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A REALIZATION THEOREM FOR SETS OF LENGTHS IN NUMERICAL MONOIDS

ALFRED GEROLDINGER AND WOLFGANG A. SCHMID

ABSTRACT. We show that for every finite nonempty subset of $\mathbb{N}_{\geq 2}$ there are a numerical monoid H and a squarefree element $a \in H$ whose set of lengths $L(a)$ is equal to L .

1. INTRODUCTION

In the last decade the arithmetic of numerical monoids has found wide interest in the literature. Since numerical monoids are finitely generated, every element of a given monoid can be written as a sum of atoms and all arithmetical invariants describing the non-uniqueness of factorizations are finite. The focus of research was on obtaining precise values for the arithmetical invariants (e.g., [2, 21, 3]), on their interplay with minimal relations of a given presentation (e.g., [8]), and also on computational aspects (e.g., [13, 14] and [11] for a software package in GAP). A further direction of research was to establish realization results for arithmetical parameters. This means to show that there are numerical monoids whose arithmetical parameters have prescribed values. So for example, it was proved only recently that every finite set (with some obvious restrictions) can be realized as the set of catenary degrees of a numerical monoid ([22]). The goal of the present note is to show a realization theorem for sets of lengths.

Let H be a numerical monoid. If $a \in H$ and $a = u_1 + \dots + u_k$, where u_1, \dots, u_k are atoms of H , then k is called a factorization length of a and the set $L(a) \subset \mathbb{N}$ of all factorization lengths is called the set of lengths of a . Further, $\mathcal{L}(H) = \{L(a) \mid a \in H\}$ denotes the system of sets of lengths of H . It is easy to see that all sets of lengths are finite nonempty and can get arbitrarily large, and it is well-known that they have a well-defined structure (see the beginning of Section 3). As a converse, we show in the present paper that for every finite nonempty set $L \subset \mathbb{N}_{\geq 2}$ there is a numerical monoid H and a squarefree element $a \in H$ such that $L(a) = L$ (Theorem 3.3). In fact, we show more precisely that the number of factorizations of each length can be prescribed. Several types of realization results for sets of lengths are known in the literature, most of them in the setting of Krull monoids (see [20, 16, 23, 12], [15, Theorem 7.4.1]). However, we know that if H is a numerical monoid, then $\mathcal{L}(H) \neq \mathcal{L}(H')$ for every Krull monoid H' (see [18, Theorem 5.5] and note that every numerical monoid is strongly primary).

It is an open problem which finite sets of positive integers can occur as sets of distances of numerical monoids. Based on our main result we can show that every finite set is contained in a set of distances of a numerical monoid (Corollary 3.4). There is a vibrant interplay between numerical monoids, and more generally affine monoids, and the associated semigroup algebras ([5, 4, 7]). In Corollary 3.5 we shift our realization result from numerical monoids to numerical semigroup algebras. In Section 4 we study the question which finite nonempty sets are sets of lengths in all proper numerical monoids (Theorem 4.1).

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2. BACKGROUND ON THE ARITHMETIC OF NUMERICAL MONOIDS

We denote by $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ the set of prime numbers, positive integers, integers, and rational numbers respectively. For $a, b \in \mathbb{Q}$, let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ be the discrete interval of integers lying between a and b . If $A, B \subset \mathbb{Z}$, then $A + B = \{a + b \mid a \in A, b \in B\}$ denotes the sumset and $kA = A + \dots + A$ is the k -fold sumset for every $k \in \mathbb{N}$. If $A = \{m_1, \dots, m_k\} \subset \mathbb{Z}$ with $m_{i-1} < m_i$ for each $i \in [2, k]$, then $\Delta(A) = \{m_i - m_{i-1} \mid i \in [2, k]\} \subset \mathbb{N}$ is the set of distances of L . Note that $\Delta(A) = \emptyset$ if and only if $|A| \leq 1$.

By a *monoid*, we mean a commutative cancellative semigroup with identity element. Let H be a monoid. Then H^\times denotes the group of invertible elements, $\mathfrak{q}(H)$ the quotient group of H , and $\mathcal{A}(H)$ the set of atoms (irreducible elements) of H . We say that H is reduced if the identity element is the only invertible element. We call $H_{\text{red}} = H/H^\times$ the reduced monoid associated to H . A *numerical monoid* is a submonoid of $(\mathbb{N}_0, +)$ whose complement in \mathbb{N}_0 is finite. Every numerical monoid is finitely generated, reduced, and its quotient group is \mathbb{Z} . For any set P , let $\mathcal{F}(P)$ denote the free abelian monoid with basis P . Then, using additive notation, every element $a \in \mathfrak{q}(\mathcal{F}(P))$ can be written uniquely in the form

$$a = \sum_{p \in P} l_p p,$$

where $l_p \in \mathbb{Z}$ for each $p \in P$, and all but finitely many l_p are equal to 0. For $a = \sum_{p \in P} l_p p \in \mathcal{F}(P)$, we set $|a| = \sum_{p \in P} l_p \in \mathbb{N}_0$ and call it the length of a .

We recall some arithmetical concept of monoids. Since our focus is on numerical monoids we use additive notation. Let H be an additively written monoid. The (additively written) free abelian monoid $Z(H) = \mathcal{F}(\mathcal{A}(H_{\text{red}}))$ is called the *factorization monoid* of H and the canonical epimorphism $\pi: Z(H) \rightarrow H_{\text{red}}$ is the factorization homomorphism. For $a \in H$ and $k \in \mathbb{N}$,

$$\begin{aligned} Z_H(a) &= Z(a) = \pi^{-1}(a + H^\times) \subset Z(H) \quad \text{is the set of factorizations of } a, \\ Z_{H,k}(a) &= Z_k(a) = \{z \in Z(a) \mid |z| = k\} \quad \text{is the set of factorizations of } a \text{ of length } k, \quad \text{and} \\ L_H(a) &= L(a) = \{|z| \mid z \in Z(a)\} \subset \mathbb{N}_0 \quad \text{is the set of lengths of } a. \end{aligned}$$

Thus, by definition, $L(a) = \{0\}$ if and only if $a \in H^\times$ and $L(a) = \{1\}$ if and only if $a \in \mathcal{A}(H)$. The monoid H is said to be atomic if $Z(a) \neq \emptyset$ for all $a \in H$ (equivalently, every non invertible element is a finite sum of atoms). We call

- $\mathcal{L}(H) = \{L(a) \mid a \in H\}$ the *system of sets of lengths* of H , and
- $\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L)$ the *set of distances* (also called *delta set*) of H .

Every numerical monoid H is atomic with finite set of distances $\Delta(H)$, and $\Delta(H) = \emptyset$ if and only if $H = \mathbb{N}_0$.

3. A REALIZATION THEOREM FOR SETS OF LENGTHS

The goal of this section is to prove our main realization theorem, namely that for every finite nonempty subset $L \subset \mathbb{N}_{\geq 2}$ there exists a numerical monoid H such that L is a set of lengths of H (Theorem 3.3). We show the existence of this monoid by an explicit recursive construction over the size of L . Instead of working with numerical monoids directly, we work in the setting of finitely generated additive submonoids of the nonnegative rationals. Additive submonoids of $(\mathbb{Q}_{\geq 0}, +)$ are called Puiseux monoids and have recently been studied in a series of papers by F. Gotti et al. (e.g., [19]). In the setting of Puiseux monoids all arithmetical concepts refer to addition and not to multiplication of rationals. In particular, an element a of a Puiseux monoid H is said to be squarefree if there are no nonzero elements $b, c \in H$ such that $a = b + b + c$.

