



**HAL**  
open science

## Looking backward: From Euler to Riemann

Athanase Papadopoulos

► **To cite this version:**

| Athanase Papadopoulos. Looking backward: From Euler to Riemann. 2017. hal-01614429

**HAL Id: hal-01614429**

**<https://hal.science/hal-01614429>**

Preprint submitted on 10 Oct 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LOOKING BACKWARD: FROM EULER TO RIEMANN

ATHANASE PAPADOPOULOS

*Il est des hommes auxquels on ne doit pas adresser d'éloges,  
si l'on ne suppose pas qu'ils ont le goût assez peu délicat  
pour goûter les louanges qui viennent d'en bas.*  
(Jules Tannery, [241] p. 102)

ABSTRACT. We survey the main ideas in the early history of the subjects on which Riemann worked and that led to some of his most important discoveries. The subjects discussed include the theory of functions of a complex variable, elliptic and Abelian integrals, the hypergeometric series, the zeta function, topology, differential geometry, integration, and the notion of space. We shall see that among Riemann's predecessors in all these fields, one name occupies a prominent place, this is Leonhard Euler. The final version of this paper will appear in the book *From Riemann to differential geometry and relativity* (L. Ji, A. Papadopoulos and S. Yamada, ed.) Berlin: Springer, 2017.

AMS Mathematics Subject Classification: 01-02, 01A55, 01A67, 26A42, 30-03, 33C05, 00A30.

Keywords: Bernhard Riemann, function of a complex variable, space, Riemannian geometry, trigonometric series, zeta function, differential geometry, elliptic integral, elliptic function, Abelian integral, Abelian function, hypergeometric function, topology, Riemann surface, Leonhard Euler, space, integration.

## CONTENTS

1. Introduction	1
2. Functions	10
3. Elliptic integrals	20
4. Abelian functions	30
5. Hypergeometric series	32
6. The zeta function	33
7. On space	40
8. Topology	47
9. Differential geometry	62
10. Trigonometric series	68
11. Integration	78
12. Conclusion	80
References	84

---

*Date:* October 10, 2017.

## 1. INTRODUCTION

More than any other branch of knowledge, mathematics is a science in which every generation builds on the accomplishments of the preceding ones, and where reading the old masters has always been a ferment for new discoveries. Examining the roots of Riemann's ideas takes us into the history of complex analysis, topology, integration, differential geometry and other mathematical fields, not to speak of physics and philosophy, two domains in which Riemann was also the heir of a long tradition of scholarship.

Riemann himself was aware of the classical mathematical literature, and he often quoted his predecessors. For instance, in the last part of his Habilitation lecture, *Über die Hypothesen, welche der Geometrie zu Grunde liegen* [231] (1854), he writes:<sup>1</sup>

The progress of recent centuries in the knowledge of mechanics depends almost entirely on the exactness of the construction which has become possible through the invention of the infinitesimal calculus, and through the simple principles discovered by Archimedes, Galileo and Newton, and used by modern physics.

The references are eloquent: Archimedes, who developed the first differential calculus, with his computations of length, area and volume, Galileo, who introduced the modern notions of motion, velocity and acceleration, and Newton, who was the first to give a mathematical expression to the forces of nature, describing in particular the motion of bodies in resisting media, and most of all, to whom is attributed a celebrated notion of space, the “Newtonian space.” As a matter of fact, the subject of Riemann's habilitation lecture includes the three domains of Newton's *Principia*: mathematics, physics and philosophy. It is interesting to note also that Archimedes, Galileo and Newton are mentioned as the three founders of mechanics in the introduction (Discours préliminaire) of Fourier's *Théorie analytique de la chaleur* ([117], p. i–ii), a work in which the latter lays down the rigorous foundations of the theory of trigonometric series. Fourier's quote and its English translation are given in §10 of the present paper. In the historical part of his Habilitation dissertation, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (On the representability of a function by a trigonometric series) [216], a memoir which precisely concerns trigonometric series, Riemann gives a detailed presentation of the history of the subject, reporting on results and conjectures by Euler, d'Alembert, Lagrange, Daniel Bernoulli, Dirichlet, Fourier and others. The care with which Riemann analyses the evolution of this field, and the wealth of historical details he gives, is another indication of the fact that he valued to a high degree the history of ideas and was aware of the first developments of the subjects he worked on. In the field of trigonometric series and in others, he was familiar with the important paths and sometimes the wrong tracks that his predecessors took for the solutions of the problems he tackled. Riemann's sense of history is also manifest in the announcement of his memoir *Beiträge zur Theorie der durch die Gauss'sche Reihe  $F(\alpha, \beta, \gamma, x)$  darstellbaren Functionen* (Contribution to the theory of functions representable by

---

<sup>1</sup>In all this paper, for Riemann's habilitation, we use Clifford's translation [232].

Gauss's series  $F(\alpha, \beta, \gamma, x)$ , published in the *Göttinger Nachrichten*, No. 1, 1857, in which he explains the origin of the problems considered, mentioning works of Wallis, Euler, Pfaff, Gauss and Kummer. There are many other examples.

Among Riemann's forerunners in all the fields that we discuss in this paper, one man fills almost all the background; this is Leonhard Euler. Riemann was an heir of Euler for what concerns functions of a complex variable, elliptic integrals, the zeta function, the topology of surfaces, the differential geometry of surfaces, the calculus of variations, and several topics in physics.

Riemann refers to Euler at several places of his work, and Euler was himself a diligent reader of the classical literature: Euclid, Pappus, Diophantus, Theodosius, Descartes, Fermat, Newton, etc. All these authors are mentioned all along his writings, and many of Euler's works were motivated by questions that grew out of his reading of them.<sup>2</sup> Before going into more details, I would like to say a few words about the lives of Euler and Riemann, highlighting analogies, but also differences between them.

Both Euler and Riemann received their early education at home, from their fathers, who were protestant ministers, and who both were hoping that their sons will become like them, pastors. At the age of fourteen, Euler attended a Gymnasium in Basel, while his parents lived in Riehen, a village near the city of Basel.<sup>3</sup> At about the same age, Riemann was sent to a Gymnasium in Hanover, away from his parents. During their Gymnasium years, both Euler and Riemann lived with their grandmothers.<sup>4</sup> They both enrolled a theological curriculum (at the Universities of Basel and Göttingen respectively), before they obtain their fathers' approval to shift to mathematics.

There are also major differences between the lives of the two men. Euler's productive period lasted 57 years (from the age of 19, when he wrote his first paper, until his death at the age of 76). His written production comprises more than 800 memoirs and 50 books. He worked on all domains of mathematics (pure and applied) and physics (theoretical and practical) that existed at his epoch. He also published on geography, navigation, machine theory, ship building, telescopes, the making of optical instruments, philosophy, theology and music theory. Besides his research books, he wrote elementary schoolbooks, including a well-known book on the art of reckoning [64]. The publication of his collected works was decided in 1907, the year of his bicentenary, the first volumes appeared in 1911, and the edition is still in progress (two volumes appeared in 2015), filling up to now more than 80 large volumes. Unlike Euler's, Riemann's life was short. He published his first work at the age of 25 and he died at the age of 39. Thus, his productive period lasted only 15 years. His collected works stand in a single slim volume. Yet, from the points of view of the originality and the

---

<sup>2</sup>Cf. for instance Euler's *Problematis cuiusdam Pappi Alexandrini constructio* (On a problem posed by Pappus of Alexandria), [97], 1780.

<sup>3</sup>Today, Riehen belongs to the Canton of the city of Basel, and it hosts the famous Beyeler foundation.

<sup>4</sup>In 1842, at the death of his grandmother, Riemann quitted Hanover and attended the Gymnasium at the Johanneum Lüneburg.

impact of their ideas, it would be unfair to affirm that either of them stands before the other. They both had an intimate and permanent relation to mathematics and to science in general. Klein writes in his *Development of mathematics in the 19th century* ([163], p. 231 of the English translation):

After a quiet preparation Riemann came forward like a bright meteor, only to be extinguished soon afterwards.

On Euler, I would like to quote André Weil, from his book on the history of number theory, *Number Theory: An approach through history from Hammurapi to Legendre* [256]. He writes, in the concluding section:

[...] Hardly less striking is the fact that Euler never abandoned a problem after it has once aroused his insatiable curiosity. Other mathematicians, Hilbert for instance, have had their lives neatly divided into periods, each one devoted to a separate topic. Not so Euler. All his life, even after the loss of his eyesight, he seems to have carried in his head the whole of the mathematics of his day, both pure and applied. Once he has taken up a question, not only did he come back to it again and again, little caring if at times he was merely repeating himself, but also he loved to cast his net wider and wider with never failing enthusiasm, always expecting to uncover more and more mysteries, more and more “herrliche proprietates” lurching just around the next corner. Nor did it matter to him whether he or another made the discovery. “*Penitus obstupui*”, he writes (“I was quite flabbergasted”: *Eu.I-21.1* in E 506|1777, cf. his last letter to Lagrange, *Eu.IV A-5.505|1775*) on learning Lagrange’s additions to his own work on elliptic integrals; after which he proceeds to improve upon Lagrange’s achievement. Even when a problem seemed to have been solved to his own satisfaction (as happened with his first proof of Fermat’s theorem  $a^p \equiv a \pmod p$ , or in 1749 with sums of two squares) he never rested in his search for better proofs, “more natural” (*Eu. I-2.510* in E 262|1755; cf. §VI), “easy” (*Eu.1-3.504* in E 522|1772; cf. §VI), “direct” (*Eu.I-2.363* in E 242|1751; cf. §VI); and repeatedly he found them.<sup>5</sup>

Let us say in conclusion that if we had to mention a single mathematician of the eighteenth century, Euler would probably be the right choice. For the nineteenth century, it would be Riemann. Gauss, who will also be mentioned many times in the present paper, is the main figure astride the two centuries.

Euler’s results are contained in his published and posthumous writings, but also in his large correspondence, available in several volumes of his *Opera Omnia*. We shall mention several times this correspondence in the present paper. It may be useful to remind the reader that at the epoch we are considering here, there were very few mathematical journals (essentially the publications of the few existing Academies of Sciences). The transmission of open problems and results among mathematicians was done largely through correspondence. On this question, let us quote the mathematician Paul Heinrich Fuss, who published the first set of letters of Euler, and who was

---

<sup>5</sup>In Weil’s book, every piece of historical information is accompanied by a precise reference. Works that attain this level of scholarship are very rare.

his great-grandson. He writes in the introduction to his *Correspondence* [119], p. xxv:<sup>6</sup>

Since sciences ceased to be the exclusive property of a small number of initiates, correspondence between scholars was taken over by the periodical publications. The progress is undeniable. However, this freeness with which ideas and discoveries were communicated in the past, in private and very confidential letters, we do not find it any more in the ripe and printed pieces of work. At that time, the life of a scholar was, in some way, all reflected in that correspondence. We see there the great discoveries being prepared and gradually developed; no link and no transition is missing; the path which led to these discoveries is followed step by step, and we can draw there some information even in the errors committed by these great geniuses who were the authors. This is sufficient to explain the interest tied to this kind of correspondence.<sup>7</sup>

In the case of Euler, particularly interesting is his correspondence with Christian Goldbach, published recently in two volumes of the *Opera Omnia* [110]. It contains valuable information on Euler's motivations and progress in several of the domains that are surveyed in the sections that follow, in particular, topology, the theory of elliptic functions and the zeta function. A few lines of biography on this atypical person are in order.

Christian Goldbach (1690–1764) was one of the first German scholars whom Euler met at the Saint Petersburg Academy of Sciences when he arrived there in 1827. He was very knowledgeable in mathematics, although he was interested in this field only in an amateurish fashion, encouraging others' works rather than working himself on specific problems. He was also a linguist and thoroughly involved in politics. Goldbach studied law at the University of Königsberg. In Russia, he became closely related to the Imperial family. In 1732, he was appointed secretary of the Saint Petersburg Academy of Sciences and in 1737 he became the administrator of that institution. In 1740, he held an important position at the Russian ministry of foreign affairs and became the official cryptographer there. Goldbach had a tremendous influence on Euler, by being attentive to his progress, by the questions he asked him on number theory, and also by motivating him to read Diophantus and Fermat. Goldbach, who was seventeen years older than Euler, became later on one of his closest friends and the godfather of his oldest son, Johann Albrecht, the only one among Euler's thirteen children who became a mathematician. Paul Heinrich Fuss writes in the introduction to his *Correspondence* [119], p. xxii:

---

<sup>6</sup>Unless otherwise stated, the translations from the French in this paper are mine.

<sup>7</sup>Depuis que les sciences ont cessé d'être la propriété exclusive d'un petit nombre d'initiés, ce commerce épistolaire des savants a été absorbé par la presse périodique. Le progrès est incontestable. Cependant, cet abandon avec lequel on se communiquait autrefois ses idées et ses découvertes, dans des lettres toutes confidentielles et privées, on ne le retrouve plus dans les pièces mûries et imprimées. Alors, la vie du savant se reflétait, pour ainsi dire, tout entière dans cette correspondance. On y voit les grandes découvertes se préparer et se développer graduellement ; pas un chaînon, pas une transition n'y manque ; on suit pas à pas la marche qui a conduit à ces découvertes, et l'on puise de l'instruction jusque dans les erreurs des grands génies qui en furent les auteurs. Cela explique suffisamment l'intérêt qui se rattache à ces sortes de correspondances.

It is more than probable that if this intimate relationship between Euler and this scholar, a relationship that lasted 36 years without interruption, hadn't been there, then the science of numbers would have never attained the degree of perfection which it owes to the immortal discoveries of Euler.<sup>8</sup>

Goldbach kept a regular correspondence with Euler, Nicolas and Daniel Bernoulli, Leibniz (in particular on music theory) and many other mathematicians.

After Goldbach and his influence on Euler, we turn to Gauss, who, among the large number of mathematicians with whom Riemann was in contact, was certainly the most influential on him.<sup>9</sup> We shall see in the various sections of the present paper that this influence was crucial for what concerns the fields of complex analysis, elliptic integrals, topology, differential geometry – the same list as for Euler's influence on Riemann – and also for what concerns his ideas on space. There are other topics in mathematics and physics which were central in the work of Riemann and where he used ideas he learned from Gauss: the Dirichlet principle, magnetism, etc.; they are addressed in several other chapters of the present book.

The first contact between Gauss and Riemann took place probably in 1846, before Gauss became officially Riemann's mentor. In that year, in a letter to his father dated November 5 and translated in [234], Riemann informs the latter about the courses he plans to follow, and among them he mentions a course by Gauss on "the theory of least squares."<sup>10</sup> During his two years stay in Berlin (1847–1849), Riemann continued to study thoroughly Gauss's papers. In another letter to his father, dated May 30, 1849, he writes (translation in [234]):

Dirichlet has arranged to me to have access to the library. Without his assistance, I fear there would have been obstacles. I am usually in the reading room by nine in the morning, to read two papers

---

<sup>8</sup>Il me semble plus que probable que si cette liaison intime entre Euler et ce savant, liaison qui dura 36 ans sans interruption, n'eût pas lieu, la science des nombres n'aurait guère atteint ce degré de perfection dont elle est redevable aux immortelles découvertes d'Euler.

<sup>9</sup>Some historians of mathematics claimed that when Riemann enrolled the University of Göttingen, as a doctoral student of Gauss, the latter was old and in poor health, and that furthermore, he disliked teaching. From this, they deduced that Gauss's influence on Riemann was limited. This is in contradiction with the scope and the variety of the mathematical ideas of Riemann for which he stated, in one way or another, but often explicitly, that he got them under the direct influence of Gauss or by reading his works. The influence of a mathematician is not measured by the time spent talking with him or reading his works. Gauss died the year after Riemann obtained his habilitation, but his imprint on him was permanent.

<sup>10</sup>The other courses are on the Cultural History of Greece and Rome, Theology, Recent Church history, General Physiology and Definite Integrals. Riemann had also the possibility to choose courses among Probability, Mineralogy and General Natural History. He adds: "The most useful to me will be mineralogy. Unfortunately it conflicts with Gauss's lecture, since it is scheduled at 10 o'clock, and so I'd be able to attend only if Gauss moved his lecture forward, otherwise it looks like it won't be possible. General Natural History would be very interesting, and I would certainly attend, if along with everything else I had enough money."

by Gauss that are not available anywhere else. I have looked fruitlessly for a long time in the catalog of the royal library for another work of Gauss, which won the Copenhagen prize, and finally just got it through Dr. Dale of the Observatory. I am still studying it.

During the same stay in Berlin, Riemann followed lectures by Dirichlet on topics related to Gauss's works. He writes to his father (letter without date, quoted in [234]):

My own course of specialization is the one with Dirichlet; he lectures on an area of mathematics to which Gauss owes his entire reputation. I have applied myself very seriously to this subject, not without success, I hope.

Regarding his written production, Riemann endorsed Gauss' principle: *pauca sed matura* (few but ripe).

Riemann, as a child, liked history. In a letter to his father, dated May 3, 1840 (he was 14), he complains about the fact that at his Gymnasium there were fewer lessons on history than on *Rechnen* (computing), cf. [234]. On August 5, 1841, he writes, again to his father, that he is the best student in history in his class. Besides history, Riemann was doing very well in Greek, Latin, and German composition (letters of February 1, 1845 and March 8, 1845). According to another letter to his father, dated April 30, 1845, it is only in 1845 that Riemann started being really attracted by mathematics. In the same letter, Riemann declares that he plans to enroll the University of Göttingen to study theology, but that in reality he must decide for himself what to do, since otherwise he "will bring nothing good to any subject."

Besides Euler and Gauss, we shall mention several other mathematicians. Needless to say, it would have been unreasonable to try to be exhaustive in this paper; the subject would need a book, and even several books. We have tried to present a few markers on the history of the major questions that were studied by Riemann, insisting only on the mathematicians whose works and ideas had an overwhelming impact on him.

The content of the rest of this paper is the following.

Section 2 is essentially an excursion into the realm of Euler's ideas on the notion of function, with a stress on algebraic functions and functions of a complex variable. Algebraic functions are multivalued, and Euler included these functions as an important element of the foundations of the field of analysis, which he laid down in his famous treatise *Introductio in analysin infinitorum* (Introduction to the analysis of the infinite) [61]. Riemann's work on what became known as Riemann surfaces was largely motivated by the desire to find a domain of definition for an algebraic multi-valued function on which it becomes single-valued.<sup>11</sup> The study of functions of a complex variable, which includes as a special case that of algebraic functions, is one of the far-reaching subjects of Riemann's investigations, and its development is one of the few most important achievements of the nineteenth century (probably the most important one).

Section 3 is concerned with elliptic integrals. These integrals constitute a class of complex functions with new interesting properties, and the work

---

<sup>11</sup>As a matter of fact, this is the origin of the use of the word "uniformization" by Riemann.



described in this section is a natural sequel to that which is reviewed in §2. We shall mention works done on this subject by Johann Bernoulli, Fagnano, Euler (who published thirty-three memoirs on elliptic integrals), Legendre, Abel and Jacobi.

Section 4 focusses on Abelian functions, a vast generalization of elliptic functions, which led to an important problem in which Riemann became interested, namely, the Jacobi inversion problem, and which he eventually solved using  $\vartheta$  functions. In fact, Abelian integrals constitute one of the major topics that Riemann worked on. He started his investigation on this subject in his doctoral dissertation [215] (1851), worked on it in his 1854 memoir [218] whose title is quite rightly “The theory of Abelian functions,” and he never stopped working and lecturing on it during the few years that were left to him. Some lecture notes and memoirs by Riemann on Abelian functions were published posthumously. In particular, his memoir *Über das Verschwinden der  $\vartheta$ -Functionen* (On the vanishing on theta functions) [225], in which he gives a solution to Jacobi’s problem of inversion for the general case of integrals of algebraic functions, is analyzed in Chapter 4 of the present volume, written by Houzel [142].

Section 5 is concerned with the so-called Gauss hypergeometric series. These series, in various forms, were studied by Euler in his *Institutiones calculi integralis* (Foundations of integral calculus), a treatise in three volumes [93] (1768–1770), and in several other papers by him, and by Gauss. The hypergeometric series is a family of functions of the form

$$1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$$

where  $\alpha, \beta, \gamma$  are parameters and where the variable is  $x$ .

Gauss considered that almost any transcendental function is obtained from a hypergeometric series by assigning special values to the parameters. By providing such a broad class of functions, the introduction of the hypergeometric series in the field of analysis opened up new paths. Besides Euler and Gauss, the predecessors of Riemann in this field include Pfaff and Kummer.

In Section 6, we deal with the zeta function. The history of this function is sometimes traced back to the work of Pietro Mengoli (1625–1686) on the problem of finding the value of the infinite series of inverses of squares of integers. Indeed, it is reasonable to assume that questions about this series were accompanied by questions about the series of inverses of cubes and other powers. But it was Euler again who studied  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  as a function of  $s$  (for  $s$  real), establishing the functional equation that it satisfies, and the relation with prime numbers. This was the starting point of Riemann’s investigations on what became later known as the Riemann zeta function.

In Section 7, we make a quick review of some works done by Riemann’s predecessors on the notion of space. This is essentially a philosophical debate, but it has a direct impact on mathematics and in particular on Riemann’s work on geometry, more especially on his habilitation dissertation. It is in his reflections on space that Riemann introduced in mathematics

the notion of *Mannigfaltigkeit*, which he borrowed from the philosophical literature. This notion reflects Riemann's multi-faced view on space, and it is an ancestor of the modern notion of manifold. Our review of space is necessarily very sketchy, since this notion is one of the most fundamental notions of philosophy, and talking seriously about it would require a whole essay. In particular, there is a lot to say on the philosophy of space in the works of Newton, Euler and Riemann and the comparison between them, but it is not possible to do it in the scope of the present paper. Our intent here is just to indicate some aspects of the notion of space as it appears in the works of these authors and those of some other philosophers, including Aristotle, Descartes and Kant, and, as much as possible in a short survey, to give some hints on the context in which they emerge.

It is also important to say that the effect of this discussion on space goes far beyond the limits of philosophy. Euler's theories of physics are strongly permeated with his philosophical ideas on space. Gauss's differential geometry was motivated by his investigations on physical space, more precisely, on geodesy and astronomy, and, more generally, by his aspiration to understand the world around him. At a more philosophical level, Gauss was an enthusiastic reader of Kant, and he criticized the latter's views on space, showing that they do not agree with the recent discoveries – his own and others' – of geometry. Riemann, in this field, was an heir of Gauss. In his work, the curvature of space (geometric space) is the expression of the physical forces that act on it. These are some of the ideas that we try to convey in Section 7 and in other sections of this paper.

Section 8 is concerned with topology. Riemann is one of the main founders of this field in the modern sense of the word, but several important topological notions may be traced back to Greek antiquity and to the later works of Descartes, Leibniz and Euler. We shall review the ideas of Leibniz, and consider in some detail the works of Descartes and Euler on the so-called Euler characteristic of a convex polyhedral surface, which in fact is nothing else but an invariant of the topological sphere, a question whose generalization is contained in Riemann's doctoral dissertation [215] and his paper on Abelian functions [218], from where one can deduce the invariants of surfaces of arbitrary genus.

Section 9 is concerned with the differential geometry of surfaces. We review essentially the works of Euler, Gauss and Riemann, but there was also a strong French school of differential geometry, operating between the times of Euler and Riemann, involving, among others, Monge and several of his students, and, closer to Riemann, Bonnet.

Section 10 is a review of the history of trigonometric series and the long controversy on the notion of function that preceded this notion. In his Habilitation memoir, Riemann describes at length this important episode of eighteenth and nineteenth century mathematics which also led to his discovery of the theory of integration, which we discuss in the next section.

In Section 11, we review some of the history of the Riemann integral. From the beginning of integral calculus until the times of Legendre, passing through Euler, integration was considered as an antiderivative. Cauchy defined the integral by limits of sums that we call now Riemann sums,

taking smaller and smaller subdivisions of the interval of integration and showing convergence to make out of that a definition of the definite integral, but he considered only integrals of continuous functions, where convergence is always satisfied. It was Riemann who developed the first general theory of integration, leading to the notion of integrable and non-integrable function.

The concluding section, §12 contains a few remarks on the importance of returning to the texts of the old masters.

Some of the historical points in our presentation are described in more detail than others; this reflects our personal taste and intimate opinion on what is important in history and worth presenting in more detail in such a quick survey. The reader will find at the end of this paper (before the bibliography) a table presenting in parallel some works of Euler and of Riemann on related matters.

## 2. FUNCTIONS

Vito Volterra, in his 1900 Paris ICM plenary lecture [251], declared that the nineteenth century was “the century of function theory.”<sup>12</sup> In the language of that epoch, the expression “function theory” refers, in the first place, to functions of a complex variable. One of the mottos, which was the result of a thorough experience in the domain, was that a function of a real variable acquires its full strength when it is complexified, that is, when it is extended to become a function of a complex variable. This idea was shared by Cauchy, Riemann Weierstrass, and others to whom we refer now.

On functions of a complex variable, we first quote a letter from Lagrange to Antonio Lorgna, an engineer and the governor of the military school at Verona who made important contributions to mathematics, physics and chemistry. The letter is dated December 20, 1777. Lagrange writes (cf. Lagrange’s *Œuvres*, [167] t. 14, p. 261):

I consider it as one of the most important steps made by Analysis in the last period, that of not being bothered any more by imaginary quantities, and to be able to submit them to calculus, in the same way as the real ones.<sup>13</sup>

Gauss, who, among other titles he carried, was one of the main founders of the theory of functions of a complex variable, was also responsible for the introduction of complex numbers in several theories. In particular, he realized their power in number theory, and he used this in his *Disquisitiones arithmeticae* (Arithmetical researches) [122] (1801), a masterpiece he wrote

---

<sup>12</sup>The title of Volterra’s lecture is: *Betti, Brioschi, Casorati : Trois analystes italiens et trois manières d’envisager les questions d’analyse* (Betti, Brioschi, Casorati: Three Italian analysts and three manners of addressing the analysis questions). In that lecture, Volterra presents three different ways of doing analysis, through the works of Betti, Brioschi and Casorati, who are considered as the founders of modern Italian mathematics. The three mathematicians had very different personalities, and contrasting approaches to analysis, but in some sense they were complementing each other. In particular, Brioschi was capable of doing very long calculations, Betti was a geometer repugnant to calculations, and Casorati was an excellent teacher and an applied mathematician.

<sup>13</sup>Je regarde comme un des pas les plus importants que l’Analyse ait faits dans ces derniers temps, de n’être plus embarrassée des quantités imaginaires et de pouvoir les soumettre au calcul comme les quantités réelles.

at the age of 24. In his second paper on biquadratic residues [121] (Sec. 30), he writes that “number theory is revealed in its entire simplicity and natural beauty when the field of arithmetic is extended to the imaginary numbers.” He explains that this means admitting integers of the form  $a + bi$ . “Such numbers,” he says, “will be called complex integers.”

In the same vein, Riemann, who had a marked philosophical viewpoint on things, writes, regarding complex functions, in §20 of his doctoral dissertation, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* [215] (Foundations of a general theory of functions of a variable complex magnitude) [215] (1851): “Attributing complex values to the variable quantities reveals a harmony and a regularity which otherwise would remain hidden.”

Finally, let us quote someone closer to us, Jacques Hadamard, from his *Psychology of invention in the mathematical field* [138]. A sentence by him which is often repeated is that “the shortest and the best way between two truths of the real domain often passes by the imaginary one.” We quote the whole passage ([138] p. 122–123):

It is Cardan, who is not only the inventor of a well-known joint which is an essential part of automobiles, but who has also fundamentally transformed mathematical science by the invention of imaginaries. Let us recall what an imaginary quantity is. The rules of algebra show that the square of any number, whether positive or negative, is a positive number: therefore, to speak of the square root of a negative number is mere absurdity. Now, Cardan deliberately commits that absurdity and begins to calculate on such “imaginary” quantities.

One would describe this as pure madness; and yet the whole development of algebra and analysis would have been impossible without that fundament – which, of course, was, in the nineteenth century, established on solid and rigorous bases. It has been written that the shortest and the best way between two truths of the real domain often passes by the imaginary one.

In the rest of this section, we review some markers in the history of functions, in particular functions of a complex variable and algebraic functions, two topics which are at the heart of Riemann’s work on Riemann surfaces, on Abelian functions, on the zeta function, and on other topics. Before that, we make a digression on the origin of the general notion of function.

It is usually considered that Euler’s *Introductio in analysin infinitorum* [61] is the first treatise in which one can find the definition of a function, according to modern standards, and where functions are studied in a systematic way. We take this opportunity to say a few words on Euler’s treatise, to which we refer several times in the rest of this paper.

The *Introductio* is a treatise in two volumes, first published in 1748, which is concerned with a variety of subjects, including (in the first volume) algebraic curves, trigonometry, logarithms, exponentials and their definitions by limits, continued fractions, infinite products, infinite series and integrals. The second volume is essentially concerned with the differential geometry of curves and surfaces. The importance of the *Introductio* lies above all in

the fact that it made analysis the branch of mathematics where one studies functions. But the *Introductio* is more than a treatise with a historical value. Two hundred and thirty years after the first edition appeared in print, André Weil considered that it was more useful for a student in mathematics to study that treatise rather than any other book on analysis. This is reported on by John Blanton who writes, in his English edition of the *Introductio* [62]:

In October, 1979, Professor André Weil spoke at the University of Rochester on the life and work of Leonhard Euler. One of his remarks was to the effect that he was trying to convince the mathematical community that students of mathematics would profit much more from a study of Euler's *Introductio in analysin infinitorum*, rather than the available modern textbooks.

The importance of this work has also been highlighted by several other mathematicians. C. B. Boyer, in his 1950 ICM communication (Cambridge, Mass.) [28], compares the impact of the *Introductio* to that of Euclid's *Elements* in geometry and to al-Khwārizmī's *Jabr* in algebra. He writes:

The most influential mathematics textbooks of ancient times (or, for that matter, of all times) is easily named. The *Elements* of Euclid, appearing in over a thousand editions, has set the pattern in elementary geometry ever since it was composed more than two and a quarter millenia ago. The medieval textbook which most strongly influenced mathematical development is not so easily selected; but a good case can be made out of *Al-jabr wal muqābala* of al-Khwārizmī, just about half as old as the *Elements*. From this Arabic work, algebra took its name and, to a great extent, its origin. Is it possible to indicate a modern textbook of comparable influence and prestige? Some would mention the *Géométrie* of Descartes, or the *Principia* of Newton or the *Disquisitiones* of Gauss; but in pedagogical significance these classics fell short of a work less known. [...] over these well known textbooks there towers another, a work which appeared in the very middle of the great textbook age and to which virtually all later writers admitted indebtedness. This was the *Introductio in analysin infinitorum* of Euler, published in two volumes in 1748. Here in effect Euler accomplished for analysis what Euclid and al-Khwārizmī had done for synthetic geometry and elementary algebra, respectively.

Even though the *Introductio* is generally given as the main reference for the introduction of functions in analysis, regarding the *usage* of functions, one can go far back into history. Tables of functions exist since the Babylonians (some of their astronomical tables survive). Furthermore, in ancient Greece, mathematicians manipulated functions, not only in the form of tables. In particular, the chord function (an ancestor of our sine function)<sup>14</sup> is used extensively in some Greek treatises. For instance, Proposition 67 of Menelaus' *Spherics* (1st-2nd century A.D.) says the following [213]:

Let  $ABC$  and  $DEG$  be two spherical triangles whose angles  $A$  and  $D$  are equal, and where  $C$  and  $G$  are either equal or their sum is

---

<sup>14</sup>The relation between chord and sine is :  $\sin x = \frac{1}{2}\text{crd } 2x$ .

equal to two right angles. Then,

$$\frac{\text{crd } 2AB}{\text{crd } 2BC} = \frac{\text{crd } 2DE}{\text{crd } 2EG}.$$

Youshkevitch, in his interesting survey [261], argues that the general idea of a dependence of a quantity upon another one is absent from Greek geometry. The author of the present paper declares that if in the above proposition of Menelaus one does not see the notion of function, and hence the general idea of a dependence of a quantity upon another one, then this author fails to know what mathematicians mean by the word function.

Leibniz and Johann I Bernoulli, who were closer to Euler, manipulated functions, even though the functions they considered were always associated with geometrical objects, generally, curves in the plane. For instance, in a memoir published in 1718 on the isoperimetry problem in the plane, [24] Bernoulli writes:

Here, we call *function* of a variable magnitude, a quantity formed in whatever manner with that variable magnitude and constants.<sup>15</sup>

The functions that Bernoulli considers in this memoir are associated to *arbitrary* curves in the plane having the same perimeter, among which Bernoulli looks for the one which bounds the greatest area. This is an example of the general idea that before Euler, analysis was tightly linked to geometry, and the study of functions consisted essentially in the study of curves associated to some geometric properties. With the *Introductio*, things became different. Analysis started to release itself from geometry, and functions were studied for themselves. Let us now make a quick review of the part of this treatise which concerns us here.

