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Narasimha Chary Bonala

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A NOTE ON TORIC DEGENERATION OF A BOTT-SAMELSON-DEMAZURE-HANSEN VARIETY

B. NARASIMHA CHARY

ABSTRACT. In this paper we study the geometry of toric degeneration of a Bott-Samelson-Demazure-Hansen (BSDH) variety, which was algebraically constructed in [Pas10] and [PK16]. We give some applications to BSDH varieties. Precisely, we classify Fano, weak Fano and log Fano BSDH varieties and their toric limits in Kac-Moody setting. We prove some vanishing theorems for the cohomology of tangent bundle (and line bundles) on BSDH varieties. We also recover the results in [PK16], by toric methods.

Keywords: Bott-Samelson-Demazure-Hansen varieties, canonical line bundle, tangent bundle and toric varieties.

1. INTRODUCTION

Bott-Samelson-Demazure-Hansen (for short, BSDH) varieties are natural desingularizations of Schubert varieties in the flag varieties. These were algebraically constructed by M. Demazure and H.C. Hansen independently by adapting a differential geometric approach from the paper of Bott and Samelson (see [BS58], [Dem74] and [Han73]). Briefly, the BSDH varieties are iterated projective line bundles, given by factoring the Schubert variety using Bruhat decomposition. These varieties depend on the given expression of the Weyl group element corresponding to the Schubert variety (see for instance [CKP15, Page 32]). We also see in this paper some properties of these varieties which depend on the given expression.

In [GK94], M. Grossberg and Y. Karshon constructed toric degenerations of BSDH varieties by complex geometric methods. In [Pas10] B. Pasquier and in [PK16] A.J. Parameswaran and P. Karuppuchamy constructed these toric degenerations algebraically. B. Pasquier used these degenerations to study the cohomology of line bundles on BSDH varieties (see [Pas10]). In [PK16], the authors studied the limiting toric variety for a simple simply connected algebraic group by geometric methods. In this paper we study the limiting toric variety of a BSDH variety in more detail by methods of toric geometry and we prove some applications to BSDH varieties. We also recover the results in [PK16] and extend them to the Kac-Moody setting. The key idea for many results in this article is that the toric limit is a ‘Bott tower’. These are studied in [Cha] and some of their properties can be transferred to BSDH varieties by using the semi-continuity theorem.

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Let G be a Kac-Moody group over the field of complex numbers (for the definition see [Kum12]). Let B be a Borel subgroup containing a fixed maximal torus T . Let W be the Weyl group corresponding to the pair (G, B, T) and let $w \in W$. Let $\tilde{w} := s_{\beta_1} \cdots s_{\beta_n}$ be an expression (possibly non-reduced) of w in simple reflections and let $Z(\tilde{w})$ be the BSDH variety corresponding to \tilde{w} (see Section 2). Let $Y_{\tilde{w}}$ be the toric limit of $Z(\tilde{w})$ constructed as in [Pas10] and [PK16] (see Section 3). We see that $Y_{\tilde{w}}$ is a Bott tower, the iterated \mathbb{P}^1 -bundle over a point $\{pt\}$ where each \mathbb{P}^1 -bundle is the projectivization of a rank 2 decomposable vector bundle (see Corollary 4.4). We prove that the ample cone $Amp(Y_{\tilde{w}})$ of $Y_{\tilde{w}}$ can be identified with a subcone of the ample cone $Amp(Z(\tilde{w}))$ of $Z(\tilde{w})$ (see Corollary 5.1).

Recall that a smooth projective variety X is called *Fano* (respectively, *weak Fano*) if its anti-canonical divisor $-K_X$ is ample (respectively, *nef* and *big*). Following [AS14], we say that a pair (X, D) of a normal projective variety X and an effective \mathbb{Q} -divisor D is *log Fano* if it is Kawamata log terminal and $-(K_X + D)$ is ample.

When G is a simple algebraic group and the expression \tilde{w} is reduced, Fanoness and weak Fanoness of the BSDH variety $Z(\tilde{w})$ are considered in [Cha17]. Here we have the following results in Kac-Moody setting. Let $\tilde{w} = s_{\beta_1} \cdots s_{\beta_i} \cdots s_{\beta_j} \cdots s_{\beta_r}$ be an expression (remember that β_k 's are simple roots). Let $\beta_{ij} := \langle \beta_j, \check{\beta}_i \rangle$, where $\check{\beta}_i$ is the co-root of β_i . Now we define some conditions on the expression \tilde{w} (see Section 6 for examples and also see [Cha, Section 1]). Define for $1 \leq i \leq r$,

$$\eta_i^+ := \{r \geq j > i : \beta_{ij} > 0\} \text{ and } \eta_i^- := \{r \geq j > i : \beta_{ij} < 0\}.$$

If $|\eta_i^+| = 1$ (respectively, $|\eta_i^+| = 2$), then let $\eta_i^+ = \{m\}$ (respectively, $\eta_i^+ = \{m_1, m_2\}$). If $|\eta_i^-| = 1$ (respectively, $|\eta_i^-| = 2$), then set $\eta_i^- = \{l\}$ (respectively, $\eta_i^- = \{l_1, l_2\}$).

- N_i^1 is the condition that
 - (i) $|\eta_i^+| = 0$, $|\eta_i^-| \leq 1$, and if $|\eta_i^-| = 1$ then $\beta_{il} = -1$; or
 - (ii) $|\eta_i^-| = 0$, $|\eta_i^+| \leq 1$, and if $|\eta_i^+| = 1$ then $\beta_{im} = 1$ and $\beta_{mk} = 0$ for all $k > m$.
- N_i^2 is the condition that
 - Case 1: Assume that $|\eta_i^+| = 0$. Then $|\eta_i^-| \leq 2$, and if $|\eta_i^-| = 1$ (respectively, $|\eta_i^-| = 2$) then $\beta_{li} = -1$ or -2 (respectively, $\beta_{il_1} = -1 = \beta_{il_2}$).
 - Case 2: If $|\eta_i^-| = 1 = |\eta_i^+|$ and $l < m$, then $\beta_{il} = -1$, $\beta_{im} = 1$ and $\beta_{mk} = 0$ for all $k > m$.
 - Case 3: Assume that $|\eta_i^+| = 1$. Then $\beta_{im} = 1$ and either it satisfies
 - (i) Case 2; or
 - (ii) $|\eta_i^-| = 0$ and $\beta_{mk} = 0$ for all $k > m$; or
 - (iii) there exists a unique $r \geq s > m$ such that $\beta_{ms} - \beta_{is} = 1$ and $\beta_{mk} - \beta_{ik} = 0$ for all $k > s$; or $\beta_{ms} - \beta_{is} = -1$ and $\beta_{is} - \beta_{ms} - \beta_{sk} = 0$ for all $k > s$.

Definition 1.1. We say the expression \tilde{w} satisfies *condition I* (respectively, *condition II*) if N_i^1 (respectively, N_i^2) holds for all $1 \leq i \leq r$. Note that $N_i^1 \implies N_i^2$ for all $1 \leq i \leq r$.

Theorem (See Lemma 6.1 and Theorem 6.3).

- (1) If \tilde{w} satisfies I, then $Y_{\tilde{w}}$ and $Z(\tilde{w})$ are Fano.
- (2) If \tilde{w} satisfies II, then $Y_{\tilde{w}}$ and $Z(\tilde{w})$ are weak Fano.

In [CKP15] and [CK17], we have obtained some vanishing results for the cohomology of tangent bundle of $Z(\tilde{w})$, when G is finite dimensional and \tilde{w} is reduced (see [CKP15, Section 3] and [CK17, Theorem 8.1]). The case \tilde{w} is non-reduced is considered in [CKP]. Here we get some vanishing results in Kac-Moody setting. Let $T_{Z(\tilde{w})}$ denote the tangent bundle of $Z(\tilde{w})$.

Corollary (see Corollary 6.6). *If \tilde{w} satisfies I, then $H^i(Z(\tilde{w}), T_{Z(\tilde{w})}) = 0$ for all $i \geq 1$. In particular, $Z(\tilde{w})$ is locally rigid.*

In [AS14], D. Anderson and A. Stapledon studied the log Fano-ness of Schubert varieties, and in [And14], log Fano-ness of BSDH varieties is studied for chosen divisors. Let D be a divisor in $Z(\tilde{w})$ with support in the boundary of $Z(\tilde{w})$. For $1 \leq i \leq r$, we define some constants f_i which again depend on the given expression \tilde{w} (for more details see Section 6).

