let  $W$  be the Weyl group corresponding to the pair  $(G, B, T)$  and let  $w \in W$ . Let  $\tilde{w} := s_{\beta_1} \cdots s_{\beta_n}$  be an expression (possibly non-reduced) of  $w$  in simple reflections and let  $Z(\tilde{w})$  be the BSDH variety corresponding to  $\tilde{w}$  (see Section 2). Let  $Y_{\tilde{w}}$  be the toric limit of  $Z(\tilde{w})$  constructed as in [Pas10] and [PK16] (see Section 3). We see that  $Y_{\tilde{w}}$  is a Bott tower, the iterated  $\mathbb{P}^1$ -bundle over a point  $\{pt\}$  where each  $\mathbb{P}^1$ -bundle is the projectivization of a rank 2 decomposable vector bundle (see Corollary 4.4). We prove that the ample cone  $Amp(Y_{\tilde{w}})$  of  $Y_{\tilde{w}}$  can be identified with a subcone of the ample cone  $Amp(Z(\tilde{w}))$  of  $Z(\tilde{w})$  (see Corollary 5.1).

Recall that a smooth projective variety  $X$  is called *Fano* (respectively, *weak Fano*) if its anti-canonical divisor  $-K_X$  is ample (respectively, *nef* and *big*). Following [AS14], we say that a pair  $(X, D)$  of a normal projective variety  $X$  and an effective  $\mathbb{Q}$ -divisor  $D$  is *log Fano* if it is Kawamata log terminal and  $-(K_X + D)$  is ample.

When  $G$  is a simple algebraic group and the expression  $\tilde{w}$  is reduced, Fanoness and weak Fanoness of the BSDH variety  $Z(\tilde{w})$  are considered in [Cha17]. Here we have the following results in Kac-Moody setting. Let  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_i} \cdots s_{\beta_j} \cdots s_{\beta_r}$  be an expression (remember that  $\beta_k$ 's are simple roots). Let  $\beta_{ij} := \langle \beta_j, \check{\beta}_i \rangle$ , where  $\check{\beta}_i$  is the co-root of  $\beta_i$ . Now we define some conditions on the expression  $\tilde{w}$  (see Section 6 for examples and also see [Cha, Section 1]). Define for  $1 \leq i \leq r$ ,

$$\eta_i^+ := \{r \geq j > i : \beta_{ij} > 0\} \text{ and } \eta_i^- := \{r \geq j > i : \beta_{ij} < 0\}.$$

If  $|\eta_i^+| = 1$  (respectively,  $|\eta_i^+| = 2$ ), then let  $\eta_i^+ = \{m\}$  (respectively,  $\eta_i^+ = \{m_1, m_2\}$ ). If  $|\eta_i^-| = 1$  (respectively,  $|\eta_i^-| = 2$ ), then set  $\eta_i^- = \{l\}$  (respectively,  $\eta_i^- = \{l_1, l_2\}$ ).

- $N_i^1$  is the condition that
  - (i)  $|\eta_i^+| = 0$ ,  $|\eta_i^-| \leq 1$ , and if  $|\eta_i^-| = 1$  then  $\beta_{il} = -1$ ; or
  - (ii)  $|\eta_i^-| = 0$ ,  $|\eta_i^+| \leq 1$ , and if  $|\eta_i^+| = 1$  then  $\beta_{im} = 1$  and  $\beta_{mk} = 0$  for all  $k > m$ .
- $N_i^2$  is the condition that
  - Case 1: Assume that  $|\eta_i^+| = 0$ . Then  $|\eta_i^-| \leq 2$ , and if  $|\eta_i^-| = 1$  (respectively,  $|\eta_i^-| = 2$ ) then  $\beta_{li} = -1$  or  $-2$  (respectively,  $\beta_{il_1} = -1 = \beta_{il_2}$ ).
  - Case 2: If  $|\eta_i^-| = 1 = |\eta_i^+|$  and  $l < m$ , then  $\beta_{il} = -1$ ,  $\beta_{im} = 1$  and  $\beta_{mk} = 0$  for all  $k > m$ .
  - Case 3: Assume that  $|\eta_i^+| = 1$ . Then  $\beta_{im} = 1$  and either it satisfies
    - (i) Case 2; or
    - (ii)  $|\eta_i^-| = 0$  and  $\beta_{mk} = 0$  for all  $k > m$ ; or
    - (iii) there exists a unique  $r \geq s > m$  such that  $\beta_{ms} - \beta_{is} = 1$  and  $\beta_{mk} - \beta_{ik} = 0$  for all  $k > s$ ; or  $\beta_{ms} - \beta_{is} = -1$  and  $\beta_{is} - \beta_{ms} - \beta_{sk} = 0$  for all  $k > s$ .

**Definition 1.1.** *We say the expression  $\tilde{w}$  satisfies condition I (respectively, condition II) if  $N_i^1$  (respectively,  $N_i^2$ ) holds for all  $1 \leq i \leq r$ . Note that  $N_i^1 \implies N_i^2$  for all  $1 \leq i \leq r$ .*

**Theorem** (See Lemma 6.1 and Theorem 6.3).

- (1) *If  $\tilde{w}$  satisfies I, then  $Y_{\tilde{w}}$  and  $Z(\tilde{w})$  are Fano.*
- (2) *If  $\tilde{w}$  satisfies II, then  $Y_{\tilde{w}}$  and  $Z(\tilde{w})$  are weak Fano.*

In [CKP15] and [CK17], we have obtained some vanishing results for the cohomology of tangent bundle of  $Z(\tilde{w})$ , when  $G$  is finite dimensional and  $\tilde{w}$  is reduced (see [CKP15, Section 3] and [CK17, Theorem 8.1]). The case  $\tilde{w}$  is non-reduced is considered in [CKP]. Here we get some vanishing results in Kac-Moody setting. Let  $T_{Z(\tilde{w})}$  denote the tangent bundle of  $Z(\tilde{w})$ .

**Corollary** (see Corollary 6.6). *If  $\tilde{w}$  satisfies I, then  $H^i(Z(\tilde{w}), T_{Z(\tilde{w})}) = 0$  for all  $i \geq 1$ . In particular,  $Z(\tilde{w})$  is locally rigid.*

In [AS14], D. Anderson and A. Stapledon studied the log Fano-ness of Schubert varieties, and in [And14], log Fano-ness of BSDH varieties is studied for chosen divisors. Let  $D$  be a divisor in  $Z(\tilde{w})$  with support in the boundary of  $Z(\tilde{w})$ . For  $1 \leq i \leq r$ , we define some constants  $f_i$  which again depend on the given expression  $\tilde{w}$  (for more details see Section 6).

**Corollary** (see Corollary 6.7). *The pair  $(Z(\tilde{w}), D)$  is log Fano if  $f_i > 0$  for all  $1 \leq i \leq r$ .*

The article is organized as follows: In section 2, we recall the construction of BSDH varieties. In Section 3, we give the algebraic construction of toric degeneration of a BSDH variety. In Section 4, we describe the limiting toric variety as an iterated  $\mathbb{P}^1$ -bundle. In Section 5, we see some vanishing results of cohomology of line bundles on BSDH varieties. Section 6 contains the results on Fano, weak Fano and log Fano properties of BSDH varieties and their toric limits. We also study the vanishing results on cohomology of tangent bundle on BSDH varieties. In Section 7, we recover the results in [PK16] by toric methods.

## 2. PRELIMINARIES

In this section we recall the construction of Bott-Samelson-Demazure-Hansen varieties (see [BK07] and [Kum12]) and we recall some definitions in toric geometry which are used in this article (for more details on toric varieties see [CLS11] and also [Ful93]). We work over the field of complex numbers throughout.

**2.1. BSDH varieties.** Let  $A = (a_{ij})_{1 \leq i, j \leq n}$  be a generalized Cartan matrix. Let  $G$  be the Kac-Moody group associated to  $A$  (see [Kum12, Chapter IV]). Fix a maximal torus  $T$  and a Borel subgroup  $B$  containing  $T$ . Let  $S := \{\alpha_1, \dots, \alpha_n\}$  be the set of all simple roots of  $(G, B, T)$ . We denote  $s_{\alpha_i}$  the simple reflection corresponding to  $\alpha_i$ . Note that the Weyl group  $W$  of  $G$  is generated by

$$\{s_{\alpha_i} : 1 \leq i \leq n\}.$$

Let  $w \in W$ , an expression  $\tilde{w}$  of  $w$  is a sequence  $(s_{\beta_1}, \dots, s_{\beta_r})$  of simple reflections  $s_{\beta_1}, \dots, s_{\beta_r}$  such that  $w = s_{\beta_1} \cdots s_{\beta_r}$ . An expression  $\tilde{w}$  of  $w$  is said to be reduced if the number  $r$  of simple reflections is minimal. In such case we call  $r$  the length of  $w$ . By abuse of notation, we also denote the expression  $\tilde{w}$  by  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$ . For  $\alpha \in S$ , we denote  $P_\alpha$ , the minimal parabolic subgroup of  $G$  generated by  $B$  and a representative of  $s_\alpha$ .