The first paper is called *On functions in general*. In this chapter, Euler states his general definition of a function, after a description of what is a variable quantity:

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.

The word “analytic” means in this context that the function is obtained by some process that uses the four operations (addition, subtraction, multiplication and division), together with root extraction, exponentials, logarithms, trigonometric functions, derivatives and integrals. Analyticity in terms of being defined by a convergent power series is not intended by this definition. The meaning of the word “analytic function” rather is “a function used in (the field of) analysis.” Concerning the notion of variable, Euler writes (§3):<sup>16</sup>

[...] Even zero and complex numbers are not excluded from the signification of a variable quantity.

Thus, functions of a complex variable are included in Euler’s *Introductio*. We note however that in this treatise, Euler, in his examples, always deals with

---

<sup>15</sup>On appelle ici *Fonction* d’une grandeur variable, une quantité composée de quelque manière que ce soit avec cette grandeur variable et des constantes. [The emphasis is Bernoulli’s]

<sup>16</sup>We are using the translation from Latin in [61].

functions that are given by formulae: polynomials, exponentials, logarithms, trigonometric functions, etc. but also infinite products and infinite sums.

After the definition of a function, we find in the *Introductio* the definition of an algebraic function. In §7, Euler writes:

Functions are divided into algebraic and transcendental. The former are those made up from only algebraic operations, the latter are those which involve transcendental operations.

And in §8:

Algebraic functions are subdivided into non-irrational and irrational functions: the former are such that the variable quantity is in no way involved with irrationality; the latter are those in which the variable quantity is affected by radical signs.

Concerning irrational functions (§9), he writes:

It is convenient to distinguish these into explicit and implicit irrational functions.

The explicit functions are those expressed with radical signs, as in the given examples. The implicit are those irrational functions which arise from the solution of equations. Thus  $Z$  is an implicit irrational function of  $z$  if it is defined by an equation such as  $Z^7 = az$  or  $Z^2 = bz^5$ . Indeed, an explicit value of  $Z$  may not be expressed even with radical signs, since common algebra has not yet developed to such a degree of perfection.

And in §10:

Finally, we must make a distinction between single-valued and multi-valued functions.

A single-valued function is one for which, no matter what value is assigned to the variable  $z$ , a single value of the function is determined. On the other hand, a multi-valued function is one such that, for some value substituted for the variable  $z$ , the function determines several values. Hence, all non-irrational functions, whether polynomial or rational, are single-valued functions, since expressions of this kind, whatever value be given to the variable  $z$ , produce a single value. However, irrational functions are all multi-valued, because the radical signs are ambiguous and give paired values. There are also among the transcendental functions, both single-valued and multi-valued functions; indeed, there are infinite-valued functions. Among these are the arcsine of  $z$ , since there are infinitely many circular arcs with the same sine.

Euler then gives examples of two-valued, three-valued and four-valued functions, and in §14 he writes:

Thus  $Z$  is a multi-valued function of  $z$  which for each value of  $z$ , exhibits  $n$  values of  $Z$  where  $n$  is a positive integer. If  $Z$  is defined by this equation

$$Z^n - PZ^n - 1 + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \dots = 0$$

[...] Further it should be kept in mind that the letters  $P, Q, R, S$ , etc. should denote single-valued functions of  $z$ . If any of them is already a multi-valued function, then the function  $Z$  will have many more values, corresponding to each value of  $z$ , than the exponent would indicate. It is always true that if some of the values

are complex, then there will be an even number of them. From this we know that if  $n$  is an odd number, there will be at least one real value of  $Z$ .

He then makes the following remarks:

If  $Z$  is a multi-valued function of  $z$  such that it always exhibits a single real value, then  $Z$  imitates a single-valued function of  $z$ , and frequently can take the place of a single-valued function.

Functions of this kind are  $P^{\frac{1}{3}}$ ,  $P^{\frac{1}{5}}$ ,  $P^{\frac{1}{7}}$ , etc. which indeed give only one real value, the others all being complex, provided  $P$  is a single-valued function of  $z$ . For this reason, an expression of the form  $P^{\frac{m}{n}}$ , whenever  $n$  is odd, can be counted as a single-valued function, whether  $m$  is odd or even. However, if  $n$  is even then  $P^{\frac{m}{n}}$  will have either no real value or two; for this reason, expressions of the form  $P^{\frac{m}{n}}$ , with  $n$  even, can be considered to be two-valued functions, provided the fraction  $\frac{m}{n}$  cannot be reduced to lower terms.

From this discussion we single out the fact that algebraic functions are considered as functions, even though they are multi-valued. They are solutions of algebraic equations. Since we are talking about history, it is good to recall that the study of such equations is an old subject that can be traced back to the work done on algebraic curves by the Greeks. In fact, Diophantus (3d century B.C.) thoroughly studied integral solutions of what is now called “Diophantine equations.” They are examples of algebraic equations.<sup>17</sup> Algebraic equations are also present in the background of the geometric work of Apollonius (3d–2d century B.C.) on conics. In that work, intersections of conics were used to find geometrical solutions of algebraic equations.<sup>18</sup> It is true however that in these works, there is no *definition* of an algebraic function as we intend it today, and in fact at that time there was no definition of function at all.

The multi-valuedness of algebraic functions gave rise to tremendous developments by Cauchy and Puiseux, and it was also a major theme in Riemann’s work, in particular in his doctoral dissertation [215] (1851) and his memoir on Abelian functions [218] (1857). In fact, the main reason for which Riemann introduced the surfaces that we call today Riemann surfaces was to find ground spaces on which multi-valued functions are defined and become single-valued. We discuss the works of Cauchy and Puiseux in relation with that of Riemann in Chapter 7 of the present volume, [192].

We note for later use that a definition of “continuity” is given in Volume 2 of the *Introductio*, where Euler says that a curve is continuous if it represents “one determinate function,” and discontinuous if it is decomposed into “portions that represent different continuous functions.” We shall see that such a notion was criticized by Cauchy (regardless of the fact that it is called “continuity”).<sup>19</sup>

<sup>17</sup>For what concerns Diophantus’ *Arithmetica*, we refer the interested reader to the recent and definitive editions [52], [53], [54], and [211] by R. Rashed.

<sup>18</sup>For a recent and definitive edition of Apollonius’ *Conics*, we refer the reader to the volumes [11], [12], [13], [14], and [15], again edited by R. Rashed.

<sup>19</sup>There are other imperfections in the *Introductio*, even though this book is one of the most interesting treatises ever written on elementary analysis.



We note finally that it is usually considered that the expression *analysis infinitorum* in the title of Euler's treatise does not refer to the field of infinitesimal analysis in the sense of Newton or Leibniz, but, rather, to the use of infinity (infinite series, infinite products, continued fractions expansions, integral representations, etc.) in analysis. Euler was also the first to highlight the zeta function, the gamma function and elliptic integrals as functions. However, it is good to recall that infinite sums were known long before Euler. For instance, Zeno of Elea (5th c. B.C.) had already addressed the question of convergence of infinite series, and to him are attributed several well-known paradoxes in which the role and the significance of infinite series and their convergence are emphasized (the paradox of Achilles and the tortoise, the arrow paradox, the paradox of the grain of millet, etc.). However, infinite series are not considered as functions in these works. Zeno's paradoxes are commented in detail in Aristotle's *Physics* [20], but also by mathematicians and philosophers from the modern period, including Bertrand Russell, Hermann Weyl, Paul Tannery and several others; cf. [237] p. 346–354, and [259] and [242].

We also recall that convergent series were used by Archimedes in his computations of areas and volumes.

Before leaving this book, let us mention that Euler establishes there a hierarchy among transcendental functions by introducing a notion close to what we call today the transcendence degree of a function.

In his later works, Euler dealt with much more general functions. For instance, in his 1755 memoir [104], entitled *Remarques sur les mémoires précédents de M. Bernoulli* (Remarks on the preceding memoirs by Mr. Bernoulli), any mechanical curve (that is any curve drawn by hand) is associated with a function.<sup>20</sup> In his *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum* (Foundations of differential calculus, with applications to finite analysis and series) [83], also published in 1755, Euler gave again a very general definition of a function (p. vi):

Those quantities that depend on others in this way, namely, those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity could be determined by another.

Likewise, in his memoir [105], *De repraesentatione superficiei sphaericae super plano* (On the representation of Spherical Surfaces onto the Plane)

---

<sup>20</sup>One may recall here that the mathematicians of Greek antiquity (Archytas of Tarentum, Hippias, Archimedes, etc.) who examined curves formulated a mechanical definition. The curves with which they dealt were not necessarily defined by equations, they were “traced by a moving point,” sometimes (in theory) using a specific mechanical device. Of some interest here would be the connections between this subject and the theory of mechanical linkages, which was extensively developed in the nineteenth century and became fashionable again in the twentieth century. A conjecture by Thurston says (roughly speaking) that any “topological curve” is drawable by a mechanical linkage. This is a vast generalization of a result of Kempe stating that any bounded piece of an algebraic curve is drawable by some linkage, cf. [161]. We refer the reader to Sossinsky's survey of this subject and its recent developments [239], in particular the solution of Thurston's conjecture.

(1777), Euler dealt with “arbitrary mappings” between the sphere and the plane. He writes:<sup>21</sup>

I take the word “mapping” in the widest possible sense; any point of the spherical surface is represented on the plane by any desired rule, so that every point of the sphere corresponds to a specified point in the plane, and inversely.

We shall consider again the question of functions, from the epoch of Euler and until the work of Riemann, in §10 concerned with trigonometric functions.

Riemann, in his doctoral dissertation, [215] (1851), also considers arbitrary functions. In fact, the dissertation starts as follows: “If we designate by  $z$  a variable magnitude, which may take successively all possible real values, then, when to each of these values corresponds a unique value of the indeterminate magnitude  $w$ , we say that  $w$  is a function of  $z$  [...]” One may also refer to the beginning of §XIX of the same dissertation, where Riemann states that the principles he is presenting are the bases of a general theory of functions which is independent of any explicit expression.

The details of the seventeenth-century debate concerning functions are rather confusing if one does not include them in their historical context. For instance, the notion of “continuity” which we alluded to and which is referred to in the debate is different from what we intend today by this word. In fact, the word “continuity,” even restricted to the works of Euler, varied in the course of time.

Cauchy, the major figure standing between Euler and Riemann for what concerns the notion of function, in his *Mémoire sur les fonctions continues* (Memoir on continuous functions) [37], starts as follows:

In the writings of Euler and Lagrange, a function is termed *continuous* or *discontinuous* according to whether the various values of this function corresponding to various values of the variable follow or not the same law, or are given or not by only one equation. It is in these terms that the continuity of functions was defined by these famous geometers, when they used to say that “the arbitrary functions, introduced by the integration of partial differential equations, may be continuous or discontinuous functions.” However, the definition which we just recalled is far from offering mathematical accuracy [...] A simple change in notation will often suffice to transform a continuous function into a discontinuous one, and conversely.<sup>22</sup>

In fact, one might consider that Euler’s definition of continuity is just one definition that is different from the new definition which Cauchy had

---

<sup>21</sup>We are using George Heines’ translation.

<sup>22</sup>Dans les ouvrages d’Euler et de Lagrange, une fonction est appelée *continue* ou *discontinue*, suivant que les diverses valeurs de cette fonction, correspondantes à diverses valeurs de la variable, sont ou ne sont pas assujetties à une même loi, sont ou ne sont pas fournies par une seule équation. C’est en ces termes que la continuité des fonctions se trouvait définie par ces illustres géomètres, lorsqu’ils disaient que “les fonctions arbitraires, introduites par l’intégration des équations aux dérivées partielles, peuvent être des fonctions continues ou discontinues.” Toutefois, la définition que nous venons de rappeler est loin d’offrir une précision mathématique [...] un simple changement de notation suffira souvent pour transformer une fonction continue en fonction discontinue, et réciproquement.

in mind (and which is the definition we use today). This would have been fine, and it would not be the only instance in mathematics where the same word is used for notions that are different, especially at different epochs. But Cauchy showed by an example that in this particular case Euler's definition is inconsistent, because the property it expresses depends on the parametrization that is used. Cauchy continues:

But the non-determinacy will cease if we substitute to Euler's definition the one I gave in Chapter II of the *Analyse algébrique*. According to the new definition, a function of the variable  $x$  will be *continuous* between two limits  $a$  and  $b$  of this variable if between two limits the function has always a value which is unique and finite, in such a way that an infinitely small increment of this variable always produces an infinitely small increment of the function itself.<sup>23</sup>

We quoted these texts in order to give an idea of the progress of the notion of continuity. We now come to the study of functions of a complex variable.

In his memoir on Abelian functions, Riemann refers explicitly to Gauss for the fact that we represent a complex magnitude  $z = x + iy$  by a point in the plane with coordinates  $x$  and  $y$ .

It is not easy to know when the theory of functions of a complex variable started, and, in fact, the answer depends on whether one studies holomorphic functions, and what properties of holomorphic functions are meant (before the epoch of Riemann, they were not known to be equivalent): angle-preservation, power series expansion, the Cauchy-Riemann equation, etc.

Euler used complex variables and the notion of conformality (angle-preservation) in his memoirs on geographical maps. He wrote three memoirs on this subject, *De repraesentatione superficiei sphaericae super plano* (On the representation of spherical surfaces on a plane) [105], *De projectione geographica superficiei sphaericae* (On the geographical projections of spherical surfaces) [106], and *De projectione geographica Deslisiana in mappa generali imperii russici usitata* (On Delisle's geographic projection used in the general map of the Russian empire) [107]. The three memoirs were published in 1777. In the development of the theory, he used complex numbers to represent angle-preserving maps. Lagrange also studied angle-preserving maps, in his memoir *Sur la construction des cartes géographiques* (On the construction of geographical maps) [166], published in 1779.

In fact, the notion of angle-preserving map can be traced back to Greek antiquity, see the survey [195]. We already recalled that Euler, in his didactical treatise *Introductio in analysin infinitorum*, refers explicitly to functions in which the variable is a complex number. De Moivre, already in 1730, considered polynomials defined on the complex plane, and it is conceivable that other mathematicians before him did the same [183]. Remmert, who, besides being a specialist of complex analysis, is a highly respected historian

---

<sup>23</sup>Mais l'indétermination cessera si à la définition d'Euler on substitue celle que j'ai donnée dans le chapitre II de l'*Analyse algébrique*. Suivant la nouvelle définition, une fonction de la variable réelle  $x$  sera continue entre deux limites  $a$  et  $b$  de cette variable, si, entre ces limites, la fonction acquiert constamment une valeur unique et finie, de telle sorte qu'un accroissement infiniment petit de la variable produise toujours un accroissement infiniment petit de la fonction elle-même.

in this field, writes in his *Theory of complex variables* [214] that the theory was born at the moment when Gauss sent a letter to Bessel, dated December 18, 1811, in which he writes:<sup>24</sup>

At the beginning I would ask anyone who wants to introduce a new function in analysis to clarify whether he intends to confine it to real magnitudes (real values of the argument) and regard the imaginary values as just vestigial – or whether he subscribes to my fundamental proposition that in the realm of magnitudes the imaginary ones  $a + b\sqrt{-1} = a + bi$  have to be regarded as enjoying equal rights with the real ones. We are not talking about practical utility here; rather analysis is, to my mind, a self-sufficient science. It would lose immeasurably in beauty and symmetry from the rejection of any fictive magnitudes. At each stage truths, which otherwise are quite generally valid, would have to be encumbered with all sorts of qualifications.

In fact, the letter also shows that at that time Gauss was already aware of the concept of complex integration, including Cauchy's integral theorem; cf. [127] Vol. 8, p. 90–92.

Cauchy, in his *Cours d'analyse* [34] (1821), starts by defining functions of real variables (p. 19), and then passes to complex variables. There are two distinct definitions in the real case, for functions of one or several variables:

When variable quantities are so tied to each other that, given the value of one of them, we can deduce the values of all the others, we usually conceive these various quantities expressed in terms of one of them, which then bears the name *independent variable*; and the other quantities expressed in terms of the independent variable are what we call functions of that variable.

When variable quantities are so tied to each other that, given the values of some of them, we can deduce the values of all the others, we usually conceive these various quantities expressed in terms of several of them, which then bear the name *independent variables*; and the remaining quantities expressed in terms of the independent variables, are what we call functions of these same variables.<sup>25</sup>

Talking about Cauchy's work on functions of a complex variable, one should also mention the Cauchy–Riemann equation as a characterization of complex analyticity, which Cauchy and Riemann introduced the same year,

---

<sup>24</sup>The translation is Remmert's; cf. [214] p. 1.

<sup>25</sup>Lorsque des quantités variables sont tellement liées entre elles que, la valeur de l'une d'elles étant donnée, on puisse en conclure les valeurs de toutes les autres, on conçoit d'ordinaire ces diverses quantités exprimées au moyen de l'une d'entre elles, qui prend alors le nom de *variable indépendante* ; et les autres quantités exprimées au moyen de la variable indépendante sont ce qu'on appelle des *fonctions* de cette variable.

Lorsque les quantités variables sont tellement liées entre elles que, les valeurs de quelques unes étant données, on puisse en conclure celles de toutes les autres, on conçoit ces diverses quantités exprimées au moyen de plusieurs d'entre elles, qui prennent alors le nom de *variables indépendantes* ; et les quantités restantes, exprimées au moyen des variables indépendantes, sont ce qu'on appelle des *fonctions* de ces mêmes variables.

1851, Cauchy in his papers [38] and [39] and Riemann in his doctoral dissertation [215]. It is important to note also that the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

without the complex character, were used by d’Alembert in 1752, in his works on fluid dynamics, *Essai d’une nouvelle théorie de la résistance des fluides* (Essay on a new theory of fluid resistance) [262] p. 27. D’Alembert showed later that functions  $u$  and  $v$  satisfying this pair of equations also satisfy Laplace’s equation:  $\Delta u = 0$  and  $\Delta v = 0$ .

The work of Cauchy is also reviewed in the chapter [192] in the present volume, written by the present author.

Riemann’s doctoral dissertation [215] is in some sense an essay on functions of a complex variable. Right at the beginning of the dissertation, Riemann states explicitly what he means by a function. He starts with functions of a real variable:

If we designate by  $z$  a variable magnitude, which may take successively all possible real values, then, if to each of these values corresponds a unique value of the indeterminate magnitude  $w$ , we say that  $w$  is a function of  $z$ .

He then talks about continuity of functions, in the modern sense of the word (as opposed to the sense that Euler gave to this word).<sup>26</sup> Then he writes:

This definition does not stipulate any law between the isolated values of the function, this is evident, because after this function has been dealt with for a given interval, the way it is extended outside this interval remains quite arbitrary.

Riemann then recalls that the possibility of using some “mathematical law” that assigns to  $w$  a value for a given value of  $z$  was proper to the functions which Euler termed *functiones continuæ*. He writes that “modern research has shown that there exist analytic expressions by which any continuous function on a given interval can be represented.” He then declares that the case of functions of a complex variable is treated differently. In fact, Riemann considers only functions of a complex variable whose derivative does not depend on the direction, that is, holomorphic functions. He makes this property part of his definition of a function of a complex variable. Thus, when he talks about a function in the complex setting, he considers only conformal maps.

Regarding Riemann’s dissertation, let us note that in a letter to his brother, dated November 26, 1851 [234], after he submitted his doctoral dissertation manuscript, he writes that Gauss took it home to examine it for a few days, and that before reading it, Gauss told him:

[Riemann speaking] for years he had been preparing an essay, on which today he is still occupied, whose subject is the same or at least in part the same as that covered by me. Already in his doctoral dissertation now 52 years ago he actually expressed the intention to write on this subject.

---

<sup>26</sup>In the *Introductio* Euler used the expression *continuous function* for a function that is “given by a formula.” This is thoroughly discussed in §10 of the present paper.

This is an instance where Gauss was aware of a theory, or part of it, long before its author; we shall mention several other such instances in what follows.

### 3. ELLIPTIC INTEGRALS

In the huge class of integrals of functions, the integrals of algebraic functions constitute the simplest and the most natural class to work with. The class of elliptic integrals (and their Abelian generalizations) which deal with such functions soon turned out to be enough tractable and at the same time very rich from the point of view of the problems that they posed. These integrals led to a huge amount of work by several prominent mathematicians, as we shall see in this section.

Riemann had several reasons to work on Abelian integrals. Motivated by lectures by Dirichlet, Jacobi and others, he worked on the open problems that these functions presented, in particular the Jacobi inversion problem.

When Riemann started his work on integrals as functions of a complex variable, this subject was already well developed. An important challenging problem that he tackled was the so-called Jacobi inversion problem which we mention below. Most of all, these functions constituted for Riemann an interesting class of non-necessarily algebraic functions of a complex variable. The double periodicity of these integrals, the multi-valuedness of their inverses, the operations that one can perform on them, constituted a treasure of examples of new functions of a complex variable, and a context in which his theory of Riemann surfaces may naturally be used.

We start by summarizing some of the main ideas and problems that concern elliptic functions that were addressed since the time of Euler.

- (1) The study of definite integrals representing arcs of conics and of lemniscates, and the comparison of their properties with those of integrals representing arcs of circles, which are computable in terms of the trigonometric functions or their inverses. We recall, by way of comparison, that whereas the integral  $\int_0^x \frac{dt}{\sqrt{1-t^2}}$  represents arc length along a circle centered at the origin, the integral  $\int_0^x \frac{dt}{\sqrt{1-t^4}}$  represents arc length along the lemniscate of polar equation  $r^2 = \cos 2\theta$ .
- (2) The search for sums and product formulae for such integrals, in the same way as there are formulae for sums and products of trigonometric functions.
- (3) The study of periods, again, in analogy with those of trigonometric functions.

In fact, some of the first questions concerning elliptic integrals can be traced back to Johann I Bernoulli who tried to use the newly discovered integral calculus to obtain formulae for lengths of arcs of conic sections and some other curves. Bernoulli found the first addition formulae for such integrals. Finding general addition theorems for elliptic integrals remained one of the major problems for the following hundred years, involving the works

of several major figures including Euler, Legendre, Abel, Jacobi and Riemann. Bernoulli also discovered that the lengths of some curves, expressed using integrals, may be expressed using infinite series [23].

Johann Bernoulli was Euler's teacher, and it is not surprising that the latter became interested in these problems early in his career. In his first paper on the subject, *Specimen de constructione aequationum differentialium sine indeterminatarum separatione* (Example of the construction of differential equations without separation of variables) [70] written in 1733, Euler gives a formula for arc lengths of ellipses. He obtains them by first writing a differential equation satisfied by these arcs. Generally speaking, Euler systematically searched for differential equations that describe the various situations that he was studying.

Between the work of Bernoulli and that of Euler, we must mention that of Fagnano, who, around the year 1716, in a study he was carrying on the lemniscate, discovered some results which Euler considered several years later as outstanding. These results included an addition formulae for a class of elliptic integrals [112], and the fact that on an ellipse or a hyperbola, one may find infinitely many pairs of arcs whose difference is expressible by algebraic means. The word used by Euler and others for such arcs (or differences of arcs) is that they are "rectifiable." Fagnano managed to reduce the rectifiability of the lemniscate to that of the ellipse and hyperbola. A few words on Fagnano are in order.

Giulio Carlo de' Toschi di Fagnano (1682–1766) was a noble Italian interested in science, who worked during several decades in isolation, away from any scientific environment. Weil's authoritative book on the history of number theory [256] starts with the following:

According to Jacobi, the theory of elliptic functions was born between the twenty-third of December 1751 and the twenty-seventh of January 1752. On the former date, the Berlin Academy of Sciences handed over to Euler the two volumes of Marchese Fagnano's *Produzioni Matematiche*, published in Pesaro in 1750 and just received from the author; Euler was requested to examine the book and draft a suitable letter of thanks. On the latter date, Euler, referring explicitly to Fagnano's work on the lemniscate, read to the Academy the first of a series of papers, eventually proving in full generality the addition and multiplication theorems for elliptic integrals.

On p. 245 of the same treatise, Weil writes:

On 23 December 1751 the two volumes of Fagnano's *produzioni Matematiche*, just published, reached the Berlin Academy and were handed over to Euler; the second volume contained reprints of pieces on elliptic integrals which appeared between 1714 and 1720 in an obscure Italian journal and had remained totally unknown. On reading these few pages Euler caught fire instantly; on 27 January 1752 he was presenting to the Academy a memoir [91] with an exposition of Fagnano's main results, to which he was already adding some of his own.

The most striking of Fagnano's results concerned transformations of the "lemniscate differential"

$$w(z) = \frac{dz}{\sqrt{1-z^4}};$$

how he had reached them was more than even Euler could guess. "Surely his discoveries would shed much light on the theory of transcendental functions," Euler wrote in 1753, "if only his procedure supplied a sure method for pursuing these investigations further; but it rests upon substitutions of a tentative character, almost haphazardly applied ..."<sup>27</sup>

In a letter dated October 17, 1730 ([110] p. 624), well before being aware of Fagnano's work, Euler informed Goldbach that "even admitting logarithms,"<sup>28</sup> he could by no means compute the integral  $\int \frac{a^2 dx}{\sqrt{a^4 - x^4}}$ , that "expresses the curve element of the rectangular elastic curve, or rectify this ellipse." Fagnano, instead of giving explicit values, established equalities between such integrals which paved the way to a new series of results by Euler and others. In a letter to Goldbach, dated May 30, 1752, that is, about six months after reading Fagnano's work, Euler writes (see [110] p. 1064): "Recently some curious integrations occurred to me." He first notes that three differential equations

$$\frac{dx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}},$$

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}},$$

and

$$\frac{dx}{\sqrt{1-x^3}} = \frac{dy}{\sqrt{1-y^3}}$$

can be integrated explicitly, and lead respectively to

$$y^2 + x^2 = c^2 + 2xy\sqrt{1-c^2},$$

$$y^2 + x^2 = c^2 + 2xy\sqrt{1-c^4} - c^2y^2$$

and

$$y^2 + x^2 + c^2x^2y^2 = 4c - 4c^2(x+y) + 2xy - 2cxy(x+y).$$

He adds that from these and other formulae of the same kind, he deduced the following theorem (see Figure ??):

If, in the quadrant  $ACB$  of an ellipse, the tangent  $VTM$  at an arbitrary point  $M$  is drawn which meets one of the axes,  $CB$ , at  $T$ , if  $TV$  is taken equal to  $CA$  and from  $V$ ,  $VN$  is drawn parallel to  $CB$ , and if finally  $CP$  is the perpendicular on the tangent through the center  $C$ , then I say the difference of the arcs  $BM$  and  $AN$  will be rectifiable, namely,  $BM - AN = MP$ .

<sup>27</sup>The reference is to Euler's memoir *Specimen novae methodi curvarum quadraturas et rectificationes aliasque quantitates transcendentes inter se comparandi* (An example of a new method for the quadrature and rectification of curves and of comparing other quantities which are transcendently related to each other) [89].

<sup>28</sup>The reference to logarithms comes from the fact that  $\frac{dt}{t}$  and some more general rational functions can be integrated using logarithms.



In the following letter to Goldbach, dated June 3rd, 1752, Euler gave a proof of this theorem and clarified a formula that Fagnano had given in his 1716 paper [111].

About five weeks after Euler received the work of Fagnano, he presented to the Berlin Academy a memoir entitled *Observationes de comparatione arcuum curvarum irrectificibilium* (Observations on the comparison of arcs of irrectifiable curves) [91] in which he expands on what he had announced in his correspondence with Goldbach, generalizing Fagnano's duplication result on the lemniscate to a general multiplication result and giving examples of arcs of an ellipse, hyperbola and lemniscate whose differences are rectifiable. This was the beginning of a systematic study by Euler of elliptic integrals. The year after, he presented to the Saint Petersburg Academy of Sciences a memoir entitled *De integratione aequationis differentialis*  $\frac{mdx}{\sqrt{1-x^4}} = \frac{ndy}{\sqrt{1-y^4}}$  (On the integration of the differential equation  $\frac{mdx}{\sqrt{1-x^4}} = \frac{ndy}{\sqrt{1-y^4}}$ ) [87] which starts with the sentence:<sup>29</sup>

When, prompted by the illustrious Count Fagnano, I first considered this equation, I found indeed an algebraic relation between the variables  $x$  and  $y$  which satisfied the equation.

Several years later, in his famous treatise *Institutiones calculi integralis* [93], Euler included a section on the addition and multiplication of integrals of the form

$$\int \frac{PdZ}{\sqrt{A + 2BZ + CZ^2 + 2DZ^3 + EZ^4}}.$$

Fagnano's works, in three volumes, were edited in 1911–1912 by Gambioli, Loria and Volterra [113].

Among the large number of memoirs that Euler wrote on elliptic integrals,<sup>30</sup> we mention the short memoir [82], *Problema, ad cuius solutionem geometrae invitantur; theorema, ad cuius demonstrationem geometrae invitantur* (A Problem, to which a geometric solution is solicited; a theorem, to which a geometric proof is solicited), published in 1754, containing his result on the rectification of the difference of two arcs of an ellipse. We also mention the memoir [90], *Demonstratio theorematis et solutio problematis in actis erud. Lipsiensibus propositorum* (Proof of a theorem and solution of a theorem proposed in the Acta Eruditorum of Leipzig) [90], in which he studies the division by 2 of an arc of ellipse. The memoir [94], entitled *Integratio aequationis*  $\frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4}} = \frac{dy}{\sqrt{\alpha+\beta y+\gamma y^2+\delta y^3+\epsilon y^4}}$  (The integration of the equation  $\frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4}} = \frac{dy}{\sqrt{\alpha+\beta y+\gamma y^2+\delta y^3+\epsilon y^4}}$ ), written

<sup>29</sup>The translation is by S. G. Langton.

<sup>30</sup>The Euler archive lists thirty-three memoirs by him under the heading "Elliptic integrals," published between 1738 and 1882. It is sometimes hard to know exactly the year where Euler wrote his memoirs. For several of them, the date of publication was much later than the date of writing, and there are several reasons for that, including the huge backlog of the publication department of the Academies of Sciences of Saint Petersburg and Berlin, the main reason being that Euler used to send them too many papers.

in 1765 and published in 1768, is mentioned by Jacobi in a letter to Legendre which we quote below.

Besides Euler, one may mention d'Alembert. In a letter dated December 29, 1746, Euler writes to his Parisian colleague (see [108] p. 251):

I read with as much profit as satisfaction your last piece with which you honored our Academy. [...] But what pleased me most in your piece is the reduction of several integral formulae to the rectification of the ellipse and the hyperbola; a matter to which I had also already given my thoughts, but I was not able to get entirely to the formula

$$\frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}}$$

and I regard your formula as a masterpiece of your expertise.<sup>31</sup>

Lagrange, whose name is associated with that of Euler in several contexts, studied elliptic integrals in his famous *Théorie des fonctions analytiques* (Theory of analytic functions) [165] (first edition 1797). In particular, he discovered a relation between Euler's addition formula and a problem in spherical trigonometry.

After Euler, d'Alembert and Lagrange, we must talk about Legendre, who investigated these integrals for almost forty years. He wrote two famous treatises on the subject, his *Exercices de calcul intégral sur divers ordres de transcendentes et sur les quadratures* (Exercises of integral calculus on various orders of transcendence and on the quadratures) [172] (1811–1816) and his *Traité des fonctions elliptiques et des intégrales eulériennes* (Treatise of elliptic functions and Eulerian integrals) [173] (1825–1828), both in three volumes. In the introduction to the latter (p. 1ff.), Legendre makes a brief history of the subject, from its birth until the moment he started working on it. According to his account, elliptic functions were first studied by MacLaurin and d'Alembert who found several formulae for integrals that represent arcs of ellipses or arcs of hyperbolas.<sup>32</sup> Legendre declares that their results were too disparate to form a theory. He then mentions Fagnano, recalling that his work was the starting point of the profound analogy between elliptic integrals and trigonometric functions. After describing Fagnano's work, Legendre talks about some of the main contributions of Euler, Lagrange and Landen on the subject. His treatise starts with a detailed study of integrals of the form  $\int \frac{Pdx}{R}$  investigated by Euler, where  $P$  is an arbitrary rational function of  $x$  and  $R = \sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}$ . The expression *Eulerian integral* contained in the title of Legendre's treatise was coined by him. He writes:

<sup>31</sup>J'ai lu avec autant de fruit que de satisfaction votre dernière pièce dont vous avez honoré notre académie. [...] Mais ce qui m'a plu surtout dans votre pièce c'est la réduction de plusieurs formules intégrales à la rectification de l'ellipse et de l'hyperbole ; matière à laquelle j'avais aussi déjà pensé, mais je n'ai pu venir à bout de la formule

$$\frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}}$$

et je regarde votre formule comme un chef-d'œuvre de votre expertise.