Corollary (see Corollary 6.7). *The pair $(Z(\tilde{w}), D)$ is log Fano if $f_i > 0$ for all $1 \leq i \leq r$.*

The article is organized as follows: In section 2, we recall the construction of BSDH varieties. In Section 3, we give the algebraic construction of toric degeneration of a BSDH variety. In Section 4, we describe the limiting toric variety as an iterated \mathbb{P}^1 -bundle. In Section 5, we see some vanishing results of cohomology of line bundles on BSDH varieties. Section 6 contains the results on Fano, weak Fano and log Fano properties of BSDH varieties and their toric limits. We also study the vanishing results on cohomology of tangent bundle on BSDH varieties. In Section 7, we recover the results in [PK16] by toric methods.

2. PRELIMINARIES

In this section we recall the construction of Bott-Samelson-Demazure-Hansen varieties (see [BK07] and [Kum12]) and we recall some definitions in toric geometry which are used in this article (for more details on toric varieties see [CLS11] and also [Ful93]). We work over the field of complex numbers throughout.

2.1. BSDH varieties. Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a generalized Cartan matrix. Let G be the Kac-Moody group associated to A (see [Kum12, Chapter IV]). Fix a maximal torus T and a Borel subgroup B containing T . Let $S := \{\alpha_1, \dots, \alpha_n\}$ be the set of all simple roots of (G, B, T) . We denote s_{α_i} the simple reflection corresponding to α_i . Note that the Weyl group W of G is generated by

$$\{s_{\alpha_i} : 1 \leq i \leq n\}.$$

Let $w \in W$, an expression \tilde{w} of w is a sequence $(s_{\beta_1}, \dots, s_{\beta_r})$ of simple reflections $s_{\beta_1}, \dots, s_{\beta_r}$ such that $w = s_{\beta_1} \cdots s_{\beta_r}$. An expression \tilde{w} of w is said to be reduced if the number r of simple reflections is minimal. In such case we call r the length of w . By abuse of notation, we also denote the expression \tilde{w} by $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$. For $\alpha \in S$, we denote P_α , the minimal parabolic subgroup of G generated by B and a representative of s_α .

Definition 2.1. Let $w \in W$ and $\tilde{w} := s_{\beta_1} \cdots s_{\beta_r}$ be an expression (not necessarily reduced) of w . The Bott-Samelson-Demazure-Hansen (for short, BSDH) variety corresponding to \tilde{w} is

$$Z(\tilde{w}) := P_{\beta_1} \times \cdots \times P_{\beta_r} / B^r,$$

where the action of B^r on $P_{\beta_1} \times \cdots \times P_{\beta_r}$ is defined by

$$(p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r) \text{ for all } p_i \in P_{\beta_i}, b_i \in B.$$

These are smooth projective varieties of dimension r . There is a natural morphism $\phi_{\tilde{w}} : Z(\tilde{w}) \rightarrow G/B$ defined by

$$[(p_1, \dots, p_r)] \mapsto p_1 \cdots p_r B.$$

If \tilde{w} is reduced, the BSDH variety $Z(\tilde{w})$ is a natural desingularization of the Schubert variety, the B -orbit closure of wB/B in G/B (see [Dem74], [Han73] and [Kum12, Chapter VIII]). We can also construct the BSDH variety as an iterated \mathbb{P}^1 -bundles. Let $\tilde{w}' := s_{\beta_1} \cdots s_{\beta_{r-1}}$. Let $f : G/B \rightarrow G/P_{\beta_r}$ be the map given by $gB \mapsto gP_{\beta_r}$ and let $p : Z(\tilde{w}') \rightarrow G/P_{\beta_r}$ be the map given by $[(p_1, \dots, p_{r-1})] \mapsto p_1 \cdots p_{r-1} P_{\beta_r}$. Then we have the following cartesian diagram (see [BK07, Page 66] and [Kum12, Chapter VII]):

$$\begin{array}{ccc} Z(\tilde{w}) = Z(\tilde{w}') \times_{G/P_{\beta_r}} G/B & \xrightarrow{\phi_{\tilde{w}}} & G/B \\ \downarrow f_{\tilde{w}} & & \downarrow f \\ Z(\tilde{w}') & \xrightarrow{p} & G/P_{\beta_r} \end{array}$$

Note that $f_{\tilde{w}}$ is a \mathbb{P}^1 -fibration and the relative tangent bundle $T_{f_{\tilde{w}}}$ of $f_{\tilde{w}}$ is $\phi_{\tilde{w}}^*(\mathcal{L}_{\beta_r})$, where \mathcal{L}_{β_r} is the homogeneous line bundle on G/B corresponding to β_r . Using the cohomology of the relative tangent bundle $T_{f_{\tilde{w}}}$ we studied the cohomology of the tangent bundle of $Z(\tilde{w})$, when G is finite dimensional and \tilde{w} is a reduced expression (see [CKP15] and [CK17]). The fibration $f_{\tilde{w}}$ comes with a natural section $\sigma_{\tilde{w}} : Z(\tilde{w}') \rightarrow Z(\tilde{w})$ induced by the projection

$$P_{\beta_1} \times \cdots \times P_{\beta_r} \rightarrow P_{\beta_1} \times \cdots \times P_{\beta_{r-1}}.$$

For the toric limits we get two natural sections, as will be explained in Section 3. For all $i \in \{1, \dots, r\}$, we denote Z_i , the divisor in $Z(\tilde{w})$ defined by

$$\{[(p_1, \dots, p_r)] \in Z(\tilde{w}) : p_i \in B\}.$$

In [LT04], N. Lauritzen and J.F. Thomsen proved that Z'_i 's forms a basis of the Picard group of $Z(\tilde{w})$ and they also proved that if \tilde{w} is a reduced expression these form a basis of the monoid of effective divisors (see [LT04, Proposition 3.5]). Recently, the effective divisors of $Z(\tilde{w})$ for \tilde{w} non-reduced case have been considered in [And14].

The BSDH variety can be described also as an iterated projective line bundle, where each projective bundle is the projectivization of certain rank 2 vector bundle (not necessarily decomposable). In Section 4, we see the toric degeneration of a BSDH variety (constructed in Section 3) is a Bott tower, the iterated \mathbb{P}^1 -bundle over a point $\{pt\}$, where each \mathbb{P}^1 -bundle is the projectivization of a rank 2 decomposable vector bundle.

2.2. Toric varieties.

Definition 2.2. *A normal variety X is called a toric variety (of dimension n) if it contains an n -dimensional torus T (i.e. $T = (\mathbb{C}^*)^n$) as a Zariski open subset such that the action of the torus on itself by multiplication extends to an action of the torus on X .*

Toric varieties are completely described by the combinatorics of the corresponding fans. We denote the fan corresponding to a toric variety by Σ and the collection of cones of dimension s in Σ by $\Sigma(s)$ for $1 \leq s \leq n$. For each cone $\sigma \in \Sigma$, we denote $V(\sigma)$, the orbit closure of the orbit corresponding to cone σ . For each $\sigma \in \Sigma$, $\sigma(1) := \sigma \cap \Sigma(1)$. For each $\rho \in \Sigma(1)$, we can associate a divisor in X , we denote it by D_ρ (see [CLS11, Chapter 4] for more details). We recall the following:

Definition 2.3.

- (1) *We say $P \subset \Sigma(1)$ is a primitive collection if P is not contained in $\sigma(1)$ for some $\sigma \in \Sigma$ but any proper subset is. Note that if Σ is simplicial, primitive collection means that P does not generate a cone in Σ but every proper subset does.*
- (2) *Let $P = \{\rho_1, \dots, \rho_k\}$ be a primitive collection in a complete simplicial fan Σ . Recall u_ρ be the primitive vector of the ray $\rho \in \Sigma$. Then $\sum_{i=1}^k u_{\rho_i}$ is in the relative interior of a cone γ_P in Σ with a unique expression*

$$\sum_{i=1}^k u_{\rho_i} - \left(\sum_{\rho \in \gamma_P(1)} c_\rho u_\rho \right) = 0. \tag{2.1}$$

where $c_\rho \in \mathbb{Q}_{>0}$. Then we call (2.1) the primitive relation of X corresponding to P .

