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Optimal control of diffusion coefficients via decoupling fields

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Abstract

We consider a diffusion control problem, where the controller totally determines the state's diffusion coefficient but has no influence on the state's drift rate. By using the Pontryagin maximum principle we characterize an optimal control in terms of the adjoint forward-backward stochastic differential equation (FBSDE), turning out to be fully coupled. We use the method of decoupling fields for proving that the adjoint FBSDE possesses a solution.

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Introduction

Let (M_t^α) be a stochastic process with controlled dynamics of the form $dM_t^\alpha = \mu(t, M_t^\alpha)dt + \alpha_t dW_t$, where μ is affine linear in its second argument and W is a one-dimensional Brownian motion. In this article we consider the control problem that consists in minimizing, among a suitable class of controls α , the target functional

$$E \left[\int_0^T f(t, M_t^\alpha, \alpha_t) dt + g(M_T^\alpha) \right], \quad (1)$$

where f and g are nice functions, in particular convex in M and α .

This control problem arises in situations where one can control a state's fluctuation intensity but not its drift, and where one aims at steering the state into a target area. To give an explicit example, M^α may describe the position of a particle in a medium with temperature α . By heating or cooling the medium the particle's fluctuations increase or decrease respectively. The function f reflects the costs involved by any temperature change.

Diffusion control problems arise also in portfolio optimization. In this context M^α can be interpreted as a portfolio value process with volatility α . A reduction of the portfolio's

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volatility involves hedging costs f . The function $-g$ can be taken to be a utility function. The optimal control of diffusion coefficients appear also in other fields of applications, see e.g. [16] for examples arising in biology.

Assuming a Markovian framework, one can choose a Hamilton-Jacobi-Bellman (HJB) approach and characterize the value function in terms of the HJB equation. In a non-Markovian framework the problem of minimizing (1) seems to be unsolved, to the best of our knowledge.

Our solution method is based on the maximum principle, leading to a probabilistic representation of the value function's sensitivity w.r.t. the controlled space variable. More precisely, we reduce the control problem to an adjoint forward-backward stochastic differential equation (FBSDE). The adjoint FBSDE turns out to be fully coupled, and hence it is a priori not clear whether a solution exists. Our idea is to use the so-called method of decoupling fields for proving that the fully coupled system possesses indeed a solution.

In order to make our idea work, we first transform the adjoint FBSDE so as to reduce the dependence of the forward diffusion coefficient on the control. The decoupling field of the transformed FBSDE can be controlled by using the convexity properties of the cost functions. More precisely, we show that the space derivative of the decoupling field is bounded and bounded away from zero. This allows to conclude existence of a solution of the adjoint equation and hence to obtain an optimal control for the problem of minimizing (1). We remark that the convexity assumptions also guarantee that the maximum principle applies in the first place.

We now explain how our solution method compares to other approaches. One can strive to obtain a probabilistic representation of the value function itself (and not only of its derivative). In the Markovian case the value function satisfies a fully non-linear PDE, allowing a representation in terms of a 2nd order BSDE (see [21]). Our approach focuses on the derivative of the value function, which satisfies a semi-linear PDE and hence has a representation in terms of a standard FBSDE.

A difficulty of the HJB approach arises from the fact that the diffusion coefficient is unbounded in the control. In order to circumvent this difficulty, [4] introduce a tailor-made modification of the viscosity solution concept. The control problem considered in [4] is more general than the problem of the present article. In contrast to [4], our approach does not require that the function g has strictly smaller growth in x than the function f in a .

Finally, a further advantage of our method compared to the HJB approach is that it allows for a non-Markovian setting, i.e. the functions f and g can additionally depend, in a progressively measurable way, on the Brownian paths.

The literature provides many examples of problems involving the optimal control of a diffusion coefficient. We do not strive to give an overview on this classical type of control problem, but select some articles that seem closest to the problem we study in the current article. A simple problem version with infinite time horizon is discussed in Example 7.6, [20]. Specific problems with a finite time horizon have been studied in [19], [9] and [10]. A reverse diffusion control problem is solved by McNamara [15]: he determines the reward functions for which a given bang-bang diffusion control is optimal. A diffusion control problem within an exponential martingale model is studied in [2].

As mentioned earlier, our approach is rooted in reducing the initial control problem to an FBSDE. It is a longstanding challenge to find conditions guaranteeing that a given fully coupled FBSDE possesses a solution. Sufficient conditions are provided e.g. in [11], [17], [14], [18], [5], [12] (see also references therein). The method of decoupling fields, developed in [6] (see also the precursor articles [13], [7] and [12]), is practically useful for determining whether a solution exists. A decoupling field describes the functional dependence of the

backward part Y on the forward component X . If the coefficients of a fully coupled FBSDE satisfy a Lipschitz condition, then there exists a maximal non-vanishing interval possessing a solution triplet (X, Y, Z) and a decoupling field with nice regularity properties. The method of decoupling fields consists in analyzing the dynamics of the decoupling field's gradient in order to determine whether the FBSDE has a solution on the whole time interval $[0, T]$. The method can be successfully applied to various problems involving coupled FBSDE: In [8] solutions to a quadratic strongly coupled FBSDE with a two-dimensional forward equation are constructed to obtain solutions to the Skorokhod embedding problem for Gaussian processes with non-linear drift. In Chapter 5 of [6] the problem of utility maximization in incomplete markets is treated for a general class of utility functions via construction of solutions to the associated coupled FBSDE. In the more recent work [3], the method is used to obtain solutions to the problem of optimal position targeting for general cost functionals.

The paper is organized as follows. In Section 1 we rigorously describe the problem and its mathematical set-up. In Section 2 we apply the maximum principle to reduce the control problem to the adjoint FBSDE. Section 3 provides a brief introduction into the method of decoupling fields. In Section 4 we first transform the adjoint FBSDE so as to dampen the dependence of the forward diffusion on the control. We then prove existence of a solution by using the method of decoupling fields. In Section 5 we illustrate the construction of a solution with an explicit example. Finally, in Section 6 we explain heuristically how the adjoint FBSDE is connected to the control problem's HJB equation.

1 Problem formulation

Let $T > 0$ be a deterministic finite time horizon. Let W be a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $(\mathcal{F}_t)_{t \in [0, T]}$ the smallest filtration satisfying the usual conditions and containing the filtration generated by W .

Let $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable such that for all $(m, a) \in \mathbb{R}^2$ the mapping $(\omega, t) \mapsto f(\omega, t, m, a)$ is progressively measurable. We make the following additional assumptions on f and g :

(C0) For every fixed pair $(\omega, t) \in \Omega \times [0, T]$ the mappings $(m, a) \mapsto f(t, m, a)$ and $m \mapsto g(m)$ are convex, with f being strictly convex in a . Note that we follow the usual convention and omit the function argument ω .

(C1) $g(\cdot)$ and $f(t, \cdot, \cdot)$ are twice continuously differentiable. Moreover, g' , f_m and f_a are Lipschitz continuous in the last two components and satisfy

$$\|(|f_m| + |f_a|)(\cdot, \cdot, 0, 0)\|_\infty < \infty \text{ and } \|g'(0)\|_\infty < \infty.$$

(C2) There exists a positive constant $\delta_l > 0$ such that $f_{aa} \geq \delta_l$ everywhere. Throughout we assume that δ_l denotes the largest constant with this property.

Notice that (C1) implies that there exists a $\delta_u \in [\delta_l, \infty)$ such that for all $(\omega, t, m, a) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}$ we have

$$f_{aa}(t, m, a) \leq \delta_u. \tag{2}$$

Moreover, (C1) implies that f_m and f_a grow at most linearly in m and a .