<sup>32</sup>See e.g. [180] and [5].

Although Euler's name is attached to almost all the important theories of integral calculus, I nevertheless thought that I was allowed to give more especially the name *Eulerian integral* to two sorts of transcendents whose properties constituted the subject of several beautiful memoirs of Euler, and form the most complete theory on definite integrals which exists up to now [...]<sup>33</sup>

After Legendre, and among the immediate predecessors of Riemann on elliptic functions, we find Abel, Jacobi, and Gauss. The last two were his teachers in Berlin and Göttingen respectively. With this work, the emphasis in the study of elliptic integrals shifted to that of their inverses. Considering inverses is naturally motivated by the analogy with trigonometric functions, as one may see by recalling that the integral  $\int_0^x \frac{dt}{\sqrt{1-t^2}}$  represents the arcsine function, and therefore, its inverse is the more tractable sine function. The periodic behavior of inverses of elliptic integrals like  $\int_0^x \frac{dt}{\sqrt{1-t^4}}$  and others, which became later one of the main questions in that theory, is in some sense a generalization of that of trigonometric functions.

Abel and Jacobi developed simultaneously the theory of elliptic integrals, and separating their results has always been a difficult task. It is also well established that Gauss discovered several results of Abel and Jacobi before them, but never published them. This is attested in his notebook and in his correspondence, published in his *Collected Works*. Gauss started his notebook in 1796, at the age of 19, and he wrote his last note there in 1814. The notes consist of 146 statements, most of them very concise, and they fill up a total of 20 pages in his *Collected Works* (vol. 10). This edition of the notebook published in Gauss's *Collected Works* is accompanied by detailed comments by Bachmann, Brendel, Dedekind, Klein, Lœwy, Schlesinger and Stäckel. There is a French translation of the notebook [126]. Among the notes contained in this diary, several concern elliptic functions. For instance, in Notes 32 and 33, Gauss studies the inverse of the lemniscate integral  $\int \frac{dx}{\sqrt{1-x^4}}$ , as a particular case of the elliptic integral  $\int \frac{dx}{\sqrt{1-x^n}}$ . In Note 53, he mentions that he is studying the general integral  $\int \frac{dx}{\sqrt{1-x^n}}$ , which was already considered by Euler in his *Institutiones calculi integralis*. In Note 54, he states that he has an easy method for determining the integral  $\int \frac{x^n dx}{1+x^m}$ , again an integral that was considered by Euler. There are several other notes on elliptic integrals in Gauss's notebook.

Jacobi read Euler's works while he was in high school. He obtained his PhD at the age of 21, and at the age of 22, he started a correspondence with Legendre, who was 74, informing him about his results on elliptic integrals. This correspondence became famous. It is reproduced in Crelle's Journal<sup>34</sup> and in Jacobi's *Collected Works*.<sup>35</sup> The beginning of this correspondence

<sup>33</sup>Quoique le nom d'Euler soit attaché à presque toutes les théories importantes du calcul intégral, cependant j'ai cru qu'il me serait permis de donner plus spécialement le nom d'*Intégrales Eulériennes* à deux sortes de transcendentes dont les propriétés ont fait le sujet de plusieurs beaux mémoires d'Euler, et forment la théorie la plus complète que l'on connaisse jusqu'à présent sur les intégrales définies [...]

<sup>34</sup>Crelle's Journal, 80 (1875), p. 205–279

<sup>35</sup>*Collected Works*, t. I, p. 385–46.

is touching. Jacobi sends his first letter to Legendre on August 5, 1827, expressing his great respect for the work of his older French colleague. He writes ([147] vol. 1, p. 390):

A young geometer dares to present you a few discoveries in the theory of elliptic functions, to which he was led by a diligent study of your beautiful writings. It is to you, Sir, that this brilliant part of analysis owes the highest degree of perfection to which it has been elevated, and it is only in following the footsteps of such a great master that the geometers will be able to push it beyond limits which have been so far prescribed. Thus, it is to thee that I must offer the following, as a fair tribute of admiration and gratefulness.<sup>36</sup>

In his response, dated November 30, 1827, Legendre, referring to one of the theorems that Jacobi communicated to him, writes ([147] vol. 1, p. 396):

I checked this theorem by my own methods and I found it perfectly correct. Even though I regret that this discovery escaped me, the joy I experienced was most vivid when I saw the significant improvement that was added to the beautiful theory of which I am the creator and which I developed almost alone during more than forty years.<sup>37</sup>

In another letter, sent on January 12, 1828, Jacobi informs Legendre about Abel's discoveries, in particular on the division of the lemniscate ([147], vol. 1, p. 401):

Since my last letter, researches of the highest importance were published on elliptic functions by a young geometer, who may be personally known to you.<sup>38</sup>

Legendre sent his response on February 9, informing his correspondent that he knew about Abel's work, but that he was happy to see it summarized in a language which was closer to his own.<sup>39</sup>

Regarding Gauss's work on elliptic functions, we mention an excerpt of the first letter from Jacobi to Legendre [147] p. 393–394:

These researches were born only very recently. However, they are not the only ones that are conducted in Germany on the same object. Mr. Gauss, when he learned about them, informed me

---

<sup>36</sup>Un jeune géomètre ose vous présenter quelques découvertes faites dans la théorie des fonctions elliptiques, auxquelles il a été conduit par la lecture assidue de vos beaux écrits. C'est à vous, Monsieur, que cette partie brillante de l'analyse doit le haut degré de perfectionnement auquel elle a été portée, et ce n'est qu'en marchant sur les vestiges d'un si grand maître, que les géomètres pourront parvenir à la pousser au-delà des bornes qui lui ont été prescrites jusqu'ici. C'est donc à vous que je dois offrir ce qui suit comme un juste tribut d'admiration et de reconnaissance.

<sup>37</sup>J'ai vérifié ce théorème par les méthodes qui me sont propres et je l'ai trouvé parfaitement exact. En regrettant que cette découverte m'ait échappée je n'en ai pas moins éprouvé une joie très vive de voir un perfectionnement si notable ajouté à la belle théorie, dont je suis le créateur, et que j'ai cultivé presque seul depuis plus de quarante ans.

<sup>38</sup>Depuis ma dernière lettre, des recherches de la plus grande importance ont été publiées sur les fonctions elliptiques de la part d'un jeune géomètre, qui peut-être vous sera connu personnellement.

<sup>39</sup>[J'avais déjà connaissance du beau travail de M. Abel inséré dans le *Journal de Crelle*. Mais vous m'avez fait beaucoup de plaisir de m'en donner une analyse dans votre langage qui est plus rapproché du mien.] ([147], t. 1, p. 407).

that he had developed, already in 1808, the cases of 3 sections, 5 sections and 7 sections, and that he found at the same time the corresponding new scales of modules. It seems to me that this information is very interesting.<sup>40</sup>

Legendre was outraged by Gauss's reaction. In his response to Jacobi, dated November 30, 1827, he writes ([147] p. 398):

How is it possible that Mr. Gauss dared telling you that most of your theorems were known to him and that he discovered them back in 1808? This excess of impudence is unbelievable from a man who has enough personal merit so as he does not need to appropriate the discoveries of others... But this is the same man who, in 1801, wanted to attribute to himself the law of reciprocity published in 1785 and who wanted, in 1809, to take hold of the method of least squares that was published in 1805.<sup>41</sup>

It was only at the publication of Gauss's *Collected Works*,<sup>42</sup> containing in particular his famous notebook, that it became clear that Gauss's assertion concerning the fact that he had discovered before Abel most of the properties of elliptic functions, including their double periodicity, was correct. One of the first results of Abel concerns integrals of arcs of lemniscate, a curve which he showed to be divisible by ruler and compass into  $n$  equal parts, for the same values of  $n$  for which the circle is divisible into  $n$  equal parts. The same result was stated without proof in Gauss's *Disquisitiones arithmeticae* [122].

Abel's first major results on elliptic functions are contained in his 1827 paper *Recherches sur les fonctions elliptiques* (Researches on elliptic functions) [1]. He explains there the double periodicity of these functions, as well as their multiplication and division properties. The analogy with circular functions is again highlighted. At the beginning of his paper, Abel talks about his famous predecessors, Euler, Lagrange and Legendre. He writes (p. 101):

The first idea of these [elliptic] functions were given by the immortal Euler, who showed that the separable equation

$$\frac{\partial x}{\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}} + \frac{\partial y}{\sqrt{\alpha + \beta y + \gamma y^2 + \delta y^3 + \epsilon y^4}} = 0$$

is algebraically integrable. After Euler, Lagrange added something, when he gave his elegant theory of the transformation of

---

<sup>40</sup>Il n'y a que très peu de temps que ces recherches ont pris naissance. Cependant elles ne sont pas les seules entreprises en Allemagne sur le même objet. M. Gauss, ayant appris de celles-ci, m'a fait dire qu'il avait développé déjà en 1808 les cas de 3 sections, 5 sections et de 7 sections, et trouvé en même temps les nouvelles échelles de modules qui s'y rapportent. Cette nouvelle, à ce qui me paraît, est bien intéressante.

<sup>41</sup>Comment se fait-il que M. Gauss ait osé vous dire que la plupart de vos théorèmes lui étaient connus et qu'il en avait fait la découverte dès 1808 ? Cet excès d'impudence n'est pas croyable de la part d'un homme qui a assez de mérite personnel pour n'avoir pas besoin de s'approprier les découvertes des autres... Mais c'est le même homme qui en 1801 voulut s'attribuer la découverte de la loi de réciprocité publiée en 1785 et qui voulut s'emparer en 1809 de la méthode des moindres carrés publiée en 1805.

<sup>42</sup>Gauss's collected works, *Carl Friedrich Gauss' Werke*, in twelve volumes, were published between 1863 and 1929.

the integral

$$\int \frac{Rdx}{\sqrt{(1-p^2x^2)(1-q^2x^2)}},$$

where  $R$  is a rational function of  $x$ . But the first, if I am not mistaken, who went thoroughly into the nature of these functions, is Mr. Legendre, who, first in a memoir on elliptic functions, and then in his excellent *Exercices de mathématiques*, developed numerous elegant properties of these functions, and showed their usefulness.<sup>43</sup>

Riemann was already interested in elliptic functions while he was a student in Berlin. Klein, in his *Development of mathematics in the 19th century* [163] (Chapter VI) writes that the latter, since the end of the 1840s, was interested in elliptic functions because this subject was fashionable in Germany. From a letter to his father, dated May 30, 1849, we know that Riemann was following in Berlin Jacobi's and Eisenstein's lectures on elliptic functions. He writes (cf. [234]): "Jacobi has just begun a series of lectures in which he leads off once again with the entire system of the theory of elliptical functions in the most advanced, but elementary way." In another letter (without date), also written in Berlin, Riemann writes: "I enrolled with five other students into a private class (Privatissimum) with Eisenstein, who was promoted in the course of this semester to a Privatdozent with a paper on the theory of elliptic functions."

We already mentioned Euler's impact on Jacobi. Eisenstein is another prominent mathematician on which Euler exerted a crucial influence. In his biography of Eisenstein [238], M. Schmitz writes that during the period 1837–1842, while he was a Gymnasium pupil, Eisenstein attended lectures by Dirichlet at the University of Berlin, and that he studied on his own Gauss's *Disquisitiones Arithmeticae* as well as papers and books by Euler and Lagrange. We quote Eisenstein, from his autobiography translated in [238]:

After I had acquired the fundamentals by private study (I never had a private tutor) I proceeded to advanced mathematics and studied, besides other books containing advanced material, the brilliant work of Euler and Lagrange about differential and integral calculus. I was able to commit this material securely to my memory and to master it entirely, because I made it a rule to compose every theory in writing as soon as I understood it.

---

<sup>43</sup>La première idée de ces fonctions a été donnée par l'immortel Euler, en démontrant que l'équation séparée

$$\frac{\partial x}{\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}} + \frac{\partial y}{\sqrt{\alpha + \beta y + \gamma y^2 + \delta y^3 + \epsilon y^4}} = 0$$

est intégrable algébriquement. Après Euler, Lagrange y a ajouté quelque chose, en donnant son élégante théorie de la transformation de l'intégrale

$$\int \frac{Rdx}{\sqrt{(1-p^2x^2)(1-q^2x^2)}},$$

où  $R$  est une fonction rationnelle de  $x$ . Mais le premier, et si je ne me trompe, le seul, qui ait approfondi la nature de ces fonctions, est M. Legendre, qui d'abord dans un mémoire sur les fonctions elliptiques, et ensuite dans ses excellents exercices de mathématiques, a développé nombre de propriétés élégantes de ces fonctions, et a montré leur application.

In his ICM communication [255], Weil declares (p. 233) that “Eisenstein fell in love with mathematics at an early age by reading Euler and Lagrange.”

We shall conclude this section with two other quotes of Weil. Before that, let us recall that elliptic integrals are studied in number theory in relation with the theory of elliptic curves. Weil writes in an essay on the history of number theory, [254], p. 15, that Fermat, in his work on number theory, had already dealt with elliptic curves (without the name), in particular in his proof of the non-existence of integer solutions for the equation  $x^4 - y^4 = z^2$ . We quote him from his book on the history of number theory that we already mentioned ([256] p. 242):

What we call now “elliptic curves” (i.e. algebraic curves of genus 1) were considered by Euler under two quite different aspects without ever showing an awareness of the connection between them, or rather of their substantial identity. On the one hand, he must surely have been familiar, from the very beginning of his career, with the traditional methods for handling Diophantine equations of genus 1. [...] On the other hand he had inherited from his predecessors, and notably from Johann Bernoulli, a keen interest in what we know as “elliptic integrals” because the rectification of the ellipse depends upon integrals of that type; they were perceived to come next to the integrals of rational functions in order of difficulty.

Eisenstein and Dirichlet were mostly interested in elliptic functions because of their use in number theory, contrary to Riemann, who, even though he was introduced to elliptic functions through Eisenstein’s lectures, was not excited by that field. Weil writes in his essays [254], p. 21:

[...] The case of Riemann is more curious. Of all the great mathematicians of the last century, he is outstanding for many things, but also, strangely enough, for his complete lack of interest for number theory and algebra. This is really striking, when one reflects how close he was, as a student, to Dirichlet and Eisenstein, and, at a later period, also to Gauss and to Dedekind who became his most intimate friend. During Riemann’s student days in Berlin, Eisenstein tried (not without some success, he fancied) to attract him to number theory. In 1855, Dedekind was lecturing in Göttingen on Galois theory, and one might think that Riemann, interested as he was in algebraic functions, might have paid some attention. But there is not the slightest indication that he ever gave any serious thoughts to such matters.

We shall mention the work of Dirichlet on number theory (in particular on the prime number theorem) in §6 below. In Chapter 8 [193] of the present book, we report on several treatises on elliptic functions that were published in France during the few decades that followed Riemann’s early work on the subject. In the next section, we review the more general Abelian functions.

#### 4. ABELIAN FUNCTIONS

A few years before Riemann started his work on elliptic functions and elliptic integrals, the general interest moved towards the more general Abelian integrals, and their inversion. The term *Abelian function*, first introduced by

Jacobi in honor of Abel, is generally given to the functions obtained by inverting an arbitrary algebraic integral or a combination of such integrals. An algebraic integral is an integral of the form  $\int R(x, y)dx$  where  $R$  is a rational function of the two variables  $x$  and  $y$  and where  $x$  and  $y$  satisfy furthermore a polynomial equation  $f(x, y) = 0$ . In his 1826 memoir submitted to the Paris Academy, Abel extended Euler's addition formula for elliptic integrals to Abelian integrals. He proved that the sum of an arbitrary number of such integrals can be written as the sum of  $p$  linearly independent integrals, to which is added an algebraic-logarithmic expression. Here  $p$  is the so-called *genus* of the algebraic curve defined by the equation  $f(x, y) = 0$ . After he learned about Abel's work, Jacobi formulated a generalized inversion problem for a system of  $p$  hyperelliptic integrals. His ideas were pursued by several mathematicians, and in particular by Riemann, who gave a solution to the inversion problem in terms of  $\vartheta$  functions.

Abel also discovered that the inverse functions of elliptic integrals are doubly periodic functions defined on the complex plane. This property was at the basis of the later introduction of group theory in the theory of elliptic curves.

In the passage from elliptic functions to Abelian functions, one must also mention Galois. The day before his death, Galois sent a letter to his friend Auguste Chevalier in which he described his thoughts, saying that one could write a memoir based on his ideas on integrals. The letter is analyzed by Picard in his article [197].<sup>44</sup> Picard writes:

All what we know about these researches is contained in what he says in this letter. Several points remain obscure in some statements of Galois; however, we can have a precise idea of some of the results he reached in the theory of integrals of algebraic functions. We thus acquire the certainty that he possessed the most essential results on Abelian integrals that Riemann was led to obtain twenty-five years later. We see without surprise Galois talking about the periods of an Abelian integral relative to an arbitrary algebraic function [...] The statements are precise; the famous author makes the classification of Abelian integrals into three kinds, and he declares that if  $n$  denotes the number of linearly independent integrals of the first kind, the number of periods is  $2n$ . The theorem relative to the parameter inversion in the integrals of the third type is clearly marked, as well as the relations with the periods of Abelian integrals. Galois also talks about a generalization of Legendre's classical equation where the periods of elliptic integrals appear, a generalization which probably led him to the

---

<sup>44</sup>This article constituted the preface to the Collected Works of Galois which were published shortly after.



important relation that was discovered later on by Weierstrass and Mr. Fuchs.<sup>45</sup>

In his paper on Abelian functions [218], Riemann establishes existence results for Abelian functions and more generally their determination in terms of the points of discontinuity and the information on the ramification at these points. It is in that paper that Riemann introduces the notion of birational equivalence and number of moduli both of which played an essential role in mathematics. In the same paper, he presents Abel's addition theorem for elliptic integrals, and he solves Jacobi's inversion problem in terms of  $p$  variable magnitudes, for a  $(2p+2)$ -connected surface. It is also in this paper that Riemann gives his well known classification of Abelian integrals into three types, a classification which depends on the existence and the nature of the singularities (poles or logarithmic). Riemann mentions in his paper several works on the inversion problem, in particular the successful attempt by Weierstrass in the case of hyperelliptic integrals.

On the work of Riemann on Abelian integrals, the reader is also referred to Chapter 4, by Houzel, in the present volume [142]. For a comprehensive survey on the work of Abel, the interested reader is referred to the article [143] by Houzel.

## 5. HYPERGEOMETRIC SERIES

The theory of the hypergeometric series is another topic which Riemann tackled and whose roots involve in an essential way the works of Euler and Gauss. Riemann's main paper on the subject is *Beiträge zur Theorie der durch die Gauss'sche Reihe  $F(\alpha, \beta, \gamma, x)$  darstellbaren Functionen* (Contribution to the theory of functions representable by Gauss's series  $F(\alpha, \beta, \gamma, x)$ ) [223], published in 1857. The work in this paper was used by Riemann later in his development of the theory of analytic differential equations. There are also fragments on the same subject published in Riemann's *Collected works*.

The hypergeometric series is a function of the form

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$$

---

<sup>45</sup>Nous ne connaissons de ces recherches que ce qu'il en dit dans cette lettre ; plusieurs points restent obscurs dans quelques énoncés de Galois, mais on peut cependant se faire une idée précise de quelques-uns des résultats auxquels il était arrivé dans la théorie des intégrales de fonctions algébriques. On acquiert ainsi la conviction qu'il était en possession des résultats les plus essentiels sur les intégrales abéliennes que Riemann devait obtenir vingt-cinq ans plus tard. Nous voyons sans étonnement Galois parler des périodes d'une intégrale abélienne relative à une fonction algébrique quelconque [...] Les énoncés sont précis ; l'illustre auteur fait la classification en trois espèces des intégrales abéliennes, et affirme que, si  $n$  désigne le nombre des intégrales de première espèce linéairement indépendantes, les périodes seront en nombre  $2n$ . Le théorème relatif à l'inversion du paramètre dans les intégrales de troisième espèce est nettement indiqué, ainsi que les relations entre les périodes des intégrales abéliennes ; Galois parle aussi d'une généralisation de l'équation classique de Legendre, où figurent les périodes des intégrales elliptiques, généralisation qui l'avait probablement conduit à l'importante relation découverte depuis par Weierstrass et par M. Fuchs.

where  $x$  is the variable.

The term “hypergeometric series” appears in Euler’s *Institutiones calculi integralis* [93] (1769), Chapter XI. The series is a solution of the so-called Euler hypergeometric differential equation which appears in Chapters VIII and XI of the same treatise. As a matter of fact, this name was given to several different but closely related objects. Euler, in one of his earliest memoir *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt* (On transcendental progressions, that is, those whose general terms cannot be given algebraically) [65], published in 1738, starts by mentioning Wallis’s “hypergeometric series”  $1! + 2! + 3! + 4! + \dots$  (without the factorial notation). The terminology here refers to the fact that in analogy with the case of geometric progressions, where each term is obtained from the preceding one by multiplying it by a constant, one defined a hypergeometric progression as a progression in which each term is obtained from the preceding one by multiplying it by a factor which increases by a unit at each step. Wallis’s papers on this subject include [252] (1, Scholium to Proposition 190) and [253] (p. 315).

Gauss mentions a hypergeometric series in his doctoral dissertation *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse* [120] (New proof of the theorem that every rational integral algebraic function of one variable can be resolved into real factors of the first or second degree) (1799).

We refer the reader to the paper [59] for a comprehensive history of the hypergeometric series.

At the beginning of his announcement of his memoir [223], Riemann states: “This memoir treats a class of functions which are useful to solve various problems in mathematical physics.” As a matter of fact, these functions are still commonly used today in mathematical physics. Riemann notes that the name *hypergeometric series* was first proposed by Pfaff, for a more general series, whereas Euler, after Wallis, used such a name for a series which is slightly different. Pfaff was Gauss’s friend, and had been his teacher. He studied this function in his book *Disquisitiones analyticae maxime ad calculum integralem et doctrinam serierum pertinentes* (Analytic investigations most relevant for integral calculus and the doctrine of series) [196] (1797). Gauss has a series of unpublished results on the hypergeometric series, which he communicated to the astronomer Bessel, who was also his friend, in a letter dated September 3, 1805. The results were used by Gauss in his later works. In his writings on the subject, Gauss used continued fractions in his study of the quotient of two hypergeometric series. He developed these ideas in his paper *Disquisitiones generales circa seriem  $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$  etc.* (General investigations on the series  $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$  etc.) [123]. The same year, he wrote another paper on the same subject which he never published but which is contained in his *Collected Works* edition [124]. Riemann, in his paper [223], proved that these fractions converge in the complex plane cut along the subset  $[2, +\infty]$  of the  $x$ -axis. In the same

memoir, he introduced in the study of the hypergeometric functions a new method, which applies to all functions that satisfy linear differential equations with algebraic coefficients. He recalls in the announcement of that memoir, published in the *Göttinger Nachrichten*, No. 1, 1857, that Euler and Gauss made a thorough study of these functions from the theoretical point of view.

In the introduction to his paper [123] Gauss declares that practically any transcendental function that appears in analysis may be obtained as a special case of the hypergeometric series. In fact, it is known that functions like  $\log(1+z)$ ,  $\arcsin z$  and several orthogonal polynomials, including Legendre polynomials and Chebyshev polynomials, can be expressed using hypergeometric functions. The so-called confluent hypergeometric function (or Kummer's function) is a limit of the hypergeometric function.

The introduction of the hypergeometric series brought a whole new class of new functions to the field of analysis which, at least in the times of Euler, consisted in the study of functions.

## 6. THE ZETA FUNCTION

This section is concerned with Riemann's article *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the number of primes less than a given magnitude) [220]. This memoir, which is only 8 pages long, changed the course of mathematics. Riemann wrote it at the occasion of his election to the Berlin Academy of Sciences, on August 11, 1859. Every newly elected member at that academy was asked to report on his most recent research, and Riemann chose this topic. A short history of the subject will show that the list of predecessors of Riemann in this field includes names which are familiar to us now: Euler, as always, then Legendre, Dirichlet and Gauss.

Riemann starts his memoir by recalling that Gauss and Dirichlet had been interested in this subject several years before him. He displays the following formula, which he recalls was noted by Euler, and which was his own departure point :

$$\prod \frac{1}{1 - \frac{1}{p^s}} = \sum \frac{1}{n^s}.$$

Here,  $p$  takes all the prime values and  $n$  all the integer values. Riemann considers the function represented by these two expressions as a function of a complex variable  $s$  as long as the two series converge, and he denotes this function by  $\zeta(s)$ .<sup>46</sup> He then gives an integral formula for this function, and he notes that this integral is "uniform" (uni-valued), that it is defined and finite for any value of  $s$  except for  $s = 1$  and that it vanishes when  $s$  is a negative odd integer.

The distribution of primes, which is the subject of Riemann's paper, may be traced back to Greek antiquity. The reader may recall that there are several results on prime numbers in Euclid's *Elements*. In particular, Proposition 20 of Book IX says that there are infinitely many primes. It is also

---

<sup>46</sup>Even though the notation  $\zeta(s)$  and the name zeta function first appear in Riemann's paper, we shall commit the usual anachronism of using the notation  $\zeta(s)$  for the series  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  even when we talk about the work done on this series before Riemann.

known that the Greeks had a method to list effectively the sequence of primes (Eratosthenes sieve). Without any doubt, the general question of the distribution of primes kept busy the mathematicians of that epoch. It is also good to recall, right at the beginning, that Euler, in his paper *Variae observationes circa series infinitas* (Various observations about infinite series) [76], showed that the series of inverses of primes,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots,$$

diverges, which in some sense is a wide generalization of the fact that the number of primes is infinite.

Euler was fascinated by the question of the distribution of primes. We quote him from a paper entitled *Découverte d'une loi tout extraordinaire des nombres par rapport à la somme de leurs diviseurs* (Discovery of a very extraordinary law of numbers in relation to the sum of their divisors) [80], written in 1747 and published in 1751:

Mathematicians tried in vain, until now, to discover some or other order in the sequence of prime numbers, and we have reasons to think that this is a mystery which human mind will never be able to penetrate. To be convinced, it suffices to take a look at the tables of prime numbers, that a few persons have taken the trouble to continue beyond one hundred thousand: one will primarily notice that there is no order and no rule there.<sup>47</sup>

Let us return now to the zeta function.

The history of the zeta function in Euler's works naturally starts with the question of the value of the sum of the series of reciprocals of squares,

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Before Euler, this series was known to be convergent, and the determination of its value was an open question whose formulation can be traced back at least to Pietro Mengoli in his treatise *Novae quadrature arithmeticae, seu de additione fractionum*<sup>48</sup> (New arithmetic quadratures, or the addition of fractions) [181] (1650). Several mathematicians worked on the problem, including Wallis, Leibniz, Stirling, de Moivre, Goldbach and several Bernoullis. In fact, the question of computing infinite sums was already a fashionable subject at that epoch. Mengoli, Huygens and Leibniz independently computed the sum of reciprocals of the triangular numbers, that is, numbers of the form  $\frac{(n)(n+1)}{2}$ . Leibniz's computation of the series of inverses of triangular numbers uses the classical "telescopic method" known to students, so its level of difficulty has nothing to do with Euler's computation of  $\zeta(2)$ . The problem of finding the value of  $\zeta(2)$  became widely known among mathematicians after it was asked explicitly by Jakob

---

<sup>47</sup>Les mathématiciens ont tâché jusqu'ici en vain à découvrir un ordre quelconque dans la progression des nombres premiers, et on a lieu de croire, que c'est un mystère auquel l'esprit humain ne saurait jamais pénétrer. Pour s'en convaincre, on n'a qu'à jeter les yeux sur les tables des nombres premiers, que quelques personnes se sont donné la peine de continuer au-delà de cent mille : et on s'apercevra d'abord qu'il ne règne aucun ordre ni règle.

<sup>48</sup>Mengoli's treatise is entirely devoted to the theory of infinite series, despite the word *quadrature* (that is, computation of areas) in the title.

Bernoulli in his series of papers *Positiones de seriebus infinitis* (Positions of an infinite series) (1689).<sup>49</sup> In the same work, Bernoulli considered the series for an arbitrary rational number  $s$ .

Euler published several papers on various aspects of the zeta function. In particular, he was the first to discover a formula establishing a relation between this series and prime numbers. It is interesting to recall that Euler has been investigating the convergence of infinite series and infinite products since his early days as a mathematician.<sup>50</sup> His first letter addressed to Goldbach, dated October 13, 1729, concerns the  $\Gamma$  function, a function that interpolates the factorials. Goldbach had asked the opinion of several mathematicians on that problem. Euler writes [110]:<sup>51</sup>

When lately I came across a few ideas that apparently could contribute to the interpolation of series having a variable law – as you are wont to call it – I took a closer look and discovered many things regarding that subject. As Mr. Bernoulli hinted that these results might please you, Sir, I decided to write to you and submit them to your judgment. For the series 1, 2, 6, 24, 120, . . . , which you have treated extensively, as I see, I have found the general term [...]

The letter ends with:

You, Sir, who have already enriched the theory of series by so many important discoveries, will therefore judge for yourself what else may be expected from this novel way to deal with series. It would certainly acquire its greatest utility and perfection if you could bring yourself to investigate how the differential calculus can be most conveniently applied to these questions. For up to now my method has the drawback that I cannot find what I want, but rather have to be content with wanting what I find.

In his paper *De summatione innumerabilium progressionum* (The summation of an innumerable progression) [66], Euler starts by giving a 7-digit approximate value of  $\zeta(2)$ , namely, 1.644934. Needless to say, such a computation needed from his part a large amount of computing, because the series converges very slowly. Before that, Wallis had given, in his *Arithmetica infinitorum* (Arithmetic of the infinite), 1655, a 3-digit approximation of that series. Goldbach and Daniel Bernoulli also gave 3-digit approximations, in 1728.<sup>52</sup> The reader may find interesting information on that subject in the correspondence between Euler, Bernoulli and Goldbach.

In 1735, Euler, who was 28 years old, obtained the summation formula for  $\zeta(2)$  and, more generally, for the infinite series  $\zeta(2\nu) = \sum_{n=1}^{\infty} \frac{1}{n^{2\nu}}$  for any positive integer  $\nu$ . He found the values  $\zeta(2) = \pi^2/6$  and  $\zeta(2\nu) = r_\nu \pi^{2\nu}$ , where  $r_\nu$  are rational numbers which are closely related to the Bernoulli numbers.

<sup>49</sup>See the comments of this work of Bernoulli in Weil's article [257] p. 4.

<sup>50</sup>One should note that power series representations of functions already appear in the works of Newton, in the 1660s.

<sup>51</sup>In this volume of the *Opera Omnia*, the letters are translated into English.

<sup>52</sup>In a letter to Goldbach, sent in 1728, Daniel Bernoulli writes that the value of the series  $\zeta(2)$  "is very nearly  $8/5$ , and Goldbach answers that  $\zeta(2) - 1$  lies between  $16233/25200$  and  $30197/46800$ ; cf. Weil [256] p. 257 for more details on this history.

In the introduction to his memoir *De summis serierum reciprocarum* (On the sums of series of reciprocals) [72] (1735), he writes:<sup>53</sup>

So much work has been done on the series  $\zeta(n)$  that it seems hardly likely that anything new about them may still turn up ... I too, in spite of repeated efforts, could achieve nothing more than approximate values for their sums ... Now, however, quite unexpectedly, I have found an elegant formula for  $\zeta(2)$ , depending upon the quadrature of the circle.<sup>54</sup>

Euler's discovery made him famous, perhaps for the first time, among mathematicians in all Europe. When the news of Euler's discovery reached the city of Basel, the first reaction of his teacher, Johann Bernoulli, was to exclaim that the most burning desire of his deceased older brother Jakob was now fulfilled. Seen all the work he has done on the subject, there is no doubt that throughout his life, Euler tried (without success) to find a formula for  $\zeta(s)$  for  $s$  an odd integer.

It was not unusual for Euler to publish several proofs of the same result, and his result on the convergence on  $\zeta(2)$  is one instance of this fact. In particular, there are proofs of this fact in his memoirs [72] (presented to the Saint Petersburg Academy on December 5, 1735 and published in 1740) and [76] (presented to the Saint Petersburg Academy on April 25, 1727 and published in 1744), and an account is given in his *Introductio* [61] (first edition 1748).

In a letter to Goldbach dated August 28, 1742 (Letter 54 in [110]), Euler expresses  $\zeta(2)$  in terms of dilogarithms. We recall that the dilogarithm function<sup>55</sup> is defined as

$$\text{Li}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

We have  $\text{Li}(1) = \zeta(2)$ . In his paper [66], presented to the Saint Petersburg Academy in 1731 and published in 1738, Euler had already used the dilogarithm function to find numerical approximations for  $\zeta(2)$ .