- (3) *For a primitive relation P , we can associate an element $r(P)$ in $N_1(X)$, where $N_1(X)$ is the real vector space of numerical classes of one-cycles in X (see [CLS11, Page 305]).*

3. TORIC DEGENERATION OF A BSDH VARIETY

In [GK94], toric degenerations of BSDH varieties were constructed by complex geometric methods. In [Pas10] and [PK16] they have given an algebraic construction for toric degeneration of a BSDH variety. We recall the algebraic construction here.

Note that the simple roots are linearly independent elements in the character group of G . Let N be the lattice of one-parameter subgroups of T . We can choose a positive integer q and an injective morphism $\lambda : \mathbb{G}_m \rightarrow T$ (i.e. $\lambda \in N$ and λ is injective) such that for all $1 \leq i \leq n$ and $u \in \mathbb{G}_m$, $\alpha_i(\lambda(u)) = u^q$ (see [Pas10, Page 2836]). When G is finite dimensional, for each one-parameter subgroup $\lambda \in N$, define

$$P(\lambda) := \{g \in G : \lim_{u \rightarrow 0} \lambda(u)g\lambda(u)^{-1} \text{ exists in } G\}.$$

The set $P(\lambda)$ is a parabolic subgroup and the unipotent radical $R_u(P(\lambda))$ of $P(\lambda)$ is given by

$$R_u(P(\lambda)) = \{g \in G : \lim_{u \rightarrow 0} \lambda(u)g\lambda(u)^{-1} \text{ is identity in } G\}.$$

Any parabolic subgroup of G is of this form (see [Spr10, Proposition 8.4.5]). Choose a one-parameter subgroup $\lambda \in N$ such that the corresponding parabolic subgroup is B . Let us define an endomorphism of G for all $u \in \mathbb{G}_m$ by

$$\tilde{\Psi}_u : G \rightarrow G, \quad g \mapsto \lambda(u)g\lambda(u)^{-1}.$$

Let \mathcal{B} be the set of all endomorphisms of B . Now define a morphism

$$\Psi : \mathbb{G}_m \rightarrow \mathcal{B} \text{ by } u \mapsto \tilde{\Psi}_u|_B.$$

This map can be extended to 0 and for all $x \in U$, $\Psi_u|_B(x)$ goes to identity when u goes to zero. Let $\mathbb{A}^1 := \text{Spec}\mathbb{C}[t]$ be the affine line over \mathbb{C} . We denote for all $u \in \mathbb{A}^1$, Ψ_u the image of u in \mathcal{B} . Note that Ψ_u is the identity on T and Ψ_0 is the projection from B to T . Let $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$ be an expression.

Definition 3.1.

(i) Let \mathcal{X} be the variety defined by

$$\mathcal{X} := \mathbb{A}^1 \times P_{\beta_1} \times \cdots \times P_{\beta_r} / B^r,$$

where the action of B^r on $\mathbb{A}^1 \times P_{\beta_1} \times \cdots \times P_{\beta_r}$ is given by

$$(u, p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (u, p_1 b_1, \Psi_u(b_1)^{-1} p_2 b_2, \dots, \Psi_u(b_{r-1})^{-1} p_r b_r).$$

(ii) For all $i \in \{1, \dots, r\}$, we denote \mathcal{Z}_i the divisor in \mathcal{X} defined by

$$\{(u, p_1, \dots, p_r) \in \mathcal{X} : p_i \in B\}.$$

Note that \mathcal{X} and \mathcal{Z}_i 's are integral. Let $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$ be the projection onto the first factor. Then we have the following theorem (see [Pas10, Proposition 1.3 and 1.4] and [PK16, Theorem 9]).

Theorem 3.2.

- (1) $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$ is a smooth projective morphism.
- (2) For all $u \in \mathbb{A}^1 \setminus \{0\}$, the fiber $\pi^{-1}(u)$ is isomorphic to the BSDH variety $Z(\tilde{w})$ such that $\pi^{-1}(u) \cap \mathcal{Z}_i$ corresponds to the divisor Z_i in $Z(\tilde{w})$.
- (3) $\pi^{-1}(0)$ is a smooth projective toric variety.

We denote $\mathcal{X}_u := \pi^{-1}(u)$ for $u \in \mathbb{A}^1$ and the limiting toric variety $\mathcal{X}_0 = \pi^{-1}(0)$ by $Y_{\tilde{w}}$.

4. CONNECTION TO BOTT TOWERS

In this section we describe the toric limit $Y_{\tilde{w}}$ as an iterated \mathbb{P}^1 -bundle. We also recall some results on Bott towers from [Cha]. Let $\{e_1^+, \dots, e_r^+\}$ be the standard basis of the lattice \mathbb{Z}^r . Define for all $i \in \{1, \dots, r\}$,

$$e_i^- := -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+, \tag{4.1}$$

where $\beta_{ij} := \langle \beta_j, \check{\beta}_i \rangle$. The following proposition will give the description of the fan of the toric variety $Y_{\tilde{w}}$ (see [Pas10, Proposition 1.4]).

Proposition 4.1.

- (1) *The fan Σ of the smooth toric variety $Y_{\tilde{w}}$ consists of the cones generated by subsets of*

$$\{e_1^+, \dots, e_r^+, e_1^-, \dots, e_r^-\}$$

and containing no subset of the form $\{e_i^+, e_i^-\}$.

- (2) *For all $i \in \{1, \dots, r\}$, Z_i^0 is the irreducible $(\mathbb{C}^*)^r$ -stable divisor in $Y_{\tilde{w}}$ corresponding to the one-dimensional cone of Σ generated by e_i^+ and these form a basis of the divisor class group of $Y_{\tilde{w}}$.*

Note that the maximal cones of Σ are generated by $\{e_i^\epsilon : 1 \leq i \leq r, \epsilon \in \{+, -\}\}$. We denote the divisor corresponding to the one-dimensional cone ρ_i^ϵ generated by e_i^ϵ by $D_{\rho_i^\epsilon}$ for $\epsilon \in \{+, -\}$. Let $\tilde{w}' := s_{\beta_1} \cdots s_{\beta_{r-1}}$. Then we get a toric morphism $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$ induced by the lattice map $\bar{f}_r : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1}$, the projection onto the first $r-1$ coordinates. We prove,

Lemma 4.2.

- (1) *$f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$ is a toric \mathbb{P}^1 -fibration with two disjoint toric sections.*
 (2) *$Y_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{Y_{\tilde{w}'}} \oplus \mathcal{L})$ for some unique line bundle \mathcal{L} on $Y_{\tilde{w}'}$.*

Proof. Let Σ' be the fan corresponding to the toric variety $Y_{\tilde{w}'}$. From the above proposition, we can see that Σ has a splitting by Σ' and $\{e_r^+, 0, e_r^-\}$. Then by [CLS11, Theorem 3.3.19],

$$f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$$

is a locally trivial fibration with the fan Σ_F of the fiber being $\{e_r^+, 0, e_r^-\}$. Since Σ_F is the fan of the projective line \mathbb{P}^1 , we conclude f_r is a toric \mathbb{P}^1 -fibration. As toric sections of the toric fibration correspond to the maximal cones in Σ_F , we get two disjoint toric sections for f_r . This proves (1).

Proof of (2): Since $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$ is \mathbb{P}^1 -fibration with a section, we see $Y_{\tilde{w}}$ is a projective bundle $\mathbb{P}(\mathcal{E})$ over $Y_{\tilde{w}'}$ corresponding to a rank 2 vector bundle \mathcal{E} on $Y_{\tilde{w}'}$ (see for example [Har77, Chapter V, Proposition 2.2, page 370]).

Recall that the sections of projective bundle $Y_{\tilde{w}} = \mathbb{P}(\mathcal{E})$ correspond to the quotient line bundles of \mathcal{E} (see [Har77, Proposition 7.12]). Since $Y_{\tilde{w}} = \mathbb{P}(\mathcal{E})$ is projective line bundle on $Y_{\tilde{w}'}$ with two disjoint sections, we see \mathcal{E} is decomposable as a direct sum of line bundles on $Y_{\tilde{w}'}$.