Let \mathcal{A} be the set of all progressively measurable $\alpha : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that $\mathbb{E} \left[\int_0^T \alpha_s^2 ds \right] < \infty$. For all $m \in \mathbb{R}$ and $\alpha \in \mathcal{A}$ we define

$$M_t^{m,\alpha} = m + \int_0^t (b_s + B_s M_s^{m,\alpha}) ds + \int_0^t \alpha_s dW_s, \quad (3)$$

where $b, B : \Omega \times [0, T] \rightarrow \mathbb{R}$ are progressively measurable and bounded processes. As outlined in the introduction, our aim is to solve the following problem:

$$\text{Minimize } J(m, \alpha) := \mathbb{E} \left[\int_0^T f(s, M_s^{m,\alpha}, \alpha_s) ds + g(M_T^{m,\alpha}) \right] \text{ over all } \alpha \in \mathcal{A}. \quad (4)$$

For simplicity we sometimes write M^α or just M instead of $M^{m,\alpha}$. In other words, for given m , the goal is to choose α from the set \mathcal{A} of admissible controls in such a way that J is minimized.

2 Reducing the problem to an FBSDE

The so-called Hamiltonian of the control problem (4) is defined by

$$H(t, m, a, y, z) := (b_t + B_t m)y + az + f(t, m, a),$$

for $t \in [0, T]$ and $(m, a, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Notice that

$$\min_{a \in \mathbb{R}} H(t, m, a, y, z) = (b_t + B_t m)y - f^*(t, m, -z), \quad (5)$$

where $f^*(t, m, \cdot)$ is the convex conjugate of $f(t, m, \cdot)$. Observe that condition (C2) guarantees that f^* assumes real values only. For the following observation we need both (C1) and (C2).

The minimum in (5) is attained at $a = f_3^*(t, m, -z)$, where f_3^* denotes the partial derivative of f^* with respect to the last component. This partial derivative exists since f is differentiable and, by (C2), the image of $f_a(t, m, \cdot)$ is the whole real line. More precisely, using Fermat's theorem applied to the minimization problem $\min_{a \in \mathbb{R}} H(t, m, a, y, z) = \min_{a \in \mathbb{R}} H(t, m, a, y, z)$ one can deduce

$$a = f_3^*(t, m, -z) = f_a^{-1}(t, m, -z),$$

where $f_a^{-1}(t, m, \cdot)$ denotes the inverse of the function $f_a(t, m, \cdot)$, which is strictly increasing.

The so-called adjoint forward-backward stochastic differential equation (FBSDE) for the control problem (4) is given by

$$\begin{aligned} M_s^m &= m + \int_0^s (b_r + B_r M_r^m) dr + \int_0^s f_3^*(r, M_r^m, -Z_r^m) dW_r, \\ Y_s^m &= g'(M_T^m) - \int_s^T Z_r^m dW_r + \int_s^T (B_r Y_r + f_m(r, M_r^m, f_3^*(r, M_r^m, -Z_r^m))) dr \end{aligned} \quad (6)$$

for all $s \in [0, T]$. To simplify the notations, when there is no ambiguity, (M^m, Y^m, Z^m) will be denoted by (M, Y, Z) . In order to rigorously define what we mean by a solution of (6) we introduce the following process spaces. For any $t_0 \in [0, T]$ we denote by $\mathcal{H}_{t_0, T}^2$ the set of all (\mathcal{F}_t) -progressively measurable processes $(X_t)_{t \in [t_0, T]}$ such that $E \int_{t_0}^T X_t^2 dt < \infty$. We define $\mathcal{H}_T^2 := \mathcal{H}_{0, T}^2$.

A solution to (6) is a triplet $(M, Y, Z) = (M^m, Y^m, Z^m)$ of progressively measurable processes such that

- M, Y and Z are processes in \mathcal{H}_T^2 ,
- M and Y are continuous processes,
- the two equations (6) are satisfied a.s. for every fixed $s \in [t, T]$.

Constructing solutions to the above FBSDE is important for the following reason:

Proposition 2.1. *If there exists a solution (M, Y, Z) of (6), then an optimal control for problem (4) is given by*

$$\alpha_s = f_3^*(s, M_s, -Z_s), \quad s \in [0, T].$$

Proof. We adapt the proof of Theorem 5.2 in [22] to our setting. For $m \in \mathbb{R}$ and $\bar{\alpha} \in \mathcal{A}$ let $\bar{M} = M^{m, \bar{\alpha}}$ be the associated state process. Let us define $\delta M := \bar{M} - M$ and $\delta \alpha := \bar{\alpha} - \alpha$. Note that $\bar{M}_0 = M_0 = m$, such that $\delta M_0 = 0$. Since g is convex we have a.s.

$$Y_T \delta M_T - Y_0 \delta M_0 = Y_T \delta M_T = g'(M_T) \delta M_T \leq g(\bar{M}_T) - g(M_T). \quad (7)$$

At the same time Itô's formula proves that

$$\begin{aligned} Y_t \delta M_t - Y_0 \delta M_0 &= \int_0^t (\delta M_s) dY_s + \int_0^t Y_s d(\delta M_s) + \int_0^t (\delta \alpha_s) Z_s ds \\ &= - \int_0^t (\delta M_s) (B_s Y_s + f_m(s, M_s, \alpha_s)) ds + \int_0^t (\delta M_s) Z_s dW_s \\ &\quad + \int_0^t Y_s B_s (\delta M_s) ds + \int_0^t Y_s (\delta \alpha_s) dW_s + \int_0^t (\delta \alpha_s) Z_s ds \\ &= \int_0^t ((\delta \alpha_s) Z_s - (\delta M_s) f_m(s, M_s, \alpha_s)) ds + \\ &\quad + \int_0^t (Y_s (\delta \alpha_s) + (\delta M_s) Z_s) dW_s, \end{aligned} \quad (8)$$

for all $t \in [0, T]$. Now note that

$$H_a(s, M_s, \alpha_s, Y_s, Z_s) = Z_s + f_a(s, M_s, f_3^*(s, M_s, -Z_s)) = Z_s - Z_s = 0.$$

Together with the convexity of H this implies

$$\begin{aligned} H(s, \bar{M}_s, \bar{\alpha}_s, Y_s, Z_s) - H(s, M_s, \alpha_s, Y_s, Z_s) &\geq H_m(s, M_s, \alpha_s, Y_s, Z_s) (\delta M_s) \\ &\quad + H_a(s, M_s, \alpha_s, Y_s, Z_s) (\delta \alpha_s) \\ &= (B_s Y_s + f_m(s, M_s, \alpha_s)) (\delta M_s). \end{aligned}$$

Thus, due to the definition of H

$$\begin{aligned} f(s, \bar{M}_s, \bar{\alpha}_s) - f(s, M_s, \alpha_s) &= H(s, \bar{M}_s, \bar{\alpha}_s, Y_s, Z_s) - H(s, M_s, \alpha_s, Y_s, Z_s) \\ &\quad - B_s (\delta M_s) Y_s - Z_s (\delta \alpha_s) \\ &\geq f_m(s, M_s, \alpha_s) (\delta M_s) - Z_s (\delta \alpha_s). \end{aligned}$$

Let τ be a $[0, T]$ -valued stopping time such that M , \bar{M} and Y are bounded on $[0, \tau]$. Formula (8), together with inequality (9), implies

$$\mathbb{E}[Y_\tau \delta M_\tau - Y_0 \delta M_0] \geq \mathbb{E} \left[\int_0^\tau (f(s, M_s, \alpha_s) - f(s, \bar{M}_s, \bar{\alpha}_s)) ds \right],$$

where all integrals are well-defined, since M , \bar{M} , Y are bounded on $[0, \tau]$, α and $\bar{\alpha}$ are square integrable and f is at most quadratic in a . Note that

$$\sup_{s \in [0, T]} |M_s|, \quad \sup_{s \in [0, T]} |\bar{M}_s| \quad \text{and} \quad \sup_{s \in [0, T]} |Y_s|$$

are square integrable. Choosing an appropriate localizing sequence $\tau_n \rightarrow T$ of stopping times we can pass to the limit using dominated convergence and obtain

$$\mathbb{E}[Y_T \delta M_T - Y_0 \delta M_0] \geq \mathbb{E} \left[\int_0^T (f(s, M_s, \alpha_s) - f(s, \bar{M}_s, \bar{\alpha}_s)) ds \right]. \quad (9)$$

Combining (9) with (7) we obtain

$$\mathbb{E}[g(\bar{M}_T) - g(M_T)] \geq \mathbb{E} \left[\int_0^T (f(s, M_s, \alpha_s) - f(s, \bar{M}_s, \bar{\alpha}_s)) ds \right].$$

This leads to

$$J(m, \bar{\alpha}) \geq J(m, \alpha),$$

which shows optimality of α . □

3 The method of decoupling fields

As mentioned above, solving (6) is crucial in constructing optimal controls. As a key result of this paper we prove in Section 4 the solvability of (6).