**Definition 2.1.** Let  $w \in W$  and  $\tilde{w} := s_{\beta_1} \cdots s_{\beta_r}$  be an expression (not necessarily reduced) of  $w$ . The Bott-Samelson-Demazure-Hansen (for short, BSDH) variety corresponding to  $\tilde{w}$  is

$$Z(\tilde{w}) := P_{\beta_1} \times \cdots \times P_{\beta_r} / B^r,$$

where the action of  $B^r$  on  $P_{\beta_1} \times \cdots \times P_{\beta_r}$  is defined by

$$(p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r) \text{ for all } p_i \in P_{\beta_i}, b_i \in B.$$

These are smooth projective varieties of dimension  $r$ . There is a natural morphism  $\phi_{\tilde{w}} : Z(\tilde{w}) \rightarrow G/B$  defined by

$$[(p_1, \dots, p_r)] \mapsto p_1 \cdots p_r B.$$

If  $\tilde{w}$  is reduced, the BSDH variety  $Z(\tilde{w})$  is a natural desingularization of the Schubert variety, the  $B$ -orbit closure of  $wB/B$  in  $G/B$  (see [Dem74], [Han73] and [Kum12, Chapter VIII]). We can also construct the BSDH variety as an iterated  $\mathbb{P}^1$ -bundles. Let  $\tilde{w}' := s_{\beta_1} \cdots s_{\beta_{r-1}}$ . Let  $f : G/B \rightarrow G/P_{\beta_r}$  be the map given by  $gB \mapsto gP_{\beta_r}$  and let  $p : Z(\tilde{w}') \rightarrow G/P_{\beta_r}$  be the map given by  $[(p_1, \dots, p_{r-1})] \mapsto p_1 \cdots p_{r-1} P_{\beta_r}$ . Then we have the following cartesian diagram (see [BK07, Page 66] and [Kum12, Chapter VII]):

$$\begin{array}{ccc} Z(\tilde{w}) = Z(\tilde{w}') \times_{G/P_{\beta_r}} G/B & \xrightarrow{\phi_{\tilde{w}}} & G/B \\ \downarrow f_{\tilde{w}} & & \downarrow f \\ Z(\tilde{w}') & \xrightarrow{p} & G/P_{\beta_r} \end{array}$$

Note that  $f_{\tilde{w}}$  is a  $\mathbb{P}^1$ -fibration and the relative tangent bundle  $T_{f_{\tilde{w}}}$  of  $f_{\tilde{w}}$  is  $\phi_{\tilde{w}}^*(\mathcal{L}_{\beta_r})$ , where  $\mathcal{L}_{\beta_r}$  is the homogeneous line bundle on  $G/B$  corresponding to  $\beta_r$ . Using the cohomology of the relative tangent bundle  $T_{f_{\tilde{w}}}$  we studied the cohomology of the tangent bundle of  $Z(\tilde{w})$ , when  $G$  is finite dimensional and  $\tilde{w}$  is a reduced expression (see [CKP15] and [CK17]). The fibration  $f_{\tilde{w}}$  comes with a natural section  $\sigma_{\tilde{w}} : Z(\tilde{w}') \rightarrow Z(\tilde{w})$  induced by the projection

$$P_{\beta_1} \times \cdots \times P_{\beta_r} \rightarrow P_{\beta_1} \times \cdots \times P_{\beta_{r-1}}.$$

For the toric limits we get two natural sections, as will be explained in Section 3. For all  $i \in \{1, \dots, r\}$ , we denote  $Z_i$ , the divisor in  $Z(\tilde{w})$  defined by

$$\{[(p_1, \dots, p_r)] \in Z(\tilde{w}) : p_i \in B\}.$$

In [LT04], N. Lauritzen and J.F. Thomsen proved that  $Z'_i$ 's forms a basis of the Picard group of  $Z(\tilde{w})$  and they also proved that if  $\tilde{w}$  is a reduced expression these form a basis of the monoid of effective divisors (see [LT04, Proposition 3.5]). Recently, the effective divisors of  $Z(\tilde{w})$  for  $\tilde{w}$  non-reduced case have been considered in [And14].

The BSDH variety can be described also as an iterated projective line bundle, where each projective bundle is the projectivization of certain rank 2 vector bundle (not necessarily decomposable). In Section 4, we see the toric degeneration of a BSDH variety (constructed in Section 3) is a Bott tower, the iterated  $\mathbb{P}^1$ -bundle over a point  $\{pt\}$ , where each  $\mathbb{P}^1$ -bundle is the projectivization of a rank 2 decomposable vector bundle.

## 2.2. Toric varieties.

**Definition 2.2.** *A normal variety  $X$  is called a toric variety (of dimension  $n$ ) if it contains an  $n$ -dimensional torus  $T$  (i.e.  $T = (\mathbb{C}^*)^n$ ) as a Zariski open subset such that the action of the torus on itself by multiplication extends to an action of the torus on  $X$ .*

Toric varieties are completely described by the combinatorics of the corresponding fans. We denote the fan corresponding to a toric variety by  $\Sigma$  and the collection of cones of dimension  $s$  in  $\Sigma$  by  $\Sigma(s)$  for  $1 \leq s \leq n$ . For each cone  $\sigma \in \Sigma$ , we denote  $V(\sigma)$ , the orbit closure of the orbit corresponding to cone  $\sigma$ . For each  $\sigma \in \Sigma$ ,  $\sigma(1) := \sigma \cap \Sigma(1)$ . For each  $\rho \in \Sigma(1)$ , we can associate a divisor in  $X$ , we denote it by  $D_\rho$  (see [CLS11, Chapter 4] for more details). We recall the following:

### Definition 2.3.

- (1) *We say  $P \subset \Sigma(1)$  is a primitive collection if  $P$  is not contained in  $\sigma(1)$  for some  $\sigma \in \Sigma$  but any proper subset is. Note that if  $\Sigma$  is simplicial, primitive collection means that  $P$  does not generate a cone in  $\Sigma$  but every proper subset does.*
- (2) *Let  $P = \{\rho_1, \dots, \rho_k\}$  be a primitive collection in a complete simplicial fan  $\Sigma$ . Recall  $u_\rho$  be the primitive vector of the ray  $\rho \in \Sigma$ . Then  $\sum_{i=1}^k u_{\rho_i}$  is in the relative interior of a cone  $\gamma_P$  in  $\Sigma$  with a unique expression*

$$\sum_{i=1}^k u_{\rho_i} - \left( \sum_{\rho \in \gamma_P(1)} c_\rho u_\rho \right) = 0. \tag{2.1}$$

where  $c_\rho \in \mathbb{Q}_{>0}$ . Then we call (2.1) the primitive relation of  $X$  corresponding to  $P$ .

- (3) *For a primitive relation  $P$ , we can associate an element  $r(P)$  in  $N_1(X)$ , where  $N_1(X)$  is the real vector space of numerical classes of one-cycles in  $X$  (see [CLS11, Page 305]).*

## 3. TORIC DEGENERATION OF A BSDH VARIETY

In [GK94], toric degenerations of BSDH varieties were constructed by complex geometric methods. In [Pas10] and [PK16] they have given an algebraic construction for toric degeneration of a BSDH variety. We recall the algebraic construction here.

Note that the simple roots are linearly independent elements in the character group of  $G$ . Let  $N$  be the lattice of one-parameter subgroups of  $T$ . We can choose a positive integer  $q$  and an injective morphism  $\lambda : \mathbb{G}_m \rightarrow T$  (i.e.  $\lambda \in N$  and  $\lambda$  is injective) such that for all  $1 \leq i \leq n$  and  $u \in \mathbb{G}_m$ ,  $\alpha_i(\lambda(u)) = u^q$  (see [Pas10, Page 2836]). When  $G$  is finite dimensional, for each one-parameter subgroup  $\lambda \in N$ , define

$$P(\lambda) := \{g \in G : \lim_{u \rightarrow 0} \lambda(u)g\lambda(u)^{-1} \text{ exists in } G\}.$$

The set  $P(\lambda)$  is a parabolic subgroup and the unipotent radical  $R_u(P(\lambda))$  of  $P(\lambda)$  is given by

$$R_u(P(\lambda)) = \{g \in G : \lim_{u \rightarrow 0} \lambda(u)g\lambda(u)^{-1} \text{ is identity in } G\}.$$

Any parabolic subgroup of  $G$  is of this form (see [Spr10, Proposition 8.4.5]). Choose a one-parameter subgroup  $\lambda \in N$  such that the corresponding parabolic subgroup is  $B$ . Let us define an endomorphism of  $G$  for all  $u \in \mathbb{G}_m$  by

$$\tilde{\Psi}_u : G \rightarrow G, \quad g \mapsto \lambda(u)g\lambda(u)^{-1}.$$

Let  $\mathcal{B}$  be the set of all endomorphisms of  $B$ . Now define a morphism

$$\Psi : \mathbb{G}_m \rightarrow \mathcal{B} \text{ by } u \mapsto \tilde{\Psi}_u|_B.$$

This map can be extended to 0 and for all  $x \in U$ ,  $\Psi_u|_B(x)$  goes to identity when  $u$  goes to zero. Let  $\mathbb{A}^1 := \text{Spec}\mathbb{C}[t]$  be the affine line over  $\mathbb{C}$ . We denote for all  $u \in \mathbb{A}^1$ ,  $\Psi_u$  the image of  $u$  in  $\mathcal{B}$ . Note that  $\Psi_u$  is the identity on  $T$  and  $\Psi_0$  is the projection from  $B$  to  $T$ . Let  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$  be an expression.