Clearly, the constructed numerical monoid heavily depends on the given set L . This is inevitable because for every fixed numerical monoid H , sets of lengths have a well-defined structure. Indeed, there is a constant $M \in \mathbb{N}_0$ (just depending on H) such that every $L \in \mathcal{L}(H)$ has the form

$$(3.1) \quad L = y + (L' \cup \{\nu d \mid \nu \in [0, l]\} \cup L'') \subset y + d\mathbb{Z},$$

where $d = \min \Delta(H)$, $y \in \mathbb{Z}$, $L' \subset [-M, -1]$, and $L'' \subset ld + [1, M]$ ([15, Theorem 4.3.6]).

We start with a technical lemma.

Lemma 3.1. *Let $k \in \mathbb{N}_{\geq 2}$. Then there exist pairwise distinct nonzero $c_1, \dots, c_k \in [-k^{k-1}, k^{k-1}]$ with $c_1 + \dots + c_k = 0$ such that for all primes $p > (k+1)k^{k-1}$ the following property holds: if $l_1, \dots, l_k \in \mathbb{N}_0$ such that $\sum_{i=1}^k l_i c_i \equiv 0 \pmod{p}$, then*

$$l_1 = \dots = l_k = 0 \text{ or } l_1 = \dots = l_k = 1 \text{ or } l_1 + \dots + l_k > k.$$

Proof. For $i \in [1, k-1]$ we define $c_i = k^{i-1}$, and we set $c_k = -\sum_{i=1}^{k-1} c_i$. Then clearly,

$$c_k = -\sum_{i=1}^{k-1} c_i = -\sum_{i=1}^{k-1} k^{i-1} = -\frac{k^{k-1} - 1}{k-1}.$$

Now we choose a prime $p > (k+1)k^{k-1}$ and $l_1, \dots, l_k \in \mathbb{N}_0$ such that $\sum_{i=1}^k l_i c_i \equiv 0 \pmod{p}$ and $\sum_{i=1}^k l_i > 0$. We may distinguish the following two cases.

CASE 1: $\sum_{i=1}^{k-1} l_i c_i \geq p$ or $l_k c_k \leq -p$.

If $p \leq \sum_{i=1}^{k-1} l_i c_i \leq (\sum_{i=1}^{k-1} l_i) c_{k-1}$, then

$$\sum_{i=1}^k l_i \geq \sum_{i=1}^{k-1} l_i \geq \frac{p}{c_{k-1}} > \frac{(k+1)k^{k-1}}{c_{k-1}} \geq k+1.$$

If $p \leq l_k |c_k|$, then

$$\sum_{i=1}^k l_i \geq l_k \geq \frac{p}{|c_k|} > \frac{(k+1)k^{k-1}}{|c_k|} \geq k+1.$$

CASE 2: $\sum_{i=1}^{k-1} l_i c_i < p$ and $l_k c_k > -p$.

Since $\sum_{i=1}^k l_i c_i \equiv 0 \pmod{p}$, we infer that $\sum_{i=1}^k l_i c_i = 0$. Suppose that there is a $j \in [1, k]$ with $l_j \geq k$. Since at least two elements of l_1, \dots, l_k are positive, it follows that $\sum_{i=1}^k l_i > k$. Suppose that $l_i \in [0, k-1]$ for all $i \in [1, k]$. Since $0 = \sum_{i=1}^k l_i c_i$, the definition of c_1, \dots, c_k implies that

$$\sum_{i=1}^{k-1} l_i k^{i-1} = \sum_{i=1}^{k-1} l_k k^{i-1}.$$

By the uniqueness of the k -adic digit expansion, we infer that $l_i = l_k$ for all $i \in [1, k-1]$. If $l_1 = 1$, then $l_1 = \dots = l_k = 1$. If $l_1 > 1$, then $l_1 + \dots + l_k = kl_1 > k$. \square

The following proposition will be our key tool to do the recursive construction step in Theorem 3.3. For every prime $p \in \mathbb{P}$, we denote by v_p the usual p -adic valuation of the rationals, that is, for $q \in \mathbb{Q} \setminus \{0\}$, $v_p(q)$ the integer j such that $q = p^j \frac{a}{b}$ with integers a, b such that $p \nmid ab$. Moreover, we set $v_p(0) = \infty$.

Proposition 3.2. *Let $k \in \mathbb{N}_{\geq 2}$ and $H \subset (\mathbb{Q}_{\geq 0}, +)$ be a finitely generated monoid with $\mathbb{N}_0 \subset H$ and $\mathcal{A}(H) \subset \mathbb{Q}_{<1}$. Then there exists a finitely generated monoid H' with $H \subset H'$ and $\mathcal{A}(H) \subset \mathcal{A}(H') \subset \mathbb{Q}_{<1}$ such that the following properties are satisfied:*

- (a) For all $u \in H$ with $u < 1$ we have $Z_H(u) = Z_{H'}(u)$.
- (b) $Z_{H'}(1) = Z_H(1) \uplus \{q_1 + \dots + q_k\}$, where q_1, \dots, q_k are pairwise distinct and $\mathcal{A}(H') = \mathcal{A}(H) \uplus \{q_1, \dots, q_k\}$.

Proof. We set

$$\mathcal{A}(H) = \left\{ \frac{a_1}{b_1}, \dots, \frac{a_s}{b_s} \right\}$$

where $a_i, b_i \in \mathbb{N}$ with $\gcd(a_i, b_i) = 1$ for all $i \in [1, s]$. Let $c_1, \dots, c_k \in [-k^{k-1}, k^{k-1}]$ such that all properties of Lemma 3.1 are satisfied. We choose a prime number $p \in \mathbb{N}$ such that

$$p \nmid \text{lcm}(b_1, \dots, b_s) \quad \text{and} \quad p > (k+1)k^{k-1},$$

and we define

$$q_i = \frac{p + c_i}{kp} \quad \text{for every } i \in [1, k].$$

By construction, we have $q_1 + \dots + q_k = 1$ and $v_p(q_i) = -1$ whence $q_i \notin H$ for all $i \in [1, k]$. We define

$$H' = [H, q_1, \dots, q_k] \subset (\mathbb{Q}_{\geq 0}, +)$$

to be the additive submonoid of nonnegative rationals generated by the elements of H and by q_1, \dots, q_k . Thus H' is generated by $\mathcal{A}(H) \cup \{q_1, \dots, q_k\}$ whence finitely generated. Since H' is reduced, [15, Proposition 1.1.7] implies that H' is atomic and

$$(3.2) \quad \mathcal{A}(H') \subset \mathcal{A}(H) \cup \{q_1, \dots, q_k\}.$$

We continue with the following assertions.

A1. $\{q_1, \dots, q_k\} \subset \mathcal{A}(H')$.

A2. Let $u \in H$ and suppose that u has a factorization $z \in Z_{H'}(u)$ which is divisible by some element from $\{q_1, \dots, q_k\}$. Then either $u > 1$ or $z = q_1 + \dots + q_k \in Z_{H'}(u)$ (whence in particular $u = 1$).