As

$$\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{L}' \otimes \mathcal{E})$$

for any line bundle \mathcal{L}' on $Y_{\tilde{w}'}$ (see [Har77, Lemma 7.9]), we can assume without loss of generality

$$\mathcal{E} = \mathcal{O}_{Y_{\tilde{w}'}} \oplus \mathcal{L}$$

for some unique line bundle \mathcal{L} on $Y_{\tilde{w}'}$. Hence $Y_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{Y_{\tilde{w}'}} \oplus \mathcal{L})$ and this completes the proof of the lemma. \square

Definition 4.3. A Bott tower of height r is a sequence of projective bundles

$$Y_r \xrightarrow{\pi_r} Y_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} Y_1 = \mathbb{P}^1 \xrightarrow{\pi_1} Y_0 = \{pt\},$$

where $Y_i = \mathbb{P}(\mathcal{O}_{Y_{i-1}} \oplus \mathcal{L}_{i-1})$ for a line bundle \mathcal{L}_{i-1} over Y_{i-1} for all $1 \leq i \leq r$ and $\mathbb{P}(-)$ denotes the projectivization (see for more details [Civ05] and also [Cha, Section 2]).

Then by definition of Bott tower and by Lemma 4.2(2) we get:

Corollary 4.4. The toric limit $Y_{\tilde{w}}$ is a Bott tower.

We have the following situation:

$$\begin{array}{ccc} \mathbb{P}^1 & & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ Z(\tilde{w}) & \xrightarrow{\quad \quad \quad} & Y_{\tilde{w}} \\ \downarrow f_{\tilde{w}} & & \downarrow f_r \\ Z(\tilde{w}') & \xrightarrow{\quad \quad \quad} & Y_{\tilde{w}'} \end{array}$$

(Red arrows indicate maps from \mathbb{P}^1 to $Z(\tilde{w})$ and $Y_{\tilde{w}}$, and from $Z(\tilde{w})$ to $Z(\tilde{w}')$ and $Y_{\tilde{w}}$ to $Y_{\tilde{w}'}$.)

Recall that the Bott towers bijectively correspond to the upper triangular matrices with integer entries (see [Civ05, Section 3]). Here the upper triangular matrix $M_{\tilde{w}}$ corresponding to $Y_{\tilde{w}}$ is given by

$$M_{\tilde{w}} = \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2r} \\ 0 & 0 & 1 & \cdots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}_{r \times r},$$

where β_{ij} 's are integers as defined before. Let $P_i := \{\rho_i^+, \rho_i^-\}$ for $1 \leq i \leq r$. Then by [Cha, Lemma 4.3], $\{P_i : 1 \leq i \leq r\}$ is the set of all primitive collections of $Y_{\tilde{w}}$. For each $1 \leq i \leq r$, we denote the cone in the definition of primitive relation (see Section 2) corresponding to P_i by γ_{P_i} . Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a toric divisor in $Y_{\tilde{w}}$ with $a_\rho \in \mathbb{Z}$ and for $1 \leq i \leq r$, define

$$d_i := (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}).$$

Then we recall the following from [Cha, Lemma 5.1]:

Lemma 4.5.

- (1) D is ample if and only if $d_i > 0$ for all $1 \leq i \leq r$.
- (2) D is numerically effective (nef) if and only if $d_i \geq 0$ for all $1 \leq i \leq r$.

Also note that the conditions *I* and *II* on \tilde{w} are same as the conditions on $M_{\tilde{w}}$ as in [Cha].

5. VANISHING RESULTS ON COHOMOLOGY OF CERTAIN LINE BUNDLES ON BSDH VARIETIES

Let X be a smooth projective variety. Recall $N^1(X)$ denote the real finite dimensional vector space of numerical classes of real divisors in X (see [Kle66, §1, Chapter IV]). The ample cone $\text{Amp}(X)$ of X is the cone in $N^1(X)$ generated by classes of ample divisors.

5.1. Ample cone of the toric limit of BSDH variety. In [LT04], the ampleness of line bundles on BSDH variety $Z(\tilde{w})$ is studied. Now we compare the ample cone of the toric limit $Y_{\tilde{w}}$ with that of the BSDH-variety $Z(\tilde{w})$ as a consequence of Theorem 3.2.

Corollary 5.1. *The ample cone $\text{Amp}(Y_{\tilde{w}})$ of $Y_{\tilde{w}}$ can be identified with a subcone of the ample cone $\text{Amp}(Z(\tilde{w}))$ of $Z(\tilde{w})$.*

Proof. By Theorem 3.2, $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$ is a smooth projective morphism with $\mathcal{X}_0 = Y_{\tilde{w}}$ and $\mathcal{X}_u = Z(\tilde{w})$ for $u \neq 0$. Let $\mathcal{L} = \{\mathcal{L}_u : u \in \mathbb{A}^1\}$ be a line bundle on $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$ with \mathcal{L}_0 is an ample line bundle on $Y_{\tilde{w}}$. Note that the ampleness of line bundle is an open condition for the proper morphism π , i.e. there exists an open subset U in \mathbb{A}^1 containing 0 such that \mathcal{L}_u is an ample line bundle on \mathcal{X}_u for all $u \in U$ (see [Laz04, Theorem 1.2.17]). Hence we can identify $\text{Amp}(Y_{\tilde{w}})$ with a subcone of $\text{Amp}(Z(\tilde{w}))$. \square

5.2. Vanishing results. In [Pas10], B. Pasquier obtained vanishing theorems for the cohomology of certain line bundles on BSDH varieties, by using combinatorics of the toric limit (see [Pas10, Theorem 0.1]). Here we see some vanishing results for the cohomology of certain line bundles on BSDH varieties. Let $1 \leq i \leq r$, define $h_i^{i-1} := -\beta_{i(i-1)}$ and

$$h_i^j := \begin{cases} 0 & \text{for } j > i. \\ 1 & \text{for } j = i. \\ -\sum_{k=j}^{i-1} \beta_{ik}(h_k^j) & \text{for } j < i. \end{cases}$$

Let $\epsilon \in \{+, -\}$. Define $\Sigma(1)^\epsilon := \{\rho_i^\epsilon : 1 \leq i \leq r\}$. Then we can write a toric divisor D in $Y_{\tilde{w}}$ as follows:

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho = \sum_{\rho \in \Sigma(1)^+} a_\rho D_\rho + \sum_{\rho \in \Sigma(1)^-} a_\rho D_\rho.$$

For $1 \leq i \leq r$, let

$$g_i := a_{\rho_i^+} + \sum_{j=i}^r a_{\rho_j^-} h_j^i.$$

Recall d_i from Section 4,

$$d_i := (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}),$$

Let $D' = \sum_{i=1}^r g_i Z_i$ be a divisor in $Z(\tilde{w})$, where Z_i is as in Section 2 for $1 \leq i \leq r$.

Lemma 5.2. *If $d_i \geq 0$ for all $1 \leq i \leq r$, then $H^j(Z(\tilde{w}), D') = 0$ for all $j > 0$.*

Proof. If $d_i \geq 0$ for all $1 \leq i \leq r$, by Lemma 4.5, $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is a *nef* divisor in $Y_{\tilde{w}}$. Then we have

$$H^j(Y_{\tilde{w}}, \sum_{\rho \in \Sigma(1)} a_\rho D_\rho) = 0 \text{ for all } j > 0 \quad (5.1)$$

(see [CLS11, Theorem 9.2.3, page 410] or [Oda88, Theorem 2.7, page 77]). Recall that by Theorem 3.2, we have

$$\mathcal{Z}_i^x = Z_i \text{ for } 0 \neq x \in k \text{ and } \mathcal{Z}_i^0 = D_{\rho_i^+}.$$

By [Cha, Corollary 3.3], we can write

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \sim \sum_{i=1}^r g_i D_{\rho_i^+}.$$

Hence by (5.1), Theorem 3.2 and by semi-continuity theorem (see [Har77, Theorem 12.8]), we get

$$H^j(Z(\tilde{w}), D') = 0 \text{ for all } j > 0.$$

□

6. FANO, WEAK FANO AND LOG FANO BSDH VARIETIES

6.1. Fano and weak Fano properties. In this section, we observe that Fano and weak Fano properties for BSDH variety $Z(\tilde{w})$ depend on the given expression \tilde{w} . We use the terminology from Section 1. First we discuss the conditions *I* and *II* with some examples. We use the ordering of simple roots as in [Hum72, Page 58].