**Definition 3.1.**

(i) Let  $\mathcal{X}$  be the variety defined by

$$\mathcal{X} := \mathbb{A}^1 \times P_{\beta_1} \times \cdots \times P_{\beta_r} / B^r,$$

where the action of  $B^r$  on  $\mathbb{A}^1 \times P_{\beta_1} \times \cdots \times P_{\beta_r}$  is given by

$$(u, p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (u, p_1 b_1, \Psi_u(b_1)^{-1} p_2 b_2, \dots, \Psi_u(b_{r-1})^{-1} p_r b_r).$$

(ii) For all  $i \in \{1, \dots, r\}$ , we denote  $\mathcal{Z}_i$  the divisor in  $\mathcal{X}$  defined by

$$\{(u, p_1, \dots, p_r) \in \mathcal{X} : p_i \in B\}.$$

Note that  $\mathcal{X}$  and  $\mathcal{Z}_i$ 's are integral. Let  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  be the projection onto the first factor. Then we have the following theorem (see [Pas10, Proposition 1.3 and 1.4] and [PK16, Theorem 9]).

**Theorem 3.2.**

- (1)  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism.
- (2) For all  $u \in \mathbb{A}^1 \setminus \{0\}$ , the fiber  $\pi^{-1}(u)$  is isomorphic to the BSDH variety  $Z(\tilde{w})$  such that  $\pi^{-1}(u) \cap \mathcal{Z}_i$  corresponds to the divisor  $Z_i$  in  $Z(\tilde{w})$ .
- (3)  $\pi^{-1}(0)$  is a smooth projective toric variety.

We denote  $\mathcal{X}_u := \pi^{-1}(u)$  for  $u \in \mathbb{A}^1$  and the limiting toric variety  $\mathcal{X}_0 = \pi^{-1}(0)$  by  $Y_{\tilde{w}}$ .

#### 4. CONNECTION TO BOTT TOWERS

In this section we describe the toric limit  $Y_{\tilde{w}}$  as an iterated  $\mathbb{P}^1$ -bundle. We also recall some results on Bott towers from [Cha]. Let  $\{e_1^+, \dots, e_r^+\}$  be the standard basis of the lattice  $\mathbb{Z}^r$ . Define for all  $i \in \{1, \dots, r\}$ ,

$$e_i^- := -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+, \tag{4.1}$$

where  $\beta_{ij} := \langle \beta_j, \check{\beta}_i \rangle$ . The following proposition will give the description of the fan of the toric variety  $Y_{\tilde{w}}$  (see [Pas10, Proposition 1.4]).

**Proposition 4.1.**

- (1) *The fan  $\Sigma$  of the smooth toric variety  $Y_{\tilde{w}}$  consists of the cones generated by subsets of*

$$\{e_1^+, \dots, e_r^+, e_1^-, \dots, e_r^-\}$$

*and containing no subset of the form  $\{e_i^+, e_i^-\}$ .*

- (2) *For all  $i \in \{1, \dots, r\}$ ,  $\mathcal{Z}_i^0$  is the irreducible  $(\mathbb{C}^*)^r$ -stable divisor in  $Y_{\tilde{w}}$  corresponding to the one-dimensional cone of  $\Sigma$  generated by  $e_i^+$  and these form a basis of the divisor class group of  $Y_{\tilde{w}}$ .*

Note that the maximal cones of  $\Sigma$  are generated by  $\{e_i^\epsilon : 1 \leq i \leq r, \epsilon \in \{+, -\}\}$ . We denote the divisor corresponding to the one-dimensional cone  $\rho_i^\epsilon$  generated by  $e_i^\epsilon$  by  $D_{\rho_i^\epsilon}$  for  $\epsilon \in \{+, -\}$ . Let  $\tilde{w}' := s_{\beta_1} \cdots s_{\beta_{r-1}}$ . Then we get a toric morphism  $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$  induced by the lattice map  $\bar{f}_r : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1}$ , the projection onto the first  $r-1$  coordinates. We prove,

**Lemma 4.2.**

- (1)  *$f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$  is a toric  $\mathbb{P}^1$ -fibration with two disjoint toric sections.*  
 (2)  *$Y_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{Y_{\tilde{w}'}} \oplus \mathcal{L})$  for some unique line bundle  $\mathcal{L}$  on  $Y_{\tilde{w}'}$ .*

*Proof.* Let  $\Sigma'$  be the fan corresponding to the toric variety  $Y_{\tilde{w}'}$ . From the above proposition, we can see that  $\Sigma$  has a splitting by  $\Sigma'$  and  $\{e_r^+, 0, e_r^-\}$ . Then by [CLS11, Theorem 3.3.19],

$$f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$$

is a locally trivial fibration with the fan  $\Sigma_F$  of the fiber being  $\{e_r^+, 0, e_r^-\}$ . Since  $\Sigma_F$  is the fan of the projective line  $\mathbb{P}^1$ , we conclude  $f_r$  is a toric  $\mathbb{P}^1$ -fibration. As toric sections of the toric fibration correspond to the maximal cones in  $\Sigma_F$ , we get two disjoint toric sections for  $f_r$ . This proves (1).

Proof of (2): Since  $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$  is  $\mathbb{P}^1$ -fibration with a section, we see  $Y_{\tilde{w}}$  is a projective bundle  $\mathbb{P}(\mathcal{E})$  over  $Y_{\tilde{w}'}$  corresponding to a rank 2 vector bundle  $\mathcal{E}$  on  $Y_{\tilde{w}'}$  (see for example [Har77, Chapter V, Proposition 2.2, page 370]).

Recall that the sections of projective bundle  $Y_{\tilde{w}} = \mathbb{P}(\mathcal{E})$  correspond to the quotient line bundles of  $\mathcal{E}$  (see [Har77, Proposition 7.12]). Since  $Y_{\tilde{w}} = \mathbb{P}(\mathcal{E})$  is projective line bundle on  $Y_{\tilde{w}'}$  with two disjoint sections, we see  $\mathcal{E}$  is decomposable as a direct sum of line bundles on  $Y_{\tilde{w}'}$ .

As

$$\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{L}' \otimes \mathcal{E})$$

for any line bundle  $\mathcal{L}'$  on  $Y_{\tilde{w}'}$  (see [Har77, Lemma 7.9]), we can assume without loss of generality

$$\mathcal{E} = \mathcal{O}_{Y_{\tilde{w}'}} \oplus \mathcal{L}$$

for some unique line bundle  $\mathcal{L}$  on  $Y_{\tilde{w}'}$ . Hence  $Y_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{Y_{\tilde{w}'}} \oplus \mathcal{L})$  and this completes the proof of the lemma.  $\square$



**Definition 4.3.** A Bott tower of height  $r$  is a sequence of projective bundles

$$Y_r \xrightarrow{\pi_r} Y_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} Y_1 = \mathbb{P}^1 \xrightarrow{\pi_1} Y_0 = \{pt\},$$

where  $Y_i = \mathbb{P}(\mathcal{O}_{Y_{i-1}} \oplus \mathcal{L}_{i-1})$  for a line bundle  $\mathcal{L}_{i-1}$  over  $Y_{i-1}$  for all  $1 \leq i \leq r$  and  $\mathbb{P}(-)$  denotes the projectivization (see for more details [Civ05] and also [Cha, Section 2]).

Then by definition of Bott tower and by Lemma 4.2(2) we get:

**Corollary 4.4.** The toric limit  $Y_{\tilde{w}}$  is a Bott tower.

We have the following situation:

$$\begin{array}{ccc} \mathbb{P}^1 & & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ Z(\tilde{w}) & \xrightarrow{\quad \quad \quad} & Y_{\tilde{w}} \\ \downarrow f_{\tilde{w}} & & \downarrow f_r \\ Z(\tilde{w}') & \xrightarrow{\quad \quad \quad} & Y_{\tilde{w}'} \end{array}$$

(Red arrows indicate maps from  $Z(\tilde{w})$  to  $Z(\tilde{w}')$  and from  $Y_{\tilde{w}}$  to  $Y_{\tilde{w}'}$ .)

Recall that the Bott towers bijectively correspond to the upper triangular matrices with integer entries (see [Civ05, Section 3]). Here the upper triangular matrix  $M_{\tilde{w}}$  corresponding to  $Y_{\tilde{w}}$  is given by

$$M_{\tilde{w}} = \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2r} \\ 0 & 0 & 1 & \cdots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}_{r \times r},$$

where  $\beta_{ij}$ 's are integers as defined before. Let  $P_i := \{\rho_i^+, \rho_i^-\}$  for  $1 \leq i \leq r$ . Then by [Cha, Lemma 4.3],  $\{P_i : 1 \leq i \leq r\}$  is the set of all primitive collections of  $Y_{\tilde{w}}$ . For each  $1 \leq i \leq r$ , we denote the cone in the definition of primitive relation (see Section 2) corresponding to  $P_i$  by  $\gamma_{P_i}$ . Let  $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$  be a toric divisor in  $Y_{\tilde{w}}$  with  $a_\rho \in \mathbb{Z}$  and for  $1 \leq i \leq r$ , define

$$d_i := (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}).$$

Then we recall the following from [Cha, Lemma 5.1]:

**Lemma 4.5.**

- (1)  $D$  is ample if and only if  $d_i > 0$  for all  $1 \leq i \leq r$ .
- (2)  $D$  is numerically effective (nef) if and only if  $d_i \geq 0$  for all  $1 \leq i \leq r$ .