Note that even under our Lipschitz assumptions, it is not trivial to show well-posedness of (6) due to its coupled nature. It is necessary to take more subtle structural properties into account to conduct the proof. Our argumentation will be based on the so-called method of decoupling fields which we will briefly sum up in this section.

For a fixed finite time horizon $T > 0$, we consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, where \mathcal{F}_0 consists of all null sets, $(W_t)_{t \in [0, T]}$ is a 1-dimensional Brownian motion and $\mathcal{F}_t := \sigma(\mathcal{F}_0, (W_s)_{s \in [0, t]})$ with $\mathcal{F} := \mathcal{F}_T$. The dynamics of an FBSDE is given by

$$\begin{aligned} X_s &= X_0 + \int_0^s \mu(r, X_r, Y_r, Z_r) dr + \int_0^s \sigma(r, X_r, Y_r, Z_r) dW_r, \\ Y_t &= \xi(X_T) - \int_t^T f(r, X_r, Y_r, Z_r) dr - \int_t^T Z_r dW_r, \end{aligned}$$

for $s, t \in [0, T]$ and $X_0 \in \mathbb{R}$, where $(\xi, (\mu, \sigma, f))$ are measurable functions such that

$$\begin{aligned} \xi &: \Omega \times \mathbb{R} \rightarrow \mathbb{R}, & \mu &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \\ \sigma &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, & f &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \end{aligned}$$

Throughout the whole section μ , σ and f are assumed to be progressively measurable with respect to $(\mathcal{F}_t)_{t \in [0, T]}$.

A decoupling field comes with an even richer structure than just a classical solution (X, Y, Z) .

Definition 3.1. Let $t \in [0, T]$. A function $u: [t, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ with $u(T, \cdot) = \xi$ a.e. is called *decoupling field* for $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if for all $t_1, t_2 \in [t, T]$ with $t_1 \leq t_2$ and any \mathcal{F}_{t_1} -measurable $X_{t_1}: \Omega \rightarrow \mathbb{R}$ there exist progressively measurable processes (X, Y, Z) on $[t_1, t_2]$ such that

$$\begin{aligned} X_s &= X_{t_1} + \int_{t_1}^s \mu(r, X_r, Y_r, Z_r) dr + \int_{t_1}^s \sigma(r, X_r, Y_r, Z_r) dW_r, \\ Y_s &= Y_{t_2} - \int_s^{t_2} f(r, X_r, Y_r, Z_r) dr - \int_s^{t_2} Z_r dW_r, \\ Y_s &= u(s, X_s), \end{aligned} \tag{10}$$

a.s. for all $s \in [t_1, t_2]$. In particular, we want all integrals to be well-defined.

Some remarks about this definition are in place.

- The first equation in (10) is called the *forward equation*, the second the *backward equation* and the third will be referred to as the *decoupling condition*.
- Note that, if $t_2 = T$, we get $Y_T = \xi(X_T)$ a.s. as a consequence of the decoupling condition together with $u(T, \cdot) = \xi$. At the same time $Y_T = \xi(X_T)$, together with the decoupling condition, implies $u(T, \cdot) = \xi$ a.e.
- If $t_2 = T$ we can say that a triplet (X, Y, Z) solves the FBSDE, meaning that it satisfies the forward and the backward equation, together with $Y_T = \xi(X_T)$. This relationship $Y_T = \xi(X_T)$ is referred to as the *terminal condition*.

For the following we need to introduce further notation.

Let $I \subseteq [0, T]$ be an interval and $u: I \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a map such that $u(s, \cdot)$ is measurable for every $s \in I$. We define

$$L_{u,x} := \sup_{s \in I} \inf \{ L \geq 0 \mid \text{for a.a. } \omega \in \Omega : |u(s, \omega, x) - u(s, \omega, x')| \leq L|x - x'| \text{ for all } x, x' \in \mathbb{R} \},$$

where $\inf \emptyset := \infty$. We also set $L_{u,x} := \infty$ if $u(s, \cdot)$ is not measurable for every $s \in I$. One can show that $L_{u,x} < \infty$ is equivalent to u having a modification which is truly Lipschitz continuous in $x \in \mathbb{R}$.

We denote by $L_{\sigma,z}$ the Lipschitz constant of σ w.r.t. the dependence on the last component z . We set $L_{\sigma,z} = \infty$ if σ is not Lipschitz continuous in z .

By $L_{\sigma,z}^{-1} = \frac{1}{L_{\sigma,z}}$ we mean $\frac{1}{L_{\sigma,z}}$ if $L_{\sigma,z} > 0$ and ∞ otherwise.

For an integrable real valued random variable F the expression $\mathbb{E}_t[F]$ refers to $\mathbb{E}[F|\mathcal{F}_t]$, while $\mathbb{E}_{t,\infty}[F]$ refers to $\text{ess sup } \mathbb{E}[F|\mathcal{F}_t]$, which might be ∞ , but is always well defined as the infimum of all constants $c \in [-\infty, \infty]$ such that $\mathbb{E}[F|\mathcal{F}_t] \leq c$ a.s. Additionally, we write $\|F\|_\infty$ for the essential supremum of $|F|$.

In practice it is important to have explicit knowledge about the regularity of (X, Y, Z) . For instance, it is important to know in which spaces the processes live, and how they react to changes in the initial value.

Definition 3.2. Let $u: [t, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a decoupling field to $(\xi, (\mu, \sigma, f))$.

1. We say u to be *weakly regular* if $L_{u,x} < L_{\sigma,z}^{-1}$ and $\sup_{s \in [t, T]} \|u(s, \cdot, 0)\|_\infty < \infty$.
2. A weakly regular decoupling field u is called *strongly regular* if for all fixed $t_1, t_2 \in [t, T]$, $t_1 \leq t_2$, the processes (X, Y, Z) arising in (10) are a.e unique and satisfy

$$\sup_{s \in [t_1, t_2]} \mathbb{E}_{t_1, \infty}[|X_s|^2] + \sup_{s \in [t_1, t_2]} \mathbb{E}_{t_1, \infty}[|Y_s|^2] + \mathbb{E}_{t_1, \infty} \left[\int_{t_1}^{t_2} |Z_s|^2 ds \right] < \infty, \quad (11)$$

for each constant initial value $X_{t_1} = x \in \mathbb{R}$. In addition they are required to be measurable as functions of (x, s, ω) and even weakly differentiable w.r.t. $x \in \mathbb{R}^n$ such that for every $s \in [t_1, t_2]$ the mappings X_s and Y_s are measurable functions of (x, ω) and even weakly differentiable w.r.t. x such that

$$\begin{aligned} \text{ess sup}_{x \in \mathbb{R}} \sup_{s \in [t_1, t_2]} \mathbb{E}_{t_1, \infty} \left[|\partial_x X_s|^2 \right] &< \infty, \\ \text{ess sup}_{x \in \mathbb{R}} \sup_{s \in [t_1, t_2]} \mathbb{E}_{t_1, \infty} \left[|\partial_x Y_s|^2 \right] &< \infty, \\ \text{ess sup}_{x \in \mathbb{R}} \mathbb{E}_{t_1, \infty} \left[\int_{t_1}^{t_2} |\partial_x Z_s|^2 ds \right] &< \infty. \end{aligned} \quad (12)$$

3. We say that a decoupling field on $[t, T]$ is *strongly regular* on a subinterval $[t_1, t_2] \subseteq [t, T]$ if u restricted to $[t_1, t_2]$ is a strongly regular decoupling field for $(u(t_2, \cdot), (\mu, \sigma, f))$.