In his memoir *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques* (Remarks on a beautiful relation between direct as well as reciprocal power series), [95], written in 1749 and published in 1768, Euler found the functional equation satisfied by the zeta function. The relation is not explicitly written by Euler but it follows from a relation he writes, as pointed out by Weil in [254] p. 10, who deduces it immediately from the following formula which Euler writes:

$$\begin{aligned} & \frac{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + 5^{n-1} - 6^{n-1} + \text{etc.}}{1 - 2^{-n} + 3^{-n} - 4^{-n} + 5^{-n} - 6^{-n} + \text{etc.}} \\ &= \frac{-1.2.3\dots(n-1)(2^n - 1)}{(2^{n-1} - 1)\pi^n} \cos \frac{n\pi}{2}. \end{aligned}$$

Weil comments on this formula:

<sup>53</sup>The translation from the Latin is by André Weil, [256] p. 261.

<sup>54</sup>Weil adds: [i.e., upon  $\pi$ ].

<sup>55</sup>This name was still not given to that function in the work of Euler mentioned.

In the left hand side, we have formally the quotient  $\zeta(1-n)\zeta(n)$ , except that Euler had written alternating signs to make the series more tractable; the effect of this is merely to multiply  $\zeta(n)$  by  $1-2^{1-n}$ , and  $\zeta(1-n)$  by  $1-2^n$ . In the right hand side we have the gamma function, which Euler had invented. Euler proves the formula for every positive integer  $n$  (using the so-called Abel summation to give a meaning to the divergent series in the numerator of the left hand side), and conjectures its validity for all  $n$ .

It was Riemann who showed later on that this equation is valid for any real number  $\neq 0, 1$ .

In his paper *Variae observationes circa series infinitas* which we already mentioned, [76], Euler found, for  $s > 1$ , the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} dx}{e^x - 1}.$$

Here  $\Gamma$  is the Euler gamma function, which is an extension of the factorial:

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du.$$

In the same paper, he obtained the following formula, valid for real  $s > 1$ :

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

where the product is over all prime numbers  $p$ . (Weil explains Euler's derivation of this formula in [256] p. 265-266.) This equality was the starting point of Riemann's investigations in his paper [220], and it became at the basis of the field called "analytic number theory." Incidentally, it gives a new proof of the fact that there are infinitely many prime numbers (taking  $s = 1$  in the formula). We note by the way that Euler gave another proof of the existence of infinitely many prime numbers, using the divergence of the harmonic series  $\sum \frac{1}{n}$ .

After Euler, the next substantial work on the zeta function,  $\zeta(s)$ , was done more than a century later, by Riemann. Indeed, in the history of number theory that he wrote, Weil considers (see [256] p. 278) that after Euler, the subject was dead, and that Riemann resurrected it. He conjectures that in 1859, Riemann started working on this subject after he seized a remark by Eisenstein, see [257] for the details. Let us summarize some of the major ideas that Riemann brought in his short paper:

- (1) Using analytic continuation, Riemann showed that the zeta function can be extended to a holomorphic function defined on the complex plane, except at the point 1 where the function has a simple pole with residue 1.
- (2) He discovered the relation between the zeros of the zeta function and the asymptotic distribution of prime numbers. In fact, Riemann gave the principal term in the asymptotic law of the so-called counting function  $\pi(x)$  which measures the number of prime numbers  $\leq x$ . More precisely, Riemann gave the formula

$$\pi(x) \sim \frac{x}{\log x}, \quad x \rightarrow \infty$$

with a sketch of a proof. The result became known as the “prime number theorem.” Complete proofs of this theorem were given later by Hadamard and de la Vallée Poussin in 1896.

- (3) Starting from the functional equation discovered by Euler – and of which Riemann provided two new proofs adapted to the newly extended function – Riemann showed that the set of zeros of the zeta function contains the even negative integers, and conjectured that all the other zeros are situated on the line  $\text{Im}(s) = \frac{1}{2}$ . This is the famous Riemann hypothesis.
- (4) Riemann obtained a new functional equation satisfied by the zeta function:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

for  $s \neq 0, 1$ .

Finding the asymptotic behavior of the prime counting function  $\pi(x)$  was, at the epoch of Riemann, one of the major problems in number theory. Legendre, Gauss and Dirichlet had already investigated this problem, and more precisely, they worked on a conjecture saying that  $\pi(x)$  is asymptotic to a function of the size of  $\frac{x}{\ln x}$ . Riemann’s main contribution was the introduction of complex analysis in this study, and his intuition that the distribution of primes is related to the zeros of the zeta function extended to the complex plane. The works by de la Vallée Poussin and Hadamard rely heavily on Riemann’s ideas, and the outlines of their proofs are based on his sketch. We talk about Hadamard’s work on the zeta function and the prime number theorem in Chapter 8 of the present volume, [193]. Let us add here a few historical notes on the counting function; it will give us the occasion to mention again the work of Legendre.

In 1798, Legendre published his *Essai sur la théorie des nombres* (Essay on number theory) [171], a long essay (about 472 pages without the tables) in which, based on numerical evidence, he proposed a conjecture on the form of the counting function  $\pi(x)$ . He writes (p. 19):

Moreover, it is likely that the rigorous formula which gives the value of  $b$  when  $a$  is very large is of the form  $b = \frac{a}{A \log a + B}$ ,  $A$  and  $B$  being constant coefficients, and  $\log a$  denoting a hyperbolic logarithm. The exact determination of these coefficients would be a curious problem, worth of training the expertise of the analysts.<sup>56</sup>

Legendre also gave an approximate value of the constant  $A(x)$ . Let us note incidentally that Legendre, in the preface to his essay, makes a short history of the development of number theory, starting with the Greeks (Euclid and Diophantus), and passing by Viète, Bachet, Fermat, Euler and Lagrange.

In the second edition of his essay (1808), Legendre formulated another conjecture, saying that there are infinitely many primes in any arithmetic progression, that is, primes of the form  $l + kn$  for any natural integer  $n$ .

<sup>56</sup>Au reste, il est vraisemblable que la formule rigoureuse qui donne la valeur de  $b$  lorsque  $a$  est très grand, est de la forme  $b = \frac{a}{A \log a + B}$ ,  $A$  et  $B$  étant des coefficients constants, et  $\log a$  désignant un logarithme hyperbolique. La détermination exacte de ces coefficients serait un problème curieux et digne d’exercer la sagacité des Analystes.



This conjecture is at the foundations of the theory of Dirichlet series, and it was at the basis of several approaches on the prime number theorem. The conjecture was proved by Dirichlet in 1837 [56], in a paper which brought new tools on how to approach the prime number theorem. In particular, Dirichlet introduced in this paper his famous  $L$ -function.

Besides Dirichlet and Legendre, one has to mention Gauss, who, at the age of 15 or 16, started an extensive investigation on the distribution of prime numbers. Based mostly on empirical data (tables of prime numbers that he compiled), he observed that the density of prime numbers around a fixed number  $x$  is inversely proportional to  $\log x$ , and he deduced that the counting function  $\pi(x)$  should be well approximated by the integral  $\int_2^x \frac{dt}{t}$ . Gauss never published this work, but he described it in an 1849 letter to his friend and former student, the astronomer J. F. Encke. Gauss, in that letter, makes a comparison between his results and those of Legendre. The letter is included in Gauss's correspondence, edited in his Complete Works, and it is also translated and commented in the article [129] by L. J. Goldstein.

Finally, one has to mention the work of Chebyshev in his two papers [42] and [43], done slightly before Riemann (the papers are published in 1851 and 1852), in which he gave precise approximate values for the prime number counting function, making use of the zeta function in the study of the counting function, as Riemann did in his 1859 paper. Chebyshev's paper [43] contains the proof of the so-called Bertrand postulate stating that for any integer  $n \geq 3$ , there exists a prime number  $p$  satisfying  $n < p < 2n$ .<sup>57</sup>

The question of the zeros of the zeta function was proposed by Hilbert in one of the problems he offered at the Paris 1900 ICM.

Riemann's memoir [220] had a major influence on several later mathematicians, including Weil, Siegel, and Selberg.

We conclude this section by quoting Weil, from an obituary article by A. Knapp [164]:

A substantial portion of Weil's research was motivated by an effort to prove the Riemann hypothesis concerning the zeroes of the Riemann zeta function. He was continually looking for new ideas from other fields that he could bring to bear on a proof. He commented on this matter in a 1979 interview:<sup>58</sup> Asked what theorem he most

---

<sup>57</sup>The work of Chebyshev deserves to be much more developed than in these few lines. Like his famous Swiss-Russian predecessor Leonhard Euler, Chebyshev published on most of the fields of pure and applied mathematics. In 1852, he made a stay in France, whose aim was essentially to visit factories and industrial plants, but during his stay he also met several French mathematicians and discussed with them. The list includes Bienaymé, Cauchy, Liouville, Hermite, Lebesgue, Poulignac, Serret and others. A detailed report on this stay, written by Chebyshev himself, is contained in his *Collected works* [45]. Chebyshev used to published in French journals and his relations with French mathematicians remained constant over the years. In 1860, he was elected corresponding member of the Paris Academy of Sciences, and in 1874 foreign member. We learn from his report that at the end of his 1852 stay in France, on his way back to Russia, Chebyshev stopped in Berlin and had several discussions with Dirichlet. It is conceivable that during that meeting the two mathematicians talked about the problems related to the prime number counting function. We refer the reader to the article [194] where some of Chebyshev's works are compared with works of Euler.

<sup>58</sup>Pour la Science, November 1979.

wished he had proved, he responded, “In the past it sometimes occurred to me that if I could prove the Riemann hypothesis, which was formulated in 1859, I would keep it secret in order to be able to reveal it only on the occasion of its centenary in 1959. Since in 1959, I have felt that I am quite far from it, I have gradually given up, not without regret.”<sup>59</sup>

One of the famous Weil conjectures is known as the “Riemann hypothesis over finite fields.”

## 7. ON SPACE

Riemann’s habilitation lecture contains a discussion on the nature of physical space and its relation with geometry. The concepts on which Riemann dwells there make it clear that the theme of space belongs to his profound thought. One of the main ideas on which he stresses is the possibility that physical space is different from the space of Euclidean geometry, a point of view that makes Riemann in some sense a predecessor of modern physics.

In speculating on space, Riemann follows a long tradition which includes the Greeks, Newton, Descartes, Kant and many others, a tradition which survived until the modern period; one may mention, among the mathematicians of the post-Riemannian period, Hermann Weyl, René Thom, Alexandre Grothendieck, and there are many others. It is therefore natural to have, in this paper, a section on space, in which, not only we review Riemann’s ideas – this is done in several chapters of the present volume – but where we mention some of the ideas on this subject that were expressed by his predecessors. Our exposition will necessarily be succinct. Writing a serious essay on the notion of space needs a whole volume.

Space is one of the first very few basic philosophico-epistemological notions. It appears at several places in the works of Aristotle: there are sections on space in the *Categories*, [18], the *Physics* [20], the *Metaphysics* [19], the treatise *On the heavens* [21], etc. Furthermore, like for many other subjects, we learn from Aristotle’s works the opinions of his predecessors on space: the Meletians, the Pythagoreans, Plato, etc.

In the *Categories* (5a, 8-14), Aristotle explains that space, like time, belongs to the category of *continuous quantity*.<sup>60</sup> In Book IV of his *Physics*, he writes about the difference between “space” and “place.” This is a fundamental distinction, with an impact in physics, and it had a huge influence on

---

<sup>59</sup>Autrefois, il m’est quelquefois venu à l’esprit que, si je pouvais démontrer l’hypothèse de Riemann, laquelle avait été formulée en 1859, je la garderais secrète pour ne la révéler qu’à l’occasion de son centenaire en 1959. Comme en 1959, je m’en sentais encore bien loin, j’y ai peu à peu renoncé, non sans regret.

<sup>60</sup>In the *Categories*, (4b 20-5b 11) Aristotle distinguishes seven different types of quantities, which he classifies as continuous and discrete. Discrete quantity comprises number and speech. Continuous quantity comprises the line, the surface, the body, time, and space. Needless to say, although this classification may appear limited from a modern point of view, it has the great merit of existing, may be for the first time. Aristotle asked the pertinent questions.

later thinkers.<sup>61</sup> The question has also implications in the history of topology. The Greek origin for the word place is *topos* (τόπος), and is translated into Latin by *situs*. The expression *analysis situs*, which was used by Leibniz and the Western founders of topology, finds its origin there.

Among the Western thinkers whose work on the theme of space emerges amid the classical philosophical monuments, we mention Galileo, Newton, Descartes, Leibniz, Huygens and Kant. Most of them are quoted by Riemann.

We start by quoting a text from Greek antiquity. This is a fragment by Archytas of Tarentum which is often referred to in the literature on Pythagorean philosophy, to show the kind of questions on space and on place that the ancient Greeks addressed, e.g., whether space is bounded or not, and the paradoxes to which this question leads (see [144] p. 541):

“But Archytas,” as Eudemus says, “used to propound the argument in this way: ‘If I arrived at the outermost edge of the heaven [that is to say at the fixed heaven], could I extend my hand or staff into what is outside or not?’ It would be paradoxical not to be able to extend it. But if I extend it, what is outside will be either body or place. It doesn’t matter which, as we will learn. So then he will always go forward in the same fashion to the limit that is supposed in each case and will ask the same question, and if there will always be something else to which his staff [extends], it is clear that it is also unlimited. And if it is a body, what was proposed has been demonstrated. If it is place, place is that in which body is or could be, but what is potential must be regarded as really existing in the case of eternal things, and thus there would be unlimited body and space.” (Eudemus, Fr. 65 Wehrli, Simplicius, In Ar. Phys. iii 4; 541)

The most basic question that was addressed by many of the philosophers of the modern period that we mentioned is probably the following: Does space have an objective existence or is it only a construction of human mind? Before trying to answer this question, or to have an opinion on it, it is helpful to make it precise what notion of space it refers to: three-dimensional physical space? the three-dimensional space of Euclidean geometry? an abstract notion of space? Other related questions are: Is Euclid’s three-dimensional geometry a pure logical construction or is it a mathematical formulation of the properties of external nature? Is the space of (theoretical) physics the same as the mathematicians’ space? Does void exist, and what function does it have? These are some of the questions which obviously obsessed Riemann, and before him, many others.

---

<sup>61</sup>This theme of space and its relation to place was particularly expanded by Aristotle’s commentators. We mention in particular the medieval Andalusian polymath Averroes (1126–1198). The third chapter of Rashed’s book *Les mathématiques infinitésimales du IXème au XIème siècle* [212] contains a critical edition together with a translation and commentaries of the treatise *On space* by the Arabic scientist Ibn al-Haytham (known in the West under the name al-Hazen) in which this author criticizes Aristotle’s theory of space developed in his *Physics*, and where he defines subsets of space by metric properties. There is also a rich discussion on the notion of space in Greek philosophy in the multi-volume encyclopedic work of P. Duhem [57], see in particular vol. I, p. 197ff.

In Descartes' doctrine, space depends on matter, therefore void cannot exist. Leibniz and Euler after him shared the same opinion. Newton had a notion of "absolute space" and "relative space." Furthermore, following the ancient Greeks, Descartes made a difference between space and place. We quote some passages from his *Principes de la philosophie* (Principles of philosophy) [50] (1644).

*Principle XIV. How place and space differ:* However, place and space are different in names, because place indicates more expressly situation than magnitude or figure, and that on the contrary, we think about that one when we talk about space; for we say that a thing entered at the place of another, even though it does not have exactly neither the same magnitude nor figure, and for that we do not mean that it occupies the same space that this other thing occupies; and when the situation is changed, we say that the place has also changed, even though it has the same magnitude and figure than before: in this sort, if we say that a thing is in some place, we only mean that it is situated in such a way with respect to other things; but if we add that it occupies a certain space, or place, then we mean that it has such magnitude and figure that it can occupy it exactly.<sup>62</sup>

*Principle XV: How the surface surrounding a body can be taken as its exterior place:* Thus, we never make a distinction between space and extent, for what regards length, width and depth; but we sometimes consider place as if it were within the thing which is placed, and sometimes also as if it were outside it. By no means the interior differs from space; but sometimes we take the exterior to be either the surface surrounding immediately the thing which is placed (and one has to notice that by surface we must not intend any part of the body surrounding it but only the extremity which is between the body which surrounds and the one which is surrounded which is only a mode or a way), or to be the surface in general, which is not part of a body rather than another one, and which always seems to be the same, provided it has the same magnitude and the same figure; because even if we see that the body that surrounds another body passes somewhere else with its surface, we are not used to say that what was surrounded by it has changed its place for this reason, it stays at the same situation regarding the other bodies that we consider as still. Thus, we say that a boat which is carried away by the stream of a river, and which is at the same time pushed away by the wind by a force

---

<sup>62</sup>*Principe XIV. Quelle difference il y a entre le lieu et l'espace :* Toutefois le lieu et l'espace sont différents en leurs noms, parce que le lieu nous marque plus expressément la situation que la grandeur ou la figure, et qu'au contraire nous pensons plutôt à celles-ci lorsqu'on nous parle de l'espace ; car nous disons qu'une chose est entrée en la place d'une autre, bien qu'elle n'en ait exactement ni la grandeur ni la figure, et n'entendons point qu'elle occupe pour cela le même espace qu'occupait cette autre chose ; et lorsque la situation est changée, nous disons que le lieu est aussi changé, quoiqu'il soit de même grandeur et de même figure qu'auparavant : de sorte que si nous disons qu'une chose est en un tel lieu, nous entendons seulement qu'elle est située de telle façon à l'égard de quelques autres choses ; mais si nous ajoutons qu'elle occupe un tel espace, ou un tel lieu, nous entendons outre cela qu'elle est de telle grandeur et de telle figure qu'elle peut le remplir tout justement.

which is so equal that it does not change its situation regarding the shores, stays at the same place, even though we see that all the surface that surrounds it changes permanently.<sup>63</sup>

Euler had also a strong philosophical background and, needless to say, a tendency for abstraction. We recall that the subject of his first public lecture, delivered at the University of Basel at the occasion of his graduation, was the comparison between the philosophical systems of Newton and Descartes. The notions of space, of motion and of force are discussed in several of his papers on physics. His most important work related to these matters is his *Mechanica*, in two volumes of 500 pages each, [63] with its systematic use of analysis (differential equations) in the field of mechanics, as opposed to Newton's geometric point of view developed in his *Principia*. In his memoir *Recherches sur l'origine des forces* (Research on the origin of forces) [78] (1750), Euler uses an argument involving a notion of "impenetrability of bodies" from which he deduces the law of shock of bodies. We also mention his *Anleitung zur Naturlehre, worin die Gründe zu Erklärung aller in der Natur sich ereignenden Begebenheiten und Veränderungen festgesetzt wedren* (Introduction to natural science establishing the fundamentals for the explanation of the events and changes that occur in nature), [99], a long memoir written in 1745, but never completed and published in 1862. Hermann Weyl says ([259] p. 42) about this memoir that Euler "in magnificent clarity summarizes the foundations of the philosophy of nature of his time." In this memoir, Euler discusses notions like the *extent* of material bodies, the infinite divisibility of these bodies, motion, space, place magnitude, aether and gravity. His memoir *Essai d'une démonstration métaphysique du principe général de l'équilibre* (Essay on a metaphysical demonstration of the general principle of equilibrium) [81] concerns again, force, equilibrium, motion and gravity. In his memoir *Réflexions sur l'espace et le temps* (Reflections on space and time) [79], he makes a comparison between the mathematicians' and the philosophers' (which he calls the "metaphysicians") points of view. He describes *position* as the relation of a body with other

---

<sup>63</sup>*Principe XV. Comment la superficie qui environne un corps peut être prise pour son lieu extérieur : Ainsi nous ne distinguons jamais l'espace d'avec l'étendue en longueur, largeur et profondeur ; mais nous considérons quelquefois le lieu comme s'il était en la chose qui est placée, et quelquefois aussi comme s'il en était dehors. L'intérieur ne diffère en aucune façon de l'espace ; mais nous prenons quelquefois l'extérieur ou pour la superficie qui environne immédiatement la chose qui est placée (et il est à remarquer que par la superficie on ne doit entendre aucune partie du corps qui environne, mais seulement l'extrémité qui est entre le corps qui environne et celui qui est environné, qui n'est rien qu'un mode ou une façon), ou bien pour la superficie en général, qui n'est point partie d'un corps plutôt que d'un autre, et qui semble toujours la même, tant qu'elle est de même grandeur et de même figure ; car encore que nous voyions que le corps qui environne un autre corps passe ailleurs avec sa superficie, nous n'avons pas coutume de dire que celui qui en était environné ait pour cela changé de place lorsqu'il demeure en la même situation à l'égard des autres corps que nous considérons comme immobiles. Ainsi nous disons qu'un bateau qui est emporté par le cours d'une rivière, et qui en même temps est repoussé par le vent d'une force si égale qu'il ne change point de situation à l'égard des rivages, demeure en même lieu, bien que nous voyions que toute la superficie qui l'environne change incessamment.*

bodies around it. He declares that the metaphysicians are wrong in claiming that the notions of space and place are abstract constructions of the mind, and he argues to show the reality of space and time. He claims that both absolute space and time, as mathematicians represent them, are real and exist beyond human imagination. He discusses inertia and the relativity of motion, the ideas of place and position, supported by notions from mechanics.

Euler's philosophical ideas, and their impact on Riemann, have not yet been seriously discussed in the literature.

Immanuel Kant is among the commanding figures that preceded Riemann on the subject of philosophy of space. As a matter of fact, space was already a major theme in Kant's *Inaugural dissertation* (1770). Kant expresses there his doctrine of the a priori nature of space and of geometric objects, that is, the belief that they are not derived from an outside experience. The following excerpt contains an expression of this point of view, which, as we shall recall, Gauss criticized later ([160] § 15, A–D):

The concept of space is not abstracted from external sensations. For I am unable to conceive of anything posited without me unless by representing it as in a place different from that in which I am, and of things as mutually outside of each other unless by locating them in different places in space. Therefore the possibility of external perceptions, as such, presupposes and does not create the concept of space, so that, although what is in space affects the senses, space cannot itself be derived from the senses.

The concept of space is a singular representation comprehending all things in itself, not an abstract and common notion containing them under itself. What are called several spaces are only parts of the same immense space mutually related by certain positions, nor can you conceive of a cubic foot except as being bounded in all directions by surrounding space.

The concept of space, therefore, is a pure intuition, being a singular concept, not made up by sensations, but itself the fundamental form of all external sensation. This pure intuition is in fact easily perceived in geometrical axioms, and any mental construction of postulates or even problems. That in space there are no more than three dimensions, that between two points there is but one straight line, that in a plane surface from a given point with a given right line a circle is describable, are not conclusions from some universal notion of space, but only discernible in space as in the concrete. Which things in a given space lie toward one side and which are turned toward the other can by no acuteness of reasoning be described discursively or reduced to intellectual marks. There being in perfectly similar and equal but incongruous solids, such as the right and the left hand, conceived of solely as to extent, or spherical triangles in opposite hemispheres, a difference rendering impossible the coincidence of their limits of extension, although for all that can be stated in marks intelligible to the mind by speech they are interchangeable, it is patent that only by pure intuition can the difference, namely, incongruity, be noticed. Geometry, therefore, uses principles not only undoubted and discursive but

falling under the mental view, and the obviousness of its demonstrations – which means the clearness of certain cognition in as far as assimilated to sensual knowledge – is not only greatest, but the only one which is given in the pure sciences, and the exemplar and medium of all obviousness in the others. For, since geometry considers the relations of space, the concept of which contains the very form of all sensual intuition, nothing that is perceived by the external sense can be clear and perspicuous unless by means of that intuition which it is the business of geometry to contemplate. Besides, this science does not demonstrate its universal propositions by thinking the object through the universal concept, as is done in intellectual disquisition, but by submitting it to the eyes in a single intuition, as is done in matters of sense.

Space is not something objective and real, neither substance, nor accident, nor relation; but subjective and ideal, arising by fixed law from the nature of the mind like an outline for the mutual co-ordination of all external sensations whatsoever. Those who defend the reality of space either conceive of it as an absolute and immense receptacle of possible things, an opinion which, besides the English, pleases most geometers, or they contend for its being the relation of existing things itself, which clearly vanishes in the removal of things and is thinkable only in actual things, as besides Leibniz, is maintained by most of our countrymen. The first inane fiction of the reason, imagining true infinite relation without any mutually related things, pertains to the world of fable. But the adherents of the second opinion fall into a much worse error. Whilst the former only cast an obstacle in the way of some rational or monumental concepts, otherwise most recondite, such as questions concerning the spiritual world, omnipresence, etc., the latter place themselves in flat opposition to the very phenomena, and to the most faithful interpreter of all phenomena, to geometry. For, not to enlarge upon the obvious circle in which they become involved in defining space, they cast forth geometry, thrown down from the pinnacle of certitude, into the number of those sciences whose principles are empirical. If we have obtained all the properties of space by experience from external relations only, geometrical axioms have only comparative universality, such as is acquired by induction. They have universality evident as far as observed, but neither necessity, except as far as the laws of nature may be established, nor precision, except what is arbitrarily made. There is hope, as in empirical sciences, that a space may some time be discovered endowed with other primary properties, perchance even a rectilinear figure of two lines.

The reader will notice that Kant talks about “geometrical axioms,” and mentions axioms of Euclidean geometry such as the fact that “between two points there is but one straight line.” Kant was by no means a mathematician, but he had a sufficient knowledge, as a philosopher, of several basic principles of mathematics.

It appears from Gauss’s correspondence, published in Volume VII of his *Collected Works* (p. 200ff.) that Gauss he meditating on the nature of space since a very young age, probably from the age of 16. It is from these

meditations that he became interested in the parallel postulate and in non-Euclidean geometry, spherical and (the hypothetical) hyperbolic. Unlike most of the geometers that preceded him, Gauss was convinced, at a very early stage of his life, that the parallel postulate was not a consequence of the others, and he spent a lot of time and energy pondering on the principles of hyperbolic geometry, a geometry resulting from the negation of the postulate.

Gauss was also thoroughly interested in philosophy, and, in particular, he read Kant. He became very critical of the latter's conception of space, exemplified in the text we just quoted as being "not something objective and real, neither substance, nor accident, nor relation, but subjective and ideal, arising by fixed law from the nature of the mind." On Kant, Gauss had the advantage of being a mathematician. In a letter to his friend Bessel, dated April 9, 1830, Gauss writes (translation from [30] p. 13):

We must confess in all humility that a number is *solely* a product of our mind. Space, on the other hand, possesses also a reality outside of our minds, the laws of which we cannot fully prescribe *a priori*.

In another letter, sent to Wolfgang Bolyai on March 6, 1832 and published in his *Collected Works*, Gauss writes, concerning the two hypotheses on the angle sum in a triangle, that it is precisely in the difficulty of this decision that "lies the clearest proof that Kant was wrong in asserting that space is just a form of our perception."

Gauss was also very critical of Kant's argument based on symmetries in the text we quoted above ("There being in perfectly similar and equal but incongruous solids, such as the right and the left hand, conceived of solely as to extent, . . . it is patent that only by pure intuition can the difference, namely, incongruity, be noticed"). We further discuss this in Chapter 6 of the present volume [191].

It is not surprising that Riemann declares, in his habilitation lecture, that, concerning his ideas on space, he is influenced by Gauss.

Riemann's ideas on space were discussed by Clifford, the first mathematician who translated into English Riemann's habilitation text, cf. [46].

## 8. TOPOLOGY

Poincaré, who is certainly the major founder of the modern field of topology,<sup>64</sup> declares in his "Analysis of his own works" (*Analyse des travaux scientifiques de Henri Poincaré faite par lui-même*), [203] p. 100, that he has two predecessors in the field, namely, Riemann and Betti. The latter, in his correspondence with his friend and colleague Placido Tardy reports on several conversations he had with Riemann on topology. Two letters from Betti to Tardy on this subject are reproduced and translated in the book

---

<sup>64</sup>We may quote P. S. Alexandrov, who declared in a talk he gave at a celebration of the centenary of Poincaré's birth [9]: "To the question of what is Poincaré's relationship to topology, one can reply in a single sentence: he created it." On Poincaré and Riemann, Alexandrov, in the same talk, says the following: "The close connection of the theory of functions of a complex variable, which Riemann has observed in embryonic form, was first understood in all its depth by Poincaré."



[207] by Pont, in the article [258] by Weil, and prior to them, by Loria in his obituary on Tardy [179].

The first of these two letters by Betti, dated October 6, 1863, starts with the following (Weil's translation): "I have newly talked with Riemann about the connectivity of spaces, and have formed an accurate idea of the matter," and he goes on explaining to his friend the notion of connectivity and that of the order of connectivity. Betti then writes:

What gave Riemann the idea of the cuts was that Gauss defined them to him, talking about other matters, in a private conversation. In his writings one finds that *analysis situs*, that is, this consideration of quantities independently from their measure, is "wichtig"; in the last years of his life he has been much concerned with a problem in *analysis situs*, namely: given a winding thread and knowing, at every one of its self-intersections, which part is above and which below, to find whether it can be unwound without making knots; this problem he did not succeed in solving except in special cases ...

The second letter, dated October 16, 1863, starts with: "Riemann proves quite easily that every space can be reduced to an SC space by means of 1-cuts and SC 2-cuts." In the same letter, Betti elaborates on this subject, giving many examples in  $n$  dimensions. He concludes the letter by noting that the number of line sections is equal to the number of periodicity moduli of an  $(n - 1)$ -integral, the number of simply connected surface sections to the number of periodicity moduli of an  $(n - 2)$ -integral, and so on.

This should make clear the parentage, for what concerns topology, from Riemann to Poincaré, potentially including Betti. In this section, we go further back in the history of topological ideas, and we review some of the important works done before Riemann in this field.

René Thom considers that topology was born in ancient Greece. He expanded on this idea in several articles, cf. [247] and [248]. This is a perfectly reasonable theory. In fact, the question depends on what sense we give to the word "topology." If the matter concerns the notions of limit and convergence, then the roots of this field are indeed in Greek antiquity, and more especially, in the writings of Zeno, which do not survive, but which were quoted by his critics and commentators, including Plato, Aristotle and Simplicius. Likewise, if the question concerns the notion of space, and the related notion of place, then the roots also are in Greek science. We already alluded to this fact in the previous section. The Greeks made a distinction between space and place and the notion of place (*situs*) is at the basis of topology. The three words place, *situs* and  $\tau\acute{o}\pi\omicron\varsigma$  are synonyms. To the best of our knowledge, a systematic investigation of the origin of topology in Greek antiquity has never been conducted. A whole book may be written on that subject. Failing to do this now, we shall start our exposition of the roots of topology with Leibniz, as it is usually done. Indeed, it is commonly accepted that the first explicit mention of topology as a mathematical field was made by him.

Even though no purely topological result can be attributed to Leibniz, he had the privilege to express for the first time, back in the seventeenth century, the need for a new branch of mathematics, which would be "a

geometry that is more general than the rigid Euclidean geometry and the analytic geometry of Descartes.” Leibniz describes his geometry as purely qualitative and concerned with the study of figures independently of their metrical properties. In a letter to Christiaan Huygens, sent on September 8, 1679 (cf. [177] p. 578–569 and [145] vol. VIII n° 2192), he writes:

After all the progress I have made in these matters, I am still not happy with Algebra, because it provides neither the shortest ways nor the most beautiful constructions of Geometry. This is why when it comes to that, I think that we need another analysis which is properly geometric or linear, which expresses to us directly *situm*, in the same way as algebra expresses *magnitudinem*. And I think that I have the tools for that, and that we might represent figures and even engines and motion in character, in the same way as algebra represents numbers in magnitude.<sup>65</sup>

In the same letter ([177] p. 570), Leibniz adds:

I found the elements of a new characteristic, completely different from Algebra and which will have great advantages for the exact and natural mental representation, although without figures, of everything that depends on the imagination. Algebra is nothing but the characteristic of undetermined numbers or magnitudes. But it does not directly express the place, angles and motions, from which it follows that it is often difficult to reduce, in a computation, what is in a figure, and that it is even more difficult to find geometrical proofs and constructions which are enough practical even when the Algebraic calculus is all done.<sup>66</sup>

Together with his letter to Huygens, Leibniz included the manuscript of an essay he wrote on the new subject. He writes, in the same letter ([177] p. 571):

But since I don't see that anybody else has ever had the same thought, which makes me fear that it might be lost if I do not get enough time to complete it, I will add here an essay which seems to me important, and which will suffice at least to rendre my aim more credible and easier to conceive, so that if something prevents

---

<sup>65</sup>Après tous les progrès que j'ai faits en ces matières, je ne suis pas encore content de l'Algèbre, en ce qu'elle ne donne ni les plus courtes voies, ni les plus belles constructions de Géométrie. C'est pourquoi lorsqu'il s'agit de cela, je crois qu'il nous faut encore une autre analyse proprement géométrique ou linéaire, qui nous exprime directement *situm*, comme l'algèbre exprime *magnitudinem*. Et je crois d'en avoir le moyen, et qu'on pourrait représenter des figures et même des machines et mouvements en caractères, comme l'algèbre représente les nombres en grandeurs. [We have modernized the French.]