Also note that the conditions *I* and *II* on  $\tilde{w}$  are same as the conditions on  $M_{\tilde{w}}$  as in [Cha].

5. VANISHING RESULTS ON COHOMOLOGY OF CERTAIN LINE BUNDLES ON BSDH VARIETIES

Let  $X$  be a smooth projective variety. Recall  $N^1(X)$  denote the real finite dimensional vector space of numerical classes of real divisors in  $X$  (see [Kle66, §1, Chapter IV]). The ample cone  $\text{Amp}(X)$  of  $X$  is the cone in  $N^1(X)$  generated by classes of ample divisors.

**5.1. Ample cone of the toric limit of BSDH variety.** In [LT04], the ampleness of line bundles on BSDH variety  $Z(\tilde{w})$  is studied. Now we compare the ample cone of the toric limit  $Y_{\tilde{w}}$  with that of the BSDH-variety  $Z(\tilde{w})$  as a consequence of Theorem 3.2.

**Corollary 5.1.** *The ample cone  $\text{Amp}(Y_{\tilde{w}})$  of  $Y_{\tilde{w}}$  can be identified with a subcone of the ample cone  $\text{Amp}(Z(\tilde{w}))$  of  $Z(\tilde{w})$ .*

*Proof.* By Theorem 3.2,  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism with  $\mathcal{X}_0 = Y_{\tilde{w}}$  and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \neq 0$ . Let  $\mathcal{L} = \{\mathcal{L}_u : u \in \mathbb{A}^1\}$  be a line bundle on  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  with  $\mathcal{L}_0$  is an ample line bundle on  $Y_{\tilde{w}}$ . Note that the ampleness of line bundle is an open condition for the proper morphism  $\pi$ , i.e. there exists an open subset  $U$  in  $\mathbb{A}^1$  containing 0 such that  $\mathcal{L}_u$  is an ample line bundle on  $\mathcal{X}_u$  for all  $u \in U$  (see [Laz04, Theorem 1.2.17]). Hence we can identify  $\text{Amp}(Y_{\tilde{w}})$  with a subcone of  $\text{Amp}(Z(\tilde{w}))$ .  $\square$

**5.2. Vanishing results.** In [Pas10], B. Pasquier obtained vanishing theorems for the cohomology of certain line bundles on BSDH varieties, by using combinatorics of the toric limit (see [Pas10, Theorem 0.1]). Here we see some vanishing results for the cohomology of certain line bundles on BSDH varieties. Let  $1 \leq i \leq r$ , define  $h_i^{i-1} := -\beta_{i(i-1)}$  and

$$h_i^j := \begin{cases} 0 & \text{for } j > i. \\ 1 & \text{for } j = i. \\ -\sum_{k=j}^{i-1} \beta_{ik}(h_k^j) & \text{for } j < i. \end{cases}$$

Let  $\epsilon \in \{+, -\}$ . Define  $\Sigma(1)^\epsilon := \{\rho_i^\epsilon : 1 \leq i \leq r\}$ . Then we can write a toric divisor  $D$  in  $Y_{\tilde{w}}$  as follows:

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho = \sum_{\rho \in \Sigma(1)^+} a_\rho D_\rho + \sum_{\rho \in \Sigma(1)^-} a_\rho D_\rho.$$

For  $1 \leq i \leq r$ , let

$$g_i := a_{\rho_i^+} + \sum_{j=i}^r a_{\rho_j^-} h_j^i.$$

Recall  $d_i$  from Section 4,

$$d_i := (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}),$$

Let  $D' = \sum_{i=1}^r g_i Z_i$  be a divisor in  $Z(\tilde{w})$ , where  $Z_i$  is as in Section 2 for  $1 \leq i \leq r$ .

**Lemma 5.2.** *If  $d_i \geq 0$  for all  $1 \leq i \leq r$ , then  $H^j(Z(\tilde{w}), D') = 0$  for all  $j > 0$ .*

*Proof.* If  $d_i \geq 0$  for all  $1 \leq i \leq r$ , by Lemma 4.5,  $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$  is a *nef* divisor in  $Y_{\tilde{w}}$ . Then we have

$$H^j(Y_{\tilde{w}}, \sum_{\rho \in \Sigma(1)} a_\rho D_\rho) = 0 \text{ for all } j > 0 \quad (5.1)$$

(see [CLS11, Theorem 9.2.3, page 410] or [Oda88, Theorem 2.7, page 77]). Recall that by Theorem 3.2, we have

$$\mathcal{Z}_i^x = Z_i \text{ for } 0 \neq x \in k \text{ and } \mathcal{Z}_i^0 = D_{\rho_i^+}.$$

By [Cha, Corollary 3.3], we can write

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \sim \sum_{i=1}^r g_i D_{\rho_i^+}.$$

Hence by (5.1), Theorem 3.2 and by semi-continuity theorem (see [Har77, Theorem 12.8]), we get

$$H^j(Z(\tilde{w}), D') = 0 \text{ for all } j > 0.$$

□

## 6. FANO, WEAK FANO AND LOG FANO BSDH VARIETIES

**6.1. Fano and weak Fano properties.** In this section, we observe that Fano and weak Fano properties for BSDH variety  $Z(\tilde{w})$  depend on the given expression  $\tilde{w}$ . We use the terminology from Section 1. First we discuss the conditions *I* and *II* with some examples. We use the ordering of simple roots as in [Hum72, Page 58].

The condition *I*:

(1) Special case:  $|\eta_i^+| = 0$  and  $|\eta_i^-| = 0$ . This condition means that the expression  $\tilde{w}$  is fully commutative without repeating the simple reflections. For example if  $G = SL(n, \mathbb{C})$  and  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_r}$ ,  $1 < r \leq n - 1$  and  $r$  is odd, then  $|\eta_i^+| = 0$  and  $|\eta_i^-| = 0$  for all  $i$ . Hence  $\tilde{w}$  satisfies the condition *I* and also observe that in this case we have

$$Y_{\tilde{w}} \simeq Z(\tilde{w}) \simeq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad (\dim(Z(\tilde{w})) \text{ times}).$$

(2) Let  $G = SL(n, \mathbb{C})$  and fix  $1 \leq j < r \leq n - 1$  such that  $j$  is even and  $r$  is odd. Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{j-3}} s_{\alpha_{j-1}} s_{\alpha_j} s_{\alpha_{j+1}} s_{\alpha_{j+3}} \cdots s_{\alpha_r}$ . Note that  $s_{\alpha_j}$  appears only once in the expression  $\tilde{w}$  and  $|\eta_i^+| = 0$  for all  $i$ . Let  $p$  be the ‘position of  $s_{\alpha_j}$ ’ in the expression  $\tilde{w}$ , then  $|\eta_i^-| = 0$  for all  $i \neq p, p - 1$  and  $|\eta_{p-1}^-| = 1 = |\eta_p^-|$  with  $\beta_{p-1p} = -1 = \beta_{pp+1}$ . Hence  $\tilde{w}$  satisfies condition *I*.

The condition *II*:

Again, let  $G = SL(n, \mathbb{C})$  and fix  $1 \leq j < r \leq n - 1$  such that  $j$  is even and  $r$  is odd. Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{j-3}} s_{\alpha_j} s_{\alpha_{j-1}} s_{\alpha_{j+1}} s_{\alpha_{j+3}} \cdots s_{\alpha_r}$  (observe that we interchanged  $s_{\alpha_j}$  and  $s_{\alpha_{j-1}}$  in the example of condition *II*). Then  $|\eta_i^+| = 0$  and  $|\eta_i^-| \leq 2$  for all  $i$ . Let  $p$  be the ‘position of  $s_{\alpha_j}$ ’ in the expression  $\tilde{w}$ , then  $|\eta_i^-| = 0$  for all  $i \neq p$  and  $|\eta_p^-| = 2$  with  $\beta_{pp+1} = -1 = \beta_{pp+1}$ . Hence  $\tilde{w}$  satisfies the condition *II* but not *I*.

Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} s_{\alpha_1}$ . Then  $|\eta_1^+| = 1$  with  $\beta_{13} = 2$ , and  $|\eta_1^-| = |\eta_2^+| = |\eta_2^-| = 0$ . Hence  $\tilde{w}_1$  satisfies *II* but not *I*.

Observe that the condition  $|\eta_i^-| = 1$  and  $\beta_{il} = -2$ , happens only in non-simply laced cases. Let  $G = SO(5, k)$  (i.e.  $G$  is of type  $B_2$ ), let  $\tilde{w}_1 = s_{\alpha_2} s_{\alpha_1}$  and  $\tilde{w}_2 = s_{\alpha_1} s_{\alpha_2}$ . Recall that we have  $\langle \alpha_1, \alpha_2 \rangle = -2$  and  $\langle \alpha_2, \alpha_1 \rangle = -1$ . Then Hence  $\tilde{w}_1$  satisfies *II* but not *I* and  $\tilde{w}_2$  satisfies *I*.