Under suitable conditions a rich existence, uniqueness and regularity theory for decoupling fields can be developed.

Assumption (SLC): $(\xi, (\mu, \sigma, f))$ satisfies *standard Lipschitz conditions* (SLC) if

1. (μ, σ, f) are Lipschitz continuous in (x, y, z) with Lipschitz constant L ,
2. $\|(|\mu| + |f| + |\sigma|)(\cdot, \cdot, 0, 0, 0)\|_\infty < \infty$,
3. $\xi: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable such that $\|\xi(\cdot, 0)\|_\infty < \infty$ and $L_{\xi,x} < L_{\sigma,z}^{-1}$.

In order to have a notion of global existence we need the following definition:

Definition 3.3. We define the maximal interval $I_{\max} \subseteq [0, T]$ of the problem given by $(\xi, (\mu, \sigma, f))$ as the union of all intervals $[t, T] \subseteq [0, T]$, such that there exists a weakly regular decoupling field u on $[t, T]$.

Note that the maximal interval might be open to the left. Also, let us remark that we define a decoupling field on such an interval as a mapping which is a decoupling field on every compact subinterval containing T . Similarly we can define weakly and strongly regular decoupling fields as mappings which restricted to an arbitrary compact subinterval containing T are weakly (or strongly) regular decoupling fields in the sense of the definitions given above.

Finally, we have global existence and uniqueness on the maximal interval:

Theorem 3.4 ([6], Theorem 5.1.11, Lemma 5.1.12 and Corollary 2.5.5). *Let $(\xi, (\mu, \sigma, f))$ satisfy SLC. Then there exists a unique strongly regular decoupling field u on I_{\max} . Furthermore, either $I_{\max} = [0, T]$ or $I_{\max} = (t_{\min}, T]$, where $0 \leq t_{\min} < T$. In the latter case we have*

$$\lim_{t \downarrow t_{\min}} L_{u(t, \cdot), x} = L_{\sigma, z}^{-1}. \quad (13)$$

Moreover, for any $t \in I_{\max}$ and any initial condition $X_t = x \in \mathbb{R}$ there is a unique solution (X, Y, Z) of the FBSDE on $[t, T]$ satisfying

$$\sup_{s \in [t, T]} \mathbb{E}[|X_s|^2] + \sup_{s \in [t, T]} \mathbb{E}[|Y_s|^2] + \mathbb{E} \left[\int_t^T |Z_s|^2 ds \right] < \infty.$$

Equality (13) allows to verify global existence, i.e. $I_{\max} = [0, T]$, via contradiction. We refer to this approach as the method of decoupling fields.

4 Transforming the FBSDE

The aim of this section is to prove that the adjoint FBSDE (6) has a solution on the whole interval $[0, T]$. To this end we first transform the FBSDE in a way that reduces the dependence of the forward diffusion on the control. We then apply the method of decoupling fields to the transformed system.

Let $\gamma = \frac{1}{\delta_u}$. We consider the auxiliary FBSDE

$$\begin{aligned} X_s^x &= x + \int_0^s (b_r + B_r M_r^x - B_r (X_r^x - M_r^x) - \gamma f_m(r, M_r^x, \tilde{Z}_r^x)) dr \\ &\quad + \int_0^s (\tilde{Z}_r^x - \gamma f_a(r, M_r^x, \tilde{Z}_r^x)) dW_r \\ M_s^x &= (Id + \gamma g')^{-1}(X_T^x) - \int_s^T \tilde{Z}_r^x dW_r - \int_s^T (b_r + B_r M_r^x) dr, \end{aligned} \quad (14)$$

for all $s \in [0, T]$.

We first show that the parameters of the FBSDE (14) satisfy the standard Lipschitz conditions.

Notice that $\gamma g'$ is non-decreasing in x since g is convex. Hence $Id + \gamma g'$ is strictly increasing with a derivative of at least 1. This implies that the inverse in x , denoted by $\xi = (Id + \gamma g')^{-1}$, exists and that the inverse is Lipschitz continuous in x with a Lipschitz constant smaller than or equal to 1. Since $g'(0)$ is essentially bounded, also $\xi(0) = (Id + \gamma g')^{-1}(0)$ is essentially bounded. Indeed,

$$|\xi(0)| = |\xi(0) - \xi((Id + \gamma g')(0))| \leq |(Id + \gamma g')(0)| \leq \gamma \|g'(0)\|_{\infty}.$$

Let $\mu(t, x, m, z) = (b_t + B_t m) - B_t(x - m) - \gamma f_m(t, m, z)$ and $\sigma(t, m, z) = z - \gamma f_a(t, m, z)$ be the drift and the diffusion coefficient of the forward equation in (14), respectively. Condition (C1) entails that μ and σ are Lipschitz continuous and that $(|\sigma| + |\mu|)(\cdot, \cdot, 0, 0)$ is bounded.

Notice that $\sigma(t, m, z)$ is differentiable in z and that the derivative takes values only in $[0, 1 - \frac{\delta_l}{\delta_u}]$. Hence the diffusion coefficient is Lipschitz continuous in z with a Lipschitz constant equal to $\frac{\delta_u - \delta_l}{\delta_u}$. In particular, the Lipschitz constant of $(Id + \gamma g')^{-1}$ is strictly smaller than $L_{\sigma, z}^{-1} = \frac{\delta_u}{\delta_u - \delta_l}$. Note that $L_{\sigma, z} = 0$ if and only if $\delta_l = \delta_u$.

To sum up, we have verified that the parameters of the FBSDE (14) satisfy the standard Lipschitz conditions (SLC). The benefit of considering (14) comes from the following observation.

Lemma 4.1. *If (X_t, M_t, \tilde{Z}_t) is a solution of (14), then $(M_t, \frac{1}{\gamma}(X_t - M_t), -f_a(t, M_t, \tilde{Z}_t))$ is a solution of (6).*

Proof. Let (X_t, M_t, \tilde{Z}_t) be a solution of (14). In particular, all three processes are in \mathcal{H}_T^2 . The linear growth condition on f_a implies that also $(f_a(t, M_t, \tilde{Z}_t))$ is in \mathcal{H}_T^2 . Thus each process in the new triplet $(M_t, \frac{1}{\gamma}(X_t - M_t), -f_a(t, M_t, \tilde{Z}_t))$ is in \mathcal{H}_T^2 . A straightforward calculation shows that the three processes of the new triplet satisfy the dynamics (6). \square

We use the method of decoupling fields for proving that there exists a solution of (14) on $[0, T]$. Since the parameters of (14) satisfy the (SLC), there exists a maximal interval I_{\max} with a weakly regular decoupling field u (see Theorem 3.4).

In the following fix $t_0 \in I_{\max}$. Let $(X, M, \tilde{Z}) = (X^{t_0, x}, M^{t_0, x}, \tilde{Z}^{t_0, x})$ be the solution of (14) on $[t_0, T]$ with initial value $x \in \mathbb{R}$ such that $M_t = u(t, X_t)$ a.s. for all $(t, x) \in [t_0, T] \times \mathbb{R}$.