Proof of A1. Assume to the contrary that there is an $i \in [1, k]$ such that $q_i \notin \mathcal{A}(H')$. Since $q_i \notin H$, it is divisible by an atom from $\mathcal{A}(H') \setminus \mathcal{A}(H) \subset \{q_1, \dots, q_k\}$, say $q_i = q_j + b$ with $j \in [1, k] \setminus \{i\}$ and $b \in H' \setminus \{0\}$. We claim that $b \notin H$. Since $0 \neq b = q_i - q_j$ and $0 \neq |c_i - c_j| \leq 2k^{k-1} < p$,

$$v_p(q_i - q_j) = v_p\left(\frac{c_i - c_j}{kp}\right) = -1$$

which implies that $b \notin H$. Thus there is an $l \in [1, k]$ such that $b = q_l + d$ with $d \in H' \subset \mathbb{Q}_{\geq 0}$. Since $p > (k+1)k^{k-1} \geq 3k^{k-1} \geq |c_i| + |c_j| + |c_l|$, it follows that

$$q_i = q_j + q_l + d \geq q_j + q_l = \frac{2p + c_j + c_l}{kp} > \frac{p + c_i}{kp} = q_i,$$

a contradiction. □[Proof of A1]

Proof of A2. Since u has a factorization which is divisible by some element from $\{q_1, \dots, q_k\}$, there are $l_1, \dots, l_k \in \mathbb{N}_0$ and $v \in H$ such that

$$u = v + \sum_{i=1}^k l_i q_i \quad \text{and} \quad \sum_{i=1}^k l_i > 0.$$

Since $v_p(u) \geq 0$ and $v_p(v) \geq 0$, it follows that

$$0 \leq v_p(u - v) = v_p\left(\sum_{i=1}^k l_i q_i\right) = v_p\left(\frac{\sum_{i=1}^k l_i p + \sum_{i=1}^k l_i c_i}{kp}\right)$$

whence $\sum_{i=1}^k l_i c_i \equiv 0 \pmod{p}$. Therefore Lemma 3.1 implies that

$$l_1 = \dots = l_k = 1 \quad \text{or} \quad \sum_{i=1}^k l_i > k.$$

If $\sum_{i=1}^k l_i > k$ and $j \in [1, k]$ with $q_j = \min\{q_1, \dots, q_k\}$, then

$$(3.3) \quad u = v + \sum_{i=1}^l l_i q_i \geq (k+1)q_j = (k+1)\frac{p+c_j}{kp} > 1,$$

where the last inequality uses that $p > (k+1)k^{k-1} \geq (k+1)|c_j|$. If $l_1 = \dots = l_k = 1$, then

$$u = \sum_{i=1}^l l_i q_i + v = q_1 + \dots + q_k + v = 1 + v.$$

Thus $v > 0$ implies $u > 1$ and $v = 0$ implies $u = 1$ and $z = q_1 + \dots + q_k$. □[Proof of **A2**]

If $u \in \mathcal{A}(H)$, then $u < 1$ by assumption and **A2** implies that u is not divisible by any element from $\{q_1, \dots, q_k\}$ and therefore $u \in \mathcal{A}(H')$. Thus we obtain that $\mathcal{A}(H) \subset \mathcal{A}(H')$ and together with **A1** and (3.2), it follows that

$$(3.4) \quad \mathcal{A}(H') = \mathcal{A}(H) \uplus \{q_1, \dots, q_k\}.$$

Thus, we have that

$$(3.5) \quad \mathcal{Z}(H) = \mathcal{F}(\mathcal{A}(H)) \subset \mathcal{F}(\mathcal{A}(H')) = \mathcal{Z}(H') \quad \text{and} \quad \mathcal{Z}_H(u) \subset \mathcal{Z}_{H'}(u)$$

for every $u \in H$. If $u < 1$, then **A2** implies that $\mathcal{Z}_H(u) = \mathcal{Z}_{H'}(u)$.

It remains to show Property (b) given in Proposition 3.2, namely that

$$\mathcal{Z}_H(1) \uplus \{q_1 + \dots + q_k\} = \mathcal{Z}_{H'}(1).$$

We see from Equation (3.5) that $\mathcal{Z}_H(1) \uplus \{q_1 + \dots + q_k\} \subset \mathcal{Z}_{H'}(1)$. Conversely, let z be a factorization of 1 in H' . Then either $z \in \mathcal{Z}_H(1)$ or z is divisible (in $\mathcal{Z}(H')$) by some element from $\{q_1, \dots, q_k\}$. In the latter case **A2** implies that $z = q_1 + \dots + q_k \in \mathcal{Z}(H')$. □

Theorem 3.3. *Let $L \subset \mathbb{N}_{\geq 2}$ be a finite nonempty set and $f: L \rightarrow \mathbb{N}$ a map. Then there exist a numerical monoid H and a squarefree element $a \in H$ such that*

$$(3.6) \quad \mathcal{L}(a) = L \quad \text{and} \quad |\mathcal{Z}_k(a)| = f(k) \quad \text{for every } k \in L.$$

Proof. Every finitely generated submonoid of $(\mathbb{Q}_{\geq 0}, +)$ is isomorphic to a numerical monoid (cf. [19, Proposition 3.2]) and the isomorphism maps squarefree elements onto squarefree elements. Thus it is sufficient to show that, for every set L and every map f as in the statement of the theorem, there is a finitely generated submonoid H of the nonnegative rationals with $\mathbb{N}_0 \subset H$ and $\mathcal{A}(H) \subset \mathbb{Q}_{< 1}$ such that the element $a = 1 \in H$ is squarefree in H and has the properties given in (3.6).

Clearly, it is equivalent to consider nonzero maps $f: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}_0$ with finite support and to find a monoid H as above such that $|\mathcal{Z}_k(1)| = f(k)$ for every $k \in \mathbb{N}_{\geq 2}$ and 1 is squarefree in H . For every nonzero map $f: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}_0$ with finite support $\sum_{k \geq 2} f(k)$ is a positive integer and we proceed by induction on this sum.

To do the base case, let $f: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}_0$ be a map with $\sum_{k \geq 2} f(k) = 1$. Let $k \in \mathbb{N}_{\geq 2}$ with $f(k) = 1$. We have to find a finitely generated monoid $H \subset (\mathbb{Q}_{\geq 0}, +)$ with $\mathcal{A}(H) \subset \mathbb{Q}_{< 1}$ and pairwise distinct atoms $q_1, \dots, q_k \in H$ such that $\mathcal{Z}_H(1) = \{q_1 + \dots + q_k\}$.

We proceed along the lines of the proof of **A2** in Proposition 3.2. Indeed, we choose $c_1, \dots, c_k \in [-k^{k-1}, k^{k-1}]$ such that all properties of Lemma 3.1 are satisfied and pick a prime number $p \in \mathbb{N}$ with $p > (k+1)k^{k-1}$. We set

$$q_i = \frac{p + c_i}{kp} \quad \text{for every } i \in [1, k]$$

and define $H = [q_1, \dots, q_k] \subset \mathbb{Q}_{\geq 0}$. By [15, Proposition 1.1.7], $\mathcal{A}(H) \subset \{q_1, \dots, q_k\}$. Since for all (not necessarily distinct) $r, s, t \in [1, k]$ we have $q_r < q_s + q_t$, it follows that $q_r \in \mathcal{A}(H)$. Thus we obtain that $\mathcal{A}(H) = \{q_1, \dots, q_k\}$. Since $q_1 + \dots + q_k = 1$, it follows that $\{q_1 + \dots + q_k\} \subset \mathcal{Z}_H(1)$. To show equality,

let $l_1, \dots, l_k \in \mathbb{N}_0$ such that $1 = \sum_{i=1}^k l_i q_i$. It follows that $\sum_{i=1}^k l_i c_i \equiv 0 \pmod{p}$. Therefore Lemma 3.1 implies that

$$l_1 = \dots = l_k = 1 \quad \text{or} \quad \sum_{i=1}^k l_i > k.$$

If $\sum_{i=1}^k l_i > k$ and $j \in [1, k]$ with $q_j = \min\{q_1, \dots, q_k\}$, then

$$(3.7) \quad 1 = \sum_{i=1}^l l_i q_i \geq (k+1)q_j = (k+1)\frac{p+c_j}{kp} > 1,$$

a contradiction. Thus $l_1 = \dots = l_k = 1$ and the claim follows.