Let  $G$  be of type  $G_2$  (with  $\langle \alpha_1, \alpha_2 \rangle = -1$  and  $\langle \alpha_2, \alpha_1 \rangle = -3$ ). Let  $\tilde{w}_1 = s_{\alpha_2} s_{\alpha_1}$  and  $\tilde{w}_2 = s_{\alpha_1} s_{\alpha_2}$ . Then Hence  $\tilde{w}_1$  satisfies *I* and  $\tilde{w}_2$  does not satisfy any of the conditions *I* or *II*.

Now we have the following result:

**Lemma 6.1.**

- (1)  $Y_{\tilde{w}}$  is Fano if and only if  $\tilde{w}$  satisfies *I*.
- (2)  $Y_{\tilde{w}}$  is weak Fano if and only if  $\tilde{w}$  satisfies *II*.

*Proof.* This follows from Corollary 4.2 and [Cha, Theorem 6.3]. □

Recall the following (see for instance [Cha, Corollary 6.2]):

**Lemma 6.2.** *Let  $X$  be a smooth projective variety and  $D$  be an effective divisor. Let  $\text{supp}(D)$  denote the support of  $D$ . If  $X \setminus \text{supp}(D)$  is affine, then  $D$  is big.*

We prove the following:

**Theorem 6.3.**

- (1) If  $\tilde{w}$  satisfies *I*, then  $Z(\tilde{w})$  is Fano.
- (2) If  $\tilde{w}$  satisfies *II*, then  $Z(\tilde{w})$  is weak Fano.

*Proof.* First recall that the canonical line bundle  $\mathcal{O}_{Z(\tilde{w})}(K_{Z(\tilde{w})})$  of  $Z(\tilde{w})$  is given by

$$\mathcal{O}_{Z(\tilde{w})}(K_{Z(\tilde{w})}) = \mathcal{O}_{Z(\tilde{w})}(-\partial Z(\tilde{w})) \otimes \mathcal{L}(-\delta),$$

where  $\partial Z(\tilde{w})$  is the boundary divisor of  $Z(\tilde{w})$  and  $\delta \in N$  such that  $\langle \delta, \check{\alpha} \rangle = 1$  for all  $\alpha \in S$ , where  $\check{\alpha}$  is the co-root of  $\alpha$  (see [Kum12, Proposition 8.1.2] and also [Ram85, Proposition 2]). Note that if  $G$  is finite dimensional,  $\delta$  is half sum of the positive roots.

By Theorem 3.2,  $\phi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism with  $\mathcal{X}_0 = Y_{\tilde{w}}$  and  $\mathcal{X}_u = Z(\tilde{w})$  for  $0 \neq u \in \mathbb{A}^1$ .

Proof of (1): By [Laz04, Theorem 1.2.17], if  $-K_{\mathcal{X}_0}$  is ample then  $-K_{\mathcal{X}_u}$  is ample for  $u \neq 0$ . By Lemma 6.1,  $-K_{Y_{\tilde{w}}}$  is ample if and only if  $\tilde{w}$  satisfies *I*. Hence we conclude that if  $\tilde{w}$  satisfies *I*, then  $Z(\tilde{w})$  is Fano.

Proof of (2): First we prove  $-K_{Z(\tilde{w})}$  is big. Let

$$Z_0 := Z(\tilde{w}) \setminus \partial Z(\tilde{w}).$$

Note that  $Z_0$  is an open affine subset of  $Z(\tilde{w})$ . Then by Lemma 6.2,  $\partial Z(\tilde{w})$  is big. Since

$$\mathcal{O}(-K_{Z(\tilde{w})}) = \mathcal{O}(\partial Z(\tilde{w})) \otimes \mathcal{L}(\delta)$$

and  $\mathcal{L}(\delta)$  is *nef*, we conclude  $-K_{Z(\tilde{w})}$  is *big*, as tensor product of a *big* and a *nef* line bundles is again a *big* line bundle. By [Laz04, Theorem 1.4.14] and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \neq 0$ , we can see that if  $-K_{\mathcal{X}_0}$  is *nef* then  $-K_{\mathcal{X}_u}$  is also *nef* for  $u \neq 0$ . Therefore, (2) follows from Lemma 6.1(2).  $\square$

There exists expressions  $\tilde{w}$  such that the BSDH variety  $Z(\tilde{w})$  is Fano (respectively, weak Fano) but the toric limit  $Y_{\tilde{w}}$  is not Fano (respectively, not weak Fano).

**Example 6.4.** Let  $G = SL(4, \mathbb{C})$ .

- (1) Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_1}$ . Then  $Z(\tilde{w}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ , which is Fano. The toric limit  $Y_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ . Since  $\tilde{w}$  does not satisfy *I*, then by Lemma 6.1,  $Y_{\tilde{w}}$  is not Fano.
- (2) Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$ . Then it can be seen  $Z(\tilde{w})$  is Fano (see [Cha17, Example 5.4]). By Lemma 6.1, the toric limit  $Y_{\tilde{w}}$  is weak Fano but not Fano.

**Example 6.5.** Let  $G = SO(5, k)$ , i.e.  $G$  is of type  $B_2$ . Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$ . By Lemma 6.1, the toric limit  $Y_{\tilde{w}}$  is not weak Fano. Also we can see  $Z(\tilde{w})$  is weak Fano but not Fano (see [Cha17, Theorem 5.3]).

**6.2. Local rigidity of BSDH varieties.** In this section we obtain some vanishing results for the cohomology of tangent bundle of the toric limit  $Y_{\tilde{w}}$  and  $Z(\tilde{w})$ . Let  $T_X$  denote the tangent bundle of  $X$ , where  $X = Y_{\tilde{w}}$  or  $Z(\tilde{w})$ . Then we have

**Corollary 6.6.**

- (1) If  $\tilde{w}$  satisfies *I*, then  $H^i(Y_{\tilde{w}}, T_{Y_{\tilde{w}}}) = 0$  for all  $i \geq 1$ . In particular,  $Y_{\tilde{w}}$  is locally rigid.
- (2) If  $\tilde{w}$  satisfies *I*, then  $H^i(Z(\tilde{w}), T_{Z(\tilde{w})}) = 0$  for all  $i \geq 1$ . In particular,  $Z(\tilde{w})$  is locally rigid.

*Proof.* Proof of (1): If  $\tilde{w}$  satisfies *I*, then by Lemma 6.1,  $Y_{\tilde{w}}$  is a Fano variety. By [BB96, Proposition 4.2], since  $Y_{\tilde{w}}$  is a smooth Fano toric variety, we get  $H^i(Y_{\tilde{w}}, T_{Y_{\tilde{w}}}) = 0$  for all  $i \geq 1$ .

Proof of (2): From Theorem 3.2,  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism with  $\mathcal{X}_0 = Y_{\tilde{w}}$  and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \in \mathbb{A}^1$ ,  $u \neq 0$ . Hence (2) follows from (1) by semi-continuity theorem (see [Har77, Theorem 12.8]).  $\square$

**6.3. Log Fano BSDH varieties.** In [And14] and [AS14] log Fanoness of Schubert varieties and BSDH varieties were studied respectively. Now we characterize the (suitably chosen)  $\mathbb{Q}$ -divisors  $D'$  in  $Z(\tilde{w})$  for which  $(Z(\tilde{w}), D')$  is log Fano. Recall that  $Z_i = \{[(p_1, \dots, p_r)] \in Z(\tilde{w}) : p_i \in B\}$  is a divisor in  $Z(\tilde{w})$  (see Section 2). Let  $\gamma_i = s_{\beta_r} \cdots s_{\beta_{i+1}}(\beta_i)$  for  $1 \leq i \leq r$ . Then,

$$\mathcal{L}(\delta) = \sum_{i=1}^r b_i Z_i \text{ with } b_i = \langle \delta, \tilde{\gamma}_i \rangle = ht(\gamma_i), \quad (6.1)$$

where  $\delta$  is as in Section 6.1 (see page 10),  $\mathcal{L}(\delta)$  is the homogeneous line bundle on  $Z(\tilde{w})$  corresponding to  $\delta$  and  $ht(\beta)$  for a root  $\beta = \sum_{i=1}^n n_i \alpha_i$ , is the height defined by  $ht(\beta) =$

$\sum_{i=1}^n n_i$  (see [MR85, Proof of Proposition 10]). When  $\tilde{w}$  is reduced,  $\gamma_i$  is a positive root and we can see the relation (6.1) from the Chevalley formula for intersection of Schubert variety by a divisor (see [AS14, Page 410] or [Che94]). It is known that

$$-K_{Z(\tilde{w})} = \sum_{i=1}^r (b_i + 1)Z_i \quad (6.2)$$

(see [MR85, Proposition 4]). Let  $D' = \sum_{i=1}^r a_i Z_i$  be a effective  $\mathbb{Q}$ -divisor in  $Z(\tilde{w})$ , with  $[D'] = 0$ , where  $[\sum_i a_i Z_i] = \sum_i [a_i] Z_i$ ,  $[x]$  is the greatest integer  $\leq x$ . Then by (6.2), we get

$$-(K_{Z(\tilde{w})} + D') = \sum_{i=1}^r (b_i + 1 + a_i)Z_i.$$

For  $1 \leq i \leq r$ , define

$$f_i := (b_i + 1 + a_i) - \sum_{\gamma_j \in \gamma_{P_i}(1)^+} c_j (b_j + 1 + a_j),$$

where  $\gamma_{P_i}(1)^+ := \gamma_{P_i}(1) \cap \{\rho_l^+ : 1 \leq l \leq r\}$  and  $\gamma_{P_i}$  is the cone as in (2.1) for the toric limit  $Y_{\tilde{w}}$ .