According to strong regularity u is weakly differentiable w.r.t. the initial value $x \in \mathbb{R}$. In the following we denote by u_x a version of the weak derivative of u w.r.t. x such that it coincides with the classical derivative at all points for which it exists and with 0 everywhere else. Moreover, the processes (X, M, \tilde{Z}) are weakly differentiable w.r.t. x . We can formally differentiate the forward and the backward equation in (14). One can verify that one can interchange differentiation and integration and that a chain rule for weak derivatives applies (see Sections A.2 and A.3 in [6]). We thus obtain that for every version $(\partial_x X, \partial_x M, \partial_x \tilde{Z}) = (\partial_x X^{t_0, x}, \partial_x M^{t_0, x}, \partial_x \tilde{Z}^{t_0, x})$ of the weak derivative, such that for every $s \in [t_0, T]$ $(\partial_x X_s, \partial_x M_s)$ is a weak derivative of (X_s, M_s) , we have for every $t \in [t_0, T]$:

$$\begin{aligned} \partial_x X_t = & 1 + \int_{t_0}^t B_s(2\partial_x M_s - \partial_x X_s) ds - \gamma(f_{mm}(s, M_s, \tilde{Z}_s)\partial_x M_s + f_{ma}(s, M_s, \tilde{Z}_s)\partial_x \tilde{Z}_s) ds \\ & + \int_{t_0}^t (\partial_x \tilde{Z}_s - \gamma f_{ma}(s, M_s, \tilde{Z}_s)\partial_x M_s - \gamma f_{aa}(s, M_s, \tilde{Z}_s)\partial_x \tilde{Z}_s) dW_s \end{aligned} \quad (15)$$

and

$$\partial_x M_t = \xi'(X_T)\partial_x X_T + \int_t^T B_s \partial_x M_s ds + \int_t^T \partial_x \tilde{Z}_s dW_s, \quad (16)$$

for $\mathbb{P} \otimes \lambda$ - almost all $(\omega, x) \in \Omega \times \mathbb{R}$.

By redefining $(\partial_x X, \partial_x M)$ as the right-hand-sides of (15) and (16) respectively, we obtain processes $(\partial_x X, \partial_x M)$ that are continuous in time for all (ω, x) but remain weak derivatives of X, M w.r.t. x . From now on, we always assume that $\partial_x X$ and $\partial_x M$ are continuous in time. We also assume that for fixed $t \in [t_0, T]$ the mappings $\partial_x X_t$ and $\partial_x M_t$ are weak derivatives of X_t and M_t w.r.t. $x \in \mathbb{R}$. In particular $\partial_x X_{t_0} = 1$ a.s. for almost all $x \in \mathbb{R}$.

In order to obtain bounds on the weak derivative u_x , we study the process $V_t := u_x(t, X_t)$, $t \in [t_0, T]$.

Recall that $M_t = u(t, X_t)$ a.s. for all $(t, x) \in [t_0, T] \times \mathbb{R}$. Therefore, for fixed $t \in [t_0, T]$, the weak derivatives of the two sides of the equation w.r.t. $x \in \mathbb{R}$ must coincide up to a $\mathbb{P} \otimes \lambda$ - null set. The chain rule for weak derivatives (see Corollary 3.2 in [1] or Lemma A.3.1. in [6]) implies, for any fixed $t \in [t_0, T]$, that we have for $\mathbb{P} \otimes \lambda$ - almost all (ω, x)

$$\partial_x M_t \mathbf{1}_{\{\partial_x X_t > 0\}} = u_x(t, X_t) \partial_x X_t \mathbf{1}_{\{\partial_x X_t > 0\}} = V_t \partial_x X_t \mathbf{1}_{\{\partial_x X_t > 0\}}. \quad (17)$$

Now, choose a fixed $x \in \mathbb{R}$ such that $\partial_x X_{t_0} = 1$ a.s., (17), (15), (16) are satisfied for almost all $(\omega, t) \in [t_0, T] \times \Omega$ and, in addition, (17) is satisfied for $t = t_0$, \mathbb{P} - almost surely. Note

that, since $\partial_x X$, $\partial_x M$ are continuous in time, (15) and (16) in fact hold for all $t \in [t_0, T]$, \mathbb{P} -almost surely.

Observe that V_t is bounded since $u_x(t, x)$ is bounded. Furthermore, according to the definition of the maximal interval, if $L_{\sigma, z} > 0$, then there exists $\varepsilon > 0$ such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ we have $|u_x(t, x)| \leq (1 - \varepsilon) \frac{\delta_u}{\delta_u - \delta_t}$. A priori, ε depends on t_0 . We will see below that it can be chosen independently of t_0 .

In the case $L_{\sigma, z} = 0$ there exists a constant K such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ we have $|u_x(t, x)| \leq K$. We will show below that K can be chosen independently of t_0 .

We now turn to the dynamics of V .

Lemma 4.2. *The process $(V_t)_{t \in [t_0, T]}$ has a time-continuous version which is an Itô process. Moreover, there exists $\widehat{Z} \in \mathcal{H}_{t_0, T}^2$ such that (V, \widehat{Z}) is the unique solution of the BSDE*

$$V_t = \xi'(X_T) - \int_t^T \widehat{Z}_s dW_s - \int_t^T \rho(s, V_s, \widehat{Z}_s) ds,$$

where $\rho : \Omega \times [t_0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} \rho(t, v, z) = & v[2B_t(1 - v) + \gamma f_{mm}(t, M_t, \widetilde{Z}_t)v + \gamma f_{ma}(t, M_t, \widetilde{Z}_t)h(t, v, z)] \\ & - z[-\gamma f_{ma}(t, M_t, \widetilde{Z}_t)v + (1 - \gamma f_{aa}(t, M_t, \widetilde{Z}_t))h(t, v, z)], \end{aligned}$$

with

$$h(t, v, z) = \frac{z - \gamma f_{ma}(t, M_t, \widetilde{Z}_t)v^2}{1 - v[1 - \gamma f_{aa}(t, M_t, \widetilde{Z}_t)]}.$$

Furthermore, $\widehat{Z} \in BMO(\mathbb{P})$, i.e. $\sup_{t \in [t_0, T]} \left\| \mathbb{E} \left[\int_t^T |\widehat{Z}_s|^2 ds \mid \mathcal{F}_t \right] \right\|_\infty < \infty$.

Proof. Let $\tau_n = T \wedge \inf\{t \geq t_0 : \partial_x X_t \leq \frac{1}{n}\}$. On $[t_0, \tau_n]$ we have $V_t = \partial_x M_t \frac{1}{\partial_x X_t}$, a.e. Hence V has a version which is an Itô process on $[t_0, \tau_n]$. We denote the Itô process decomposition by

$$V_t = u_x(t_0, x) + \int_{t_0}^t \widehat{Z}_s dW_s + \int_{t_0}^t \kappa_s ds, \quad t \in [t_0, \tau_n].$$

The product formula yields, on $[t_0, \tau_n]$,

$$\begin{aligned} d(V_t \partial_x X_t) = & V_t (B_t (2\partial_x M_t - \partial_x X_t) - \gamma (f_{mm}(t, M_t, \widetilde{Z}_t) \partial_x M_t + f_{ma}(t, M_t, \widetilde{Z}_t) \partial_x \widetilde{Z}_t)) dt \\ & + \partial_x X_t \kappa_t dt + \widehat{Z}_t (\partial_x \widetilde{Z}_t - \gamma f_{ma}(t, M_t, \widetilde{Z}_t) \partial_x M_t - \gamma f_{aa}(t, M_t, \widetilde{Z}_t) \partial_x \widetilde{Z}_t) dt \\ & + \left(\partial_x X_t \widehat{Z}_t + V_t [\partial_x \widetilde{Z}_t - \gamma f_{ma}(t, M_t, \widetilde{Z}_t) \partial_x M_t - \gamma f_{aa}(t, M_t, \widetilde{Z}_t) \partial_x \widetilde{Z}_t] \right) dW_t. \end{aligned}$$