Now let $N \in \mathbb{N}_{\geq 2}$ and suppose that the assertion holds all nonzero maps $f: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}_0$ with finite support and with $\sum_{k \geq 2} f(k) < N$. Let $f_0: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}_0$ with $\sum_{k \geq 2} f_0(k) = N$. We choose an element $k_0 \in \mathbb{N}_{\geq 2}$ with $f_0(k_0) \neq 0$ and define a map $f_1: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}_0$ as $f_1(k_0) = f_0(k_0) - 1$ and $f_1(k) = f_0(k)$ for all $k \in \mathbb{N}_{\geq 2} \setminus \{k_0\}$. By the induction hypothesis, there exists a finitely generated monoid $H_1 \subset (\mathbb{Q}_{\geq 0}, +)$ with $\mathbb{N}_0 \subset H_1$ and $\mathcal{A}(H_1) \subset \mathbb{Q}_{< 1}$ such that $|\mathcal{Z}_{H_1, k}(1)| = f_1(k)$ for every $k \in \mathbb{N}_{\geq 2}$ and 1 is squarefree in H_1 . By Proposition 3.2 there exist a finitely generated monoid $H_0 \subset (\mathbb{Q}_{\geq 0}, +)$ such that

$$\mathcal{Z}_{H_0}(1) = \mathcal{Z}_{H_1}(1) \uplus \{q_1 + \dots + q_{k_0}\},$$

where q_1, \dots, q_{k_0} are pairwise distinct and $\mathcal{A}(H_0) = \mathcal{A}(H_1) \uplus \{q_1, \dots, q_{k_0}\} \subset \mathbb{Q}_{< 1}$. Since q_1, \dots, q_{k_0} are pairwise distinct and since 1 was squarefree in H_1 , it follows that 1 is squarefree in H_0 . Moreover, $\mathcal{Z}_{H_0, k}(1) = \mathcal{Z}_{H_1, k}(1)$ for all $k \in \mathbb{N}_{\geq 2} \setminus \{k_0\}$ and $\mathcal{Z}_{H_0, k_0}(1) = \mathcal{Z}_{H_1, k_0}(1) \uplus \{q_1 + \dots + q_{k_0}\}$. In particular, we have $|\mathcal{Z}_{H_0, k}(1)| = f_0(k)$ for every $k \in \mathbb{N}_{\geq 2}$. \square

We continue with a corollary on sets of distances. Let H be an atomic monoid with nonempty set of distances $\Delta(H)$. Then it is easy to verify that $\min \Delta(H) = \gcd \Delta(H)$, and the question is which finite sets D with $\min D = \gcd D$ can be realized as a set of distances in a given class of monoids or domains. The question has an affirmative answer in the class of finitely generated Krull monoids ([17]). If H is a numerical monoid generated by two atoms, say $\mathcal{A}(H) = \{n_1, n_2\}$, then $\Delta(H) = \{|n_2 - n_1|\}$ whence every singleton occurs as a set of distances of a numerical monoid. There are periodicity results on individual sets $\Delta(L(a))$ for elements in a numerical monoid ([9]), but the only realization result beyond the simple observation above is due to Colton and Kaplan ([10]). They show that every two-element set D with $\min D = \gcd D$ can be realized as the set of distances of a numerical monoid. As a consequence of Theorem 3.3 we obtain that every finite set is contained in the set of distances of a numerical monoid (this was achieved first by explicit constructions in [6, Corollary 4.8]).

Corollary 3.4. *For every finite nonempty subset $D \subset \mathbb{N}$ there is a numerical monoid H such that $D \subset \Delta(H)$.*

Proof. Let $D = \{d_1, \dots, d_k\} \subset \mathbb{N}$ be a finite nonempty subset. By Theorem 3.3 there is a numerical monoid H such that $L = \{2, 2+d_1, 2+d_1+d_2, \dots, 2+d_1+\dots+d_k\} \in \mathcal{L}(H)$ whence $D = \Delta(L) \subset \Delta(H)$. \square

Let K be a field and H a numerical monoid. The semigroup algebra

$$K[H] = \left\{ \sum_{h \in H} a_h X^h \mid a_h \in K \text{ for all } h \in H \text{ and almost all } a_h \text{ are zero} \right\} \subset K[X]$$

is a one-dimensional noetherian domain and its integral closures $K[X]$ is a finitely generated module over $K[H]$. Thus $K[H]$ is weakly Krull, $\text{Pic}(K[H])$ is finite if K is finite whence it satisfies all arithmetical finiteness results established for weakly Krull Mori domains with finite class group (see [15] for basic information). However, all results on $\mathcal{L}(K[H])$ so far depend on detailed information on the Picard group and the distribution of height one prime ideals not containing the conductor in the Picard group.

Corollary 3.5. *Let K be a field, $L \subset \mathbb{N}_{\geq 2}$ a finite nonempty set, and $f: L \rightarrow \mathbb{N}$ a map. Then there is a numerical monoid H and a squarefree element $g \in K[H]$ such that*

$$\mathsf{L}_{K[H]}(g) = L \quad \text{and} \quad |\mathsf{Z}_{K[H],k}(g)| = f(k) \quad \text{for every } k \in L.$$

Proof. By Theorem 3.3 there is a numerical monoid H and a squarefree element $c \in H$ having the required properties. Clearly, the additive monoid H is isomorphic to the multiplicative monoid of monomials

$$H' = \{X^h \mid h \in H\} \subset K[H].$$

Since $K[H]^\times = K^\times$, the monoid $H'' = \{cX^h \mid h \in H, c \in K^\times\} \subset K[H]$ is a divisor-closed submonoid and $H''_{\text{red}} \cong H'$. Thus for every $k \in L$ and the element $g = X^c \in K[H]$ we obtain that

$$|\mathsf{Z}_{K[H],k}(X^c)| = |\mathsf{Z}_{H'',k}(X^c)| = |\mathsf{Z}_{H',k}(X^c)| = |\mathsf{Z}_{H,k}(c)| = f(k)$$

whence the assertion follows. \square

4. COMPARING THE SYSTEMS $\mathcal{L}(H)$ OF NUMERICAL MONOIDS H

In the previous section we showed that for every finite nonempty subset $L \subset \mathbb{N}_{\geq 2}$ there is a numerical monoid H such that $L \in \mathcal{L}(H)$. One might even go a step further and ask which sets L are sets of lengths in many or even in all numerical monoids.