<sup>66</sup>J'ai trouvé quelques éléments d'une nouvelle caractéristique, tout à fait différente de l'Algèbre, et qui aura de grands avantages pour représenter à l'esprit exactement et au naturel, quoique sans figures, tout ce qui dépend de l'imagination. L'Algèbre n'est autre chose que la caractéristique des nombres indéterminés ou des grandeurs. Mais elle n'exprime pas directement la situation, les angles et les mouvements, d'où vient qu'il est souvent difficile de réduire dans un calcul ce qui est dans la figure, et qu'il est encore plus difficile de trouver des démonstrations et des constructions géométriques assez commodes lors même que le calcul d'Algèbre est tout fait.

its realization now, it will serve as a monument for posterity and give the possibility to somebody else to finish it.<sup>67</sup>

He then explains in more detail his vision of this new domain of mathematics, and where it stands with respect to algebra and geometry, giving several examples of a formalism to denote loci, showing how this formalism expresses statements such that the intersection of two spherical surfaces is a circle, and the intersection of two planes is a line.

Leibniz' letter ends with the words ([175] p. 25):

I have only one remark to add, namely, that I see that it is possible to extend the characteristic to things which are not subject to imagination. But this is too important and it would lead us too far for me to be able to explain myself on that in a few words.<sup>68</sup>

When Leibniz started his correspondence with Huygens, the latter was already a well established scientist whose achievements were behind him, and it was not easy to convince him of the usefulness of a new theory. Huygens thought that the theory was too abstract and he remained skeptical about it. He was above all a geometer working on concrete geometrical problems.

One may recall that when Leibniz sent him the above letter, Huygens was considered as a world authority in geometry and physics. He was settled in Paris since 15 years, and he was a leading member of the *Académie Royale des Sciences*. Leibniz had studied mathematics with Huygens, who was seventeen years older than him, and he considered him as his mentor. Huygens responded to Leibniz in a letter dated November 22, 1679 ([177] p. 577):

I have examined carefully what you are asking me regarding your new characteristic, and to be frank with you, I cannot not conceive the fact that you have so much expectations from what you spread on me. Because your example of places concerns only realities which were already perfectly known, and the proposition saying that the intersection of a plane and a spherical surface makes the circumference of a circle does not follow clearly. Finally, I cannot see in what way you can apply your characteristic to which you seem you want to reduce all these different matters, like the quadratures, the invention of curves by the properties of tangents, the irrational roots of equations, Diophantus' problems, the shortest and the most beautiful constructions of the geometric problems. And what still appears to me stranger than anything else, the invention and the explanation of machines. I say it to you unsuspectingly, in my opinion this is only wishful thinking, and I need other proofs in order to believe that there could be some reality in what you present. I would nevertheless restrain myself

---

<sup>67</sup>Mais comme je ne remarque pas que quelqu'autre ait jamais eu la même pensée, ce qui me fait craindre qu'elle ne se perde, si je n'y ai pas le temps de l'achever, j'ajouterai ici un essai qui me paraît considérable, et qui suffira au moins à rendre mon dessein plus croyable et plus aisé à concevoir, afin que si quelque hasard en empêche la perfection à présent, ceci serve de monument à la postérité, et donne lieu à quelque autre d'en venir à bout.

<sup>68</sup>Je n'ai qu'une remarque à ajouter, c'est que je vois qu'il est possible d'étendre la caractéristique jusqu'aux choses, qui ne sont pas sujettes à l'imagination ; mais cela est trop important et va trop loin pour que je me puisse expliquer là-dessus en peu de paroles.

from saying that you are mistaken, knowing the subtlety and the deepness of your mind. I only beg you that the magnificence of the things you are searching won't let you postpone from giving us those which you already found, like this Arithmetic Quadrature you discovered, concerning the roots of the equations beyond the cubical, if you are still satisfied with it.<sup>69</sup>

In another letter dated January 11, 1680 ([177] p. 584) Huygens writes:

For what concerns the effects of your characteristic, I see that you insist on being persuaded of them, but as you say yourself, the examples will be more important than reasonings. This is why I am asking you much simpler examples, but capable of overcoming my incredulity, because that of the places, I confess, does not seem to me of that sort.<sup>70</sup>

The essay that Leibniz sent did not obtain Huygens' backing and it remained hidden among other manuscripts in Huygens' estate. It was published for the first time in 1833, and drew the attention of several nineteenth-century mathematicians, including Grassmann (1809–1877), the founder of the theory of vector spaces, who realized its importance for the new field of topology. There are two recent editions of this text, both included in doctoral dissertations, by J. Acheverría [176] (1995), in France, and by de Risi, [235] (2007), in Germany. The two dissertations contain other texts by Leibniz on the same subject.

Leibniz used several names for the new field, including *analysis situs*, *geometria situs*, *characteristica situs*, *characteristica geometrica*, *analysis geometrica*, *speciosa situs*, etc.

The first mathematician who worked consciously on topological questions is Euler. These questions include the definition and the invariance of the Euler characteristic of a convex polyhedron, the problem known as that of the Königsberg seven bridges, another question related to the Knight's tour

---

<sup>69</sup>J'ai examiné attentivement ce que vous me demandez touchant votre nouvelle caractéristique, mais pour vous l'avouer franchement, je ne conçois pas parce que vous m'en étalez, que vous y puissiez fonder de si grandes espérances. Car votre exemple des Lieux ne regarde que des vérités qui nous étaient déjà fort connues, et la proposition de ce que l'intersection d'un plan et d'une surface sphérique fait la circonférence d'un cercle, s'y conclut assez obscurément. Enfin, je ne vois point de quel biais vous pourriez appliquer votre caractéristique à toutes ces choses différentes qu'il semble que vous y vouliez réduire, comme les quadratures, l'invention des courbes par la propriétés des tangentes, les racines irrationnelles des Équations, les problèmes de Diophante, les plus courtes et plus belles constructions des problèmes géométriques. Et ce qui me paraît encore le plus étrange, l'invention et l'explication des machines. Je vous le dis ingénument, ce ne sont là à mon avis que de beaux souhaits, et il me faudrait d'autres preuves pour croire qu'il y eût de la réalité dans ce que vous avancez. Je n'ai pourtant garde de dire que vous vous abusiez, connaissant d'ailleurs la subtilité et profondeur de votre esprit. Je vous prie seulement que la grandeur des choses que vous cherchez ne vous fasse point différer de nous donner celles que vous avez déjà trouvées, comme est cette Quadrature Arithmétique et que vous avez découvert pour les racines des équations au-delà du cube, si vous en êtes content vous-même.

<sup>70</sup>Pour ce qui est des effets de votre caractéristique, je vois que vous persistez à en être persuadé, mais, comme vous dites vous-même, les exemples toucheront plus que les raisonnements. C'est pourquoi je vous en demande des plus simples, mais propres à convaincre mon incrédulité, car celui des lieux, je l'avoue, ne me paraît pas de cette sorte.

on the chessboard, and a musical question concerning a graph known as the *speculum musicum*. This graph was introduced in Euler's *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae* (A attempt at a new theory of music, exposed in all clearness according to the most well-founded principles of harmony) [71]. Its vertices are the twelve notes of the chromatic scale, and the edges connect two elements which differ by a fifth or a major third with the property that one may traverse all the edges of the graph passing exactly once by each note. The article [190] is a detailed survey of the work of Euler on these questions. In the present section, we start by reviewing the work of Euler on the question of the seven bridges of Königsberg. This work shows that Euler considered himself as the direct heir of Leibniz for what concerns the field of topology. We shall then describe in detail the works of Euler and Descartes on the Euler characteristic, a question which is directly related to the topological classification of surfaces, which was one of Riemann's major achievements in topology. We recall that Euler formulated this result for a surface which is the boundary of a convex polyhedron having  $F$  faces,  $A$  edges and  $S$  vertices; the formula is then:

$$F - A + S = 2.$$

We start with the problem of the Königsberg bridges.

In the eighteenth century, the city of Königsberg<sup>71</sup> consisted of four quarters separated by branches of the river Pregel and related by seven bridges. The famous "problem of the seven bridges of Königsberg" asks for a path in that city that starts at a given point and returns to the same point after crossing once and only once each of the seven bridges. At the time of Euler, this was a popular question among the inhabitants of Königsberg.

Euler showed that such a path does not exist. He presented his solution to the Saint Petersburg Academy of Sciences on August 26, 1735, and in the same year he wrote a memoir on the solution of a more general problem entitled *Solutio problematis ad geometriam situs pertinentis* (Solution of a problem relative to the geometry of position) [74]. Euler learned about the problem from a letter, dated March 7, 1736, sent to him by Carl Leonhard Gottlieb Ehler, one of his friends who was the mayor of Danzig<sup>72</sup> and who had worked as an astronomer in Berlin. Euler solved the problem just after he received the letter. In a letter dated March 13, 1736, written to Giovanni Marioni, an Italian astronomer working at the court of Vienna, Euler declares that he became interested in this question because he realized that the problem could not be solved using geometry, algebra or combinatorics, and that therefore he wondered whether "it belonged to the 'geometry of position,' (*geometria situs*) which Leibniz has so much sought for." In the same letter, Euler announced to Marioni that after some thought, he found a proof which applies not only to that case, but to all similar problems.

In the introduction of his paper, Euler writes (translation from [29]):

---

<sup>71</sup>Today, the city of Königsberg, called Kaliningrad, is part of a Russian exclave between Poland and Lithuania on the Baltic Sea.

<sup>72</sup>Today, Danzig is the city of Gdansk, in Poland.

In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the geometry of position. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position, especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.

Euler's work on this problem is commented in several articles and books.

We now come to Euler's polyhedron formula, and we start with Descartes.

Long before Euler came out with his formula  $F - A + S = 2$  relating the faces ( $F$ ), edges ( $A$ ) and vertices ( $S$ ) of a convex polyhedron, Descartes obtained an equivalent result, with a geometric proof, involving the solid angles and the dihedral angles between the faces. Described in modern terms, Descartes' proof consists in computing in two different manners the total curvature of the boundary of the polyhedron. Descartes wrote that proof at the age of 25, but did not publish it. The story of Descartes' manuscript is interesting and we recall it now.

Descartes' manuscript was discovered in Hanover, among Leibniz's estate. The latter had copied Descartes' proof during a stay in Paris, in 1675 or 1676, presumably with the intention of publishing it. The original manuscript of Descartes, which carries the title *Progymnasmata de solidorum elementis* (Preparatory exercises to the elements of solids) [47] is mentioned in an inventory of papers which Descartes left in some chests in Stockholm, the city where he died. The copy, made by Leibniz, carries the same title, with the additional mention *excerpta ex manuscripto Cartesii* (Excerpt from a manuscript of Descartes). After the manuscript was discovered, a French translation was published by Foucher de Careil in 1859, in a volume of unpublished works of Descartes. This publication contained errors, because Foucher, who did it, was not a mathematician. The edition is nevertheless interesting, and in the introduction to the volume [115], Foucher recalls the story of the discovery. The story is also told by Adam in the commentaries of the volume of the Adam-Tannery edition of Descartes' works containing this theorem ([51] tome X, p. 257–263).

In 1890, Jonquières presented to the Paris Académie des Sciences two Comptes Rendus notes entitled *Sur un point fondamental de la théorie des polyèdres* (On a fundamental property of the theory of polyhedra) [149] and *Note sur le théorème d'Euler dans la théorie des polyèdres* (Note on the theorem of Euler on the theory of polyhedra) [150], without being aware of the work of Descartes on this subject. After Jordan pointed out the existence of the work of Descartes in Foucher's edition, Jonquières published

other Comptes Rendus notes on the work of Descartes, cf. [151] [152] [153]. Poincaré, in his celebrated first memoir on *Analysis situs* [201] attributes to Jonquières the generalization of Euler's theorem to non-necessarily convex polyhedral surfaces.

There is a relatively recent (1987) critical edition of Descartes' *Progymnasmata* with a French translation, with notes and commentaries, by P. Costabel [47].

Euler reported on his work on polyhedra in his correspondence with Goldbach. In a letter dated November 14, 1750, Euler informs his friend of the following two results which he refers to as Theorems 6 and 11 respectively:

6. In any solid enclosed by plane surfaces the sum of the number of faces and the number of solid angles is greater by 2 than the number of edges.

11. The sum if all planar angles equals four times as many right angles as the number of solid angles, decreased by 8.

The term "solid enclosed by plane surfaces" refers to a convex polyhedron. The first result is the Euler characteristic formula, and the second one is a form of the Gauss-Bonnet theorem for the sphere. Euler writes:

I am surprised that these general properties in stereometry have not been noticed by anybody, as far as I know, but still more that the most important of them, viz., Th. 6 and Th. 11, are so hard to prove; indeed I still cannot prove them in a way that satisfies me.

In the same letter, Euler gives several examples where the two theorems are satisfied.

In his memoir [85], entitled *Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita* (Proof of some of the properties of solid bodies enclosed by planes) and written one year after [84], Euler gave proofs of the two results. In the introduction of [85], he declares that his polyhedron formula is part of a more general research he is conducting on polyhedra. In fact, in the letter to Goldbach we mentioned, Euler announces a result on volumes of simplices in terms of their side lengths (a three-dimensional analogue of Heron's formula for the area of triangles), which he proves later in his paper *Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita* [85]. Euler writes:<sup>73</sup>

Although I had uncovered many properties which are common to all bodies enclosed by plane faces and which seemed to be completely analogous to those which are commonly included among the first principles of rectilinear plane figures, still, not without a great deal of surprise did I realize that the most important of those principles were so recondite that all the time and effort spent looking for a proof of them had been fruitless. Nor, when I consulted my friends, who are otherwise extremely versed in these matters and with whom I had shared those properties, were they able to shed any light from which I could derive these missing proofs. After the consideration of many types of solids I came to the point where I understood that the properties which I had perceived in

---

<sup>73</sup>Translation by C. Frances and D. Richeson.

them clearly extended to all solids, even if it was not possible for me to show this in a rigorous proof. Thus, I thought that those properties should be included in that class of truths which we can, at any rate, acknowledge, but which it is not possible to prove.

One advantage of Euler's proof, compared to the one of Descartes, is that it shows in a clear way the combinatorial aspect of the problem, highlighting the notion of edges and faces of the polyhedron.

The proof that Euler gives in [85] is based on an induction on the number of solid angles, reducing them by one at each step. He writes:

These proofs are in no way inferior to those proofs used in Geometry except that here due to the nature of solids one must use more imagination, in as much as solids are being depicted on a flat surface.

At the same time, Euler was laying down the foundations of combinatorial topology. He writes (Scholion to Proposition 4):

I admit that I have thus brought to light only the first principles of Solid Geometry, on which this science should be built as it develops further. No doubt it contains many outstanding qualities of solids of which we are so far completely ignorant.[...]

Legendre, in his *Éléments de géométrie* (Note XII) published in 1794 [174], gave a complete proof of Euler's theorem based on geometry. This proof is considered as one of the simplest, and it is repeated in more modern works, e.g. in Hopf [141].

A large number of mathematicians commented on Euler's polyhedron formula, expanding some arguments in Euler's proofs, giving new proofs, and sometimes comparing Euler's work with that of Descartes. To show the diversity of these works, we mention the papers by Andreiev [10], Bertrand [26], Bougaïev [22], Brianchon [27], Catalan [31], Cauchy [32], [33], Feil [114], Gergonne, [131], Grunert, [136], Jonquières [149], [150], [151], [152], [153], [155], [156], [157], Jordan [158], [159], Lebesgue [169], Lhuillier [178], Poincaré [200], [201], Poinot [204], [205], Prouhet [209], [210], Steiner, [240], Valat [250] and Thiel [245]. We shall quote some of these works below. We also mention that in 1858, the Paris *Académie des Sciences* proposed as a subject for the 1861 *Grand prix*: "To improve, in some important point, the geometric theory of polyherda." Möbius participated and presented a memoir (but did not get the prize).

In 1811, Cauchy brought out a purely combinatorial proof of that theorem. In this proof, one starts by deleting a face of the polyhedron and reduces the problem to another one concerning a planar polygon.<sup>74</sup> In his article *Recherches sur les polyèdres* (Researches on polyhedra) [32], published in 1813, Cauchy writes:

Euler has determined, in the Petersburg *Mémoires*, year 1758, the relation that exists between the various elements that compose the surface of a polyhedron; and Mr. Legendre, in his *Éléments de Géométrie*, proved in a much simpler manner Euler's theorem, by considerations of spherical polygons. Having been led by some researches to a new proof of that theorem, I reached a theorem

---

<sup>74</sup>A similar proof is given by Hilbert and Cohn-Vossen [139] p. 290.



which is much more general than the one of Euler, whose statement is the following:

Theorem. If we decompose<sup>75</sup> a polyhedron in as many others as we wish, by taking at will new vertices in the interior, and if we represent by  $P$  the number of new polyhedra thus formed, by  $S$  the total number of vertices, including those of the initial polyhedron, by  $F$  the total number of faces, and by  $A$  the total number of edges, then we will have

$$S + F = A + P - 1,$$

that is, the sum of the number of vertices and that of faces will overpass by one the sum of the number of edges that of polyhedra.<sup>76</sup>

Poinsot [204], in 1858, published a proof of Euler's formula using some of Cauchy's arguments. He writes: "This relation, which Euler was the first to prove, does not hold only for convex polyhedra, as one might think, but for polyhedra of any kind." In fact, this statement needs some explanation. We are used today to the fact that Euler's formula is valid for polyhedra which are homeomorphic to a sphere. This notion did not exist at that time, neither the word, nor the idea. One had to wait for that to the work of Jordan, who set up the precise hypotheses under which Euler's formula is valid. In his article [156], he writes that Euler's theorem is valid for polyhedra which he calls "simple," or "Eulerian," that is, polyhedra ([156] p. 35) "such that any contour drawn on the surface which does not traverse itself divides this surface into two separate regions; a category that contains as a particular case convex polyhedra."<sup>77</sup> A few pages later (p. 38), Jordan makes the following commentary: "It would have been easy to show that if we can draw on a polyhedron  $\lambda$  different contours which do not intersect each other and which do not divide the surface into separate parts, we would have  $S + F = A + 2 - 2\lambda$ ."<sup>78</sup> In fact, Jordan had extracted the notion we

<sup>75</sup>Cauchy "decomposes" the polyhedron by taking new vertices in the interior of the three-dimensional polyhedron (and not on the boundary surface)

<sup>76</sup>Euler a déterminé, dans les *Mémoires* de Pétersbourg, année 1758, la relation qui existe entre les différents éléments qui composent la surface d'un polyèdre ; et M. Legendre, dans ses *Éléments de Géométrie*, a démontré d'une manière beaucoup plus simple le théorème d'Euler, par la considération des polygones sphériques. Ayant été conduit par quelques recherches à une nouvelle démonstration de ce théorème, je suis parvenu à un théorème plus général que celui d'Euler et dont voici l'énoncé :

Théorème. Si l'on décompose un polyèdre en tant d'autres que l'on voudra, en prenant à volonté dans l'intérieur de nouveaux sommets ; que l'on représente par  $P$  le nombre de nouveaux polyèdres ainsi formés, par  $S$  le nombre total de sommets, y compris ceux du premier polyèdre, par  $F$  le nombre total de faces, et par  $A$  le nombre total des arêtes, on aura

$$S + F = A + P - 1,$$

c'est-à-dire que la somme faite du nombre des sommets et de celui des faces surpassera d'une unité la somme faite du nombre des arêtes et de celui des polyèdres.

<sup>77</sup>[...] tels que tout contour fermé tracé sur leur surface et ne se traversant pas lui-même divise cette surface en deux régions séparées ; catégorie qui renferme comme cas particulier les polyèdres convexes.

<sup>78</sup>Il serait aisé de démontrer que si l'on peut tracer sur un polyèdre  $\lambda$  contours différents, ne se coupant pas mutuellement et ne divisant pas la surface en parties séparées, on aura  $S + F = A + 2 - 2\lambda$ .

call today “topological surface of finite type,” to which the general theory applies, cf. [155] p. 86:

A surface is said to be of type  $(m, n)$  if it is bounded by  $m$  closed contours and if furthermore we can draw on it  $n$  closed contours that do not intersect themselves nor mutually, without dividing it into two distinct regions.<sup>79</sup>

Then Jordan makes the relation with the polyhedra to which Euler’s formula applies: “The polyherda of kind  $(0, 0)$  are nothing but those which I called *Eulerian*.”<sup>80</sup>

It is interesting to read Lebesgue’s comments on some proof of Euler’s theorem, because it gives us some hints of how the subject of topology was viewed in those days. Lebesgues’ comments are written in 1924 ([169] p. 319):

I don’t agree at all with those who pretend to attribute Euler’s theorem to Descartes. Descartes did not state the theorem; he did not see it. Euler perceived it and he fully understood its character. For Euler, the description of the form of a polyhedron must precede the use of the measures of its elements, and this is why he set his theorem as a fundamental theorem. For him, like for us, this is a theorem of enumerative *Analysis situs*; therefore he tried to find it by considerations independent of any metrical theory, that in effect belong to what we call the field of *Analysis situs*. And this is why he left to Legendre the honor of finding a rigorous proof. None of us who had read a little bit of Euler and who were amazed by his prodigious technical masterliness will doubt, even for one second, that if Euler had thought of putting aside his theorem and deducing it from one of its metric corollaries, he would have easily succeeded. (It should be noted that Euler does not at all restrict his researches to convex polyhedra.) It seems to me, on the contrary, that the fact that Descartes passed so closely to the theorem without seeing it, emphasizes Euler’s credit. (At least, this is what we believe, because Descartes employed in his notebook some algebraic characters which he used before knowing Viète’s characters.) But Leibniz, who found Descartes’ notebook enough interesting to copy it, who realized that Descartes’ geometry does not apply to questions involving order and position relations, who dreamed of constructing the algebra of these relations and who in advance gave it the name *Analysis situs*, did not notice, in Descartes’ notebook Euler’s theorem which is so fundamental in *Analysis situs*. This theorem really belongs to Euler. As for the proof, one could, may be with a little bit of unfairness, call it the proof of Legendre and Descartes. This proof is metrical, and it is fair to blame it for the fact that it uses notions that are foreign to *Analysis situs*. But one should not exaggerate the value of this grievance.<sup>81</sup>

---

<sup>79</sup>Une surface sera dite d’espèce  $(m, n)$  si elle est limitée par  $m$  contours fermés et si l’on peut d’autre part y tracer  $n$  contours fermés ne se coupant pas eux-mêmes ni mutuellement, sans la partager en deux régions distinctes.

<sup>80</sup>Les polyèdres de l’espèce  $(0, 0)$  ne sont autres que ceux que j’ai appelés *eulériens*.

<sup>81</sup>Je ne suis pas du tout d’accord avec ceux qui prétendent attribuer à Descartes le théorème d’Euler. Descartes n’a pas énoncé le théorème ; il ne l’a pas vu. Euler l’a aperçu et en a bien compris le caractère. Pour Euler, la description de la forme d’un polyèdre

We now give a quick review of some work of Gauss on topology, another field in which his impact on Riemann was huge.

Gauss was interested in applications of *Geometria situs* (a term he used in his writings), in particular in astronomy, geodesy and electromagnetism. In astronomy, he addressed the question of whether orbits of celestial bodies may be linked (cf. his short treatise entitled *Über die Grenzen der geocentrischen Orter der Planeten*). From his work on geodesy, we mention his letter to Schumacher, 21 Nov. 1825, (from Gauss's *Werke* vol. VIII, p. 400):

Some time ago I started to take up again a part of my general investigations on curved surfaces, which shall become the foundation of my projected work on higher geodesy. [...] Unfortunately, I find that I will have to go very far afield [...]. One has to follow the tree down to all its root threads, and some of this costs me week-long intense thought. Much of it even belongs to geometria situs, an almost unexploited field.

From Gauss's *Nachlaß*, we know that he worked on a combinatorial theory of knot projections, during the year 1825, and again in 1844. (Gauss's *Werke*, Vol. VIII, p. 271–286). We already mentioned at the beginning of this section, that we learn from a letter sent by Betti to Tardy that the idea of analyzing a surface by performing successive cuts was given to Riemann by Gauss, in a private conversation. Besides Riemann, Gauss had two students who worked on topology and who were certainly influenced by him: Listing and Möbius.

Riemann introduced the fundamental topological notions for surfaces: connectedness, degree of connectivity, the classification of closed surfaces by their genus. He developed this theory for the purpose of using it in his work on the theory of functions of a complex variable. In his memoir on Abelian functions, he talks about *analysis situs*, referring to Leibniz:

---

doit précéder l'utilisation des mesures de ses éléments et c'est pourquoi il a posé son théorème comme théorème fondamental. C'est, pour lui comme pour nous, un théorème d'*Analysis situs* énumérative ; aussi a-t-il cherché à le démontrer par des considérations indépendantes de toute propriété métrique, appartenant bien à ce que nous appelons le domaine de l'*Analysis situs*. Et c'est pourquoi il a laissé à Legendre l'honneur d'en trouver la preuve rigoureuse ; aucun de ceux qui ont quelque peu lu Euler, et qui ont été stupéfaits de sa prodigieuse virtuosité technique, ne doutera un seul instant que si Euler avait pensé à faire passer son théorème au second plan et à le déduire d'un de ses corollaires métriques, il n'y eût facilement réussi. (Il convient d'ajouter qu'Euler ne restreint nullement ses recherches aux polyèdres convexes.) Que Descartes soit passé si près du théorème sans le voir me paraît au contraire souligner le mérite d'Euler. Encore peut-on dire que Descartes était jeune quand il s'occupait de ces questions. (C'est du moins ce que l'on croit, parce que Descartes a employé dans son cahier certains caractères cossiques qu'il utilisait avant de connaître les notations de Viète.) Mais Leibniz qui a trouvé le cahier de Descartes assez intéressant pour le copier, qui a reconnu que la géométrie de Descartes ne s'appliquait pas aux questions où interviennent des relations d'ordre et de position, qui a rêvé de construire l'algèbre de ces relations et l'a nommée à l'avance *Analysis situs*, n'a pas aperçu, dans le cahier de Descartes, le théorème d'Euler si fondamental en *Analysis situs*. Le théorème appartient bien à Euler ; quant à la démonstration, on pourrait, un peu injustement peut-être, la dénommer démonstration de Legendre et Descartes. Cette démonstration est métrique ; il est juste de lui reprocher de faire appel à des notions étrangères à l'*Analysis situs*. Mais il ne faudrait pas s'exagérer la valeur de ce grief.

In the study of functions obtained by the integration of exact differentials, a few theorems of analysis situs are almost essential. Under that name, which was used by Leibniz, although may be in a slightly different sense, it is permitted to designate the theory of continuous magnitudes which studies these magnitudes, not as independent of their position and measurable with respect to each other, but by disregarding all idea of measure and studying them only for what regards their relation of position and inclusion. I intend to treat this subject later, in a way that is completely independent of any measure.

In his habilitation dissertation, Riemann mentions the possibility of working in the new field of topology, talking about the notion of “place.” We quote this cryptic passage:

Measure consists in the superposition of the magnitudes to be compared; it therefore requires a means of using one magnitude as the standard for another. In the absence of this, two magnitudes can only be compared when one is a part of the other; in which case we can only determine the more or less and not the how much. The researches which can in this case be instituted about them form a general division of the science of magnitude in which magnitudes are regarded not as existing independently of position and not as expressible in terms of a unit, but as regions in a manifoldness. Such researches have become a necessity for many parts of mathematics, e.g., for the treatment of many-valued analytical functions; and the want of them is no doubt a chief reason for which the celebrated theorem of Abel and the achievements of Lagrange, Pfaff, Jacobi for the general theory of differential equations, have so long remained unfruitful. Out of this general part of the science of extended magnitude in which nothing is assumed but what is contained in the notion of it, it will suffice for the present purpose to bring into prominence two points; the first of which relates to the construction of the notion of a multiply extended manifoldness, the second relates to the reduction of determinations of place in a given manifoldness to determinations of quantity, and will make clear the true character of an  $n$ -fold extent.

He also describes the passage from one dimension to another:

If in the case of a notion whose specialisations form a continuous manifoldness, one passes from a certain specialisation in a definite way to another, the specialisations passed over form a simply extended manifoldness, whose true character is that in it a continuous progress from a point is possible only on two sides, forward or backwards. If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, then all the specialisations so obtained form a doubly extended manifoldness. In a similar manner one obtains a triply extended manifoldness, if one imagines a doubly extended one passing over in a definite way to another entirely different; and it is easy to see how this construction may be continued. If one regards the variable object instead of the determinable notion of

it, this construction may be described as a composition of a variability of  $n + 1$  dimensions out of a variability of  $n$  dimensions and a variability of one dimension.

Riemann's ideas on topology are explained in some sections of Klein's booklet [162]. For instance, §8 carries the title *Classification of closed surfaces according to the value of the integer  $p$* . Let us comment on a passage from Klein's booklet [162] concerning the classification of surfaces, as an example of his point of view on topology. We know that topology, which was an emerging subject, plays an important role in Riemann surface theory. We already mentioned that Riemann introduced several major notions on surface topology. Klein tried to make a more systematic exposition of these ideas. His book [162] contains a chapter in which the classification of closed surfaces according to genera is presented. On p. 32, he writes:

That it is impossible to represent surfaces having different  $p$ 's<sup>82</sup> upon one another, the correspondence being uniform, seems evident. It is not meant, however, that this kind of geometrical certainty needs no further investigation; cf. the explanations of G. Cantor (*Crelle*, t. LXXXIV. pp. 242 *et seq.*). But these investigations are meanwhile excluded from consideration in the text, since the principle there insisted upon is to base all reasonings ultimately on intuitive relations.

Klein then states the converse: between any two surfaces of the same genus it is possible to find a "uniform correspondence." He declares that this statement is more difficult to prove, and he refers to the 1866 article by Jordan [154].<sup>83</sup> This paper is an important milestone in the history of topology, because it contains the first attempt to classify surfaces up to homeomorphism, although there was no precise definition of homeomorphism yet.<sup>84</sup> Jordan's aim, in his paper, is to prove the following theorem, which he states in the introduction:

In order for two surfaces or pieces of flexible and inextensible surfaces to be applied onto each other without tear and duplication, the following is necessary and sufficient:

- (1) That the number of separated contours that respectively bound these portions of surfaces be the same. (If the surfaces considered are closed, this number is zero.)
- (2) That the maximal number of closed contours which do not self-intersect or intersect each other that we can draw on each

---

<sup>82</sup>We recall that  $p$  denotes the genus.

<sup>83</sup>Camille Jordan (1838-1922), who is mostly known for his results on the topology of surfaces and on group theory, also worked on function theory in the sense of Riemann. The title of the second part of his doctoral thesis is: "On periods of inverse functions of integrals of algebraic differentials." The subject was proposed to him by Puiseux, whom we mention in this paper concerning uniformization. Jordan is among the first who tried to study the ideas of Galois, and he is also among the first who introduced group theory in the study of differential equations.

<sup>84</sup>The word "homeomorphism" was introduced by Poincaré in his article [201] but with a meaning that is different from the one it has today. There is a definition of homeomorphism in the 1909 article by Hadamard [137], as being a one-to-one continuous map. This is not correct, unless Hadamard meant, by "one-to-one continuous", "one-to-one bi-continuous." We refer the reader to the paper [184] on the rise and the development of the notion of homeomorphism. This paper contains several quotes, some of which are very intriguing.

of the two surfaces without cutting it into two separate regions be the same on both sides.

Jordan gives the following “definition” of two surfaces being “applicable onto each other.” For a modern reader, this definition may seem fuzzy, but one has to remember that this paper was written in the heroic epoch of the foundations of modern topology, that the notion of homeomorphism seems extremely natural for us today, but that it was not so in those times. Jordan writes:

We shall rely on the following principle, which we can consider as evident, and take it if necessary as a definition: *Two surfaces  $S, S'$  are applicable onto each other if we can decompose them into infinitely small elements such that to any contiguous elements of  $S$  correspond contiguous elements of  $S'$ .*

Besides Klein’s booklet, several books and treatises explaining Riemann’s ideas appeared in the decades that followed Riemann’s work. We mention as examples Neumann’s *Vorlesungen über Riemann’s Theorie der Abel’schen Integrale* (Lectures on Riemann’s theory of Abelian integrals) [187], Picard’s *Traité d’Analyse* [198], Appell and Goursat’s *Théorie des fonctions algébriques et de leurs intégrales* (Theory of Algebraic functions and their integrals) [16], and there are others. The last two treatises, together with several other French books on the theory of functions of a complex variable, are reviewed in Chapter 8 of the present volume [193].