Recall that if  $X$  is smooth and  $D$  is a normal crossing divisor, the pair  $(X, D)$  is log Fano if and only if  $[D] = 0$  and  $-(K_X + D)$  is ample (see [KM08, Lemma 2.30, Corollary 2.31 and Definition 2.34]).

We prove,

**Corollary 6.7.** *The pair  $(Z(\tilde{w}), D')$  is log Fano if  $f_i > 0$  for all  $1 \leq i \leq r$ .*

*Proof.* By definition of  $D'$ , the pair  $(Z(\tilde{w}), D')$  is log Fano if and only if  $-(K_{Z(\tilde{w})} + D')$  is ample. Now we prove  $-(K_{Z(\tilde{w})} + D')$  is ample if  $f_i > 0$  for all  $1 \leq i \leq r$ . Recall that  $D_{\rho_i^+}$  is the divisor corresponding to  $\rho_i^+ \in \Sigma(1)$  and  $\mathcal{Z}_i^x = \pi^{-1}(x) \cap \mathcal{Z}_i$  for  $x \in k$  (see Section 2 and Section 3).

By Theorem 3.2, we have

$$\mathcal{Z}_i^x = Z_i \text{ for } x \neq 0 \text{ and } \mathcal{Z}_i^0 = D_{\rho_i^+}. \quad (6.3)$$

Assume that  $f_i > 0$  for all  $1 \leq i \leq r$ . By (6.3) and by semicontinuity (see [Laz04, Theorem 1.2.7]) to prove  $(Z(\tilde{w}), D')$  is log Fano it is enough to prove

$$\sum_{i=1}^r (b_i + 1 + a_i) D_{\rho_i^+} \text{ is ample.}$$

By Lemma 4.5, we see that  $\sum_{i=1}^r (b_i + 1 + a_i) D_{\rho_i^+}$  is ample if and only if

$$f_i = ((b_i + 1 + a_i) - \sum_{\gamma_j \in \gamma_{P_i}(1)^+} c_j (b_j + 1 + a_j)) > 0 \text{ for all } 1 \leq i \leq r.$$

Hence we conclude that  $(Z(\tilde{w}), D')$  is log Fano.  $\square$

## 7. MORE RESULTS ON THE TORIC LIMIT

In this section we are going to recover the results of [PK16] by using methods of toric geometry. In [PK16], they have assumed that  $G$  is a simple algebraic group. In our situation  $G$  is a Kac-Moody group. Recall the following:

- (1)  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$  and  $\tilde{w}' = s_{\beta_1} \cdots s_{\beta_{r-1}}$ .
- (2) The toric morphism  $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$  is induced by the lattice map  $\bar{f}_r : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1}$ , the projection onto the first  $r - 1$  coordinates.

As we discussed in Section 3, there are two disjoint toric sections for the  $\mathbb{P}^1$ -fibration  $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$  (see Lemma 4.2).

**Definition 7.1.**

- (1) **Schubert and non-Schubert sections:** We call the section corresponding to the maximal cone  $\rho_r^+$  (respectively,  $\rho_r^-$ ) in  $\Sigma_F$  (the fan of the fiber of  $f_r$ ) by ‘Schubert section  $\sigma_{r-1}^0$ ’ (respectively, ‘non-Schubert section  $\sigma_{r-1}^1$ ’).
- (2) **Schubert point:** Let  $\sigma \in \Sigma$  be the maximal cone generated by  $\{e_1^+, \dots, e_r^+\}$ . We call the point in  $Y_{\tilde{w}}$  corresponding to the maximal cone  $\sigma$  by ‘Schubert point’.
- (3) **Schubert line:** We call the fiber of  $f_r$  over the Schubert point by ‘Schubert line  $L_r$ ’.

Note that these definitions agree with that of in [PK16, Section 4]. Now onwards we denote  $\tilde{w} = (1, \dots, r)$  (respectively,  $\tilde{w}' = (1, \dots, r - 1)$ ) for the expression  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$  (respectively,  $\tilde{w}' = s_{\beta_1} \cdots s_{\beta_{r-1}}$ ). Let  $I = (i_1, \dots, i_m)$  be a subsequence of  $\tilde{w}$ . Inductively we define the curve  $L_I$  corresponding to  $I$ . Let  $L_{I'}$  be the curve in  $Y_{\tilde{w}'}$  corresponding to the subsequence  $I' = (i_1, \dots, i_{m-1})$  of  $I$ . Then define

$$L_I := \sigma_{r-1}^1(L_{I'}) \text{ and } \sigma_{r-1}^0(L_{I'}) = L_{I'}.$$

Recall some more notations. Let  $X$  be a smooth projective variety, we define

$$N_1(X)_{\mathbb{Z}} := \left\{ \sum_{\text{finite}} a_i C_i : a_i \in \mathbb{Z}, C_i \text{ irreducible curve in } X \right\} / \equiv$$

where  $\equiv$  is the numerical equivalence, i.e.  $Z \equiv Z'$  if and only if  $D \cdot Z = D \cdot Z'$  for all divisors  $D$  in  $X$ . We denote by  $[C]$  the class of  $C$  in  $N_1(X)_{\mathbb{Z}}$ . Let  $N_1(X) := N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$ . It is a well known fact that  $N_1(X)$  is a finite dimensional real vector space dual to  $N^1(X)$  (see [Kle66, Proposition 4, §1, Chapter IV]). We have the following result:

**Lemma 7.2.** *The classes of Schubert lines  $L_j$ ,  $1 \leq j \leq r$  form a basis of  $N_1(Y_{\tilde{w}})$ .*

*Proof.* Proof is by induction on  $r$ . Assume that the result is true for  $r - 1$ . Since  $Y_{\tilde{w}}$  is a projective bundle over  $Y_{\tilde{w}'}$  (see Lemma 4.2), then by [Bar71, Lemma 1.1],

$$L_r \text{ and } \sigma_{r-1}^0(L_j) \text{ for } 1 \leq j \leq r$$

(the image of  $L_j$  by the Schubert section in  $Y_{\tilde{w}}$ ) form a basis of  $N_1(Y_{\tilde{w}})$ . By definition of  $L_I$ , we have

$$\sigma_{r-1}^0(L_j) = L_j \text{ for } 1 \leq j \leq r-1$$

and hence the result follows.  $\square$

Let  $1 \leq j \leq r$ . Let  $\mathcal{D} := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and } \epsilon_l = + \text{ for all } l\}$ . Let  $\mathcal{D}'_j := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and } \epsilon_l = + \text{ for all } l \neq j; \epsilon_j = -\}$ .

**Lemma 7.3.** *Fix  $1 \leq j \leq r$ . Then the Schubert line  $L_j$  is given by*

$$L_j = V(\tau_j), \text{ with } \tau_j = \sigma \cap \sigma'_j,$$

*intersection of two maximal cones in  $\Sigma$ , where  $\sigma$  (respectively,  $\sigma'_j$ ) is generated by  $\mathcal{D}$  (respectively,  $\mathcal{D}'_j$ ).*

*Proof.* Let us consider the expression  $\tilde{w}_j = s_{\beta_1} \cdots s_{\beta_j}$  for  $1 \leq j < r$ . Let  $\Sigma_j$  be the fan of the toric variety  $Y_{\tilde{w}_j}$ . By Lemma 4.2,

$$f_j : Y_{\tilde{w}_j} \rightarrow Y_{\tilde{w}_{j-1}}$$

is a  $\mathbb{P}^1$ -fibration induced by  $\bar{f}_j : \mathbb{Z}^j \rightarrow \mathbb{Z}^{j-1}$  the projection onto the first  $j-1$  factors. Also note that the Schubert point in  $Y_{\tilde{w}_{j-1}}$  corresponds to the maximal cone generated by

$$\{e_l^+ : 1 \leq l \leq j-1\}$$

and the fan of the fiber is given by  $\{e_j^+, 0, e_j^-\}$ . Let  $\sigma_j$  (respectively,  $\sigma'_j$ ) be the cone generated by

$$\{e_l^+ : 1 \leq l \leq j\}$$

(respectively,

$$\{e_l^+ : 1 \leq l \leq j-1\} \cup \{e_j^-\} \text{ ).}$$

Then by definition of Schubert line  $L_j$ , we can see that  $L_j$  is the curve in  $Y_{\tilde{w}_j}$  given by

$$L_j = V(\tau_j), \text{ where } \tau_j \in \Sigma_j \text{ and } \tau_j = \sigma_j \cap \sigma'_j.$$

Since the Schubert section of  $f_k$  for  $(j \leq k \leq r)$  corresponds to  $e_k^+$ , we see

$$\sigma_r^0 \circ \cdots \circ \sigma_{j+1}^0(L_j),$$

by abuse of notation we also denote it again by  $L_j$  in  $Y_{\tilde{w}}$ , is given by

$$L_j = V(\tau_j) \text{ with } \tau = \sigma \cap \sigma'_j,$$

where  $\sigma$  and  $\sigma'_j$  are as described in the statement. This completes the proof of the lemma.  $\square$