The drift and diffusion coefficients coincide with the coefficients in (16). This implies, using straightforward transformations, the definition of h and the property $V_t \partial_x X_t = \partial_x M_t$,

$$\partial_x \widetilde{Z}_t = \partial_x X_t h(t, V_t, \widehat{Z}_t)$$

and

$$\begin{aligned} \kappa_t = & \frac{1}{\partial_x X_t} [B_t \partial_x M_t \\ & - V_t (B_t (2\partial_x M_t - \partial_x X_t) - \gamma (f_{mm}(t, M_t, \widetilde{Z}_t) \partial_x M_t + f_{ma}(t, M_t, \widetilde{Z}_t) \partial_x \widetilde{Z}_t)) \\ & - \widehat{Z}_t (\partial_x \widetilde{Z}_t - \gamma f_{ma}(t, M_t, \widetilde{Z}_t) \partial_x M_t - \gamma f_{aa}(t, M_t, \widetilde{Z}_t) \partial_x \widetilde{Z}_t)] \\ = & \rho(t, V_t, \widehat{Z}_t). \end{aligned}$$

We next show that the denominator of h is bounded away from zero. Assume first $L_{\sigma,z} > 0$. Then for all v with $|v| \leq (1-\varepsilon)\frac{\delta_u}{\delta_u - \delta_l}$ we have $|v(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))| \leq |v|\frac{\delta_u - \delta_l}{\delta_u} \leq (1-\varepsilon)$, and hence the denominator of $h(t, V_t, \widehat{Z}_t)$ is greater than or equal to ε . In the case where $L_{\sigma,z} = 0$, the denominator is equal to 1.

It remains to show that $\tau := \lim_{n \rightarrow \infty} \tau_n = T$ a.s. To this end note that $\partial_x X_t$ satisfies, on $[t_0, \tau)$, the linear SDE

$$d\partial_x X_t = \alpha_t \partial_x X_t dt + \beta_t \partial_x X_t dW_t,$$

where

$$\alpha_t := B_t(2V_t - 1) - \gamma(f_{mm}(t, M_t, \tilde{Z}_t)V_t + f_{ma}(t, M_t, \tilde{Z}_t)h(t, V_t, \widehat{Z}_t))$$

and

$$\beta_t := h(t, V_t, \widehat{Z}_t) - \gamma f_{ma}(t, M_t, \tilde{Z}_t)V_t - \gamma f_{aa}(t, M_t, \tilde{Z}_t)h(t, V_t, \widehat{Z}_t).$$

Consequently,

$$\partial_x X_{t \wedge \tau_n} = \exp\left(\int_{t_0}^{t \wedge \tau_n} \left(\alpha_s - \frac{1}{2}\beta_s^2\right) ds + \int_{t_0}^{t \wedge \tau_n} \beta_s dW_s\right).$$

Note that α_t and β_t are both bounded by $C(1 + |\widehat{Z}_t|)$ for some sufficiently large $C > 0$. We claim that \widehat{Z} has a bounded BMO-norm with a bound which does not depend on n . Indeed, the pair (V, \widehat{Z}) can be interpreted as a solution of a quadratic BSDE on $[0, \tau_n]$. Since V is bounded, standard results imply that \widehat{Z} has a bounded BMO-norm (see e.g. Theorem A.1.11. in [6] for details).

Now if $(\lim_{n \rightarrow \infty} \partial_x X_{\tau_n})(\omega) = 0$ for some ω , then $\left(\int_{t_0}^{\tau} |\widehat{Z}_t|^2 dt\right)(\omega) = \infty$ must hold for the same ω . This, however, is false for almost all ω , due to \widehat{Z} being a BMO-process on $[t_0, \tau)$. In other words, the continuous process $\partial_x X$ does not reach 0 with probability 1 and, therefore, $\lim_{n \rightarrow \infty} \tau_n = T$ a.s.

In particular $V_t = \partial_x M_t \frac{1}{\partial_x X_t}$ a.e. and V has a time-continuous version. \square

In the following we assume that V refers to the time-continuous version of Lemma 4.2.

Lemma 4.3. *There exists a \mathbb{P} -equivalent probability measure Q and a Q -BM W^Q such that (V, \widehat{Z}) is the unique solution of the BSDE*

$$V_t = \xi'(X_T) - \int_t^T \widehat{Z}_s dW_s^Q - \int_t^T \left(2B_s V_s(1 - V_s) + \gamma f_{mm}(s, M_s, \tilde{Z}_s)V_s^2 - \gamma^2 \frac{f_{ma}^2(s, M_s, \tilde{Z}_s)}{1 - V_t(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))} V_t^3 \right) ds. \quad (18)$$

Proof. Note that

$$\begin{aligned} \rho(t, v, z) = & 2B_t v(1 - v) + \gamma f_{mm}(t, M_t, \tilde{Z}_t)v^2 - \gamma^2 \frac{f_{ma}^2(s, M_s, \tilde{Z}_s)}{1 - v(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))} v^3 \\ & - z\psi(t, v, z), \end{aligned}$$

where

$$\psi(t, v, z) = -\gamma v \frac{f_{ma}(s, M_s, \tilde{Z}_s)}{1 - v(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))} - \gamma f_{ma}(t, M_t, \tilde{Z}_t)v + (1 - \gamma f_{aa}(t, M_t, \tilde{Z}_t))h(t, v, z).$$

Since V is bounded, there exists a constant $C > 0$ such that $|\psi(t, V_t, \widehat{Z}_t)| \leq C(1 + |\widehat{Z}_t|)$ for all $t \in [t_0, T]$. Consequently $\psi(t, V_t, \widehat{Z}_t) \in BMO(\mathbb{P})$. In particular there exists a probability measure $Q \sim \mathbb{P}$ such that

$$\frac{dQ}{d\mathbb{P}} = \exp \left(\int_{t_0}^T \psi(t, V_t, \widehat{Z}_t) dW_t - \frac{1}{2} \int_{t_0}^T \psi^2(t, V_t, \widehat{Z}_t) dt \right).$$

By Girsanov's theorem $W_t^Q := W_t - \int_{t_0}^t \psi(s, V_s, \widehat{Z}_s) ds$, $t \in [t_0, T]$, is a Brownian motion w.r.t. Q . Finally, observe that V satisfies (18). \square

Lemma 4.4. *For all $t \in [t_0, T]$ we have a.s. $q \leq V_t \leq 1$, where*

$$q := \frac{\exp(-T(2\|B\|_\infty + \gamma\|f_{mm}\|_\infty))}{1 + \gamma\|g''\|_\infty} \in (0, 1).$$

Proof. Let $\alpha(s, y) = -2B_t y(1 - y) - \gamma f_{mm}(s, M_s, \tilde{Z}_s) y^2 + \gamma^2 \frac{f_{ma}^2(s, M_s, \tilde{Z}_s)}{1 - y(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))} y^3$ be the generator of the BSDE (18).

In the case $L_{\sigma, z} > 0$ one can modify α to a Lipschitz continuous generator by replacing y with

$$\left(y \vee \left(-(1 - \varepsilon) \frac{\delta_u}{\delta_u - \delta_l} \right) \right) \wedge (1 - \varepsilon) \frac{\delta_u}{\delta_u - \delta_l}.$$

Indeed, (V, \widehat{Z}) solves also the modified BSDE since $|V_s| \leq (1 - \varepsilon) \frac{\delta_u}{\delta_u - \delta_l}$. Notice that for all v with $|v| \leq (1 - \varepsilon) \frac{\delta_u}{\delta_u - \delta_l}$ we have $|v(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))| \leq |v| \frac{\delta_u - \delta_l}{\delta_u} \leq (1 - \varepsilon)$, which further implies that

$$v \mapsto \frac{1}{1 - v(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))}$$

takes values only in $[1, \frac{1}{\varepsilon}]$ and is Lipschitz continuous.