Among the other important topological notions that were introduced before Riemann and that were used by him, we must mention the notion of *homotopy of paths* and its use in complex analysis (in particular, in the theory of line integrals), especially by Cauchy and Puiseux. This is discussed in detail in Chapter 7 of the present volume [192]. Cauchy published his first work on the subject in 1825 [36]. This is also a topic on which Gauss was a forerunner, but he did not publish anything about it. This is attested in his letter to Bessel, December 18, 1811, published in Volume VIII of his *Collected works* (p. 90–92), a letter in which Gauss makes the important remark that if one defines integrals along paths in the complex plane, then the value obtained may depend on the path.

Regarding the history of Riemann’s ideas on topology, we could have also commented on his predecessors regarding the notion of the discreteness and continuity of space, but this would have taken us too far. We make a few remarks on this matter in Chapter 6 of the present volume [191].

We end this section by quoting Alexander Grothendieck, from his *Récoltes et semailles* (Harvesting and Sowing),<sup>85</sup> commenting on Riemann’s reflections on this theme ([134] Chapter 2 §2.20):

It must be already fifteen or twenty years ago since, skimming the modest volume that constitutes Riemann’s complete works, I was struck by a remark which he made “incidentally.” He observes

---

<sup>85</sup>The complete title is: *Récoltes et semailles : Réflexions et témoignage sur un passé de mathématicien* (Harvesting and Sowing: Reflections and testimony on the past of a mathematician). This is a long manuscript by Grothendieck in which he meditates on his past as a mathematician and where he presents without any compliance his vision of the mathematical milieu in which he evolved, especially the decline in morals, for what concerns intellectual honesty.

there that it might be that the ultimate structure of space is “discrete,” and that the “continuous” representations which we make of it may be an oversimplification (which may turn out to be excessive on the long run...) of a more complex reality; that for human thought, “the continuous” was easier to grasp than “the discontinuous,” and that it serves us, subsequently, as an “approximation” for apprehending the discontinuous. This is an amazingly penetrating remark expressed by a mathematician, at a time where the Euclidean model of physical space was not questioned. In a strictly logical sense, it was rather the discontinuous which, traditionally, served as a technical mode of approaching the continuous.

Moreover, the mathematical developments of the last decades showed an even more intimate symbiosis between continuous and discontinuous structures, which was not yet visualized in the first half of this century. The fact remains that finding a model which is “satisfactory” (or, if need be, a collection of such models, linked in the most possible satisfying way...), whether the latter is “continuous,” “discrete,” or of a “mixed” nature – such a task will surely involve a great conceptual imagination, and a consummate intuition, in order to apprehend and disclose mathematical structures of a new type. This kind of imagination and “intuition” seems to me a rare object, not only among physicists (where Einstein and Schrödinger seem to be among the rare exceptions), but even among mathematicians (and I am talking in full knowledge of the facts).

To summarize, I foresee that the long-awaited renewal (if ever it comes...) will rather come from someone who has the soul of a mathematician, who is well informed about the great problems of physics, rather than from a physicist. But above all, we need a man having the “philosophical openness” required to grasp the crux of the problem. The latter is not at all of a technical nature, but it is a fundamental problem of “natural philosophy.”<sup>86</sup>

---

<sup>86</sup>Il doit y avoir déjà quinze ou vingt ans, en feuilletant le modeste volume constituant l’œuvre complète de Riemann, j’avais été frappé par une remarque de lui “en passant.” Il y fait observer qu’il se pourrait bien que la structure ultime de l’espace soit “discrète”, et que les représentations “continues” que nous en faisons constituent peut-être une simplification (excessive, peut-être, à la longue...) d’une réalité plus complexe ; que pour l’esprit humain, “le continu” était plus aisé à saisir que “le discontinu”, et qu’il nous sert, par suite, comme une “approximation” pour appréhender le discontinu. C’est là une remarque d’une pénétration surprenante dans la bouche d’un mathématicien, à un moment où le modèle euclidien de l’espace physique n’avait jamais été mis en cause ; au sens strictement logique, c’est plutôt le discontinu qui, traditionnellement, a servi comme mode d’approche technique vers le continu.

Les développements en mathématique des dernières décennies ont d’ailleurs montré une symbiose bien plus intime entre structures continues et discontinues, qu’on ne l’imaginait encore dans la première moitié de ce siècle. Toujours est-il que de trouver un modèle “satisfaisant” (ou, au besoin, un ensemble de tels modèles, se “raccordant” de façon aussi satisfaisante que possible...), que celui-ci soit “continu,” “discret” ou de nature “mixte” – un tel travail mettra en jeu sûrement une grande imagination conceptuelle, et un flair consommé pour appréhender et mettre à jour des structures mathématiques de type nouveau. Ce genre d’imagination ou de “flair” me semble chose rare, non seulement parmi les physiciens (où Einstein et Schrödinger semblent avoir été parmi les rares exceptions), mais même parmi les mathématiciens (et là je parle en pleine connaissance de cause).

## 9. DIFFERENTIAL GEOMETRY

In this section, we shall review some milestones in the history of differential geometry, concerning especially the study of geodesics and of curvature, from its beginning until the work of Riemann.

Differential geometry starts with the study of differentiable curves. The notion of curvature of planar curves already appears in works of Newton and of Johann I and Jakob Bernoulli. We mentioned, in §2, Johann Bernoulli's paper [24] published in 1718 on the isoperimetry problem in the plane.

Differential geometry is also the study of curvature. In 1744, Euler published a book [73] in which he sets the bases of the calculus of variations. The title is *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti* (Method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense). In that book, several applications of the new methods are presented, among them isoperimetry problems, the problem of finding the shape of the brachystochrone, the study of the catenoid, and that of finding geodesics between two points on a surface. With this work of Euler, the methods of differential calculus, more precisely those of finding minima and maxima, were suddenly generalized to the realm of a variable moving in an infinite dimensional space (even though the expression "infinite dimensional" was not there yet), namely, in the question of looking for curves of minimal length or satisfying other geometric properties, among all curves joining two points.

Riemann's differential geometry is essentially about curvature, actually, the curvature of space, and we must talk now about the history of curvature, which starts with curvature of curves and surfaces. The history starts again with Euler.

Volume II of Euler's *Introductio in analysin infinitorum* [61], published in 1748, is concerned with the differential geometry of space curves and surfaces. Curves are given there by parametric equations of the form  $x = x(t), y = y(t), z = z(t)$ , and surfaces by parametric equations of the form  $x = x(u, v), y = y(u, v), z = z(u, v)$ . It is possible that this is the first time where such a parametric representation of surfaces appears in print. About twenty years after the first edition of this treatise was published, Euler wrote a memoir entitled *Recherches sur la courbure des surfaces* (Researches on the curvature of surfaces) [92] (1767), another work which transformed the subject. The aim of this memoir was to introduce and study the curvature at a point on a surface. Euler's idea, which is very natural, was to introduce a notion of curvature at a point of a differentiable surface based on the curvature of curves that pass through that point. His intuition was that to understand curvature at a point of a surface, it suffices to study the curvature of curves that are intersections of that surface with Euclidean planes. Moreover, he showed that it is sufficient to consider the planes that

---

Pour résumer, je prévois que le renouvellement attendu (s'il doit encore venir ...) viendra plutôt d'un mathématicien dans l'âme, bien informé des grands problèmes de la physique, que d'un physicien. Mais surtout, il y faudra un homme ayant "l'ouverture philosophique" pour saisir le nœud du problème. Celui-ci n'est nullement de nature technique, mais bien un problème fondamental de "philosophie de la nature."



are perpendicular to the surface, that is, the planes containing the normal vector to the surface at that point. Each such curve has an osculating circle, and the collection of radii of these circles contains all the information about the curvature of the surface at that point. Furthermore, Euler proved that at any given point on the surface, the maximal curvature and the minimal curvature associated to the normal planes determine all the other normal curvatures. To be more precise, given a point on the surface and a tangent vector  $v$  at that point, let us call *normal curvature* through  $v$  the curvature of a curve obtained by intersecting the surface with a plane containing the vector  $v$  and the normal vector at that point. The *maximal* and *minimal normal curvature* at the given point are the maximum and minimum of the normal curvatures taken over all the normal planes at that point. Likewise, the *normal curvature radius* at the given point in the direction of the vector  $v$  is the curvature radius of the associated curve. We have a similar notion of *maximal* and *minimal normal curvature radii* at the given point. Euler showed that the directions of the planes that realize these extremal curvatures (except in very special cases) are orthogonal to each other, and he proved that at a given point, if  $\rho_1$  and  $\rho_2$  are the maximal and minimal normal curvature radii respectively, then the normal curvature radius  $\rho$  of the normal section through an arbitrary tangent vector  $v$  is given by the equation

$$\rho = \frac{2\rho_1\rho_2}{(\rho_1 + \rho_2) - (\rho_1 - \rho_2)\cos(2\varphi)},$$

where  $\varphi$  is the angle between  $v$  and the tangent vector to the normal plane with maximal curvature radius.

It is usual to write Euler's equation in the following form:

$$\frac{1}{\rho} = \frac{\cos^2 \varphi}{\rho_1} + \frac{\sin^2 \varphi}{\rho_2}.$$

We note that  $\rho_1$  and  $\rho_2$  may also take negative values and that Euler's equation has also a meaning when  $\rho_1$  or  $\rho_2$  is infinite; in the latter case, the curvature  $\frac{1}{\rho}$  is zero for all  $\varphi$ . There is a classical local classification of differentiable surfaces at a point in terms of the signs of  $\rho_1$  and  $\rho_2$ .

Euler writes ([92], Réflexion VI, p. 143):

Thus, the judgement of the curvature of surfaces, however complicated it seems at the beginning, is reduced for each point to the knowledge of two osculating radii, one of which is the largest and the other the smallest at that element. These two objects determine entirely the nature of the curvature, displaying for us the curvature of all the possible sections that are perpendicular to the proposed element.<sup>87</sup>

There are other memoirs by Euler on the differential geometry of surfaces. We mention his *Solutio trium problematum difficiliorum ad methodum tangentium inversam pertinentium* (Solution of three rather difficult problems

---

<sup>87</sup>Ainsi le jugement sur la courbure des surface, quelque compliqué qu'il ait paru au commencement, se réduit pour chaque élément à la connaissance de deux rayons osculateurs, dont l'un est le plus grand et l'autre le plus petit dans cet élément ; ces deux choses déterminent entièrement la nature de la courbure en nous découvrant la courbure de toutes les sections possibles qui sont perpendiculaires sur l'élément proposé.

pertaining to the inverse method of tangents) [75] published in 1826, that is, several years after Euler's death. In this memoir, Euler addresses "inverse problems" in differential geometry, e.g., to reconstruct curves from information on their tangents. We also mention Euler's *De solidis quorum superficiem in planum explicare licet* (On solids whose entire surface can be unfolded onto a plane) [96] in which for the first time the notion of a surface developable on the plane is introduced. This notion was thoroughly investigated in the later works of Monge and his students that we mention below, and much later by Eugenio Beltrami. This paper [96] on developable surfaces also addresses a so-called "inverse problem," namely, the question of giving a characterization of the surfaces that are developable.

Gauss's development of differential geometry attained a high degree of perfection in the 1820s, motivated by his works on geography, astronomy and geodesy. He was probably the first to formulate the question of finding the properties of surfaces which are independent of their embedding in 3-space. After Euler in [92] highlighted the role of the maximal and minimal curvature at a point of a surface, it was Gauss's idea to take the product of these quantities as a measure of the curvature at that point, and to show that the result is an isometry invariant of the surface. This is (expressed in modern terms) the content of Gauss's *Theorema egregium*. The result is contained in Gauss's *Disquisitiones generales circa superficies curvas* (General investigations on curved surfaces) [125] (1828). In the abstract he wrote for his *Disquisitiones* [125] (1828), Gauss writes (translation from [125] p. 48):

[...] These theorems lead to the consideration of the theory of curved surfaces from a new point of view, where a wide and still wholly uncultivated field is open to investigation. If we consider surfaces not as boundaries of bodies, but as bodies of which one dimension vanishes, and if at the same time we conceive them as flexible but not extensible, we see that two essentially different relations must be distinguished, namely, on the one hand, those that presuppose a definite form of the surface in space; on the other hand, those that are independent of the various forms that the surface may assume. This discussion is concerned with the latter. In accordance with what has been said, the measure of curvature belongs to this case. But it is easily seen that the consideration of figures constructed upon the surface, their angles, their areas and their integral curvatures, the joining of the points by means of shortest lines, and the like, also belong to this case. All such investigations must start from this, that the very nature of the curved surface is given by means of the expression of any linear element in the form  $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$ .

Gauss, in the same memoir, used the so-called Gauss map from a surface to the unit sphere, defined by sending the normal unit normal vector at a point to the corresponding point on the sphere and showing that one can recover the curvature of the surface, which he had defined as the product of the minimal and maximal curvatures. The curvature, using the Gauss map, is obtained by taking the ratio of the area of the image of the Gauss map

by the area of the surface (the definition of the curvature at a point needs a passage to the infinitesimal level).

Riemann's most important articles on differential geometry are his habilitation lecture *Über die Hypothesen welche der Geometrie zu Grunde liegen* (1854) which is mentioned several times in the present paper and in the other chapters of this book, and the *Commentatio mathematica, qua respondere tentatur quaestioni ab Ill<sup>ma</sup> Academia Parisiensi propositae* (A mathematical note attempting to answer a question posed by the distinguished Paris Academy), a memoir which he wrote in 1861, at the occasion of his participation to a competition prize set by the Paris Academy of Sciences, and which we consider in some detail in Chapter 6 of the present volume [191]. These two memoirs are unusual for opposite reasons: the first one lacks of formulae, and the second one is full of them. The second memoir contains the explicit form of the object which we call today the Riemann tensor.

In his habilitation lecture, Riemann makes a clear reference to Gauss's *Disquisitiones* as one of his major sources of inspiration, a work which he includes however in a broad philosophical discussion on magnitude, measure, quantity and the possibility of geometric representation. It is always good to read Riemann:

Having constructed the notion of a manifoldness of  $n$  dimensions, and found that its true character consists in the property that the determination of position in it may be reduced to  $n$  determinations of magnitude, we come to the second of the problems proposed above, viz. the study of the measure-relations of which such a manifoldness is capable, and of the conditions which suffice to determine them. These measure-relations can only be studied in abstract notions of quantity, and their dependence on one another can only be represented by formulæ. On certain assumptions, however, they are decomposable into relations which, taken separately, are capable of geometric representation; and thus it becomes possible to express geometrically the calculated results. In this way, to come to solid ground, we cannot, it is true, avoid abstract considerations in our formulæ, but at least the results of calculation may subsequently be presented in a geometric form. The foundations of these two parts of the question are established in the celebrated memoir of Gauss, *Disquisitiones generales circa superficies curvas*.

For the case of surfaces, he writes:

In the idea of surfaces, together with the intrinsic measure-relations in which only the length of lines on the surfaces is considered, there is always mixed up the position of points lying out of the surface. We may, however, abstract from external relations if we consider such deformations as leave unaltered the length of lines – i.e., if we regard the surface as bent in any way without stretching, and treat all surfaces so related to each other as equivalent. Thus, for example, any cylindrical or conical surface counts as equivalent to a plane, since it may be made out of one by mere bending, in which the intrinsic measure-relations remain, and all theorems about a plane – therefore the whole of planimetry – retain their validity. On the other hand they count as essentially different from the sphere,

which cannot be changed into a plane without stretching. According to our previous investigation the intrinsic measure-relations of a twofold extent in which the line-element may be expressed as the square root of a quadric differential, which is the case with surfaces, are characterized by the total curvature. Now this quantity in the case of surfaces is capable of a visible interpretation, viz., it is the product of the two curvatures of the surface, or multiplied by the area of a small geodesic triangle, it is equal to the spherical excess of the same. The first definition assumes the proposition that the product of the two radii of curvature is unaltered by mere bending; the second, that in the same place the area of a small triangle is proportional to its spherical excess. To give an intelligible meaning to the curvature of an  $n$ -fold extent at a given point and in a given surface-direction through it, we must start from the fact that a geodesic proceeding from a point is entirely determined when its initial direction is given. According to this we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface-direction; this surface has at the given point a definite curvature, which is also the curvature of the  $n$ -fold continuum at the given point in the given surface-direction.

In these passages, Riemann summarizes his ideas on the general metric on (what became known later on as) an  $n$ -dimensional differentiable manifold, defined by a quadratic form on each tangent space, a broad generalization of Gauss's investigations on surfaces in which the quadratic form determines the metric, permits to calculate distances, angles and the curvature at any point. The curvature is the product of two quantities and is invariant by bending. The quadratic form represents the square of the line element. With these tools, one can study geodesic triangles on surfaces, prove that a geodesic is determined by its initial vector, generalize these matters to immersed surfaces, etc.

One may also include in Riemann's list of works on differential geometry his two papers on minimal surfaces [221] and [222]. They are reviewed in Chapter 5 of the present volume, [260]. We also note that in his doctoral dissertation defended in 1880 in Paris and written under the supervision of Bonnet (cf. [189]), Niewenglowski explains that Riemann, in his work on minimal surfaces, was inspired by Bonnet; cf. also the comments in Chapter 8 of the present volume [193]. Again, minimal surfaces first appear in the work of Euler (cf. Chapter V, §47 of Euler's first treatise on the calculus of variations, [73]).

In a longer survey on Riemann's predecessors in the field of differential geometry, one would have analyzed the works of several French mathematicians who stand between Euler and Gauss, in particular Gaspard Monge (1764–1818), Jean-Baptiste Meusnier (1754–1793), Siméon-Denis Poisson (1781–1840), Charles Dupin (1784–1873), Olinde Rodrigues (1795–1851) and there are others. We only mention some of these works.

Monge, who was the founder of a famous school on projective and differential geometry, continued Euler's work on developable surfaces, cf. [185] and [186]. He worked in particular with two orthogonal line fields that are defined by Euler's minimal and maximal directions of curvature radii, and

he coined the expression *umbilical point* for points where the two curvature radii have the same value. (On the sphere, every point is umbilical.) Monge expressed several times in his writings his debt to Euler. In [186], he writes:

Having resumed this matter, at the occasion of a memoir that Mr. Euler gave in the 1771 volume of the Petersburg Academy on developable surfaces, in which this famous geometer gives formulae to determine whether a given surface may or may not be applied onto a plane, I reached results on the same subject which seem to me much simpler, and whose usage is much easier.<sup>88</sup>

Poisson was a student of Lagrange and Laplace. He wrote a memoir entitled *Mémoire sur la courbure des surfaces* (Memoir on the curvature of surfaces) [206] (1832) in which he studied singular points of the curvature. We mention by the way that there are several points in the work of Poisson that are related to Riemann's works, in particular, concerning definite integrals and Fourier series.

Meusnier was a student of Monge. He gave a formula for the curvature of a curve obtained by intersecting a surface by a non-normal section, in terms of that of the normal sections at the given point that were considered by Euler. His paper on the subject carries almost the same title as Euler's paper [92], *Mémoire sur la courbure des surfaces* [182] (1785). In this paper, Meusnier writes (p. 478):

Mr. Euler treated the same matter in a very beautiful memoir, printed in 1760 among those of the Berlin Academy. This famous geometer addresses the question in a manner which is different from the one which we just presented. He makes the curvature of an element of the surface dependent on that of the various sections that one can perform on it by cutting it with planes.<sup>89</sup>

Meusnier's work is surveyed by Darboux in the paper [49].

Dupin is mostly known for the so-called Dupin indicatrix, a geometric device which characterizes the shape of a surface at a given point and which turned out to be related to the Gaussian curvature at that point. His famous work on geometry bears the title *Développements de géométrie, avec des applications à la stabilité des vaisseaux, aux déblais et remblais, au défilement, à l'optique, etc. pour faire suite à la géométrie descriptive et à la géométrie analytique de M. Monge : Théorie*. (Developments in geometry, with applications to the stability of vessels, cuts and fills, scrolling, optics, etc. as a sequel to the descriptive and analytic geometry of Mr. Monge: Theory) [58], 1813.

---

<sup>88</sup>Ayant repris cette matière, à l'occasion d'un mémoire que M. Euler a donné dans le volume 1771 de l'Académie de Pétersbourg, sur les surfaces développables, et dans lequel cet illustre géomètre donne des formules pour reconnaître si une surface courbe proposée jouit ou non de la propriété de pouvoir être appliquée sur un plan, je suis parvenu à des résultats qui me semblent beaucoup plus simples, et d'un usage bien plus facile pour le même sujet.

<sup>89</sup>M. Euler a traité la même matière dans un fort beau mémoire, imprimé en 1760 parmi ceux de l'Académie de Berlin. Cet illustre géomètre envisage la question d'une manière différente de celle que nous venons d'exposer ; il fait dépendre la courbure d'un élément de surface de celle des différentes sections qu'on peut y faire en le coupant par des plans.

Rodrigues introduced, before Gauss, the Gauss map, and showed that at a given point the ratio of the area of the image to the area on the surface is equal (at the infinitesimal level) to the product of the two principal curvatures (those defined by Euler), cf. [236]. This result pre-dates that of Gauss, but the fact that curvature is an isometry invariant is however absent from Rodrigues' work.

Finally, it is fair to mention that Riemann's work in high dimensions was prepared by works of other mathematicians done in higher dimensions, and we mention in particular Jacobi [146] on the reduction of pairs of quadratic forms, Grassmann on higher-dimensional linear algebra [132], and Cayley [41] on higher-dimensional analytic geometry.

## 10. TRIGONOMETRIC SERIES

Riemann's habilitation dissertation (*Habilitationsschrift*) is concerned with trigonometric series. Its title is *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (On the representability of a function by a trigonometric series) [216]. The dissertation is divided into two parts. The first part is a survey of the history of the problem of representing by a trigonometric series a function which is arbitrary, but which – according to Riemann's words at the beginning of his memoir – is “given graphically.” It is important to note the last fact, since, for instance, a function which is discontinuous at a dense set of points cannot be given graphically. Dealing with the most general functions is part of the subject of the second part of Riemann's memoir.

We shall report on the historical part of Riemann's memoir. It involves several mathematicians, in particular Euler, although not directly his work on trigonometric series. We therefore note right away that Euler uses trigonometric series in his two memoirs on astronomy *Recherches sur la question des inégalités du mouvement de Saturne et de Jupiter* (Researches on the question of the inequalities in the motion of Saturn and Jupiter) [77], and *De motu corporum coelestium a viribus quibuscunque perturbato* (On the movement of celestial bodies perturbed by any number of forces) [86], both presented for competitions proposed by the Paris Académie des Sciences, in 1748 and 1756 respectively. The two memoirs are analyzed in the paper [130] in which the authors show that Euler was much more concerned with convergence of series than what is claimed in several books and articles on his work.

The starting point of Riemann's historical survey is the controversy between Euler and d'Alembert which originated in the publication in 1747 of a memoir by the latter, *Recherches sur la courbe que forme une corde tendue mise en vibrations* (Researches on the curve formed by a taut string subject to vibrations) [8]. In this memoir, d'Alembert gave the partial differential equation that represents the motion of a point on a vibrating string subject to small vibrations:

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}.$$

In this equation,  $t$  represents time,  $\alpha$  is a constant and  $y = y(t)$  is the oscillation of the string at a point whose coordinate is  $x$  along the string.

The main problem, after this discovery, was to characterize the functions that are solutions of that equation.

One first obvious (but wrong) guess for a necessary condition on the solution is that it should be order-two differentiable. However, it was soon realized that this condition is too restrictive. Understanding the exact nature of the solutions of the vibrating string equation led to a fierce controversy which involved some of the most brilliant mathematicians of the eighteenth century. Among them are Euler, d'Alembert, Lagrange and Daniel Bernoulli. There is a large amount of primary literature concerned with this debate, including several memoirs by each of these mathematicians and the correspondence between them. Let us recall a few points of the history of that controversy.

In the memoir [8], d'Alembert wrote that the general solution to the wave equation is a function of the form

$$y(x, t) = \frac{1}{2} (\phi(x + \alpha t) + \phi(x - \alpha t)),$$

where  $\phi$  is an “arbitrary” periodic function whose period is the double of the length of the string. The problem was to give a meaning to the adjective “arbitrary.”

At the beginning of his memoir [8], d'Alembert declares that he will show that the problem admits infinitely many other solutions than the usual one represented by the sine curve (which he calls *compagne de la cycloïde allongée*). But from his point of view (like from Euler's one) the only acceptable functions were those given by a formula (functions which, as we recall, were termed “analytic” by Euler). The reason is that it was considered that the powerful methods of analysis can be applied only to such functions.

Euler published an article in the next volume of the *Memoirs* (1748) [68] in which he gave an exposition of d'Alembert's results but where he expressed a different point of view on the nature of the solution of the wave equation.<sup>90</sup> He claimed that a solution is not necessarily given by a formula, but that it might be “discontinuous” in the sense that it could be a concatenation of functions defined on smaller intervals on which the restriction of the function is defined by formulae. We already mentioned this notion of “discontinuity” in §2 of the present paper. His assertion was supported by physical evidence, more precisely, by the fact that the initial form of a string, in a musical instrument that is pinched in the usual manner, is a concatenation of two segments with a corner at their intersection. More than that, Euler pointed out that one may give an arbitrary initial form to the string, and therefore the solution may be arbitrary. Euler's paper introduced some doubts concerning the assertion made by d'Alembert that the solution must be twice differentiable and given by a formula. D'Alembert, who disagreed with Euler's claim, published the following year a memoir in which he confirmed his initial ideas. The rest of the controversy on the notion of function is very interesting and there are several articles on this

---

<sup>90</sup>Euler published a Latin and a French version of his memoir, which appeared in the years 1749 and 1748 respectively. (The title of the French version, *Sur la vibration des cordes, traduit du latin*, although it was published first, shows that it was written after the Latin one.)

subject. We recommend in particular the introduction, by Youschkevitch and Taton, of Volume V of Series IV A of Euler's *Opera omnia* containing Euler's correspondence with Clairaut, d'Alembert and Lagrange [262].

The difficulty of defining a general notion of function is never too much emphasized. We mention in this respect that in 1787, that is, four years after Euler's death, the Academy of Sciences of Saint-Petersburg proposed, as a competition question, to write an essay on the nature of an arbitrary function.<sup>91</sup> The prize went to the Alsatian mathematician Louis-François-Antoine Arbogast (1759–1803), who, in his *Mémoire sur la nature des fonctions arbitraires qui entrent dans les intégrales des équations aux différentielles partielles* (Memoir on the nature of arbitrary functions that appear in the integrals of partial differential equations), [17] (1791), adopted Euler's point of view: he accepted discontinuous functions in the sense Euler defined them, as solutions of partial differential equations. It is interesting to note that in the description of that problem, the Academy starts with the physical problem of vibrating strings:

The problem of the vibrating strings is without doubt one of the most famous problems of applied mathematics. The most celebrated geometers of our time, who solved it, have argued on the legitimacy of their solution, without ever being able to convince each other. It is not that it is difficult to reduce the problem itself to pure analysis, but as it has given the first occasion to treat three-variable differential equations which give, by integrating them, arbitrary and varying functions, the important question which divided the points of view of these great men is whether these functions are entirely arbitrary, whether they represent all the arbitrary curves and surfaces, formed by a voluntary motion of the hand, or whether they include only those that are comprised under an algebraic or transcendental equation. Besides the fact that on that decision depends the way of terminating the dispute on vibrating strings, the same question on the nature of arbitrary functions re-emerges each time an arbitrary problem leads to differential equations with three or more variables: this happens even very often, not only when we treat subjects of sublime mechanics, but most of all in the theory of fluid motion: in such a way that one cannot rigorously sustain that such a problem has been solved before setting precisely the nature of of arbitrary functions. The Academy invites then all the geometers to decide:

*Whether arbitrary functions, to which we are led by integrating equations with one or several variables, represent arbitrary curves or surfaces, either algebraic or transcendental, or mechanical, discontinuous or produced by a voluntary motion of the hand; or whether these functions only contain continuous curves represented by an algebraic or transcendental equation.*<sup>92</sup>

<sup>91</sup>*Histoire de l'Académie Impériale des Sciences, année 1787*, p. 4.

<sup>92</sup>Le problème des cordes vibrantes est sans contredit un des plus fameux problèmes de la mathématique appliquée. Les plus célèbres géomètres de notre temps, qui l'ont résolu, se sont disputés sur la légitimité de leurs solutions, sans avoir jamais pu se convaincre l'un l'autre. Ce n'est pas que le problème en lui-même ne soit pas facilement réduit à l'analyse pure ; mais comme il a été le premier qui ait donné occasion de traiter des équations différentielles à trois variables, par l'intégration desquelles on parvient à des



We now come to the problem of trigonometric series.

Brook Taylor, in his 1713 memoir entitled *De motu nervi tensi* (On the motion of a tense string) [243] (cf. also his *Methodus incrementorum directa et inversa* (Direct and Indirect Methods of Incrementation), [244] (first edition 1715), showed that the vibration problem admits as a solution the sine and cosine functions. For several reasons which we shall mention below, it was tempting to conjecture then that the general solution of the problem is obtained by taking an infinite sum of trigonometric functions. This was done by Daniel Bernoulli (1700–1782).

In 1753, Bernoulli wrote a memoir on the vibration of strings. Bernoulli had already thought about this question for several years. In his approach to it, like in the other physical problems he considered, Bernoulli was an adept of Leibniz' calculus, rather than Euler's geometric methods (which were adopted by d'Alembert). As a physicist, the mathematical notion of function was not a central theme in his research, and from his point of view, the function representing the solution of the question was simply identified with the shape of the vibrating string. While Taylor had considered each trigonometric solution individually, that is, he noticed that functions of the form

$$y(x, t) = \sin \frac{n\pi x}{l} \cos \frac{n\pi \alpha t}{l}$$

are solutions of the wave equation, Bernoulli stated that the general solution was an infinite sum of such functions. Thus, he added to the debate the question of the convergence of trigonometric series. In the meanwhile, d'Alembert published a first supplement to his memoir *Sur les vibrations des cordes sonores* (On the vibration of sonorous strings) [4] in which, referring to Daniel Bernoulli's work, he writes:

The question is not to *conjecture*, but to *prove*, and it would be dangerous (although, to tell the truth, this misfortune is unlikely to happen) that this kind of proof which is so odd enters into geometry. The only thing which seems surprising is that such reasonings are used in way of a proof by a famous mathematician.<sup>93</sup>

---

fonctions arbitraires et variables, la question importante qui partagea les avis de ces grands hommes fut : si ces fonctions sont entièrement arbitraires ? si elles représentent toutes les courbes et surfaces quelconques, formées par un mouvement volontaire de la main ? ou si elles ne renferment que celles qui sont comprises sous une équation soit algébrique soit transcendante ? Outre que c'est de cette décision que dépend le moyen de terminer cette dispute sur les cordes vibrantes, la même question sur la nature des fonctions arbitraires renaît toutes les fois qu'un problème quelconque conduit à des équations différentielles à trois et plusieurs variables : ce qui arrive même bien souvent, non seulement lorsqu'on traite des sujets de la mécanique sublime, mais surtout des mouvements des fluides : de sorte qu'on ne saurait soutenir rigoureusement qu'un pareil problème ait été résolu, avant qu'on ait fixé exactement la nature des fonctions arbitraires. L'Académie invite donc tous les géomètres de décider:

*Si les fonctions arbitraires, auxquelles on parvient par l'intermédiaire des équations à trois ou plusieurs variables, représentent des courbes ou surfaces quelconques, soit algébriques soit transcendantes, soit mécaniques, discontinues, ou produites par un mouvement volontaire de la main ; ou si ces fonctions renferment seulement des courbes continues représentées par une équation algébrique ou transcendante ?*

<sup>93</sup>Il ne s'agit pas de *conjecturer*, mais de *démontrer*, et il serait dangereux (quoi qu'à la vérité ce malheur soit peu à craindre) qu'un genre de démonstration si singulier

After the publication of Bernoulli's memoir, Euler wrote a new memoir in which he generalizes Bernoulli's result, *Remarques sur les mémoires précédents de M. Bernoulli* (Remarks on the preceding memoirs by Mr. Bernoulli) [104] (1755). He also confirms his own intuition that a solution of the vibrating string equation may be an arbitrary function. Today, we know that, in some sense, the solution he proposed is identical to that of Bernoulli, but the relation between trigonometric series and arbitrary functions was not yet discovered.