These questions have already been answered in Krull monoids with finite class group and prime divisors in all classes, and we briefly recall the situation (see [18, Section 3] for details and further references). To begin with, it clearly makes sense to restrict our attention to monoids which are not half-factorial i.e., to monoids H having some $L \in \mathcal{L}(H)$ with $|L| > 1$. Let H be a Krull monoid with finite class group G and prime divisors in all classes. Then H is half-factorial if and only if $|G| < 3$. If $|G| > 4$, then the standing conjecture is that $\mathcal{L}(H) \neq \mathcal{L}(H')$ for all Krull monoids H' having prime divisors in all classes and class group G' not being isomorphic to G . For every finite nonempty subset $L \subset \mathbb{N}_{\geq 2}$ there are finitely many abelian groups, say G_1, \dots, G_n such that L is a set of lengths in $\mathcal{L}(H)$ provided that the class group G of H is not isomorphic to any of the groups G_1, \dots, G_n . Furthermore, for the intersection of all systems $\mathcal{L}(S)$ of non-half-factorial Krull monoids S with prime divisors in all classes, we have

$$\bigcap \mathcal{L}(S) = \{y + 2k + [0, k] \mid y, k \in \mathbb{N}_0\}.$$

The situation is quite different for numerical monoids. Clearly, numerical monoids are isomorphic if and only if they are equal, and \mathbb{N}_0 is the only numerical monoid which is half-factorial. It was a surprising result by Chapman et al. that there are distinct numerical monoids H_1 and H_2 such that $\mathcal{L}(H_1) = \mathcal{L}(H_2)$ ([1]). In contrast to this, here we prove that there are only precisely three sets which are sets of lengths in all numerical monoids. Note that in every (additively written) atomic monoid H with zero-element 0_H and with $H \neq H^\times$ we have $\{0\} = \mathsf{L}_H(0_H)$ (by our convention) and that $\{1\} = \mathsf{L}_H(u)$ for every $u \in \mathcal{A}(H)$.

Theorem 4.1. *We have*

$$\bigcap \mathcal{L}(H) = \{\{0\}, \{1\}, \{2\}\},$$

where the intersection is taken over all numerical monoids $H \subsetneq \mathbb{N}_0$. More precisely, for every $t \in \mathbb{N}_{\geq 6}$ we have

$$\bigcap_{|\mathcal{A}(H)|=t} \mathcal{L}(H) = \{\{0\}, \{1\}, \{2\}\},$$

and for every $s \in [2, 5]$ we have

$$\bigcap_{|\mathcal{A}(H)|=s} \mathcal{L}(H) = \{\{0\}, \{1\}, \{2\}, \{3\}\},$$

where the intersections are taken over all numerical monoids H with the given properties.

Proof. By the observation above, we know that $\{0\}$ and $\{1\}$ are elements of each of the intersections. If H is a numerical monoid with $\mathcal{A}(H) = \{n_1, \dots, n_t\}$, where $t \in \mathbb{N}_{\geq 2}$ and $1 < n_1 < \dots < n_t$, then $\mathsf{L}(2n_1) = \{2\}$. Thus $\{2\}$ is an element of each of the intersections as well.

For $m \in \mathbb{N}_{\geq 2}$ and $d \in \mathbb{N}$, let $H_{m,d}$ be the numerical monoid generated by $\{1 + (m-1)d, 1 + md, \dots, 1 + (2m-2)d\}$; note that this is a numerical monoid because $\gcd(1 + (m-1)d, \dots, 1 + (2m-2)d) = 1$, and $1 + (2m-2)d < 2(1 + (m-1)d)$ guarantees that each of the generating elements is an atom. By [6, Theorem 3.9]

$$\Delta(H_{m,d}) = \{d\}.$$

Thus, for distinct d and d' , we get that $\mathcal{L}(H_{m,d}) \cap \mathcal{L}(H_{m,d'})$ cannot contain sets of cardinality greater than 1, in other words this intersection is a subset of $\{\{k\} \mid k \in \mathbb{N}_0\}$. This implies that each of the intersections in the statement of our result is contained in $\{\{k\} \mid k \in \mathbb{N}_0\}$.

To complete the proof of our result, it suffices to establish the following assertions.

- A1.** For every $m \geq 2$ and for every $k \geq 4$, there is a numerical monoid H with $|\mathcal{A}(H)| = m$ such that $\{k\} \notin \mathcal{L}(H)$.
- A2.** For every $m \geq 6$, there is a numerical monoid H with $|\mathcal{A}(H)| = m$ such that $\{3\} \notin \mathcal{L}(H)$.
- A3.** If $|\mathcal{A}(H)| = 3$, then $\{3\} \in \mathcal{L}(H)$.
- A4.** If $|\mathcal{A}(H)| = 4$, then $\{3\} \in \mathcal{L}(H)$.
- A5.** If $|\mathcal{A}(H)| = 5$, then $\{3\} \in \mathcal{L}(H)$.

Proof of A1. Let $m \geq 2$ and let H be the numerical monoid generated by $A = [m, 2m-1]$; note that $\mathcal{A}(H) = A$. First, we assert that it suffices to show that $\{4\} \notin \mathcal{L}(H)$. Let $k \geq 5$, and let $a \in H$ with $k \in \mathsf{L}(a)$, say, $a = a_1 + \dots + a_k$ with $a_i \in \mathcal{A}(H)$. Assuming $\{4\} \notin \mathcal{L}(H)$, it follows that $a' = a_1 + a_2 + a_3 + a_4$ has a factorization $a' = a'_1 + \dots + a'_l$ with $a'_i \in \mathcal{A}(H)$ and $l \neq 4$. Then, $a'_1 + \dots + a'_l + a_5 + \dots + a_k$ is a factorization of lengths $l + k - 4$ of a , whence $\mathsf{L}(a) \neq \{k\}$.

Now, let $a \in H$ with $4 \in \mathsf{L}(a)$. This means that a is in the 4-fold sumset of A , that is $a \in 4A = [4m, 8m-4]$. If $a \geq 5m$, then $a - m \in 4A$ and $4 \in \mathsf{L}(a - m)$. Thus $5 \in 1 + \mathsf{L}(a - m) \subset \mathsf{L}(a)$, showing that $\mathsf{L}(a) \neq \{4\}$. If $a \leq 5m - 1$, then $a - (m+1) \in [2m, 4m-2] = 2A$ and $2 \in \mathsf{L}(a - (m+1))$. Thus $3 \in 1 + \mathsf{L}(a - (m+1)) \subset \mathsf{L}(a)$, and again $\mathsf{L}(a) \neq \{4\}$. □[Proof of A1]

Proof of A2. Let $m \geq 6$ and let H be the numerical monoid generated by

$$A = \{m\} \cup [m+3, 2m-1] \cup \{2m+1, 2m+2\}.$$

We note that $\mathcal{A}(H) = A$. For the 2-fold, 3-fold, and 4-fold sumsets of A we obtain that

$$\begin{aligned} 2A &= \{2m\} \cup [2m+3, 4m+4], \\ 3A &= \{3m\} \cup [3m+3, 6m+6], \quad \text{and} \\ 4A &= \{4m\} \cup [4m+3, 8m+8]. \end{aligned}$$

which implies that $3A \subset 2A \cup 4A$. Thus for every $a \in H$ with $3 \in \mathsf{L}(a)$ it follows that $\mathsf{L}(a) \cap \{2, 4\} \neq \emptyset$. □[Proof of A2]

Proof of A3. Assume to the contrary that there exists a numerical monoid H with three atoms, say $\mathcal{A}(H) = \{n_1, n_2, n_3\}$ with $1 < n_1 < n_2 < n_3$, such that $\{3\} \notin \mathcal{L}(H)$. Since $3 \in \mathsf{L}(2n_1 + n_2)$, the element $2n_1 + n_2$ must have a further factorization length. Since $2n_1 + n_2$ cannot be a multiple of n_1 , it follows that $\max \mathsf{L}(2n_1 + n_2) = 3$. Thus, $2 \in \mathsf{L}(2n_1 + n_2)$ and it follows that $2n_1 + n_2 = 2n_3$. Similarly, we infer that $3n_1$ must have a factorization of length 2. Since $3n_1 < 2n_1 + n_2 = 2n_3$, it follows that $3n_1 \in \{2n_2, n_2 + n_3\}$.