In the case $L_{\sigma, z} = 0$ the generator α becomes Lipschitz continuous by replacing y with $(y \vee -K) \wedge K$.

Now consider the cut-off function $c(v) = ((v \vee 0) \wedge 1)$ and set

$$\begin{aligned} \check{\alpha}(s, v) &= -2B_t c(v)(1 - c(v)) - \gamma f_{mm}(s, M_s, \tilde{Z}_s) c(v)^2 \\ &\quad + \gamma^2 \frac{f_{ma}^2(s, M_s, \tilde{Z}_s)}{1 - c(v)(1 - \gamma f_{aa}(s, M_s, \tilde{Z}_s))} c(v)^3. \end{aligned}$$

Observe that

$$\begin{aligned} \check{\alpha}(s, v) &\leq -2B_t c(v)(1 - c(v)) - \gamma f_{mm}(s, M_s, \tilde{Z}_s) c(v)^2 + \gamma \frac{f_{ma}^2(s, M_s, \tilde{Z}_s)}{f_{aa}(s, M_s, \tilde{Z}_s)} c(v)^2 \\ &\leq -2B_t c(v)(1 - c(v)) - \gamma c(v)^2 \frac{1}{f_{aa}(s, M_s, \tilde{Z}_s)} \det(D^2 f)(s, M_s, \tilde{Z}_s) \\ &\leq -2B_t c(v)(1 - c(v)). \end{aligned}$$

Let (\check{V}, \check{Z}) be the solution of the BSDE with parameters $(\xi'(X_T), \check{\alpha})$. The comparison theorem, applied to (\check{V}, \check{Z}) and the BSDE with parameters $(1, -2B_t c(1 - c))$, implies $\check{V}_t \leq 1$.

Next, we estimate \check{V} from below. Notice that

$$\check{\alpha}(s, v) \geq -2\|B\|_\infty c(v)(1 - c(v)) - \gamma\|f_{mm}\|_\infty c(v)^2 \geq -(2\|B\|_\infty + \gamma\|f_{mm}\|_\infty) c(v).$$

Moreover, $\xi'(x) = \frac{1}{1+\gamma g''(\xi(x))} \geq \frac{1}{1+\gamma\|g''\|_\infty}$. A comparison with the BSDE with parameters

$$\left(\frac{1}{1 + \gamma\|g''\|_\infty}, -(2\|B\|_\infty + \gamma\|f_{mm}\|_\infty) c \right)$$

yields $\check{V}_t \geq q$.

Notice that this further entails that (\check{V}, \check{Z}) is also a solution of the BSDE (18). Uniqueness implies that $\check{V} = V$. \square

Recall that the diffusion coefficient in the forward part of (14), given by $\sigma(t, m, z) = z - \gamma f_a(t, m, z)$, is Lipschitz continuous in z with Lipschitz constant $\frac{\delta_u - \delta_l}{\delta_u}$. Lemma 4.4 implies that $0 < q \leq u_x(t_0, x) = V_{t_0} \leq 1 < \frac{\delta_u}{\delta_u - \delta_l}$.

Note that we have chosen a version of V such that $V_t = \frac{\partial_x M_t}{\partial_x X_t}$ for all $t \in [t_0, T]$. Moreover, for $t = t_0$, we have $V_{t_0} = \frac{\partial_x M_{t_0}}{1} = u_x(t_0, x)$, a.s.

Since x was chosen arbitrarily outside a λ -null set, we have that the $u_x(t_0, \cdot)$ is essentially bounded by 1. Since the bound does not depend on t_0 , by Theorem 3.4 it must hold that $I_{\max} = [0, T]$. Moreover, the following holds true:

Proposition 4.5. *There exists a unique weakly regular decoupling field u for (14) on $[0, T]$. Moreover, for every $t \in [0, T]$ we have $q \leq u_x(t, x) \leq 1$ for almost all $(\omega, x) \in \Omega \times \mathbb{R}$.*

We can now formulate the main result of the section.

Theorem 4.6. *Let $m \in \mathbb{R}$. Then*

1. *for all $t \in [0, T]$ the function $u(t, \cdot)$ has a Lipschitz continuous inverse $u^{-1}(t, \cdot)$, P -a.s.,*
2. *there exists a solution (M, Y, Z) of (6),*
3. *the function $\nu(t, m) := \frac{1}{\gamma}(u^{-1}(t, m) - m)$ is a decoupling field for (6), i.e. $Y_t = \nu(t, M_t)$ for all $t \in [0, T]$,*
4. *an optimal control of problem (4) is given by*

$$\alpha_t^* = \tilde{Z}_t,$$

where \tilde{Z} is part of the solution triplet of (14) with initial value $x = u^{-1}(0, m)$.

Proof. Since $u_x(t, \cdot)$ is bounded away from zero and bounded from above by 1, $u(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a bijection. Therefore, for every $m \in \mathbb{R}$ there is an $x \in \mathbb{R}$ such that $M_0 = u(0, x) = m$. For this $x \in \mathbb{R}$ there exists according to Theorem 3.4 a unique triplet (X, M, \tilde{Z}) such that

$$\sup_{s \in [0, T]} \mathbb{E}[|X_s|^2] + \sup_{s \in [0, T]} \mathbb{E}[|M_s|^2] + \mathbb{E} \left[\int_0^T |\tilde{Z}_s|^2 ds \right] < \infty$$

and such that FBSDE (14) is satisfied. Finally, using Lemma 4.1 we obtain a solution to FBSDE (6).

A straightforward calculation shows that for all $t \in [0, T]$ we have $\nu(t, M_t) = Y_t$. Finally, using Proposition 2.1 we obtain that $\tilde{Z}_t = f_a^{-1}(t, M_t, -Z_t)$ is an optimal control for problem (4). \square

5 An illustrating example

In general one can not expect the FBSDE (6) to possess a solution in closed form. For some examples, however, one can calculate the process V explicitly. This allows then to obtain a candidate for the decoupling field of FBSDE (14) and hence to derive a solution of (6) in closed form. In the following we illustrate such an explicit construction for a particular choice of f and g .

Let \bar{a}_t be a bounded and progressively measurable process, $l > 0$ and $L \geq 0$. Suppose that $f(t, m, a) = lm^2 + (a - \bar{a}_t)^2$. Furthermore, let $b = 0$, $B \in \mathbb{R}$ and $g(m) = Lm^2$. We can choose $\delta_u = \delta_l = 2$, and hence $\gamma = \frac{1}{2}$.

In this example we interpret M_t as the state of a particle in a medium with temperature α_t at time t . \bar{a}_t is the natural temperature process. Any cooling or heating entails quadratic costs. Problem (4) corresponds to the aim of steering the particle as close as possible to zero while keeping the costs for a temperature control low.

Notice that $\xi(x) = \frac{x}{1+L}$ and $\xi'(x) = \frac{1}{1+L}$. Moreover, the function ρ defined in Lemma 4.2 is given by

$$\rho(t, v) = 2Bv + (l - 2B)v^2.$$

Therefore, the process (V_t) is the solution of the Riccati equation

$$V'(t) = 2BV(t) + (l - 2B)V^2(t), \quad V(T) = \frac{1}{1+L}.$$

By solving the Riccati equation we obtain explicit expressions for V_t .

1. case: $B = 0$.

$$V_t = \frac{1}{l(T-t) + 1 + L}$$

2. case: $B \neq 0$.

$$V_t = \frac{e^{-2B(T-t)}}{1 + L + \frac{l-2B}{2B}(1 - e^{-2B(T-t)})}$$

Note that V_t does not depend on the spatial variable x . In particular, the decoupling field u of (14) is affine linear in x . This implies that M is affine linear in X .