The condition *I*:

(1) Special case: $|\eta_i^+| = 0$ and $|\eta_i^-| = 0$. This condition means that the expression \tilde{w} is fully commutative without repeating the simple reflections. For example if $G = SL(n, \mathbb{C})$ and $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_r}$, $1 < r \leq n - 1$ and r is odd, then $|\eta_i^+| = 0$ and $|\eta_i^-| = 0$ for all i . Hence \tilde{w} satisfies the condition *I* and also observe that in this case we have

$$Y_{\tilde{w}} \simeq Z(\tilde{w}) \simeq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad (\dim(Z(\tilde{w})) \text{ times}).$$

(2) Let $G = SL(n, \mathbb{C})$ and fix $1 \leq j < r \leq n - 1$ such that j is even and r is odd. Let $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{j-3}} s_{\alpha_{j-1}} s_{\alpha_j} s_{\alpha_{j+1}} s_{\alpha_{j+3}} \cdots s_{\alpha_r}$. Note that s_{α_j} appears only once in the expression \tilde{w} and $|\eta_i^+| = 0$ for all i . Let p be the ‘position of s_{α_j} ’ in the expression \tilde{w} , then $|\eta_i^-| = 0$ for all $i \neq p, p - 1$ and $|\eta_{p-1}^-| = 1 = |\eta_p^-|$ with $\beta_{p-1p} = -1 = \beta_{pp+1}$. Hence \tilde{w} satisfies condition *I*.

The condition *II*:

Again, let $G = SL(n, \mathbb{C})$ and fix $1 \leq j < r \leq n - 1$ such that j is even and r is odd. Let $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{j-3}} s_{\alpha_j} s_{\alpha_{j-1}} s_{\alpha_{j+1}} s_{\alpha_{j+3}} \cdots s_{\alpha_r}$ (observe that we interchanged s_{α_j} and $s_{\alpha_{j-1}}$ in the example of condition *II*). Then $|\eta_i^+| = 0$ and $|\eta_i^-| \leq 2$ for all i . Let p be the ‘position of s_{α_j} ’ in the expression \tilde{w} , then $|\eta_i^-| = 0$ for all $i \neq p$ and $|\eta_p^-| = 2$ with $\beta_{pp+1} = -1 = \beta_{pp+1}$. Hence \tilde{w} satisfies the condition *II* but not *I*.

Let $\tilde{w} = s_{\alpha_1} s_{\alpha_3} s_{\alpha_1}$. Then $|\eta_1^+| = 1$ with $\beta_{13} = 2$, and $|\eta_1^-| = |\eta_2^+| = |\eta_2^-| = 0$. Hence \tilde{w}_1 satisfies *II* but not *I*.

Observe that the condition $|\eta_i^-| = 1$ and $\beta_{il} = -2$, happens only in non-simply laced cases. Let $G = SO(5, k)$ (i.e. G is of type B_2), let $\tilde{w}_1 = s_{\alpha_2} s_{\alpha_1}$ and $\tilde{w}_2 = s_{\alpha_1} s_{\alpha_2}$. Recall that we have $\langle \alpha_1, \alpha_2 \rangle = -2$ and $\langle \alpha_2, \alpha_1 \rangle = -1$. Then Hence \tilde{w}_1 satisfies *II* but not *I* and \tilde{w}_2 satisfies *I*.

Let G be of type G_2 (with $\langle \alpha_1, \alpha_2 \rangle = -1$ and $\langle \alpha_2, \alpha_1 \rangle = -3$). Let $\tilde{w}_1 = s_{\alpha_2} s_{\alpha_1}$ and $\tilde{w}_2 = s_{\alpha_1} s_{\alpha_2}$. Then Hence \tilde{w}_1 satisfies *I* and \tilde{w}_2 does not satisfy any of the conditions *I* or *II*.

Now we have the following result:

Lemma 6.1.

- (1) $Y_{\tilde{w}}$ is Fano if and only if \tilde{w} satisfies *I*.
- (2) $Y_{\tilde{w}}$ is weak Fano if and only if \tilde{w} satisfies *II*.

Proof. This follows from Corollary 4.2 and [Cha, Theorem 6.3]. □

Recall the following (see for instance [Cha, Corollary 6.2]):

Lemma 6.2. *Let X be a smooth projective variety and D be an effective divisor. Let $\text{supp}(D)$ denote the support of D . If $X \setminus \text{supp}(D)$ is affine, then D is big.*

We prove the following:

Theorem 6.3.

- (1) If \tilde{w} satisfies *I*, then $Z(\tilde{w})$ is Fano.
- (2) If \tilde{w} satisfies *II*, then $Z(\tilde{w})$ is weak Fano.

Proof. First recall that the canonical line bundle $\mathcal{O}_{Z(\tilde{w})}(K_{Z(\tilde{w})})$ of $Z(\tilde{w})$ is given by

$$\mathcal{O}_{Z(\tilde{w})}(K_{Z(\tilde{w})}) = \mathcal{O}_{Z(\tilde{w})}(-\partial Z(\tilde{w})) \otimes \mathcal{L}(-\delta),$$

where $\partial Z(\tilde{w})$ is the boundary divisor of $Z(\tilde{w})$ and $\delta \in N$ such that $\langle \delta, \check{\alpha} \rangle = 1$ for all $\alpha \in S$, where $\check{\alpha}$ is the co-root of α (see [Kum12, Proposition 8.1.2] and also [Ram85, Proposition 2]). Note that if G is finite dimensional, δ is half sum of the positive roots.

By Theorem 3.2, $\phi : \mathcal{X} \rightarrow \mathbb{A}^1$ is a smooth projective morphism with $\mathcal{X}_0 = Y_{\tilde{w}}$ and $\mathcal{X}_u = Z(\tilde{w})$ for $0 \neq u \in \mathbb{A}^1$.

Proof of (1): By [Laz04, Theorem 1.2.17], if $-K_{\mathcal{X}_0}$ is ample then $-K_{\mathcal{X}_u}$ is ample for $u \neq 0$. By Lemma 6.1, $-K_{Y_{\tilde{w}}}$ is ample if and only if \tilde{w} satisfies *I*. Hence we conclude that if \tilde{w} satisfies *I*, then $Z(\tilde{w})$ is Fano.

Proof of (2): First we prove $-K_{Z(\tilde{w})}$ is big. Let

$$Z_0 := Z(\tilde{w}) \setminus \partial Z(\tilde{w}).$$

Note that Z_0 is an open affine subset of $Z(\tilde{w})$. Then by Lemma 6.2, $\partial Z(\tilde{w})$ is big. Since

$$\mathcal{O}(-K_{Z(\tilde{w})}) = \mathcal{O}(\partial Z(\tilde{w})) \otimes \mathcal{L}(\delta)$$

and $\mathcal{L}(\delta)$ is *nef*, we conclude $-K_{Z(\tilde{w})}$ is *big*, as tensor product of a *big* and a *nef* line bundles is again a big line bundle. By [Laz04, Theorem 1.4.14] and $\mathcal{X}_u = Z(\tilde{w})$ for $u \neq 0$, we can see that if $-K_{\mathcal{X}_0}$ is *nef* then $-K_{\mathcal{X}_u}$ is also *nef* for $u \neq 0$. Therefore, (2) follows from Lemma 6.1(2). \square

There exists expressions \tilde{w} such that the BSDH variety $Z(\tilde{w})$ is Fano (respectively, weak Fano) but the toric limit $Y_{\tilde{w}}$ is not Fano (respectively, not weak Fano).

Example 6.4. Let $G = SL(4, \mathbb{C})$.

- (1) Let $\tilde{w} = s_{\alpha_1} s_{\alpha_1}$. Then $Z(\tilde{w}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, which is Fano. The toric limit $Y_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. Since \tilde{w} does not satisfy *I*, then by Lemma 6.1, $Y_{\tilde{w}}$ is not Fano.
- (2) Let $\tilde{w} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$. Then it can be seen $Z(\tilde{w})$ is Fano (see [Cha17, Example 5.4]). By Lemma 6.1, the toric limit $Y_{\tilde{w}}$ is weak Fano but not Fano.

Example 6.5. Let $G = SO(5, k)$, i.e. G is of type B_2 . Let $\tilde{w} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$. By Lemma 6.1, the toric limit $Y_{\tilde{w}}$ is not weak Fano. Also we can see $Z(\tilde{w})$ is weak Fano but not Fano (see [Cha17, Theorem 5.3]).

6.2. Local rigidity of BSDH varieties. In this section we obtain some vanishing results for the cohomology of tangent bundle of the toric limit $Y_{\tilde{w}}$ and $Z(\tilde{w})$. Let T_X denote the tangent bundle of X , where $X = Y_{\tilde{w}}$ or $Z(\tilde{w})$. Then we have

Corollary 6.6.