Let  $\tau$  be a cone of dimension  $r-1$  which is a wall, that is  $\tau = \sigma \cap \sigma'$  for some  $\sigma, \sigma' \in \Sigma$  of dimension  $r$ . Let  $\sigma$  (respectively,  $\sigma'$ ) be generated by  $\{u_{\rho_1}, u_{\rho_2}, \dots, u_{\rho_r}\}$  (respectively, by  $\{u_{\rho_2}, \dots, u_{\rho_{r+1}}\}$ ) and let  $\tau$  be generated by  $\{u_{\rho_2}, \dots, u_{\rho_r}\}$ . Then we get a linear relation,

$$u_{\rho_1} + \sum_{i=2}^r b_i u_{\rho_i} + u_{\rho_{r+1}} = 0 \tag{7.1}$$



The relation (7.1) called **wall relation** and we have

$$D_\rho \cdot V(\tau) = \begin{cases} b_i & \text{if } \rho = \rho_i \text{ and } i \in \{2, 3, \dots, r\} \\ 1 & \text{if } \rho = \rho_i \text{ and } i \in \{1, r+1\} \\ 0 & \text{otherwise} \end{cases} \quad (7.2)$$

(see [CLS11, Proposition 6.4.4 and eq. (6.4.6) page 303]). We prove the following (see [PK16, Proposition 33]):

**Proposition 7.4.** *Let  $1 \leq j \leq r$  and let  $L_j$  be the Schubert line in  $Y_{\tilde{w}}$ . Then,*

$$K_{Y_{\tilde{w}}} \cdot L_j = -2 - \sum_{k>j} \beta_{kj}.$$

*Proof.* By definition of  $e_j^-$ , we have

$$e_j^+ + e_j^- + \sum_{k>j} \beta_{kj} e_k^+ = 0. \quad (7.3)$$

By Lemma 7.3, we have  $L_j = V(\tau)$ , with  $\tau = \sigma \cap \sigma'$  where  $\sigma$  (respectively,  $\sigma'$ ) is generated by

$$\{e_l^{\epsilon_l} : 1 \leq l \leq r, \epsilon_l = + \text{ for all } l\}$$

(respectively,

$$\{e_l^{\epsilon_l} : \epsilon_l = + \text{ for } 1 \leq l \leq r \text{ and } l \neq j, \epsilon_j = -\} \quad ).$$

Hence (7.3) is the *wall relation* for the curve  $L_j$ . Then by (7.2), we see that

$$D_\rho \cdot L_j = \begin{cases} 1 & \text{if } \rho = \rho_j^+ \text{ or } \rho_j^- \\ \beta_{kj} & \text{if } \rho = \rho_k^+ \text{ and } k > j \\ 0 & \text{otherwise.} \end{cases}$$

Since  $K_{Y_{\tilde{w}}} = -\sum_{\rho \in \Sigma(1)} D_\rho$ , we get

$$K_{Y_{\tilde{w}}} \cdot L_j = -2 - \sum_{k>j} \beta_{kj}.$$

This completes the proof of the proposition.  $\square$

Now onwards we denote the subsequence  $(i_1, \dots, i_m)$  by  $I_{i_1}$ . Let  $\mathcal{D}_{i_1}'' := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and}$

$$\epsilon_l = \begin{cases} + & \text{if } l \notin I_{i_1} \setminus \{i_1\} \\ - & \text{if } l \in I_{i_1} \end{cases} \quad \}.$$

Let  $\mathcal{D}_{i_1}''' := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and}$

$$\epsilon_l = \begin{cases} + & \text{if } l \notin I_{i_1} \\ - & \text{if } l \in I_{i_1} \end{cases} \quad \}.$$

**Proposition 7.5.** *The curve  $L_{I_{i_1}}$  is given by*

$$L_{I_{i_1}} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1},$$

where  $\sigma_{i_1}$  (respectively,  $\sigma'_{i_1}$ ) is the cone generated by  $\mathcal{D}''_{i_1}$  (respectively,  $\mathcal{D}'''_{i_1}$ ).

*Proof.* As in the proof of Lemma 7.3, we start with  $j = i_1$  and  $L_{i_1}$  is the Schubert line in  $Y_{\tilde{w}_{i_1}}$ . By Lemma 7.3, we have

$$L_{i_1} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1}.$$

By definition of  $L_I$ , we have

$$\sigma_{i_2-1}^0 \circ \cdots \circ \sigma_{i_1+1}^0(L_{i_1}) = L_{i_1} \text{ in } Y_{\tilde{w}_{i_2-1}}$$

and

$$\sigma_{i_2}^1 \circ \sigma_{i_2-1}^0 \circ \cdots \circ \sigma_{i_1+1}^0(L_{i_1}) = L_{\{i_1, i_2\}} \text{ in } Y_{\tilde{w}_{i_2}}.$$

By repeating the process we conclude that

$$L_{I_{i_1}} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1},$$

where  $\sigma_{i_1}$  and  $\sigma'_{i_1}$  are as described in the statement. This completes the proof of the proposition.  $\square$

Recall  $NE(X)$  is the real convex cone in  $N_1(X)$  generated by classes of irreducible curves. The Mori cone  $\overline{NE}(X)$  is the closure of  $NE(X)$  in  $N_1(X)$  and it is a strongly convex cone of maximal dimension (see for instance [CLS11, Chapter 6, page 293]). Now we describe the Mori cone of the toric limit  $Y_{\tilde{w}}$  in terms of the curves  $L_{I_{i_j}}$ 's defined above. For this we need the following notation (see also [Cha]). Fix  $1 \leq i \leq r$ . Define:

- (1) Let  $r \geq j > j_1 = i \geq 1$  and define  $a_{1,j} := \beta_{j_1 j}$ .
- (2) Let  $r \geq j_2 > j_1$  be the least integer such that  $a_{1,j} > 0$ , then define for  $j > j_2$

$$a_{2,j} := \beta_{i j_2} \beta_{j_2 j} - \beta_{i j}.$$

- (3) Let  $k > 2$  and let  $r \geq j_k > j_{k-1}$  be the least integer such that  $a_{k-1,j} < 0$ , then inductively, define for  $j > j_k$

$$a_{k,j} := -a_{k-1,j_k} \beta_{j_k j} + a_{k-1,j}.$$

- (4) Let  $\tilde{I}_i := \{i = j_1, \dots, j_m\}$ .

**Example 7.6.** *Let  $G = SL(5, \mathbb{C})$  and let  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_7} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ . Let  $i = 1$ . Then  $j_1 = 1$  and (1)  $a_{1,2} = \beta_{12} = \langle \beta_2, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$ ; (2)  $a_{1,3} = \beta_{13} = \langle \beta_3, \check{\beta}_1 \rangle = \langle \alpha_3, \check{\alpha}_2 \rangle = -1$ ;*

*(3)  $a_{1,4} = \beta_{14} = \langle \beta_4, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$ ; (4)  $a_{1,5} = \beta_{15} = \langle \beta_5, \check{\beta}_1 \rangle = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$ ;*

*(5)  $a_{1,6} = \beta_{16} = \langle \beta_6, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$ ; (6)  $a_{1,7} = \beta_{17} = \langle \beta_7, \check{\beta}_1 \rangle = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$ .*

*Then by definition of  $j_2$ , we have  $j_2 = 5$  and (1)  $a_{2,6} = \beta_{15} \beta_{56} - \beta_{16} = \langle \beta_5, \check{\beta}_1 \rangle \langle \beta_6, \check{\beta}_5 \rangle - \langle \beta_6, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$ ; (2)  $a_{2,7} = \beta_{15} \beta_{57} - \beta_{17} = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$ . Then by definition of  $j_3$ , we have  $j_3 = 6$  and  $a_{3,7} = -a_{2,6} \beta_{67} + a_{2,7} = -(\langle \beta_6, \check{\beta}_5 \rangle)(\langle \beta_7, \check{\beta}_6 \rangle) + (\langle \beta_7, \check{\beta}_5 \rangle) = -(-1)(-1) + (2) = 1$ . Therefore, we get  $\tilde{I}_1 = \{1, 5, 6\}$ .*

**Example 7.7.** We use Example 7.6, for  $i = 1$ , we have  $I_1 = \{1, 5, 6\}$ . Then

$$\mathcal{D}_1'' = \{e_1^+, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\} \text{ and } \mathcal{D}_1''' = \{e_1^-, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

Fix  $1 \leq i \leq r$ . Let

$$I_i := \tilde{I}_i = \{i = j_1, j_2, \dots, j_m\}$$

where  $j_k$ 's are as above. With this notation we prove the following (see [PK16, Theorem 22]):

**Theorem 7.8.** *The set  $\{L_{I_i} : 1 \leq i \leq r\}$  of classes of curves forms a basis of  $N_1(Y_{\bar{w}})_{\mathbb{Z}}$  and every torus invariant curve in  $N_1(Y_{\bar{w}})$  lie in the cone generated by  $\{L_{I_i} : 1 \leq i \leq r\}$ .*