Note that in this example the FBSDE (14) takes the form

$$\begin{aligned} X_t &= x + \int_0^t ((2B - l)M_s - BX_s)ds + \int_0^t \bar{a}_s dW_s, \\ M_t &= \frac{X_T}{1+L} - \int_t^T \tilde{Z}_s dW_s - \int_t^T BM_s ds. \end{aligned} \tag{19}$$

Assuming $M_t = V_t X_t$, the drift of the forward equation is linear in X , which allows to solve the equation explicitly.

Proposition 5.1. *The solution of (19) is given by*

$$\begin{aligned} X_t &= \left(x + \int_0^t H_s^{-1} \bar{a}_s dW_s \right) H_t, \\ M_t &= V_t X_t, \\ \tilde{Z}_t &= \bar{a}_t V_t, \end{aligned}$$

where $H_t = e^{\int_0^t [(2B-l)V_s - B] ds}$. In particular, the decoupling field for (19) is given by $u(t, x) = V_t x$. Moreover an optimal control of problem (4) is given by

$$\alpha_t^* = \bar{a}_t V_t$$

(note that the optimal control does not depend on the initial condition).

Proof. Let X , M and \tilde{Z} be defined as in Proposition 5.1. A straightforward calculation shows that X satisfies the dynamics

$$\begin{aligned} dX_t &= [(2B-l)V_t - B]X_t dt + \bar{a}_t dW_t \\ &= ((2B-l)M_t - BX_t)dt + \bar{a}_t dW_t. \end{aligned}$$

We now verify that M satisfies the backward equation in (19). First notice that M satisfies the terminal condition $M_T = \frac{X_T}{1+L}$. The product formula implies

$$\begin{aligned} dM_t &= V_t dX_t + X_t dV_t \\ &= V_t [(2B-l)V_t - B]X_t dt + V_t \bar{a}_t dW_t + X_t (2BV_t + (l-2B)V_t^2) dt \\ &= BM_t dt + \tilde{Z}_t dW_t. \end{aligned}$$

Hence (X, M, \tilde{Z}) satisfies (19). □

6 Linking the decoupling field to the HJB equation

In this section we try to explain the link between the FBSDE and the HJB approach for solving (4). Our arguments are mainly heuristic.

Let us assume that b, B, f, g do not depend on ω . Consider (6) and assume that there exists a differentiable function $w : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $w(t, M_t^m) = Y_t^m$ a.s. for all initial values $m \in \mathbb{R}$ and all $t \in [0, T]$. Assuming that w is continuously differentiable in time and twice continuously differentiable in space, we can apply the Itô formula to $w(t, M_t)$:

$$\begin{aligned} Y_t = w(t, M_t) &= w(0, m) + \int_0^t w_t(s, M_s) ds + \int_0^t w_m(s, M_s) f_3^*(s, M_s, -Z_s) dW_s \\ &\quad + \int_0^t w_m(s, M_s) (b_s + B_s M_s) ds + \frac{1}{2} \int_0^t w_{mm}(s, M_s) (f_3^*(s, M_s, -Z_s))^2 ds. \end{aligned}$$

At the same time the backward equation in (6) yields

$$Y_t = Y_0 + \int_0^t Z_s dW_s - \int_0^t (B_s Y_s + f_m(s, M_s, f_3^*(s, M_s, -Z_s))) ds.$$

Comparing the martingale and the finite variation parts leads to

$$\begin{aligned} Z_s &= w_m(s, M_s) f_3^*(s, M_s, -Z_s), \\ w_t(s, M_s) + w_m(s, M_s) (b_s + B_s M_s) + \frac{1}{2} w_{mm}(s, M_s) (f_3^*(s, M_s, -Z_s))^2 \\ &= -(B_s w(s, M_s) + f_m(s, M_s, f_3^*(s, M_s, -Z_s))). \end{aligned}$$

The first equation is equivalent to $w_m(s, M_s)\alpha_s + f_a(s, M_s, \alpha_s) = 0$, where $\alpha_s := f_3^*(s, M_s, -Z_s)$. Assume that $w_m(s, M_s) \geq 0$ and let $\varphi : [0, T] \times \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ be the function such that $\alpha_s = \varphi(s, M_s, w_m(s, M_s))$. Then we obtain

$$\begin{aligned} w_t(s, M_s) + w_m(s, M_s)(b_s + B_s M_s) + \frac{1}{2}w_{mm}(s, M_s) (\varphi(s, M_s, w_m(s, M_s)))^2 \\ = -B_s w(s, M_s) - f_m(s, M_s, \varphi(s, M_s, w_m(s, M_s))). \end{aligned}$$

Thus, it is natural to expect w to satisfy the PDE

$$\begin{aligned} w_t(s, m) + w_m(s, m)(b_s + B_s m) + \frac{1}{2}w_{mm}(s, m) (\varphi(s, m, w_m(s, m)))^2 \\ = -B_s w(s, m) - f_m(s, m, \varphi(s, m, w_m(s, m))). \end{aligned}$$

Now let us look at the classical HJB approach to problem (4): The associated HJB equation for the value function $v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ has the form

$$v_t(s, m) + (b_s + B_s m)v_m(s, m) - \sup_{a \in \mathbb{R}} \left[-\frac{1}{2}a^2 v_{mm}(s, m) - f(s, m, a) \right] = 0.$$

Assuming $v_{mm} \geq 0$, the optimal a in the above expression is given by $\varphi(s, m, v_{mm}(s, m))$, which is straightforward to verify using Fermat's theorem. We can rewrite the HJB equation as

$$\begin{aligned} v_t(s, m) + (b_s + B_s m)v_m(s, m) + \frac{1}{2}v_{mm}(s, m) (\varphi(s, m, v_{mm}(s, m)))^2 \\ = -f(s, m, \varphi(s, m, v_{mm}(s, m))). \end{aligned}$$

Note that $w(T, \cdot) = g'$ and $v(T, \cdot) = g$. It is, therefore, natural to conjecture that $w = v_m$ holds true. We verify this by assuming that v is three times continuously differentiable and by differentiating the HJB equation for v w.r.t. m , thus obtaining an equation for $v_m =: \bar{w}$:

$$\begin{aligned} \bar{w}_t(s, m) + \bar{w}_m(s, m)(b_s + B_s m) + \frac{1}{2}\bar{w}_{mm}(s, m) (\varphi(s, m, \bar{w}_m(s, m)))^2 + B_s \bar{w}(s, m) \\ + \bar{w}_m(s, m)\varphi(s, m, \bar{w}_m(s, m)) (\varphi_m(s, m, \bar{w}_m(s, m)) + \varphi_3(s, m, \bar{w}_m(s, m))\bar{w}_{mm}(s, m)) \\ = -f_m(s, m, \varphi(s, m, \bar{w}_m(s, m))) \\ - f_a(s, m, \varphi(s, m, \bar{w}_m(s, m))) (\varphi_m(s, m, \bar{w}_m(s, m)) + \varphi_3(s, m, \bar{w}_m(s, m))\bar{w}_{mm}(s, m)), \end{aligned}$$

where φ_3 is the derivative w.r.t. the last component. Due to the definition of φ the above simplifies to

$$\begin{aligned} \bar{w}_t(s, m) + \bar{w}_m(s, m)(b_s + B_s m) + \frac{1}{2}\bar{w}_{mm}(s, m) (\varphi(s, m, \bar{w}_m(s, m)))^2 \\ = -B_s \bar{w}(s, m) - f_m(s, m, \varphi(s, m, \bar{w}_m(s, m))). \end{aligned}$$

Indeed, w and \bar{w} satisfy the same PDE, which makes it plausible to believe that it is in fact the same object.

Now note that while v satisfies a fully non-linear PDE, the PDE satisfied by its spatial derivative w is quasi-linear and hence is easier to analyze. In particular, the quasi-linearity of w allows to reduce the problem to the FBSDE (6). Although the FBSDE is strongly coupled, it still crucially simplifies the problem. Indeed, as we have seen in Section 4, the method of decoupling fields allows to prove existence, uniqueness and regularity.

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