- (1) If \tilde{w} satisfies *I*, then $H^i(Y_{\tilde{w}}, T_{Y_{\tilde{w}}}) = 0$ for all $i \geq 1$. In particular, $Y_{\tilde{w}}$ is locally rigid.
- (2) If \tilde{w} satisfies *I*, then $H^i(Z(\tilde{w}), T_{Z(\tilde{w})}) = 0$ for all $i \geq 1$. In particular, $Z(\tilde{w})$ is locally rigid.

Proof. Proof of (1): If \tilde{w} satisfies *I*, then by Lemma 6.1, $Y_{\tilde{w}}$ is a Fano variety. By [BB96, Proposition 4.2], since $Y_{\tilde{w}}$ is a smooth Fano toric variety, we get $H^i(Y_{\tilde{w}}, T_{Y_{\tilde{w}}}) = 0$ for all $i \geq 1$.

Proof of (2): From Theorem 3.2, $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$ is a smooth projective morphism with $\mathcal{X}_0 = Y_{\tilde{w}}$ and $\mathcal{X}_u = Z(\tilde{w})$ for $u \in \mathbb{A}^1$, $u \neq 0$. Hence (2) follows from (1) by semi-continuity theorem (see [Har77, Theorem 12.8]). \square

6.3. Log Fano BSDH varieties. In [And14] and [AS14] log Fanoness of Schubert varieties and BSDH varieties were studied respectively. Now we characterize the (suitably chosen) \mathbb{Q} -divisors D' in $Z(\tilde{w})$ for which $(Z(\tilde{w}), D')$ is log Fano. Recall that $Z_i = \{[(p_1, \dots, p_r)] \in Z(\tilde{w}) : p_i \in B\}$ is a divisor in $Z(\tilde{w})$ (see Section 2). Let $\gamma_i = s_{\beta_r} \cdots s_{\beta_{i+1}}(\beta_i)$ for $1 \leq i \leq r$. Then,

$$\mathcal{L}(\delta) = \sum_{i=1}^r b_i Z_i \text{ with } b_i = \langle \delta, \tilde{\gamma}_i \rangle = ht(\gamma_i), \quad (6.1)$$

where δ is as in Section 6.1 (see page 10), $\mathcal{L}(\delta)$ is the homogeneous line bundle on $Z(\tilde{w})$ corresponding to δ and $ht(\beta)$ for a root $\beta = \sum_{i=1}^n n_i \alpha_i$, is the height defined by $ht(\beta) =$

$\sum_{i=1}^n n_i$ (see [MR85, Proof of Proposition 10]). When \tilde{w} is reduced, γ_i is a positive root and we can see the relation (6.1) from the Chevalley formula for intersection of Schubert variety by a divisor (see [AS14, Page 410] or [Che94]). It is known that

$$-K_{Z(\tilde{w})} = \sum_{i=1}^r (b_i + 1)Z_i \quad (6.2)$$

(see [MR85, Proposition 4]). Let $D' = \sum_{i=1}^r a_i Z_i$ be a effective \mathbb{Q} -divisor in $Z(\tilde{w})$, with $[D'] = 0$, where $[\sum_i a_i Z_i] = \sum_i [a_i] Z_i$, $[x]$ is the greatest integer $\leq x$. Then by (6.2), we get

$$-(K_{Z(\tilde{w})} + D') = \sum_{i=1}^r (b_i + 1 + a_i)Z_i.$$

For $1 \leq i \leq r$, define

$$f_i := (b_i + 1 + a_i) - \sum_{\gamma_j \in \gamma_{P_i}(1)^+} c_j (b_j + 1 + a_j),$$

where $\gamma_{P_i}(1)^+ := \gamma_{P_i}(1) \cap \{\rho_l^+ : 1 \leq l \leq r\}$ and γ_{P_i} is the cone as in (2.1) for the toric limit $Y_{\tilde{w}}$.

Recall that if X is smooth and D is a normal crossing divisor, the pair (X, D) is log Fano if and only if $[D] = 0$ and $-(K_X + D)$ is ample (see [KM08, Lemma 2.30, Corollary 2.31 and Definition 2.34]).

We prove,

Corollary 6.7. *The pair $(Z(\tilde{w}), D')$ is log Fano if $f_i > 0$ for all $1 \leq i \leq r$.*

Proof. By definition of D' , the pair $(Z(\tilde{w}), D')$ is log Fano if and only if $-(K_{Z(\tilde{w})} + D')$ is ample. Now we prove $-(K_{Z(\tilde{w})} + D')$ is ample if $f_i > 0$ for all $1 \leq i \leq r$. Recall that $D_{\rho_i^+}$ is the divisor corresponding to $\rho_i^+ \in \Sigma(1)$ and $\mathcal{Z}_i^x = \pi^{-1}(x) \cap \mathcal{Z}_i$ for $x \in k$ (see Section 2 and Section 3).

By Theorem 3.2, we have

$$\mathcal{Z}_i^x = Z_i \text{ for } x \neq 0 \text{ and } \mathcal{Z}_i^0 = D_{\rho_i^+}. \quad (6.3)$$

Assume that $f_i > 0$ for all $1 \leq i \leq r$. By (6.3) and by semicontinuity (see [Laz04, Theorem 1.2.7]) to prove $(Z(\tilde{w}), D')$ is log Fano it is enough to prove

$$\sum_{i=1}^r (b_i + 1 + a_i) D_{\rho_i^+} \text{ is ample.}$$

By Lemma 4.5, we see that $\sum_{i=1}^r (b_i + 1 + a_i) D_{\rho_i^+}$ is ample if and only if

$$f_i = ((b_i + 1 + a_i) - \sum_{\gamma_j \in \gamma_{P_i}(1)^+} c_j (b_j + 1 + a_j)) > 0 \text{ for all } 1 \leq i \leq r.$$

Hence we conclude that $(Z(\tilde{w}), D')$ is log Fano. \square

7. MORE RESULTS ON THE TORIC LIMIT

In this section we are going to recover the results of [PK16] by using methods of toric geometry. In [PK16], they have assumed that G is a simple algebraic group. In our situation G is a Kac-Moody group. Recall the following:

- (1) $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$ and $\tilde{w}' = s_{\beta_1} \cdots s_{\beta_{r-1}}$.
- (2) The toric morphism $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$ is induced by the lattice map $\bar{f}_r : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1}$, the projection onto the first $r - 1$ coordinates.

As we discussed in Section 3, there are two disjoint toric sections for the \mathbb{P}^1 -fibration $f_r : Y_{\tilde{w}} \rightarrow Y_{\tilde{w}'}$ (see Lemma 4.2).

Definition 7.1.

- (1) **Schubert and non-Schubert sections:** We call the section corresponding to the maximal cone ρ_r^+ (respectively, ρ_r^-) in Σ_F (the fan of the fiber of f_r) by ‘Schubert section σ_{r-1}^0 ’ (respectively, ‘non-Schubert section σ_{r-1}^1 ’).
- (2) **Schubert point:** Let $\sigma \in \Sigma$ be the maximal cone generated by $\{e_1^+, \dots, e_r^+\}$. We call the point in $Y_{\tilde{w}}$ corresponding to the maximal cone σ by ‘Schubert point’.
- (3) **Schubert line:** We call the fiber of f_r over the Schubert point by ‘Schubert line L_r ’.