*Proof.* By [Cha, Proposition 4.16], for  $1 \leq i \leq r$  the curve  $r(P_i)$  (see Section 2 for the definition of  $r(P_i)$ ) is given by

$$r(P_i) = [V(\tau_i)],$$

where  $\tau_i = \sigma_i \cap \sigma'_i$  and  $\sigma_i$  (respectively,  $\sigma'_i$ ) is generated by  $\mathcal{D}_i''$  (respectively,  $\mathcal{D}_i'''$ ). From Proposition 7.5, we see that the class of the curve  $L_{I_i}$  is  $r(P_i)$  in  $N_1(Y_{\bar{w}})_{\mathbb{Z}}$ . By [Cha, Theorem 4.7], we have

$$\overline{NE}(Y_{\bar{w}}) = \sum_{i=1}^r \mathbb{R}_{\geq 0} r(P_i).$$

Also by [Cha, Corollary 4.8], the set  $\{r(P_i) : 1 \leq i \leq r\}$  forms a basis of  $N_1(Y_{\bar{w}})_{\mathbb{Z}}$ . Hence we conclude the assertion of the theorem.  $\square$

We recall some definitions: Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$  and let  $K$  be a (closed) cone in  $V$ . A subcone  $Q$  in  $K$  is called extremal if  $u, v \in K, u + v \in Q$  then  $u, v \in Q$ . A face of  $K$  is an extremal subcone. A one-dimensional face is called an extremal ray. Note that an extremal ray is contained in the boundary of  $K$ . Then we have (see [PK16, Theorem 30]):

**Corollary 7.9.** *The extremal rays of the toric limit  $Y_{\bar{w}}$  are precisely the curves  $L_{I_i}$  for  $1 \leq i \leq r$ .*

*Proof.* This follows from the proof of the Theorem 7.8.  $\square$

Let  $X$  be a smooth projective variety. An extremal ray  $R$  in the Mori cone  $\overline{NE}(X) \subset N_1(X)$  is called Mori if  $R \cdot K_X < 0$ , where  $K_X$  is the canonical divisor in  $X$ . Recall that  $\overline{NE}(Y_{\bar{w}})$  is a strongly convex rational polyhedral cone of maximal dimension in  $N_1(Y_{\bar{w}})$ . We have the following (see [PK16, Theorem 35]):

**Corollary 7.10.** *Fix  $1 \leq i \leq r$ , the class of curve  $L_{I_i}$  is Mori ray if and only if either  $|\gamma_{P_i}(1)| = 0$ , or  $|\gamma_{P_i}(1)| = 1$  with  $c_j = 1$  for  $\gamma_j \in \gamma_{P_i}(1)$ .*

*Proof.* Since  $Y_{\bar{w}}$  is a Bott tower (see Corollary 4.2), then the result follows from Proposition 7.5 and [Cha, Theorem 8.1].  $\square$

Now we prove a general result for smooth projective toric varieties,

**Lemma 7.11.** *Let  $X$  be a smooth projective toric variety of dimension  $r$ . Then  $X$  is Fano if and only if every extremal ray is Mori.*

*Proof.* By [CLS11, Theorem 6.3.20] (Toric Cone Theorem), we have

$$\overline{NE}(X) = \sum_{\tau \in \Sigma(r-1)} \mathbb{R}_{\geq 0}[V(\tau)]. \quad (7.4)$$

If  $X$  is Fano, then by definition,  $-K_X$  is ample. By toric Kleiman criterion for ampleness [CLS11, Theorem 6.3.13], we can see that  $-K_X \cdot V(\tau) > 0$  for all  $\tau \in \Sigma(r-1)$ . Then  $K_X \cdot V(\tau) < 0$  for all  $\tau \in \Sigma(r-1)$ . In particular, every extremal ray is Mori.

Conversely, let  $\mathbb{R}_{\geq 0}[V(\tau)]$  be an extremal ray, by assumption it is a Mori ray. Then by definition of a Mori ray, we have  $K_X \cdot V(\tau) < 0$ . This implies  $-K_X \cdot V(\tau) > 0$ . By (7.4),  $\overline{NE}(X)$  is a polyhedral cone and hence the extremal rays generate the cone  $\overline{NE}(X)$ . Hence we see that  $-K_X \cdot C > 0$  for all classes of curves  $[C]$  in  $\overline{NE}(X)$ . Again by toric Kleiman criterion for ampleness, we conclude that  $-K_X$  is ample and hence  $X$  is Fano.  $\square$

Then we have the following (see [PK16, Corollary 36]):

**Corollary 7.12.** *The toric limit  $Y_{\bar{w}}$  is Fano if and only if every extremal ray in  $\overline{NE}(Y_{\bar{w}})$  is Mori.*

**Acknowledgements:** I would like to thank Michel Brion for valuable discussions and many critical comments.

## REFERENCES

- [And14] D. Anderson, *Effective divisors on Bott-Samelson varieties*, arXiv preprint arXiv:1501.00034 (2014).
- [AS14] D. Anderson and A. Stapledon, *Schubert varieties are log Fano over the integers*, Proceedings of the American Mathematical Society **142** (2014), no. 2, 409–411.
- [Bar71] C.M. Barton, *Tensor products of ample vector bundles in characteristic  $p$* , American Journal of Mathematics **93** (1971), no. 2, 429–438.
- [BB96] F. Bien and M. Brion, *Automorphisms and local rigidity of regular varieties*, Compositio Mathematica **104** (1996), no. 1, 1–26 (eng).
- [BS58] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, American Journal of Mathematics (1958), 964–1029.
- [BK07] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, vol. 231, Springer Science & Business Media, 2007.
- [Cha] B.N. Chary, *On Mori cone of Bott towers*, preprint, arxiv.org/abs/1706.02139.
- [Cha17] ———, *On Fano and weak Fano Bott-Samelson-Demazure-Hansen varieties*, to appear in Journal of Pure and Applied Algebra, <https://doi.org/10.1016/j.jpaa.2017.10.006>.
- [CK17] B.N. Chary and S.S. Kannan, *Rigidity of a Bott-Samelson-Demazure-Hansen variety for  $PSp(2n, \mathbb{C})$* , Journal of Lie Theory **27** (2017), 435–468.
- [CKP] B.N. Chary, S.S. Kannan, and A.J. Parameswaran, *Automorphism group of a Bott-Samelson-Demazure-Hansen variety for non reduced case*, in preparation.
- [CKP15] ———, *Automorphism group of a Bott-Samelson-Demazure-Hansen variety*, Transformation Groups **20** (2015), no. 3, 665–698.

- [Che94] C. Chevalley, *Sur les décompositions cellulaires des espaces  $G/B$* , Algebraic Groups and their Generalizations: Classical Methods, American Mathematical Society (1994), 1–23.
- [Civ05] Y. Civan, *Bott towers, crosspolytopes and torus actions*, Geometriae Dedicata **113** (2005), no. 1, 55–74.
- [CLS11] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric varieties*, Graduate studies in mathematics, American Mathematical Soc., 2011.
- [Dem74] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Annales scientifiques de l'École Normale Supérieure, vol. 7, Société mathématique de France, 1974, pp. 53–88.
- [Ful93] W. Fulton, *Introduction to toric varieties*, no. 131, Princeton University Press, 1993.
- [GK94] M. Grossberg, Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Mathematical Journal **76** (1994), no. 1, 23–58.
- [Han73] H. C. Hansen, *On cycles in flag manifolds.*, Mathematica Scandinavica **33** (1973), 269–274.
- [Har77] R. Hartshorne, *Algebraic geometry*, vol. 52, Springer Science & Business Media, 1977.
- [Hum72] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, vol. 9, Springer Science & Business Media, 1972.
- [Kle66] S.L. Kleiman, *Toward a numerical theory of ampleness*, Annals of Mathematics (1966), 293–344.
- [KM08] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, vol. 134, Cambridge university press, 2008.
- [Kum12] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, vol. 204, Springer Science & Business Media, 2012.
- [LT04] N. Lauritzen, J. F. Thomsen, et al., *Line bundles on Bott-Samelson varieties*, Journal of Algebraic Geomeometry **13** (2004), 461–473.
- [Laz04] R.K. Lazarsfeld, *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, vol. 48, Springer Science & Business Media, 2004.
- [MR85] V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Annals of Mathematics, (2) **122** (1985), no. 1, 27–40.
- [Oda88] T. Oda, *Convex bodies and algebraic geometry- An introduction to the theory of toric varieties*, vol. (3)15, Springer-Verlag, Berlin Heidelberg NewYork, 1988.
- [PK16] A.J. Parameswaran and P. Karuppuchamy, *Toric degeneration of Bott-Samelson-Demazure-Hansen varieties*, arXiv preprint arXiv:1604.01998 (2016).
- [Pas10] B. Pasquier, *Vanishing theorem for the cohomology of line bundles on Bott-Samelson varieties*, Journal of Algebra **323** (2010), no. 10, 2834–2847.
- [Ram85] A Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Inventiones mathematicae **80** (1985), no. 2, 283–294.
- [Spr10] T. A. Springer, *Linear algebraic groups*, Springer Science & Business Media, 2010.

B. NARASIMHA CHARY, INSTITUT FOURIER, UMR 5582 DU CNRS, UNIVERSITÉ DE GRENOBLE ALPES, CS 40700, 38058, GRENOBLE CEDEX 09, FRANCE., EMAIL: NARASIMHA-CHARY.BONALA@UNIV-GRENOBLE-APLES.FR