Suppose that $3n_1 = n_2 + n_3$. Then, using the just established equalities, $n_2 - n_1 = (2n_1 + n_2) - 3n_1 = 2n_3 - (n_2 + n_3) = n_3 - n_2 =: d$. Thus $n_2 = n_1 + d$ and $n_3 = n_1 + 2d$ which implies that $3n_1 = n_2 + n_3 = 2n_1 + 3d$ whence $n_1 = 3d$. Since $\gcd(n_1, n_2, n_3) = 1$, it follows that $d = 1$ whence $(n_1, n_2, n_3) = (3, 4, 5)$. However, since $\mathsf{L}(11) = \{3\}$, we obtain a contradiction.

Suppose that $3n_1 = 2n_2$. Then $n_2 - n_1 = 2(n_3 - n_2)$, say $n_3 - n_2 = d$. Then $n_2 = n_1 + 2d$, $3n_1 = 2n_1 + 4d$ whence $n_1 = 4d$, $n_2 = 6d$, and $n_3 = 7d$. Since $\gcd(n_1, n_2, n_3) = 1$, it follows that $d = 1$ whence $(n_1, n_2, n_3) = (4, 6, 7)$. However, since $L(15) = \{3\}$, we obtain a contradiction. \square [Proof of **A3**]

Proof of A4. Assume to the contrary that there exists a numerical monoid H with four atoms, say $\mathcal{A}(H) = \{n_1, n_2, n_3, n_4\}$ with $1 < n_1 < n_2 < n_3 < n_4$, such that $\{3\} \notin \mathcal{L}(H)$. Then as in **A3** we obtain $2 \in L(3n_1)$ and $2 \in L(2n_1 + n_2)$ which implies

$$(4.1) \quad 2n_1 + n_2 \geq 2n_3.$$

If $2n_1 + n_3$ would have a factorization of length at least four, then

$$2n_1 + n_3 \geq n_1 + 3n_2 > (2n_1 + n_2) + n_2 \stackrel{(4.1)}{\geq} 2n_3 + n_2,$$

a contradiction. Thus $2 \in L(2n_1 + n_3)$ which implies $2n_1 + n_3 \in \{n_2 + n_4, 2n_4\}$ and hence

$$(4.2) \quad 2n_1 + n_3 \geq n_2 + n_4.$$

If $2n_1 + n_4$ would have a factorization of length two, then $2n_1 + n_4 \leq 2n_3$ but

$$2n_1 + n_4 > 2n_1 + n_2 \stackrel{(4.1)}{\geq} 2n_3, \quad \text{a contradiction.}$$

Therefore, $2n_1 + n_4$ has a factorization of length at least four which implies that

$$(4.3) \quad \begin{aligned} 2n_1 + n_4 &\geq n_1 + 3n_2 = (2n_1 + n_2) + n_2 + (n_2 - n_1) \\ &\stackrel{(4.1)}{\geq} 2n_3 + n_2 + (n_2 - n_1) = (2n_1 + n_3) + (n_3 - n_1) + 2(n_2 - n_1) \\ &\stackrel{(4.2)}{\geq} n_2 + n_4 + (n_3 - n_1) + 2(n_2 - n_1). \end{aligned}$$

Consequently, we infer that

$$2n_1 \geq n_2 + (n_3 - n_1) + 2(n_2 - n_1) \quad \text{whence} \quad 3n_1 \geq n_2 + n_3 + 2(n_2 - n_1) > n_2 + n_3$$

which implies that $3n_1 \in \{2n_3, n_2 + n_4, n_3 + n_4, 2n_4\}$. If $3n_1 = 2n_3$, then $2n_1 + n_2 \geq n_3 + n_4$ whence $2n_1 + n_3 = 2n_4$ and if $3n_1 \geq n_2 + n_4$, then $2n_1 + n_3 > n_2 + n_4$ whence $2n_1 + n_3 = 2n_4$. Thus in any case we have $2n_1 + n_3 = 2n_4$ and we can improve the last inequality in (4.3) whence

$$2n_1 + n_4 \geq 2n_4 + (n_3 - n_1) + 2(n_2 - n_1).$$

Therefore, $2n_1 \geq n_4 + (n_3 - n_1) + 2(n_2 - n_1)$ and adding n_1 we obtain that $3n_1 \geq n_4 + n_3 + 2(n_2 - n_1) > n_3 + n_4$. This implies that $3n_1 = 2n_4$, a contradiction to $2 \in L(2n_1 + n_2)$. \square [Proof of **A4**]

Proof of A5. Again, assume to the contrary that there exists a numerical monoid H with five atoms, say $\mathcal{A}(H) = \{n_1, n_2, n_3, n_4, n_5\}$ with $1 < n_1 < n_2 < n_3 < n_4 < n_5$, such that $\{3\} \notin \mathcal{L}(H)$. Then as in **A4** we obtain that 2 is an element of $L(3n_1)$, of $L(2n_1 + n_2)$, and of $L(2n_1 + n_3)$. Moreover,

$$(4.4) \quad 2n_1 + n_2 \geq 2n_3 \quad \text{and} \quad 2n_1 + n_3 \geq n_2 + n_4.$$

We proceed to show that $2 \in L(2n_1 + n_4)$. Assume not. Then, $L(2n_1 + n_4)$ contains an element greater than or equal to 4 and it follows that $2n_1 + n_4 \geq n_1 + 3n_2$. Similarly to **A4** we get that, using (4.4),

$$\begin{aligned} 2n_1 + n_4 &\geq n_1 + 3n_2 \\ &= (2n_1 + n_2) + n_2 + (n_2 - n_1) \\ &\geq 2n_3 + n_2 + (n_2 - n_1) \end{aligned}$$

whence $n_4 \geq n_3 + (n_3 - n_1) + 2(n_2 - n_1)$. In combination with $2n_1 + n_3 \geq n_2 + n_4$, that is, $n_3 \geq n_4 + n_2 - 2n_1$, we get that $n_4 \geq n_4 + (n_3 - n_1) + 3(n_2 - n_1) - n_1$. Equivalently, $n_1 \geq (n_3 - n_1) + 3(n_2 - n_1)$ and $5n_1 \geq n_3 + 3n_2$. This yields $3n_1 > n_2 + n_3$. Moreover, $2n_1 + n_4 \geq n_1 + 3n_2$, means $n_1 + n_4 \geq 3n_2$, and this implies $2n_1 + n_2 < 3n_2 \leq n_1 + n_4$. Thus, $2n_1 + n_2 \in \{n_2 + n_3, 2n_3\}$. Yet, since $2n_1 + n_2 > 3n_1 > n_2 + n_3$

this is a contradiction, both $2n_1 + n_2$ and $3n_1$ would need to equal $2n_3$. This contradiction shows that $\max L(2n_1 + n_4) < 4$, and whence $2 \in L(2n_1 + n_4)$.