Note that these definitions agree with that of in [PK16, Section 4]. Now onwards we denote $\tilde{w} = (1, \dots, r)$ (respectively, $\tilde{w}' = (1, \dots, r - 1)$) for the expression $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$ (respectively, $\tilde{w}' = s_{\beta_1} \cdots s_{\beta_{r-1}}$). Let $I = (i_1, \dots, i_m)$ be a subsequence of \tilde{w} . Inductively we define the curve L_I corresponding to I . Let $L_{I'}$ be the curve in $Y_{\tilde{w}'}$ corresponding to the subsequence $I' = (i_1, \dots, i_{m-1})$ of I . Then define

$$L_I := \sigma_{r-1}^1(L_{I'}) \text{ and } \sigma_{r-1}^0(L_{I'}) = L_{I'}.$$

Recall some more notations. Let X be a smooth projective variety, we define

$$N_1(X)_{\mathbb{Z}} := \left\{ \sum_{\text{finite}} a_i C_i : a_i \in \mathbb{Z}, C_i \text{ irreducible curve in } X \right\} / \equiv$$

where \equiv is the numerical equivalence, i.e. $Z \equiv Z'$ if and only if $D \cdot Z = D \cdot Z'$ for all divisors D in X . We denote by $[C]$ the class of C in $N_1(X)_{\mathbb{Z}}$. Let $N_1(X) := N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$. It is a well known fact that $N_1(X)$ is a finite dimensional real vector space dual to $N^1(X)$ (see [Kle66, Proposition 4, §1, Chapter IV]). We have the following result:

Lemma 7.2. *The classes of Schubert lines L_j , $1 \leq j \leq r$ form a basis of $N_1(Y_{\tilde{w}})$.*

Proof. Proof is by induction on r . Assume that the result is true for $r - 1$. Since $Y_{\tilde{w}}$ is a projective bundle over $Y_{\tilde{w}'}$ (see Lemma 4.2), then by [Bar71, Lemma 1.1],

$$L_r \text{ and } \sigma_{r-1}^0(L_j) \text{ for } 1 \leq j \leq r$$

(the image of L_j by the Schubert section in $Y_{\tilde{w}}$) form a basis of $N_1(Y_{\tilde{w}})$. By definition of L_I , we have

$$\sigma_{r-1}^0(L_j) = L_j \text{ for } 1 \leq j \leq r-1$$

and hence the result follows. \square

Let $1 \leq j \leq r$. Let $\mathcal{D} := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and } \epsilon_l = + \text{ for all } l\}$. Let $\mathcal{D}'_j := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and } \epsilon_l = + \text{ for all } l \neq j; \epsilon_j = -\}$.

Lemma 7.3. *Fix $1 \leq j \leq r$. Then the Schubert line L_j is given by*

$$L_j = V(\tau_j), \text{ with } \tau_j = \sigma \cap \sigma'_j,$$

intersection of two maximal cones in Σ , where σ (respectively, σ'_j) is generated by \mathcal{D} (respectively, \mathcal{D}'_j).

Proof. Let us consider the expression $\tilde{w}_j = s_{\beta_1} \cdots s_{\beta_j}$ for $1 \leq j < r$. Let Σ_j be the fan of the toric variety $Y_{\tilde{w}_j}$. By Lemma 4.2,

$$f_j : Y_{\tilde{w}_j} \rightarrow Y_{\tilde{w}_{j-1}}$$

is a \mathbb{P}^1 -fibration induced by $\bar{f}_j : \mathbb{Z}^j \rightarrow \mathbb{Z}^{j-1}$ the projection onto the first $j-1$ factors. Also note that the Schubert point in $Y_{\tilde{w}_{j-1}}$ corresponds to the maximal cone generated by

$$\{e_l^+ : 1 \leq l \leq j-1\}$$

and the fan of the fiber is given by $\{e_j^+, 0, e_j^-\}$. Let σ_j (respectively, σ'_j) be the cone generated by

$$\{e_l^+ : 1 \leq l \leq j\}$$

(respectively,

$$\{e_l^+ : 1 \leq l \leq j-1\} \cup \{e_j^-\} \text{).}$$

Then by definition of Schubert line L_j , we can see that L_j is the curve in $Y_{\tilde{w}_j}$ given by

$$L_j = V(\tau_j), \text{ where } \tau_j \in \Sigma_j \text{ and } \tau_j = \sigma_j \cap \sigma'_j.$$

Since the Schubert section of f_k for $(j \leq k \leq r)$ corresponds to e_k^+ , we see

$$\sigma_r^0 \circ \cdots \circ \sigma_{j+1}^0(L_j),$$

by abuse of notation we also denote it again by L_j in $Y_{\tilde{w}}$, is given by

$$L_j = V(\tau_j) \text{ with } \tau = \sigma \cap \sigma'_j,$$

where σ and σ'_j are as described in the statement. This completes the proof of the lemma. \square

Let τ be a cone of dimension $r-1$ which is a wall, that is $\tau = \sigma \cap \sigma'$ for some $\sigma, \sigma' \in \Sigma$ of dimension r . Let σ (respectively, σ') be generated by $\{u_{\rho_1}, u_{\rho_2}, \dots, u_{\rho_r}\}$ (respectively, by $\{u_{\rho_2}, \dots, u_{\rho_{r+1}}\}$) and let τ be generated by $\{u_{\rho_2}, \dots, u_{\rho_r}\}$. Then we get a linear relation,

$$u_{\rho_1} + \sum_{i=2}^r b_i u_{\rho_i} + u_{\rho_{r+1}} = 0 \tag{7.1}$$

The relation (7.1) called **wall relation** and we have

$$D_\rho \cdot V(\tau) = \begin{cases} b_i & \text{if } \rho = \rho_i \text{ and } i \in \{2, 3, \dots, r\} \\ 1 & \text{if } \rho = \rho_i \text{ and } i \in \{1, r+1\} \\ 0 & \text{otherwise} \end{cases} \quad (7.2)$$

(see [CLS11, Proposition 6.4.4 and eq. (6.4.6) page 303]). We prove the following (see [PK16, Proposition 33]):

Proposition 7.4. *Let $1 \leq j \leq r$ and let L_j be the Schubert line in $Y_{\tilde{w}}$. Then,*

$$K_{Y_{\tilde{w}}} \cdot L_j = -2 - \sum_{k>j} \beta_{kj}.$$

Proof. By definition of e_j^- , we have

$$e_j^+ + e_j^- + \sum_{k>j} \beta_{kj} e_k^+ = 0. \quad (7.3)$$

By Lemma 7.3, we have $L_j = V(\tau)$, with $\tau = \sigma \cap \sigma'$ where σ (respectively, σ') is generated by

$$\{e_l^{\epsilon_l} : 1 \leq l \leq r, \epsilon_l = + \text{ for all } l\}$$

(respectively,

$$\{e_l^{\epsilon_l} : \epsilon_l = + \text{ for } 1 \leq l \leq r \text{ and } l \neq j, \epsilon_j = -\} \quad).$$

Hence (7.3) is the *wall relation* for the curve L_j . Then by (7.2), we see that

$$D_\rho \cdot L_j = \begin{cases} 1 & \text{if } \rho = \rho_j^+ \text{ or } \rho_j^- \\ \beta_{kj} & \text{if } \rho = \rho_k^+ \text{ and } k > j \\ 0 & \text{otherwise.} \end{cases}$$

Since $K_{Y_{\tilde{w}}} = -\sum_{\rho \in \Sigma(1)} D_\rho$, we get

$$K_{Y_{\tilde{w}}} \cdot L_j = -2 - \sum_{k>j} \beta_{kj}.$$

This completes the proof of the proposition. \square

Now onwards we denote the subsequence (i_1, \dots, i_m) by I_{i_1} . Let $\mathcal{D}_{i_1}'' := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and}$

$$\epsilon_l = \begin{cases} + & \text{if } l \notin I_{i_1} \setminus \{i_1\} \\ - & \text{if } l \in I_{i_1} \end{cases} \quad \}.$$

Let $\mathcal{D}_{i_1}''' := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and}$

$$\epsilon_l = \begin{cases} + & \text{if } l \notin I_{i_1} \\ - & \text{if } l \in I_{i_1} \end{cases} \quad \}.$$

Proposition 7.5. *The curve $L_{I_{i_1}}$ is given by*

$$L_{I_{i_1}} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1},$$

where σ_{i_1} (respectively, σ'_{i_1}) is the cone generated by \mathcal{D}''_{i_1} (respectively, \mathcal{D}'''_{i_1}).