In his memoir, Euler starts by declaring that, without any doubt, Bernoulli developed the theory of formation of sound infinitely better than any other scientist before him, that his predecessors stopped at the mechanical determination of the motion of a tight string without any thorough investigation of the nature of sound, and that it was still not understood how a single string can emit several sounds at the same time. He then expresses his doubts about the fact that Bernoulli's infinite series of sines could be the general solution of the problem. He writes that it is impossible for the curve made by a vibrating string to be constituted by an infinite number of trochoids (which is the name he used for the sine curves). He declares that there are infinitely many curves that are not included in that solution.

The solution of this problem was given by Fourier and completed by Dirichlet and Riemann, in the following century, as we shall discuss below. In the same paper, Euler insists on the fact that the general solution of the equation of the vibrating string cannot be given by a formula. He mentions his own conflict with d'Alembert, saying that he wishes very much that the latter explains why he is mistaken. Based on partial differential calculus, Euler gives a new explanation of the fact that by varying the initial shape of a string, any function becomes admissible as a solution of the problem.

Between November 1, 1759 and the end of the same year, Euler presented three memoirs, [100], [101] and [102], on the propagation of sound. In these memoirs, he studies respectively the propagation in one, two and three dimensions. The differential equations that describe this propagation are the same as those which describe the vibration of strings. Euler mentions the limitations of the works of "Taylor, Bernoulli and some others." Despite the fact that the debate on the vibrating string had already lasted many years, the relation between the scientists working on that subject was still very tense.

In a later memoir, titled *Mémoire sur les vibrations des cordes d'une épaisseur inégale* (Memoir of the vibration of strings of uneven width) [25] (1767), Bernoulli gave an additional reason for the use of an infinite sum (p. 283):

When a string makes several vibrations at the same time, of whatever number, and in whatever order, the absolute curvature will always be expressed by the general equation

$$y = \alpha \sin \frac{x}{l} \pi + \beta \sin \frac{2x}{l} \pi + \gamma \sin \frac{3x}{l} \pi + \text{etc.}$$

---

s'introduisît en Géométrie. Ce qui pourra seulement paraître surprenant, c'est que de pareils raisonnements soient employés comme démonstratifs par un mathématicien célèbre.

and since the number of arbitrary coefficients is infinite, one can make the curve pass by whatever number of points of positions that we wish, which indicates that all the curves belong to this case, provided we do not oppose the hypotheses. And it would be opposing them if we do not treat the quantities  $y$ ,  $dy$  and  $ddy$  as infinitely smaller, at every point of the curve, than the quantities  $x$ ,  $dx$  and  $\frac{dx^2}{l}$ .<sup>94</sup>

The interested reader may skim the volume of Euler's collected works containing the correspondence between Euler and Lagrange, [109], and the volume containing the correspondence between Lagrange and d'Alembert, [168], not only in order to understand more deeply this multi-faced controversy, but also in order to feel the cultural and scientific atmosphere in Europe during that period. Let us quote, as examples, two excerpts related to the discussion around the solution of the wave equation. In a letter to Lagrange, dated Octobre 1759, Euler writes:

I was pleased to learn that you agree with my solution relative to the vibrating strings, which d'Alembert tried hard to refute using various sophisms, for the only reason that he did not propose it himself. He announced that he will publish an overwhelming proof of it; I don't know whether he did it. He thinks he will be able to impress people by his half-scholar eloquence. I doubt that he can seriously play such a role, unless he is profoundly blinded by self-esteem. He wanted to insert in our *Memoirs*, not a proof, but a simple declaration according to which my solution was very deficient. On my side, I proposed a new proof which has all the required rigor.<sup>95</sup>

In a letter to d'Alembert, dated March 20, 1765, Lagrange writes:

Concerning [our discussion] on vibrating strings, it is now reduced to a point which, it seems to me, escapes any analysis. Moreover, I found, by a completely direct way, that if we admit in the initial figure the conditions that you ask, the solution reduces to the one of Mr. Bernoulli, namely,  $y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \dots$ , and it is difficult for me to believe that this is the only one that can be

---

<sup>94</sup>Lorsque la corde fait à la fois plusieurs espèces de vibration, quel qu'en soit le nombre, et de quelque ordre qu'elles soient, la courbure absolue sera toujours exprimée par cette équation générale

$$y = \alpha \sin \frac{x}{l} \pi + \beta \sin \frac{2x}{l} \pi + \gamma \sin \frac{3x}{l} \pi + \text{etc.}$$

et comme le nombre des coefficients arbitraires est infini, on peut faire passer la courbe par tant de points donnés de position qu'on voudra, ce qui marque que toutes les courbes se trouvent dans ce cas, pourvu qu'on ne fasse pas violence aux hypothèses. Et ce serait leur faire violence, si on ne faisait pas les quantités  $y$ ,  $dy$  et  $ddy$  comme infiniment plus petites dans tous les points de la courbe, que les quantités  $x$ ,  $dx$  et  $\frac{dx^2}{l}$ .

<sup>95</sup>J'ai appris avec plaisir que vous approuviez ma solution relative aux cordes vibrantes, que d'Alembert s'est efforcé de réfuter par divers sophismes, et ceci pour l'unique raison qu'il ne l'a pas proposée lui-même. Il a annoncé qu'il en publierait une accablante réfutation ; j'ignore s'il l'a fait. Il croit qu'il pourra jeter de la poudre aux yeux avec son éloquence de demi-savant. Je doute qu'il joue ce rôle sérieusement, à moins qu'il ne soit profondément aveuglé par l'amour-propre. Il a voulu insérer dans nos Mémoires non une démonstration, mais une simple déclaration suivant laquelle ma solution était très défectueuse ; pour ma part, j'ai proposé une nouvelle démonstration possédant toute la rigueur voulue.

found in nature. Besides, the phenomena of sound propagation can be explained only if we admit discontinuous functions, as I proved in my second dissertation.<sup>96</sup>

Let us also quote Nicolaus Fuss, the famous biographer of Euler,<sup>97</sup> from his *Éloge* [118]:

The controversy between Messrs. Euler, d'Alembert & Bernoulli regarding the motion of the vibrating strings can be of interest only to professional geometers. Mr. D. Bernoulli, who was the first to develop the physical part which concerns the production of sound generated by this motion, thought that Taylor's solution was sufficient to explain it. Messrs. Euler and d'Alembert, who had exhausted, in this difficult matter, everything exquisite and profound that an analytic mind may possess, showed that the solution of Mr. Bernoulli, extracted from Taylor's Trochoids, is not general, and that it is even deficient. This controversy, which lasted a long time, with all the consideration that such famous men owe to each other, gave rise to a quantity of excellent memoirs; it really ended only at the death of Bernoulli.<sup>98</sup>

D'Alembert eventually accepted functions that are discontinuous (in the sense of Euler) as solutions of partial differential equations; cf. his 1780 memoir entitled *Sur les fonctions discontinues* (On discontinuous functions), published in [3] (t. VIII, §VI) in which he formulates a *Règle sur les fonctions discontinues qui peuvent entrer dans l'intégration des équations aux dérivées partielles* (Rule on discontinuous functions that may enter into the integration of partial differential equations). We refer the interested reader to the papers [262], [246], [261] and [148] for more on the history of the subject.

The confirmation of Bernoulli's conjecture followed from Fourier's manuscript *Théorie de la propagation de la chaleur dans les solides* (Theory of

---

<sup>96</sup>à l'égard de [notre discussion] sur les cordes vibrantes, elle est maintenant réduite à un point qui échappe, ce me semble, à l'Analyse. Au reste, j'ai trouvé par une voie tout à fait directe qu'en admettant dans la figure initiale les conditions que vous y exigez, la solution se réduit à celle de M. Bernoulli, savoir :  $y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \dots$ , et j'ai peine à croire que celle-ci soit la seule qui puisse avoir lieu dans la nature. D'ailleurs, les phénomènes de la propagation du son ne peuvent s'expliquer qu'en admettant les fonctions discontinues, comme je l'ai prouvé dans ma seconde dissertation.

<sup>97</sup>Nicolaus Fuss (1755–1826) was first hired as Euler's secretary, then he became successively his favorite student, his closest friend, his collaborator and colleague at the Saint Petersburg Academy of Sciences, and eventually his grandson-in-law (the husband of Euler's grand-daughter Albertine).

<sup>98</sup>La controverse entre MM. Euler, d'Alembert & Bernoulli au sujet du mouvement des cordes vibrantes ne peut intéresser proprement que les Géomètres de profession. M. D. Bernoulli, qui fut le premier à en développer la partie physique qui regarde la formation du son engendré par ce mouvement, crut la solution de Taylor suffisante de l'expliquer. MM. Euler et d'Alembert, qui avaient épuisé, dans cette matière difficile, tout ce que l'esprit analytique a de sublime & de profond, firent voir que la solution de M. Bernoulli, tirée des Trochoïdes Tayloriennes, n'est pas générale, qu'elle est même insuffisante. Cette controverse qui a été continuée longtemps, avec tous les égards que des hommes aussi illustres se doivent mutuellement, a donné naissance à quantité d'excellents mémoires ; elle n'a fini proprement qu'à la mort de M. Bernoulli.

heat propagation in solids),<sup>99</sup> [116] read to the Academy in 1807, that is, twenty-five years after Bernoulli's death. The manuscript carries the subtitle: "Mémoire sur la propagation de la chaleur avec notes séparées sur cette propagation – sur la convergence des séries  $\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$  &c." (Memoir on the propagation of heat, with separate notes on that propagation – on the convergence of the series  $\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$  etc.). Let us quote an excerpt ([133] p. 183):

It follows from my researches on this object that the arbitrary functions, even discontinuous, can always be represented by the sine or cosine expansions of multiple arcs, and that the integrals which contain these developments are precisely as general as those where arbitrary functions of multiple arcs enter. A conclusion that the celebrated Euler has always rejected.<sup>100</sup>

In 1811, the Paris *Académie des sciences* proposed a competition whose title was: *Donner la théorie mathématique des lois de la propagation de la chaleur et comparer les résultats de cette théorie à des expériences exactes* (To give the mathematical theory of the laws of propagation of heat and to compare the results of this theory with exact experiences). Fourier submitted for the prize a very extensive work which included his 1807 manuscript. The jury of the competition consisted of Lagrange, Laplace, Maus, Haüy and Legendre. Darboux, in his review [48], quotes part of the report on the work of Fourier:

This piece contains genuine differential equations of heat transmission, either in the interior of bodies, or at their surface. And what is new in the subject, added to its importance, has led the Class to crown this treatise, while noting however that the manner with which the author arrives at his equation is not exempt of difficulties, and that his analysis, to integrate them, still leaves something to be desired, either relative to the generality, or even from the point of view of rigor.<sup>101</sup>

The sum of Fourier's work on the propagation of heat was collected in his masterpiece, *Théorie analytique de la chaleur* (Analytic theory of heat) [117], published in 1822. The following result is stated at the end of Chapter III of this memoir, as a summary of what has been done (art. 235, p. 258):

It follows from all that was proved in this section, concerning the series expansion of trigonometric functions, that if we propose a

---

<sup>99</sup>This memoir, read to the Academy 1807, was never published, until it was edited with comments by Grattan-Guinness, see [133].

<sup>100</sup>Il résulte de mes recherches sur cet objet que les fonctions arbitraires même discontinues peuvent toujours être représentées par les développements en sinus ou cosinus d'arcs multiples, et que les intégrales qui contiennent ces développements sont précisément aussi générales que celles où entrent les fonctions arbitraires d'arcs multiples. Conclusion que le célèbre Euler a toujours repoussée.

<sup>101</sup>Cette pièce renferme de véritables équations différentielles de la transmission de la chaleur, soit à l'intérieur des corps, soit à leur surface ; et la nouveauté du sujet, jointe à son importance, a déterminé la Classe à couronner cet Ouvrage, en observant cependant que la manière dont l'Auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du côté de la rigueur.

function  $fx$ , whose value is represented on a given interval, from  $x = 0$  to  $x = X$ , by the ordinate of a curves line drawn arbitrarily, one can always expand this function as a series which will contain only the sines, or the cosines, or the sines and the cosines of multiple arcs, or only the cosines of odd multiples.<sup>102</sup>

Trigonometric functions were essential for the solution of the heat equation, as they were for the wave equation at the time of Bernoulli. What is important for our topic here is that trigonometric series became an essential tool in the field of complex analysis, independently of the heat flow. Fourier writes in the same section (art. 235 p. 258):

We cannot entirely solve the fundamental questions of the theory of heat without reducing to this form the functions that represent the initial state of temperatures.

These trigonometric series, ordered according to the cosines or sines of the multiples of the arc, pertain to elementary analysis, like the series whose terms contain successive powers of the variable. The coefficients of the trigonometric series are definite areas, and those of power series are fractions given by differentiation, in which one also attributes to the variable a definite value.<sup>103</sup>

Fourier then summarizes several properties of these series, including the integral formulae for the coefficients, and he also states the following:

The series, ordered according to the cosines or the sines of the multiple arcs, are always convergent, that is, when we give to the variable an arbitrary non-imaginary value, the sum of the terms converges more and more to a unique and fixed limit, which is the value of the expanded function.<sup>104</sup>

It is interesting to recall that Fourier, in his *Théorie analytique de la chaleur* quotes Archimedes, Galileo and Newton, the three scientists mentioned by Riemann in the last part of his Habilitation lecture, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*. Fourier writes ([117], p. i–ii):

---

<sup>102</sup>Il résulte de tout ce qui a été démontré dans cette section, concernant le développement des fonctions en séries trigonométriques, que si l'on propose une fonction  $fx$ , dont la valeur est représentée dans un intervalle déterminé, depuis  $x = 0$  jusqu'à  $x = X$ , par l'ordonnée d'une ligne courbe tracée arbitrairement on pourra toujours développer cette fonction en une série qui ne contiendra que les sinus, ou les cosinus, ou les sinus et les cosinus des arcs multiples, ou les seuls cosinus des multiples impairs.

<sup>103</sup>On ne peut résoudre entièrement les questions fondamentales de la théorie de la chaleur, sans réduire à cette forme les fonctions qui représentent l'état initial des températures.

Ces séries trigonométriques, ordonnées selon les cosinus ou les sinus des multiples de l'arc, appartiennent à l'analyse élémentaire, comme les séries dont les termes contiennent les puissances successives de la variable. Les coefficients des séries trigonométriques sont des aires définies, et ceux des séries de puissances sont des fonctions données par la différenciation, et dans lesquelles on attribue aussi à la variable une valeur définie.

<sup>104</sup>Les séries ordonnées selon les cosinus ou les sinus des arcs multiples sont toujours convergentes, c'est-à-dire qu'en donnant à la variable une valeur quelconque non imaginaire, la somme des termes converge de plus en plus vers une seule limite fixe, qui est la valeur de la fonction développée.

The knowledge that the most ancient people could have acquired in rational mechanics did not reach us, and the history of that science, if we except the first theorems on harmony, does not go back further than the the discoveries of Archimedes. This great geometer explained the mathematical principles of the equilibrium of solids and of fluids. About eighteen centuries passed before Galileo, the first inventor of the dynamical theories, discovered the laws of motions of massive bodies. Newton encompassed in that new science all the system of the universe.<sup>105</sup>

Riemann claims in his memoir on trigonometric series that the question of finding conditions under which a function can be represented by a trigonometric series was completely settled in the work of Dirichlet, “for all cases that one encounters in nature. [...] The questions to which Dirichlet’s researches do not apply do not occur in nature.” We quote Dirichlet, from his 1829 memoir *Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données* (On the convergence of trigonometric series used to represent an arbitrary function between two given bounds) [55] (1829), in which he gives a solution to the convergence problem, and where he also mentions the work of Cauchy on the same problem. His paper starts as follows:

The series of sines and cosines, by means of which one can represent an arbitrary function on a given interval, enjoy among other remarkable properties the one of being convergent. This property has not escaped the attention of the famous geometer who opened a new field for the applications of analysis, introducing there the way of expressing the arbitrary functions which are our subject here. It is contained in the memoir containing his first researches on heat. But, to my knowledge, nobody up to now gave a general proof. I know only on this subject a work due to Mr. Cauchy and which is part of his *Mémoires de l’Académie des sciences de Paris pour l’année 1823*. The author of this work confesses himself that his proof fails for certain functions for which convergence is nevertheless indisputable. A careful examination of that memoir led me to the belief that the proof which is presented there is insufficient even for the cases to which the authors thinks it applies.<sup>106</sup>

---

<sup>105</sup>Les connaissances que les plus anciens peuples avaient pu acquérir dans la mécanique rationnelle ne nous sont pas parvenues, et l’histoire de cette science, si l’on excepte les premiers théorèmes sur l’harmonie, ne remonte point au-delà des découvertes d’Archimède. Ce grand géomètre expliqua les principes mathématiques de l’équilibre des solides et des fluides. Il s’écoula environ dix-huit siècles avant que Galilée, premier inventeur des théories dynamiques, découvrit les lois du mouvement des corps graves. Newton embrassa dans cette science nouvelle tout le système de l’univers.

<sup>106</sup>Les séries de sinus et de cosinus, au moyen desquelles on peut représenter une fonction arbitraire dans un intervalle donné, jouissent entre autres propriétés remarquables aussi de celles d’être convergentes. Cette propriété n’avait pas échappé au géomètre illustre qui a ouvert une nouvelle carrière aux applications de l’analyse, en y introduisant la manière d’exprimer les fonctions arbitraires dont il est question ; elle se trouve énoncée dans le Mémoire qui contient ses premières recherches sur la chaleur. Mais personne, que je sache, n’en a donné jusqu’à présent une démonstration générale. Je ne connais sur cet objet qu’un travail dû à M. Cauchy et qui fait partie des Mémoires de l’Académie des sciences de Paris

Riemann, after quoting the work of Dirichlet, gives two reasons for investigating the cases to which Dirichlet's methods do not apply directly. The first reason is that sorting out these questions will bring more clarity and precision to the principles of infinitesimal calculus. The second reason is that Fourier series will be useful not only in physics, but also in number theory. Riemann says that in this field, it is precisely the functions which Dirichlet did not consider that seem to be the most important.

The so-called Dirichlet conditions for a function defined on the interval  $[0, 2\pi]$  to have a Fourier trigonometric expansion is now classical. Picard, in [199] (p. 8) writes that Dirichlet's memoir on Fourier series remained a model of rigor. We conclude this section by quoting Riemann. Talking about Dirichlet's work on trigonometric series, he writes:

This work of Dirichlet gave a firm foundation to a great number of important analytic researches. Highlighting a point on which Euler was mistaken, he succeeded in clearing out a question that had occupied so many eminent geometers for more than seventy years.

Riemann, in the same memoir, developed his integration theory in order to build a general theory for Fourier series, in particular for functions which have an infinite number of discontinuity points. This is the subject of the next section.

## 11. INTEGRATION

The second part of Riemann's memoir on trigonometric functions [216] carries the title "On the notion of definite integral and on the scope of its applicability." The relation between integration and trigonometric series is based on Fourier's formulae which give the coefficients of a trigonometric series in the form of integrals. Riemann starts the second part of the memoir by formulating a question: "First of all, what do we mean by

$$\int_a^b f(x)dx?"$$

The rest of the memoir is the answer to this question.

We explained at length that one of the fundamental questions in eighteenth century mathematics was "*What is a function,*" and how this question led to a celebrated controversy. Riemann had to deal with the same question in his memoir on trigonometric series, more than a century after the controversy started, and he gave it the definitive answer. The problem in Riemann's memoir was addressed in a new context, namely, his integration theory, which was developed in a few pages at the end of that memoir. More particularly, the question became in that context: "*What are the functions that can be integrated?*" and in particular, whether the known functions were sufficient for the theory that became known as the Riemann integral or whether a new class of functions was needed.

---

pour l'année 1823. L'auteur de ce travail avoue lui-même que sa démonstration tombe en défaut pour certaines fonctions pour lesquelles la convergence est pourtant incontestable. Un examen attentif du mémoire cité m'a porté à croire que la démonstration qui y est exposée n'est pas même suffisante pour les cas auxquels l'auteur la croit applicable.



We recall that since Newton and Leibniz, and passing by Euler, integration was defined as an anti-derivative. Cauchy started an approach to integrals as limits of sums associated to partitions of the interval of definition, that is, sums of the form

$$\sum_{k=1}^{\infty} f(x_{k-1})(x_k - x_{k-1}),$$

cf. e.g. Cauchy's *Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal* (Summary of lectures on infinitesimal calculus given at the *École Royale Polytechnique*) (1823) [35]. In Cauchy's setting, the limit always exists because he considered only continuous functions. It was soon realized that the definition may apply to more general functions. Dirichlet, in his work on trigonometric functions, used Cauchy's theory applied to discontinuous functions. Riemann states in his memoir that Cauchy's integration theory involves some random definitions which cannot make it a universal theory.

Riemann answered the question of how far discontinuity is allowed. He was led to the most general functions, which he termed "integrable." In §VII, VIII and IX of his memoir, he applies his new integration theory to the problem of representing arbitrary functions by trigonometric series. The main results are stated in three theorems in §VIII, and the propositions concerning the representation of functions by trigonometric series are contained in §IX. §X and XI contain results on the behavior of the coefficients of a trigonometric series. The last sections (§XII and XIII) concern particular cases, more precisely, cases where the Fourier series is not convergent.

It is curious that Riemann mentions Cauchy several times in this memoir on trigonometric series, but he never refers to him in his dissertation on the theory of functions of a complex variable.

There is a section on the history of integration in Lebesgue's book [170]. In particular, Lebesgue summarizes Cauchy's theory, as well as an unpublished work of Dirichlet on the subject, which reached us through a description of Lipschitz ([170] p. 9). Dirichlet's work applies to functions with an infinite number of discontinuity points, but forming a non-dense subset. Riemann, using series, constructed functions to which the preceding techniques do not apply and which may still be integrated. These functions of Riemann do not have a graphical representation. We are far from Euler's "arbitrary drawable function" which, indeed, he thought exceeded the power of the calculus (by not being differentiable).

Chapter 2 of Lebesgue's treatise is a survey of the Riemann integral. This theory allows one to prove theorems such as the fact that a uniformly convergent integrable sequence of functions is an integrable function (Lebesgue p. 30), and that a uniformly convergent series of integrable functions may be integrated term by term. Lebesgue also mentions the work of Darboux, involving the notions of upper and lower limits. He then presents his own geometric theory (as opposed to the analytic theory of Riemann), based on set theory and measure theory. There are more comments on Lebesgue's integration theory in Chapter 8 of the present volume [193].

To conclude this section, let us mention that Riemann's ideas about the general notion of function in relation with integration theory underwent several developments in the twentieth century (one may think about the difficulties in the introduction of general measurable functions).

Riemann's memoir on trigonometric series was published 13 years after it was written. It was translated into French by Darboux and Houël.

It is interesting to note that trigonometric series are used in the proof of the so-called Poincaré lemma, a lemma which plays an essential role in the proof of the modern version of the Riemann–Roch theorem which is presented in Chapter 13 of the present volume.

## 12. CONCLUSION

In the preceding sections, we reviewed part of the historical origins of Riemann's mathematical works. One should write another article about the roots of his ideas in physics and philosophy. The intermingling between the old and new ideas of physics and philosophy is yet another subject. In this respect, and since the present book is also about relativity, we quote Kurt Gödel from his article *A remark about the relationship between relativity theory and idealistic philosophy* [128]. Talking about the insight that this theory brings into the nature of time, he writes:

In short, it seems that one obtains an unequivocal proof for the view of those philosophers who, like Parmenides, Kant and the modern idealists, deny the objectivity of change and consider change as an illusion or an appearance due to our special mode of perception.

It would be stating the obvious to say that mathematicians should read the works of mathematicians from the past, not only the recent past, but most of all the founders of the theories they are working on. Yet, very few do it. I would like to conclude the present paper by quoting some pre-eminent mathematicians who expressed themselves on this question. I start with Chebyshev.

We learn from his biographer in [208] that Chebyshev's thoroughly studied the works of Euler, Lagrange, Gauss, Abel, and other pre-eminent mathematicians. The biographer also writes that, in general, Chebyshev was not interested in reading the mathematical works of his contemporaries, considering that spending time on that would prevent him of having original ideas.

On the importance of reading the old masters, we quote again André Weil, from his 1978 ICM talk ([255] p. 235):

From my own experience I can testify about the value of suggestions found in Gauss and in Eisenstein. Kummer's congruences for Bernoulli numbers, after being regarded as little more than a curiosity for many years, have found a new life in the theory of  $p$ -adic and  $L$ -functions, and Fermat's ideas on the use of the infinite descent in the study of Diophantine equations of genus one have proved their worth in contemporary work on the same subject.

Among the more recent mathematicians, I would like to quote again Grothendieck. During his apprenticeship, like most of us, Grothendieck was

not encouraged to read ancient authors. He writes, in *Récoltes et semailles* (Chapter 2, § 2.10):

In the teaching I received from my elders, historical references were extremely rare, and I was nurtured, not by reading authors which were slightly ancient, nor even contemporary, but only through communication, face to face or through correspondence with others mathematicians, and starting with those who were older than me.<sup>107</sup>

In the same work, we read (Chapter 2, §2.5):

I personally feel that I belong to a lineage of mathematicians whose spontaneous mission and joy is to constantly construct new houses. [...] I am not strong in history, and if I were asked to give names of mathematicians in that lineage, I can think spontaneously of Galois and Riemann (in the past century) and Hilbert (at the beginning of the present century).<sup>108</sup>

Grothendieck's attitude towards mathematics and mathematicians changed drastically at the time he decided to quit the mathematical milieu, in 1970, twenty years after he obtained his first job, putting an end to an extraordinarily productive working life and to his relation with his contemporary mathematicians. One thing which is not usually mentioned about him is that his writings, during the period that followed, contain many references to mathematicians of the past, to whom Grothendieck expresses his debt, and among them stands Riemann. In his *Récoltes et semailles* [134], Grothendieck writes (Chap. 2, §2.5):

Most mathematicians are led to confine themselves in a conceptual framework, in a “*Universe*,” which is fixed once and for all – essentially, the one they found “ready-made” at the time they were students. They are like the heirs of a big and completely furnished beautiful house, with its living rooms, kitchens and workshops, its kitchen set and large equipment, with which there is, well, something to cook and to tinker. How this house was progressively constructed, over the generations, and why and how such and such tool (and not another) was conceived and shaped, why the rooms are fit out in such a manner here, and in another manner there – these are as many questions as these heirs will never think to ask. That is the “Universe,” the “given” in which we must live, that's it! Something which will seem great (and, most often, we are far from having discovered all the rooms), **familiar**<sup>109</sup> at the same time, and, most of all, **unchanging**.<sup>110</sup>

---

<sup>107</sup>Dans l'enseignement que j'ai reçu de mes aînés, les références historiques étaient rarissimes, et j'ai été nourri, non par la lecture d'auteurs tant soit peu anciens ni même contemporains, mais surtout par la communication, de vive voix ou par lettres interposées, avec d'autres mathématiciens, à commencer par mes aînés.

<sup>108</sup>Je me sens faire partie, quant à moi, de la lignée des mathématiciens dont la vocation spontanée et la joie est de construire sans cesse des maisons nouvelles. [...] Moi qui ne suis pas fort en histoire, si je devais donner des noms de mathématiciens dans cette lignée-là, il me vient spontanément ceux de Galois et de Riemann (au siècle dernier) et celui de Hilbert (au début du siècle présent).

<sup>109</sup>The emphasis is Grothendieck's.

<sup>110</sup>La plupart des mathématiciens sont portés à se cantonner dans un cadre conceptuel, dans un “*Univers*” fixé une fois pour toutes – celui, essentiellement, qu'ils ont trouvé “tout

We conclude with Grothendieck's reference to Riemann. In his *Sketch of a program*, [135], he writes (p. 240 of the English translation):<sup>111</sup>

Whereas in my research before 1970, my attention was systematically directed towards objects of maximal generality, in order to uncover a general language adequate for the world of algebraic geometry [...] here I was brought back, via objects so simple that a child learns them while playing, to the beginnings and origins of algebraic geometry, familiar to Riemann and his followers!

*Acknowledgements.* — I would like to thank Vincent Alberge, Jeremy Gray and Marie-Pascale Hautefeuille who read a preliminary version of this paper and suggested corrections.

---

fait” au moment où ils ont fait leurs études. Ils sont comme les héritiers d'une grande et belle maison toute installée, avec ses salles de séjour et ses cuisines et ses ateliers, et sa batterie de cuisine et un outillage à tout venant, avec lequel il y a, ma foi, de quoi cuisiner et bricoler. Comment cette maison s'est construite progressivement, au cours des générations, et pourquoi et comment ont été conçus et façonnés tels outils (et pas d'autres...), pourquoi les pièces sont aménagées de telle façon ici, et de telle autre là – voilà autant de questions que ces héritiers ne songeraient pas à se demander jamais. C'est ça “l'Univers”, le “donné” dans lequel il faut vivre, un point c'est tout ! Quelque chose qui paraît grand (et on est loin, le plus souvent, d'avoir fait le tour de toutes ses pièces), mais **familier** en même temps, et surtout: **immuable**.

<sup>111</sup>The English translation is by Lochak and Schneps.