We consider the possible factorizations of $2n_1 + n_4$ of length 2. The factorization must not contain n_1 or n_4 . Moreover, $2n_1 + n_4$ is strictly greater than $2n_1 + n_2 \geq 2n_3$ and $2n_1 + n_3 \geq n_2 + n_4$. Thus, $2n_1 + n_4 \in \{n_2 + n_5, n_3 + n_5, 2n_5\}$ and we distinguish these three cases.

CASE 1. $2n_1 + n_4 = n_2 + n_5$. Since $2n_1 + n_3 < 2n_1 + n_4 = n_2 + n_5$ and since by (4.4) we have $2n_1 + n_3 \geq n_2 + n_4$, it follows that $2n_1 + n_3 \in \{n_2 + n_4, 2n_4\}$. We distinguish the two cases.

Case 1.1. $2n_1 + n_3 = n_2 + n_4$. Since $2n_3 \leq 2n_1 + n_2 < 2n_1 + n_3 = n_2 + n_4$, we get that $2n_1 + n_2 = 2n_3$. Considering differences we get that $n_4 - n_3 = n_5 - n_4$ and moreover $n_3 - n_2 = (n_4 - n_3) + (n_2 - n_3)$. Thus, $n_4 - n_3 = 2(n_3 - n_2)$. We set $d = n_3 - n_2$. We have that $3n_1 \in \{2n_2, n_2 + n_3\}$. We distinguish the two cases.

Case 1.1.1. $3n_1 = 2n_2$. Considering differences we get $n_2 - n_1 = 2(n_3 - n_2) = 2d$. Consequently, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 5d, n_1 + 7d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 4d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (4, 6, 7, 9, 11)$. Thus, $n_1 + n_3 = n_5$, a contradiction.

Case 1.1.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 4d, n_1 + 6d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 3d$ whence $n_1 + n_2 = n_4$, a contradiction.

Case 1.2. $2n_1 + n_3 = 2n_4$. We get that $2n_1 + n_2 \in \{2n_3, n_3 + n_4\}$. We distinguish the two cases.

Case 1.2.1. $2n_1 + n_2 = 2n_3$. Considering differences we get that $n_3 - n_2 = 2(n_4 - n_3)$. Moreover, $n_4 - n_3 = (n_5 - n_4) + (n_2 - n_4)$. Thus, setting $d = n_4 - n_3$ we have $n_3 - n_2 = 2d$ and $n_5 - n_4 = 4d$. We have that $3n_1 \in \{2n_2, n_2 + n_3\}$ and distinguish cases.

Case 1.2.1.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = 2(n_3 - n_2) = 4d$. It follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 4d, n_1 + 6d, n_1 + 7d, n_1 + 11d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 8d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (8, 12, 14, 15, 19)$. We check that $L(35) = \{3\}$, a contradiction.

Case 1.2.1.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_3 - n_2$. It follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 4d, n_1 + 5d, n_1 + 9d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 6d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (6, 8, 10, 11, 15)$. We check that $L(27) = \{3\}$, a contradiction.

Case 1.2.2. $2n_1 + n_2 = n_3 + n_4$. It follows that $3n_1 \in \{2n_2, n_2 + n_3, n_2 + n_4, 2n_3\}$. We distinguish cases.

Case 1.2.2.1. $3n_1 = 2n_2$. Considering differences we get that $n_3 - n_2 = n_4 - n_3 =: d$. Moreover, $n_2 - n_1 = (n_4 - n_2) + (n_3 - n_2) = 3d$ and $n_4 - n_1 = n_5 - n_2$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 3d, n_1 + 4d, n_1 + 5d, n_1 + 8d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 6d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (6, 9, 10, 11, 14)$. We check that $L(26) = \{3\}$, a contradiction.

Case 1.2.2.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_3 - n_2 = n_4 - n_3 =: d$ and $n_2 - n_1 = n_4 - n_2 = 2d$. Moreover, $n_4 - n_1 = n_5 - n_3$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 4d, n_1 + 7d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 5d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 1.2.2.3. $3n_1 = n_2 + n_4$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 =: d$ and $n_3 - n_2 = n_4 - n_3 = d$. Moreover, $n_4 - n_1 = n_5 - n_4 = 3d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 6d)$. From $3n_1 = n_2 + n_4$ we infer that $n_1 = 4d$ whence $n_1 + n_3 = n_5$, a contradiction.

Case 1.2.2.4. $3n_1 = 2n_3$. Considering differences we get that $n_2 - n_1 = n_4 - n_3 =: d$ and $n_3 - n_2 = n_4 - n_3 = d$. Moreover, $n_4 - n_1 = n_5 - n_3 + (n_2 - n_3)$ and thus $n_5 - n_3 = (n_4 - n_1) + (n_3 - n_2) = 4d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 6d)$. From $3n_1 = 2n_3$ we infer that $n_1 = 4d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (4, 5, 6, 7, 10)$. Thus, $n_1 + n_3 = n_5$, a contradiction.

CASE 2. $2n_1 + n_4 = n_3 + n_5$. Since $2n_1 + n_3 < 2n_1 + n_4 = n_3 + n_5$, it follows that $2n_1 + n_3 \in \{n_2 + n_4, 2n_4, n_2 + n_5\}$. We distinguish the three cases.

Case 2.1. $2n_1 + n_3 = n_2 + n_4$. We get that $2n_1 + n_2 = 2n_3$. Considering differences we get that $n_4 - n_3 = (n_3 - n_2) + (n_5 - n_4)$ and $n_3 - n_2 = (n_4 - n_3) + (n_2 - n_3)$. It follows that $n_4 - n_3 = 2(n_3 - n_2)$ and $n_5 - n_4 = (n_4 - n_3) - (n_3 - n_2)$. We set $d = n_3 - n_2$ to get $n_4 - n_3 = 2d$ and $n_5 - n_4 = d$. We infer that $3n_1 \in \{2n_2, n_2 + n_3\}$ and distinguish the two cases.

Case 2.1.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = 2(n_3 - n_2) = 2d$. It follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 5d, n_1 + 6d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 4d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 2.1.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 = d$. It follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 4d, n_1 + 5d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 3d$ whence $n_1 + n_2 = n_4$, a contradiction.

Case 2.2. $2n_1 + n_3 = 2n_4$. We infer that $2n_1 + n_2 \in \{2n_3, n_3 + n_4\}$ and distinguish the two cases.

Case 2.2.1. $2n_1 + n_2 = 2n_3$. Considering differences we get that $n_3 - n_2 = 2(n_4 - n_3)$. Moreover, $n_4 - n_3 = n_5 - n_4 + (n_3 - n_4)$ and thus $n_5 - n_4 = 2(n_4 - n_3)$. We infer that $3n_1 \in \{2n_2, n_2 + n_3, n_2 + n_4\}$ and distinguish three cases. Set $d = n_4 - n_3$.

Case 2.2.1.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = 2(n_3 - n_2) = 4d$. Thus, it follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 4d, n_1 + 6d, n_1 + 7d, n_1 + 9d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 8d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (8, 12, 14, 15, 17)$. We check that $L(33) = \{3\}$, a contradiction.