Proof. As in the proof of Lemma 7.3, we start with $j = i_1$ and L_{i_1} is the Schubert line in $Y_{\tilde{w}_{i_1}}$. By Lemma 7.3, we have

$$L_{i_1} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1}.$$

By definition of L_I , we have

$$\sigma_{i_2-1}^0 \circ \cdots \circ \sigma_{i_1+1}^0(L_{i_1}) = L_{i_1} \text{ in } Y_{\tilde{w}_{i_2-1}}$$

and

$$\sigma_{i_2}^1 \circ \sigma_{i_2-1}^0 \circ \cdots \circ \sigma_{i_1+1}^0(L_{i_1}) = L_{\{i_1, i_2\}} \text{ in } Y_{\tilde{w}_{i_2}}.$$

By repeating the process we conclude that

$$L_{I_{i_1}} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1},$$

where σ_{i_1} and σ'_{i_1} are as described in the statement. This completes the proof of the proposition. \square

Recall $NE(X)$ is the real convex cone in $N_1(X)$ generated by classes of irreducible curves. The Mori cone $\overline{NE}(X)$ is the closure of $NE(X)$ in $N_1(X)$ and it is a strongly convex cone of maximal dimension (see for instance [CLS11, Chapter 6, page 293]). Now we describe the Mori cone of the toric limit $Y_{\tilde{w}}$ in terms of the curves $L_{I_{i_j}}$'s defined above. For this we need the following notation (see also [Cha]). Fix $1 \leq i \leq r$. Define:

- (1) Let $r \geq j > j_1 = i \geq 1$ and define $a_{1,j} := \beta_{j_1 j}$.
- (2) Let $r \geq j_2 > j_1$ be the least integer such that $a_{1,j} > 0$, then define for $j > j_2$

$$a_{2,j} := \beta_{i j_2} \beta_{j_2 j} - \beta_{i j}.$$

- (3) Let $k > 2$ and let $r \geq j_k > j_{k-1}$ be the least integer such that $a_{k-1,j} < 0$, then inductively, define for $j > j_k$

$$a_{k,j} := -a_{k-1,j_k} \beta_{j_k j} + a_{k-1,j}.$$

- (4) Let $\tilde{I}_i := \{i = j_1, \dots, j_m\}$.

Example 7.6. *Let $G = SL(5, \mathbb{C})$ and let $\tilde{w} = s_{\beta_1} \cdots s_{\beta_7} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$. Let $i = 1$. Then $j_1 = 1$ and (1) $a_{1,2} = \beta_{12} = \langle \beta_2, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$; (2) $a_{1,3} = \beta_{13} = \langle \beta_3, \check{\beta}_1 \rangle = \langle \alpha_3, \check{\alpha}_2 \rangle = -1$;*

(3) $a_{1,4} = \beta_{14} = \langle \beta_4, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$; (4) $a_{1,5} = \beta_{15} = \langle \beta_5, \check{\beta}_1 \rangle = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$;

(5) $a_{1,6} = \beta_{16} = \langle \beta_6, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$; (6) $a_{1,7} = \beta_{17} = \langle \beta_7, \check{\beta}_1 \rangle = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$.

Then by definition of j_2 , we have $j_2 = 5$ and (1) $a_{2,6} = \beta_{15} \beta_{56} - \beta_{16} = \langle \beta_5, \check{\beta}_1 \rangle \langle \beta_6, \check{\beta}_5 \rangle - \langle \beta_6, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$; (2) $a_{2,7} = \beta_{15} \beta_{57} - \beta_{17} = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$. Then by definition of j_3 , we have $j_3 = 6$ and $a_{3,7} = -a_{2,6} \beta_{67} + a_{2,7} = -(\langle \beta_6, \check{\beta}_5 \rangle)(\langle \beta_7, \check{\beta}_6 \rangle) + (\langle \beta_7, \check{\beta}_5 \rangle) = -(-1)(-1) + (2) = 1$. Therefore, we get $\tilde{I}_1 = \{1, 5, 6\}$.

Example 7.7. We use Example 7.6, for $i = 1$, we have $I_1 = \{1, 5, 6\}$. Then

$$\mathcal{D}_1'' = \{e_1^+, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\} \text{ and } \mathcal{D}_1''' = \{e_1^-, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

Fix $1 \leq i \leq r$. Let

$$I_i := \tilde{I}_i = \{i = j_1, j_2, \dots, j_m\}$$

where j_k 's are as above. With this notation we prove the following (see [PK16, Theorem 22]):

Theorem 7.8. *The set $\{L_{I_i} : 1 \leq i \leq r\}$ of classes of curves forms a basis of $N_1(Y_{\bar{w}})_{\mathbb{Z}}$ and every torus invariant curve in $N_1(Y_{\bar{w}})$ lie in the cone generated by $\{L_{I_i} : 1 \leq i \leq r\}$.*

Proof. By [Cha, Proposition 4.16], for $1 \leq i \leq r$ the curve $r(P_i)$ (see Section 2 for the definition of $r(P_i)$) is given by

$$r(P_i) = [V(\tau_i)],$$

where $\tau_i = \sigma_i \cap \sigma'_i$ and σ_i (respectively, σ'_i) is generated by \mathcal{D}_i'' (respectively, \mathcal{D}_i'''). From Proposition 7.5, we see that the class of the curve L_{I_i} is $r(P_i)$ in $N_1(Y_{\bar{w}})_{\mathbb{Z}}$. By [Cha, Theorem 4.7], we have

$$\overline{NE}(Y_{\bar{w}}) = \sum_{i=1}^r \mathbb{R}_{\geq 0} r(P_i).$$

Also by [Cha, Corollary 4.8], the set $\{r(P_i) : 1 \leq i \leq r\}$ forms a basis of $N_1(Y_{\bar{w}})_{\mathbb{Z}}$. Hence we conclude the assertion of the theorem. \square

We recall some definitions: Let V be a finite dimensional vector space over \mathbb{R} and let K be a (closed) cone in V . A subcone Q in K is called extremal if $u, v \in K, u + v \in Q$ then $u, v \in Q$. A face of K is an extremal subcone. A one-dimensional face is called an extremal ray. Note that an extremal ray is contained in the boundary of K . Then we have (see [PK16, Theorem 30]):

Corollary 7.9. *The extremal rays of the toric limit $Y_{\bar{w}}$ are precisely the curves L_{I_i} for $1 \leq i \leq r$.*

Proof. This follows from the proof of the Theorem 7.8. \square

Let X be a smooth projective variety. An extremal ray R in the Mori cone $\overline{NE}(X) \subset N_1(X)$ is called Mori if $R \cdot K_X < 0$, where K_X is the canonical divisor in X . Recall that $\overline{NE}(Y_{\bar{w}})$ is a strongly convex rational polyhedral cone of maximal dimension in $N_1(Y_{\bar{w}})$. We have the following (see [PK16, Theorem 35]):

Corollary 7.10. *Fix $1 \leq i \leq r$, the class of curve L_{I_i} is Mori ray if and only if either $|\gamma_{P_i}(1)| = 0$, or $|\gamma_{P_i}(1)| = 1$ with $c_j = 1$ for $\gamma_j \in \gamma_{P_i}(1)$.*

Proof. Since $Y_{\bar{w}}$ is a Bott tower (see Corollary 4.2), then the result follows from Proposition 7.5 and [Cha, Theorem 8.1]. \square

Now we prove a general result for smooth projective toric varieties,

Lemma 7.11. *Let X be a smooth projective toric variety of dimension r . Then X is Fano if and only if every extremal ray is Mori.*

Proof. By [CLS11, Theorem 6.3.20] (Toric Cone Theorem), we have

$$\overline{NE}(X) = \sum_{\tau \in \Sigma(r-1)} \mathbb{R}_{\geq 0}[V(\tau)]. \quad (7.4)$$

If X is Fano, then by definition, $-K_X$ is ample. By toric Kleiman criterion for ampleness [CLS11, Theorem 6.3.13], we can see that $-K_X \cdot V(\tau) > 0$ for all $\tau \in \Sigma(r-1)$. Then $K_X \cdot V(\tau) < 0$ for all $\tau \in \Sigma(r-1)$. In particular, every extremal ray is Mori.

Conversely, let $\mathbb{R}_{\geq 0}[V(\tau)]$ be an extremal ray, by assumption it is a Mori ray. Then by definition of a Mori ray, we have $K_X \cdot V(\tau) < 0$. This implies $-K_X \cdot V(\tau) > 0$. By (7.4), $\overline{NE}(X)$ is a polyhedral cone and hence the extremal rays generate the cone $\overline{NE}(X)$. Hence we see that $-K_X \cdot C > 0$ for all classes of curves $[C]$ in $\overline{NE}(X)$. Again by toric Kleiman criterion for ampleness, we conclude that $-K_X$ is ample and hence X is Fano. \square

Then we have the following (see [PK16, Corollary 36]):

Corollary 7.12. *The toric limit $Y_{\bar{w}}$ is Fano if and only if every extremal ray in $\overline{NE}(Y_{\bar{w}})$ is Mori.*

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B. NARASIMHA CHARY, INSTITUT FOURIER, UMR 5582 DU CNRS, UNIVERSITÉ DE GRENOBLE ALPES, CS 40700, 38058, GRENOBLE CEDEX 09, FRANCE., EMAIL: NARASIMHA-CHARY.BONALA@UNIV-GRENOBLE-APLES.FR