Case 2.2.1.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 = 2d$. Thus, it follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 4d, n_1 + 5d, n_1 + 7d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 6d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (6, 8, 10, 11, 13)$. We check that $L(25) = \{3\}$, a contradiction.

Case 2.2.1.3. $3n_1 = n_2 + n_4$. Considering differences we get that $n_2 - n_1 = (n_3 - n_2) + (n_3 - n_4) = d$. Thus, it follows that $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 3d, n_1 + 4d, n_1 + 6d)$. From $3n_1 = n_2 + n_4$ we infer that $n_1 = 5d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 2.2.2. $2n_1 + n_2 = n_3 + n_4$. Considering differences we get that $n_3 - n_2 = n_4 - n_3 =: d$. Moreover, $n_4 - n_3 = n_5 - n_4 + (n_3 - n_4)$ and thus $n_5 - n_4 = 2(n_4 - n_3) = 2d$. We observe that $3n_1 \in \{2n_2, n_2 + n_3, n_2 + n_4, 2n_3\}$ and distinguish cases.

Case 2.2.2.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = n_4 - n_2 + (n_3 - n_2) = 3d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 3d, n_1 + 4d, n_1 + 5d, n_1 + 7d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 6d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (6, 9, 10, 11, 13)$. We check that $L(25) = \{3\}$, a contradiction.

Case 2.2.2.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_4 - n_2 = 2d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 4d, n_1 + 6d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 5d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (5, 7, 8, 9, 11)$. We check that $L(21) = \{3\}$, a contradiction.

Case 2.2.2.3. $3n_1 = n_2 + n_4$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 5d)$. From $3n_1 = n_2 + n_4$ we infer that $n_1 = 4d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 2.2.2.4. $3n_1 = 2n_3$. Considering differences we get that $n_2 - n_1 = n_4 - n_3 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 5d)$. From $3n_1 = 2n_3$ we infer that $n_1 = 4d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 2.3. $2n_1 + n_3 = n_2 + n_5$. We obtain that $2n_1 + n_2 \in \{2n_3, n_3 + n_4, 2n_4\}$ and distinguish the three cases.

Case 2.3.1. $2n_1 + n_2 = 2n_3$. Considering differences we get $n_4 - n_3 = n_3 - n_2 =: d$. Moreover, $n_4 - n_2 = n_5 - n_3 = 2d$ and therefore $n_5 - n_4 = d$. We infer that $3n_1 \in \{2n_2, n_2 + n_3, n_2 + n_4\}$ and distinguish the three cases.

Case 2.3.1.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = 2(n_3 - n_2) = 2d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 4d, n_1 + 5d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 4d$ whence $2n_1 = n_4$, a contradiction.

Case 2.3.1.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 4d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 3d$ whence $2n_1 = n_4$, a contradiction.

Case 2.3.1.3. $3n_1 = n_2 + n_4$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 - (n_4 - n_3) = 0$, a contradiction.

Case 2.3.2. $2n_1 + n_2 = n_3 + n_4$. Considering differences we get that $n_4 - n_3 = n_3 - n_2 =: d$ and $n_5 - n_4 = n_4 - n_2 = 2d$. We have $3n_1 \in \{2n_2, n_2 + n_3, n_2 + n_4, 2n_3\}$. We distinguish the four cases.

Case 2.3.2.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = n_4 - n_2 + (n_3 - n_2) = 3d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 3d, n_1 + 4d, n_1 + 5d, n_1 + 7d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 6d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (6, 9, 10, 11, 13)$. We check that $L(25) = \{3\}$, a contradiction.

Case 2.3.2.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_4 - n_2 = 2d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 4d, n_1 + 6d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 5d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (5, 7, 8, 9, 11)$. We check that $L(21) = \{3\}$, a contradiction.

Case 2.3.2.3. $3n_1 = n_2 + n_4$. Considering differences we get that $n_2 - n_1 = n_3 - n_2 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 5d)$. From $3n_1 = n_2 + n_4$ we infer that $n_1 = 4d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 2.3.2.4. $3n_1 = 2n_3$. Considering differences we get that $n_2 - n_1 = n_4 - n_3 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 5d)$. From $3n_1 = 2n_3$ we infer that $n_1 = 4d$ whence $n_1 + n_2 = n_5$, a contradiction.

Case 2.3.3. $2n_1 + n_2 = 2n_4$. Considering differences we get that $n_4 - n_3 = n_3 - n_2 =: d$ and $n_3 - n_2 = (n_5 - n_4) + (n_2 - n_4)$, that is, $n_5 - n_4 = n_4 - n_2 + (n_3 - n_2) = 3d$. We have $3n_1 \in \{2n_2, n_2 + n_3, n_2 + n_4, 2n_3, n_3 + n_4\}$. We distinguish the five cases.

Case 2.3.3.1. $3n_1 = 2n_2$. Considering differences we get that $n_2 - n_1 = 2(n_4 - n_2) = 4d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 4d, n_1 + 5d, n_1 + 6d, n_1 + 9d)$. From $3n_1 = 2n_2$ we infer that $n_1 = 8d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (8, 12, 13, 14, 17)$. We check that $L(31) = \{3\}$, a contradiction.

Case 2.3.3.2. $3n_1 = n_2 + n_3$. Considering differences we get that $n_2 - n_1 = n_4 - n_2 + (n_4 - n_3) = 3d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 3d, n_1 + 4d, n_1 + 5d, n_1 + 8d)$. From $3n_1 = n_2 + n_3$ we infer that $n_1 = 7d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (7, 10, 11, 12, 15)$. We check that $L(27) = \{3\}$, a contradiction.

Case 2.3.3.3. $3n_1 = n_2 + n_4$. Considering differences we get that $n_2 - n_1 = n_4 - n_2 = 2d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 4d, n_1 + 7d)$. From $3n_1 = n_2 + n_4$ we infer that $n_1 = 6d$. We get $d = 1$, and $(n_1, n_2, n_3, n_4, n_5) = (6, 8, 9, 10, 13)$. We check that $L(23) = \{3\}$, a contradiction.

Case 2.3.3.4. $3n_1 = 2n_3$. Considering differences we get that $n_2 - n_1 = 2(n_4 - n_3) = 2d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + 2d, n_1 + 3d, n_1 + 4d, n_1 + 7d)$. From $3n_1 = 2n_3$ we infer that $n_1 = 6d$, and we conclude as in the preceding case.

Case 2.3.3.5. $3n_1 = n_3 + n_4$. Considering differences we get that $n_2 - n_1 = n_4 - n_3 = d$. Thus, $(n_1, n_2, n_3, n_4, n_5) = (n_1, n_1 + d, n_1 + 2d, n_1 + 3d, n_1 + 6d)$. From $3n_1 = n_3 + n_4$ we infer that $n_1 = 5d$ whence $n_1 + n_2 = n_5$, a contradiction.

CASE 3. $2n_1 + n_4 = 2n_5$. It follows that $2n_1 > n_5$. We consider $2n_1 + n_5$. Since $2n_5 < 2n_1 + n_5 < 4n_1$. The first inequality shows that $L(2n_1 + n_5)$ cannot contain 2, the second shows that $L(2n_1 + n_5)$ cannot contain 4 or any larger element. Thus, $L(2n_1 + n_5) = \{3\}$. □[Proof of **A5**] □

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