Topic	Euler	Riemann
Functions of a complex variable	<ul style="list-style-type: none"> <li>• Introductio in analysin infinitorum (1748)</li> <li>• De repraesentatione superficiei sphaericae super plano (1777)</li> </ul>	<ul style="list-style-type: none"> <li>• Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (1851)</li> <li>• Theorie der Abel'schen Functionen (1857)</li> </ul>
Elliptic and Abelian integrals	<ul style="list-style-type: none"> <li>• Specimen de constructione aequationum differentialium sine indeterminatarum separatione (1738)</li> <li>• Observationes de comparatione arcuum curvarum irrectificibilium (1761)</li> <li>• De integratione aequationis differentialis <math>\frac{m dx}{\sqrt{1-x^4}} = \frac{n dy}{\sqrt{1-y^4}}</math> (1761)</li> </ul>	<ul style="list-style-type: none"> <li>• Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (1851)</li> <li>• Theorie der Abel'schen Functionen (1857)</li> <li>• Über das Verschwinden der <math>\vartheta</math>-Functionen (1857)</li> </ul>
Hypergeometric series	<ul style="list-style-type: none"> <li>• De summatione innumerabilium progressionum (1738)</li> <li>• Institutionum calculi integralis volumen secundum (1769)</li> <li>• Specimen transformationis singularis serierum (1778)</li> </ul>	<ul style="list-style-type: none"> <li>• Beiträge zur Theorie der durch die Gauss'sche Reihe <math>F(\alpha, \beta, \gamma, x)</math> darstellbaren Functionen (1857)</li> </ul>
The zeta function	<ul style="list-style-type: none"> <li>• Variarum observationes circa series infinitas (1744)</li> <li>• Remarques sur un beau rapport entre les series des puissances tant directes que reciproques (1749)</li> </ul>	<ul style="list-style-type: none"> <li>• Über die Anzahl der Primzahlen unter einer gegebenen Grösse (1859)</li> </ul>
Integration	<ul style="list-style-type: none"> <li>• Institutionum calculi integralis (1768–1770)</li> </ul>	<ul style="list-style-type: none"> <li>• Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe (1854)</li> </ul>
Space and philosophy of nature	<ul style="list-style-type: none"> <li>• Anleitung zur Naturlehre (1745)</li> <li>• Reflexions sur l'espace et le temps (1748)</li> </ul>	<ul style="list-style-type: none"> <li>• Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (1851)</li> </ul>
Topology	<ul style="list-style-type: none"> <li>• Solutio problematis ad geometriam situs pertinentis (1741)</li> <li>• Elementa doctrinae solidorum (1758)</li> <li>• Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusasunt praedita (1758)</li> </ul>	<ul style="list-style-type: none"> <li>• Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (1851)</li> <li>• Theorie der Abel'schen Functionen (1857)</li> <li>• Über die Hypothesen, welche der Geometrie zu Grunde liegen (1854)</li> </ul>
Differential geometry	<ul style="list-style-type: none"> <li>• Introductio in analysin infinitorum (1748)</li> <li>• Recherches sur la courbure des surfaces (1767)</li> </ul>	<ul style="list-style-type: none"> <li>• Über die Hypothesen, welche der Geometrie zu Grunde liegen (1854)</li> <li>• Commentatio mathematica, qua respondere tentatur quaestioni ab III<sup>ma</sup> Academia Parisiensi propositae (1861)</li> <li>• Ein Beitrag zu den Untersuchungen über die flüssigen Bewegung eines gleichartigen Ellipsoides (1861)</li> <li>• Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung (1867)</li> </ul>
Trigonometric series	<ul style="list-style-type: none"> <li>• Recherches sur la question des inegalités du mouvement de Saturne et de Jupiter (1748)</li> </ul>	<ul style="list-style-type: none"> <li>• Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe (1854)</li> </ul>
Acoustics	<ul style="list-style-type: none"> <li>• Dissertatio physica de sono (1727)</li> <li>• Sur la vibration des cordes (1748)</li> </ul>	<ul style="list-style-type: none"> <li>• Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite (1860)</li> </ul>

## REFERENCES

- [1] N. H. Abel, Recherches sur les fonctions elliptiques, *Journal für die reine und angewandte Mathematik*, 2 (1827), 101–181.
- [2] L. V. Ahlfors, Quasiconformal mappings and their applications. *Lect. on Modern Math.* 2 (1964) 151–164.
- [3] d’Alembert, Jean le Rond, *Opuscules mathématiques*, Paris, Claude-Antoine Jombert, 1780.
- [4] d’Alembert, Jean le Rond, Recherches sur les vibrations des cordes sonores & Supplément, *Opuscules mathématiques*, Paris, 1761, tome 1, p. 1-73.
- [5] d’Alembert, Jean le Rond, Recherches sur le calcul intégral, *Mémoires de l’Académie des sciences de Berlin*, 2, 1746, p. 182–192.
- [6] d’Alembert, Jean le Rond, Suite des recherches sur la courbe que forme une corde tendue mise en vibration (1747), *Mémoires de l’Académie des sciences de Berlin* 3, 1749, p. 220–249.
- [7] d’Alembert, Jean le Rond, Addition aux recherches sur la courbe que forme une corde tendue mise en vibration (1750), *Mémoires de l’Académie des sciences de Berlin* 6, 1752, p. 355–360.
- [8] d’Alembert, Jean le Rond, Recherches sur la courbe que forme une corde tendue mise en vibration (1747), *Mémoires de l’Académie des sciences de Berlin* 3, 1749, p. 214–219.
- [9] P. S. Alexandrov, Poincaré and topology (speech given at the celebration session at the International Congress of Mathematicians in honor of the centenary of Poincaré’s birth). *Uspekhi Mat. Nauk* 27 (1972) 1(163) pp. 147-158, *Russian Math. Surveys* 34 (6) (1979), 267-302; 35 (3) (1980), 315-358. With an appendix by V. A. Zorin.
- [10] K. A. Andreiev, Démonstration d’une propriété générale des polyèdres, *Société Mathématique de Moscou*, 6 (1873), 457–466.
- [11] Apollonius : Les Coniques, tome 1.1 : Livre I, ed. R. Rashed, commentaire historique et mathématique, édition et traduction du texte arabe, de Gruyter, 2008, 666 p.
- [12] Apollonius : Les Coniques, tome 2.2 : Livre IV, ed. R. Rashed, commentaire historique et mathématique, édition et traduction du texte arabe, de Gruyter, 2009, 319 p..
- [13] Apollonius : Les Coniques, tome 3 : Livre V, ed. R. Rashed, commentaire historique et mathématique, édition et traduction du texte arabe, de Gruyter, 2008, 550 p.
- [14] Apollonius : Les Coniques, tome 4 : Livres VI et VII, ed. R. Rashed, commentaire historique et mathématique, édition et traduction du texte arabe, *Scientia Graeco-Arabica*, vol. 1.4, de Gruyter, 2009, 572 p.
- [15] Apollonius : Les Coniques, tome 2.1 : Livres II et III, ed. R. Rashed, commentaire historique et mathématique, édition et traduction du texte arabe, de Gruyter, 2010, 682 p.
- [16] P. Appell and É. Goursat, *Théorie des fonctions algébriques et de leurs intégrales : Étude des fonctions analytiques sur une surface de Riemann*, Paris, Gauthier-Villars, 1895.
- [17] L.-F.-A. Arbogast, Mémoire sur la nature des fonctions arbitraires qui entrent dans les intégrales des équations aux différentielles partielles, *Académie impériale des sciences, Saint-Petersbourg*, 1791.
- [18] Aristotle, Categories, In *The Complete Works of Aristotle: The Revised Oxford Translation* (J. Barnes, editor), Volume 1, 1-27, Translated by J. L. Ackrill, Princeton University Press, Princeton, 1984.
- [19] Aristotle, the Metaphysics, In *The Complete Works of Aristotle: The Revised Oxford Translation* (J. Barnes, editor), Volume 2, 1552-1728, Translated by W. D. Ross, Princeton University Press, Princeton, 1984.
- [20] Aristotle, the Physics, In *The Complete Works of Aristotle: The Revised Oxford Translation* (J. Barnes, editor), Volume 1, 315-446. Translated by R. P. Hardie and R. K. Gaye, Princeton University Press, Princeton, 1984.

- [21] Aristotle, On the Heavens. In *The Complete Works of Aristotle: The Revised Oxford Translation* (J. Barnes, editor), Volume 1, p. 447-511. Translated by J. L. Stocks. Princeton University Press, Princeton, 1984.
- [22] N. V. Bougaïev, *Théorème d'Euler sur les polyèdres ; propriété d'un réseau géométrique plan*, Société mathématique de Moscou, Moscou, 1867, p. 87-92.
- [23] J. Bernoulli, *Commercium philosophicum et mathematicum*, 1698, I, published Bousquet, by Lausanne et Genève, 1745.
- [24] J. Bernoulli, Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes sur les isopérimètres, avec une nouvelle méthode courte et facile de les résoudre sans calcul, laquelle s'étend aussi à d'autres problèmes qui ont rapport à ceux-là, *Mém. Acad. Royale des Sciences de Paris*, 1718, p. 100 (Paris edition), p. 123 (Dutch edition). *Opera*, Lausanne and Geneva, Vol. 2, p. 253ff.
- [25] D. Bernoulli, Mémoire sur les vibrations des cordes d'une épaisseur inégale, *Histoire de l'Académie Royale des Sciences et des Belles-Lettres de Berlin* 87 (1767), 281-306.
- [26] J.-L.-F. Bertrand, Observations aux remarques de M. Prouhet sur un passage des œuvres inédites de Descartes, *Comptes rendus des séances de l'Académie des sciences de Paris*, 50, 1860, p. 781.
- [27] C. Brianchon, Théorème nouveau sur les polyèdres. *Journal de l'École Polytechnique*. Paris, 15 (1837), 317-319.
- [28] C. B. Boyer, *The Foremost Textbook of Modern Times*. International Congress of Mathematicians, Cambridge, Mass., 1950. Also published in the *American Mathematical Monthly*, 58 (4), 1951, 223-226.
- [29] N. L. Biggs, K. Lloyd and R. J. Wilson, *Graph Theory: 1736-1936*, Clarendon Press, Oxford, 1976.
- [30] P. Carus, *The foundations of mathematics: A contribution to the philosophy of geometry*, Open Court, Chicago, 1908.
- [31] E. Catalan, Mémoire sur la théorie des polyèdres, *Journal de l'École Polytechnique* 24 (1865), 1-71.
- [32] A.-L. Cauchy, Recherches sur les polyèdres, Premier mémoire, *Journal de l'École Polytechnique*, 9, 1813, p. 68-86 ; *Œuvres complètes*, 2<sup>e</sup> série, tome 1, p. 7-25.
- [33] A.-L. Cauchy, Sur les polygones et les polyèdres, Second mémoire, *Journal de l'École Polytechnique*, 9, 1813, p. 87-98 ; *Œuvres complètes*, 2<sup>e</sup> série, tome 1, p. 26-38.
- [34] A.-L. Cauchy, Cours d'analyse de l'École Royale Polytechnique, 1<sup>re</sup> partie. Analyse algébrique. Imprimerie Royale, Paris, 1821. *Œuvres complètes*, série 2, tome III.
- [35] A.-L. Cauchy, Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal, tome I, Paris, 1923. In *Œuvres*, série 2, tome IV, 5-261.
- [36] A.-L. Cauchy, Mémoire sur les intégrales définies prises entre des limites imaginaires, Imprimerie Royale, Paris, 1825, *Œuvres complètes*, série 2, tome XV, p. 41-89.
- [37] A.-L. Cauchy, Mémoire sur les fonctions continues, *C. R.*, t. XVIII, 116-129 (22 janvier 1844), *Œuvres complètes*, série 1, tome 8, p. 145-160.
- [38] A.-L. Cauchy, Sur les fonctions de variables imaginaires, *Comptes Rendus Acad. Sc. Paris* 32 (1851) 207-215, *Œuvres*, série I tome XI, 301-304.
- [39] A.-L. Cauchy, Sur les fonctions monotypiques et monogènes, *Comptes Rendus Acad. Sc. Paris* 32 (1851) 484-487. *Œuvres*, série 1, tome XI, 376-380.
- [40] A.-L. Cauchy, *Sept leçons de physique générale*, Paris, bureau du journal Les Mondes et Gauthier-Villars, 1868.
- [41] A. Cayley, Chapters in the analytic geometry of (n) dimensions, *Cambridge Mathematical Journal A* (1843) 119-127. Reprinted in *Collected mathematical papers of Arthur Cayley*, Vol. 1, Univ. Press, Cambridge (1889-1897), pp. 55-62
- [42] P. L. Chebyshev, Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée, *J. Math. Pures Appl.* (1) 17 (1852), 341-365.
- [43] P. L. Chebyshev, Mémoire sur les nombres premiers, *J. Math. Pures Appl.* (1) 17 (1852), 366-390.
- [44] P. L. Chebyshev, Rapport du professeur extraordinaire de l'université de Saint-Petersbourg Tchebychef sur son voyage à l'étranger. Reprinted in P. L. Tchebycheff, *Œuvres* [45], Vol. 2, p. vii-xiii.

- [45] P. L. Chebyshev, Œuvres, edited by A. Markoff and N. Sonin, Imprimerie de l'Académie Impériale des Sciences, Saint Petersburg, 2 volumes, 1899-1907.
- [46] W. K. Clifford, On the space-theory of matter. Proceedings of the Cambridge Philosophical Society 2 (1870), 157–158. Reprinted in: The Concepts of Space and Time, M. Capek (ed.), Volume 22 of the series Boston Studies in the Philosophy of Science, 295–296.
- [47] R. Descartes, *Exercices pour les éléments des solides : essai en complément d'Euclide ; Progymnasmatata de solidorum elementis*, édition critique avec introduction, traduction, notes et commentaire par Pierre Costabel, Presses universitaires de France, Paris, 1987.
- [48] G. Darboux, Review of Fourier's *Œuvres*, Tome I. Bulletin des Sciences Mathématiques, 2e série, t. XII, Mars 1888, 57–59.
- [49] G. Darboux, Notice historique sur le général Meusnier, Mémoires de l'Académie des Sciences de l'Institut de France, lue à la séance annuelle du 20 décembre 1901, p. I–XXXVIII.
- [50] R. Descartes, Les principes de la philosophie. First edition, in Latin, Amsterdam, 1644, French edition, translation by L'abbé Picot, First published in 1647. In: *Œuvres de Descartes*, ed. V. Cousin, Tome III, Paris, Levrault, 1824.
- [51] R. Descartes, *Œuvres complètes*, publiées par C. Adam et P. Tannery, 1897-1913, 11 vol. ; réimpression : Bibliothèque des textes philosophiques, Vrin, Paris, 1996.
- [52] Diophante : Les Arithmétiques, Livre IV, vol. 3, ed. R. Rashed, Collection des Universités de France, Paris, Les Belles Lettres, 487 p. (1984).
- [53] Diophante : Les Arithmétiques, Livres V, VI, VII, vol 4, ed. R. Rashed, Collection des Universités de France, Paris, Les Belles Lettres, 451 p. (1984).
- [54] Les Arithmétiques de Diophante : Lecture historique et mathématique, ed. R. Rashed and Ch. Houzel, de Gruyter, 629 p. (2013).
- [55] G. L. Dirichlet, Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données, Journal für die reine und angewandte Mathematik, 4 (1829), 157–169.
- [56] P. G. Dirichlet (Lejeune), Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält, Abh. Kgl. Preuss. Akad. Wiss. Berlin (gelesen in der Akad. der Wiss. 1837), 45–81, 1839. Also in: Werke, Bd. 1, (1889), 313–342.
- [57] P. Duhem, Le système du monde; histoire des doctrines cosmologiques de Platon à Copernic, Hermann, Paris, 10 volumes, 1913–1959.
- [58] Ch. Dupin, Développements de géométrie, avec des applications à la stabilité des vaisseaux, aux déblais et remblais, au défilement, à l'optique, etc. pour faire suite à la géométrie descriptive et à la géométrie analytique de M. Monge : Théorie. Courcier, Paris, 1813.
- [59] J. Dutka, The early history of the hypergeometric function, Archive for History of Exact Sciences, 31 (1984), No. 1, 15–34.
- [60] G. Enstöm, Bibl. Math. III, 12 (1912).
- [61] L. Euler, *Introductio in analysin infinitorum*, First edition: Lausannae: Apud Marcum-Michaelem Bousquet & socios., 1748. *Opera omnia*, Series 1, vol. VIII. English translation by J. T. Blanton, 2 vol., Springer-Verlag, New York, 1988, 1990.
- [62] L. Euler, Introduction to Analysis of the Infinite, English translation of [61] by J. T. Blanton, 2 vol., Springer-Verlag, New York, 1988.
- [63] L. Euler, *Mechanica*, 2 volumes, first edition, 1736 *Opera Omnia*, Series 2, Volumes 1 and 2.
- [64] L. Euler, Rechenkunst, Anmerck ungen uber die zeitungen, 1738. *Opera Omnia*, Series 3, Volume 2, p. 1–304
- [65] L. Euler, De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt, Commentarii academiae scientiarum Petropolitanae 5 (1738), 36–57. *Opera Omnia*, Series 1, Volume 14, p. 1–24



- [66] L. Euler, De summatione innumerabilium progressionum Commentarii academiae scientiarum Petropolitanae 5 (1738), 91–105. *Opera Omnia*, Series 1, Volume 14, p. 25–41.
- [67] L. Euler, *Dissertatio physica de sono*, E. and J. R. Thurneisen Brothers, Basel, 1727. *Opera omnia*, Series 3, vol. 1, 181–196.
- [68] L. Euler, Sur la vibration des cordes (1748), traduit du latin, *Mémoires de l'Académie des sciences de Berlin* 4, 1750, p. 69–85. *Opera omnia*, Series 2, vol. 10, p. 63–77.
- [69] L. Euler, Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus, Commentarii academiae scientiarum Petropolitanae 6, 1738, 103–107. *Opera Omnia*, Series 1, Volume 2, p. 1–5.
- [70] L. Euler, Specimen de constructione aequationum differentialium sine indeterminatarum separatione, Commentarii academiae scientiarum Petropolitanae 6 (1738) 168–174. *Opera Omnia*, Series 1, Volume 20, p. 1–7.
- [71] L. Euler, Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae, Saint Petersburg Academy, 1739, 263 pages. *Opera Omnia*, Series 3, Volume 1, p. 197–427.
- [72] L. Euler, De summis serierum reciprocarum, Commentarii academiae scientiarum Petropolitanae 7, 1740, 123–134. *Opera Omnia*, Series 1, Volume 14, p. 73–86.
- [73] L. Euler, Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti, Lausanne and Geneva, 1744 *Opera Omnia*, Series 1, Volume 24.
- [74] L. Euler, Solutio problematis ad geometriam situs pertinentis, Commentarii academiae scientiarum Petropolitanae 8, 1741, 128–140 ; *Opera omnia*, Series 1, vol. 7, p. 1–10.
- [75] L. Euler, Solutio trium problematum difficiliorum ad methodum tangentium inversam pertinentium, Mémoires de l'académie des sciences de Saint-Pétersbourg 10, 1826 (16–26), *Opera Omnia*, Series 1, Volume 29, p. 320–333.
- [76] L. Euler, Varias observationes circa series infinitas, Commentarii academiae scientiarum Petropolitanae 9 (1744) 160–188. In *Opera Omnia*, Series 1, Volume 14, p. 217–244.
- [77] L. Euler, Recherches sur la question des inégalités du mouvement de Saturne et de Jupiter, Pièce qui a remporté le prix de l'Académie Royale des Sciences (1748), 1749, 1–123 *Opera Omnia* Series 2, Volume 25, p. 45–157
- [78] L. Euler, Recherches sur l'origine des forces, Mémoires de l'académie des sciences de Berlin 6 (1752) 419–447. *Opera Omnia* Series 2, Volume 5, p. 109–131.
- [79] L. Euler, Réflexions sur l'espace et le temps, Mémoires de l'Académie des sciences de Berlin 4, (1750), 324–333. *Opera Omnia*, Series 3, Volume 2, p. 376–383
- [80] L. Euler, Découverte d'une loi tout extraordinaire des nombres par rapport à la somme de leurs diviseurs, Bibliothèque impartiale 3, 1751, 10–31. *Opera Omnia*, Series 1, Volume 2, p. 241–253.
- [81] L. Euler, Essai d'une démonstration métaphysique du principe général de l'équilibre, Mémoires de l'Académie des sciences de Berlin 7, 1753, 246–254 *Opera Omnia*, Series 2, Volume 5, pp. 250–256.
- [82] L. Euler, Problema, ad cuius solutionem geometrae invitantur; theorema, ad cuius demonstrationem geometrae invitantur, Nova acta eruditorum, 1754, p. 40.
- [83] L. Euler, Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum, in two books, first published in 1755. *Opera Omnia*, Series 1, Volume 10. English translation of Book I, by J. D. Blanton, Springer Verlag, 2000.
- [84] L. Euler, Elementa doctrinae solidorum, Novi Commentarii academiae scientiarum Petropolitanae 4, 1758, p. 109–140 ; *Opera omnia*, Series 1, vol. 26, p. 71–93.
- [85] L. Euler, *Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita*, *Novi Commentarii academiae scientiarum Petropolitanae* 4, 1758, p. 140–160 ; *Opera omnia*, Series 1, vol. 26, p. 94–108.
- [86] L. Euler, De motu corporum coelestium a viribus quibuscunque perturbato, Novi Commentarii academiae scientiarum Petropolitanae 4, 1758, 161–196. *Opera Omnia*, Series 2, Volume 25, p. 175–209.  
*Opera Omnia*, Series 1, Volume 20, p. 56–57.

- [87] L. Euler, De integratione aequationis differentialis  $\frac{mdx}{\sqrt{1-x^4}} = \frac{ndy}{\sqrt{1-y^4}}$ , Novi Commentarii academiae scientiarum Petropolitanae 6, 1761, 37–57/ *Opera Omnia*, Series 1, Volume 20, p. 58–79 .
- [88] L. Euler, Observationes de comparatione arcuum curvarum irrectificibilium, Novi Commentarii academiae scientiarum Petropolitanae 6, 1761, 58–84 *Opera Omnia*, Series 1, Volume 20, p. 80–107.
- [89] L. Euler, Specimen novae methodi curvarum quadraturas et rectificationes aliasque quantitates transcendentes inter se comparandi, Novi Commentarii academiae scientiarum Petropolitanae 7 (1761), 83–127. *Opera Omnia*, Series 1, Volume 20, p.108–152.
- [90] L. Euler, Demonstratio theorematis et solutio problematis in actis erud. Lipsiensibus propositorum, Novi Commentarii academiae scientiarum Petropolitanae 7, 1761, 128–162. *Opera Omnia* Series 1, Volume 20, p. 201–234.
- [91] L. Euler, Observationes de comparatione arcuum curvarum irrectificibilium, Novi Commentarii academiae scientiarum Petropolitanae 6, 1761, 58–84 *Opera Omnia*, Series 1, Volume 20, p. 80–107.
- [92] L. Euler, Recherches sur la courbure des surfaces. Memoires de l'académie de science de Berlin, **16** (1767), p. 119–143.
- [93] L. Euler, Institutionum calculi integralis, 3 volumes, original versions, 1768, 1769, 1770. *Opera Omnia*, Series 1, Volume 11, 12, 13.
- [94] L. Euler, Integratio aequationis  $\frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4}} = \frac{dy}{\sqrt{\alpha+\beta y+\gamma y^2+\delta y^3+\epsilon y^4}}$ , Novi Commentarii academiae scientiarum Petropolitanae 12 (1768) 3–16. *Opera Omnia*, Series 1, Volume 20, p. 302–317
- [95] L. Euler, Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques, Mémoires de l'académie des sciences de Berlin 17 (1768), 83–106. *Opera Omnia*, Series 1, Volume 15, p. 70–90.
- [96] L. Euler, De solidis quorum superficiem in planum explicare licet, Novi Commentarii academiae scientiarum Petropolitanae 16, (1772), 3–34. *Opera Omnia*, Series 1, Volume 28, p. 161–186.
- [97] L. Euler, Problematis cuiusdam Pappi Alexandrini constructio, Acta Academiae Scientiarum Imperialis Petropolitinae 4, (1783) 91–96. *Opera Omnia*, Series 1, Volume 26, p. 237–242.
- [98] L. Euler, Methodus succinctior comparationes quantitatum transcendentium in forma  $\frac{P\partial z}{\sqrt{A = 2BZ + CZZ + 2DZ^3 + EZ^4}}$ , Institutiones calculi integralis 4, (1794) 504–524. *Opera Omnia*, Series 1, Volume 21, pp. 207–236.
- [99] L. Euler, Anleitung zur Naturlehre, *Opera Postuma* 2, 1862, 449–560, and *Opera Omnia* Series 3, Volume 1, p. 16–180.
- [100] L. Euler, De la propagation du son (1759), *Mémoires de l'Académie des sciences de Berlin*, 15, p. 185–209, 1766. *Opera omnia*, Series 3, vol. 1, p. 428–451.
- [101] L. Euler, Supplément aux recherches sur la propagation du son (1759), *Mémoires de l'Académie des sciences de Berlin*, 15, p. 210–240, 1766 ; *Opera omnia*, Series 3, vol. 1, p. 452–483.
- [102] L. Euler, Continuation des recherches sur la propagation du son (1759), *Mémoires de l'Académie des sciences de Berlin*, 15, p. 241–264, 1766. *Opera omnia*, Series 3, vol. 1, p. 484–507.
- [103] L. Euler, Éclaircissements plus détaillés sur la génération et la propagation du son et sur la formation de l'écho (1765), *Mémoires de l'Académie des sciences de Berlin*, 21, p. 335–363, 1767. *Opera omnia*, Series 3, vol. 1, p. 540–567.
- [104] L. Euler, Remarques sur les mémoires précédents de M. Bernoulli, Mémoires de l'académie des sciences de Berlin 9, 1755, 196–222. *Opera omnia*, Series 2, Volume 10, 233–254
- [105] L. Euler, De repraesentatione superficiae sphaericae super plano, Acta Academiae Scientiarum Imperialis Petropolitinae 1777, 1778, pp. 107–132. *Opera Omnia*: Series 1, Volume 28, pp. 248–275.

- [106] L. Euler, De projectione geographica superficiei sphaericae, *Acta Academiae Scientiarum Imperialis Petropolitinae* 1777, 1778, pp. 133-142, *Opera Omnia* Series 1, Volume 28, pp. 276-287.
- [107] L. Euler, De projectione geographica Deslisiana in mappa generali imperii russici usitata, *Acta Academiae Scientiarum Imperialis Petropolitinae* 1777, 1778, pp. 143-153, *Opera Omnia* Series 1, Volume 28.
- [108] L. Euler, In: Correspondance de Leonhard Euler avec C. Clairaut, J. d'Alembert et J.-L. Lagrange, Euler's *Opera omnia* Ser. IVA, vol. V, ed. A. Juskevich and R. Taton, 1980, Birkhäuser.
- [109] L. Euler and J.-L. de Lagrange, *Correspondance de Lagrange avec Euler. Opera omnia*, Series IV, vol.1, Birkhäuser, Boston - Basel, 24 nov. 1759.
- [110] L. Euler, In: Correspondence of Leonhard Euler with Christian Goldbach, Birkhäuser, ed. Franz Lemmermeyer and Martin Mattmüller, Leonhadi Euleri *Opera Omnia*, Series IVA, vol. 4 (two parts) Basel, 2015.
- [111] G. C. Fagnano, Teorema da cui si deduce una nuova misura degli Archi Ellittici, Iperbolici, e Cicloidali: *Giornale de' Letterati d'Italia* XXVI (1716), p. 266-279. In: *Produzioni Matematiche*, Gavelli: Pesaro, 1750, t. II, p. 336-342.
- [112] G. C. Fagnano, Metodo per misurare la lemniscata *Giorn. de' Letterati d'Italia*, 1718.
- [113] G. C. Fagnano, In: *Opere matematiche del marchese G. C. de' Toschi di Fagnano*, edited by D. Gambioli, G. Loria and V. Volterra, 3 volumes, Società italiana per il progresso delle scienze, 1911-1912.
- [114] M. Feil, Über Euler'sche Polyeder, *Sitzungsberichte der Kgl. Akademie der Wissenschaften in Wien* 93, 1886, p. 869-898.
- [115] L.-A. Foucher de Careil (dir.), *Œuvres inédites de Descartes*, précédées d'une Introduction sur la Méthode, 2 vol., Lagrange, Paris, 1860.
- [116] J. Fourier, Théorie de la propagation de la chaleur dans les solides, Mémoire lu le 21 décembre 1807, présenté à l'Académie des Sciences, 29 octobre, 1809. Edited in [133].
- [117] J. Fourier, *Théorie analytique de la chaleur*, Paris, Firmin Didot, 1822. In *Œuvres*, ed. G. Darboux, Tome 1, 1888.
- [118] N. I. Fuss, Éloge de Monsieur Léonard Euler, lu à l'Académie Impériale des Sciences dans son assemblée du 23 octobre 1783. St-Pétersbourg, 1783. Reprint available at Kessinger Publishing Legacy's Reprints.
- [119] P. H. Fuss, Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIe siècle, Saint-Pétersbourg, 1843. Reprint: Johnson Reprint Corp., New York and London, 1968.
- [120] C. F. Gauss, Demonstratio nova theorematis omnem functionem algebraicam rationally integrabilem unius variabilis in factores reales primi vel secundi gradus resolvi posse. Quam pro obtinendis summis in philosophia honoribus inclito philosophorum ordini Academiae Iuliae Carolinae, Helmstadii : apud C. G. Fleckeisen, 1799.
- [121] C. F. Gauss, Theoria residuorum biquadraticorum. Commentatio prima et secunda, Göttingen Dieterich, 1828 and 1832.
- [122] C. F. Gauss, *Disquisitiones Arithmeticae*, Gehr. Fleischer, Leipzig, 1801, English translation by A. A. Clarke, Yale University Press, 1965,
- [123] C. F. Gauss, *Disquisitiones generales circa serium*  $1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$  etc. pars prior. Königliche Gesellschaft der Wissenschaften, Göttingen, 1812. In: *Werke*, Volume 3, p. 123-162.
- [124] C. F. Gauss, Determinatio seriei nostrae per aequationem differentialem secundi ordinis. *Gauss's Werke* vol. 3, 207-230.
- [125] C. F. Gauss, General Investigations of Curved Surfaces. Translated from the Latin and German by A. Hiltebeitel and J. Morehead, Princeton University Library, Princeton, 1902. New edition with an Introduction and Notes by P. Pesic, Dover, 2005.
- [126] C. F. Gauss, translation of his notebook: *Le Journal mathématique de Gauss: Traduction française annotée* par P. Eymard and J. P. Lafon. *Revue d'histoire des sciences et de leurs applications*, Vol. 9, No. 1 (Janvier-Mars 1956), pp. 21-51
- [127] C. F. Gauss, *Werke*, Königliche Gesellschaft der Wissenschaften, Göttingen, 1900,

- [128] K. Gödel, A remark about the relationship between relativity theory and idealistic philosophy, In: *Collected works of Kurt Gödel*, Vol. II, Oxford University Press, New York, 1990, p. 202–207.
- [129] L. J. Goldstein, A history of the prime number theorem, *American Mathematical Monthly*, 80, (1973), No. 6, 599–615.
- [130] L. Ahrndt Golland and R. W. Golland, Euler’s troublesome series: An early example of the use of trigonometric series, *Historia Mathematica*, 20 (1993), 54–67.
- [131] J. D. Gergonne, Sur le théorème d’Euler relatif aux polyèdres, *Annales de Gergonne* 19, 1828, p. 333.
- [132] H. Grassmann, Die Lineale Ausdehnungslehre ein neuer Zweig der Mathematik: Dargestellt und durch Anwendungen, O. Wigand, Leipzig, 1844.
- [133] I. Grattan-Guinness (in collaboration with J. R. Ravetz), Joseph Fourier (1768–1830), A survey of his life and work, based on a critical edition of his monograph on the propagation of heat, presented to the Institut de France in 1807, MIT Press, Cambridge, Massachusetts, and London, 1972.
- [134] A. Grothendieck, Récoltes et semailles : Réflexions et témoignage sur un passé de mathématicien, manuscript, 1983-1986, to appear as a book.
- [135] A. Grothendieck, Esquisse d’un programme (Sketch of a program), unpublished manuscript (1984), English translation by P. Lochak and L. Schneps in *Geometric Galois actions, vol. 1, “Around Grothendieck’s Esquisse d’un Programme”* (L. Schneps and P. Lochak, ed.) London Math. Soc. Lecture Note Ser. vol. 242, pp. 5-48, Cambridge Univ. Press, Cambridge, 1997.
- [136] J. A. Grunert, Einfacher Beweis der von Cauchy und Euler gefundenen Sätze von Figurennetzen und Polyedren, *Journal für die reine und angewandte Mathematik*, Berlin, 2, 1827, p. 367.
- [137] J. Hadamard, Notions élémentaires sur la géométrie de situation, *Ann. de math.*, 4e série, (1909) t. 9, pp. 193-235.
- [138] J. Hadamard, The psychology of invention in the mathematical field, Princeton, University Press, First ed. 1954.
- [139] D. Hilbert and S. et Cohn-Vossen, Anschauliche Geometrie, 1932. English translation: *Geometry and the Imagination*, Chelsea, New York, 1952.
- [140] P. Hilton and J. Pedersen, The Euler Characteristic and Pólya’s Dream, *American Mathematical Monthly* 103 (2), 1996, p. 121-131.
- [141] H. Hopf, *Differential Geometry in the Large – 1956 Lectures Notes*, Lectures Notes in Mathematics 1000, Springer-Verlag, Heidelberg-Berlin, 1983.
- [142] C. Houzel, Riemann’s Memoir *Über das Verschwinden der  $\vartheta$ -Functionen*, In: *From Riemann to differential geometry and relativity* (L. Ji, A. Papadopoulos and S. Yamada, ed.) Berlin: Springer, 2017.
- [143] C. Houzel, The Work of Niels Henrik Abel, In: *The legacy of Niels Henrik Abel – The Abel Bicentennial* (O. A. Laudal and R. Piene, ed.), Oslo 2002, Springer Verlag, 2004.
- [144] C. Huffman, Archytas of Tarentum: Pythagorean, Philosopher and Mathematician King. Cambridge University Press, Cambridge, 2005.
- [145] Ch. Huygens, *Œuvres complètes*, publiées par la Société hollandaise des sciences, Martinus Nijhoff, La Haye, 1888-1950.
- [146] Jacobi C.-G.-J. De binis quibus libet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integrilium multiplicium. *Journal für die reine und angewandte Mathematik. Journal de Crelle*. Berlin, 12 (1834), 1–69.
- [147] C. G. J. Jacobi, *Gesammelte Werke*, Edited by C. W. Borchardt, Herausgegeben auf Veranlassung der königlich preussischen Akademie der Wissenschaften, 8 volumes, 1881–1891. New edition, Cambridge Library Collection.
- [148] P. Jehel, Une lecture moderne d’un mémoire d’Euler: les Éclaircissements plus détaillés sur la génération et la propagation du son et sur la formation de l’écho. (A modern reading of a memoir of Euler: Detailed clarifications concerning the generation and propagation of sound and the formation of echoes). In: *Leonhard Euler*,

- Mathématicien, musicien et théoricien de la musique (X. Hascher and A. Papadopoulos, ed.), 275–300. Collection Sci. Musique Ser. Études, CNRS Éditions, Paris, 2015.
- [149] E.-J.-Ph. Jonquières (Fauque de), Sur un point fondamental de la théorie des polyèdres, *Comptes rendus des séances de l'Académie des sciences de Paris*, 110, 1890, p. 110-115.
- [150] E.-J.-Ph. Jonquières (Fauque de), Note sur le théorème d'Euler dans la théorie des polyèdres, *Comptes rendus des séances de l'Académie des sciences de Paris*, 110, 1890, p. 169-173.
- [151] E.-J.-Ph. Jonquières (Fauque de), Note sur un Mémoire de Descartes longtemps inédit, et sur les titres de son auteur à la priorité d'une découverte dans la théorie des polyèdres ; *Comptes rendus des séances de l'Académie des sciences de Paris*, 110, 1890, p. 261-266.
- [152] E.-J.-Ph. Jonquières (Fauque de), Écrit posthume de Descartes sur les polyèdres, *Comptes rendus des séances de l'Académie des sciences de Paris*, 110, 1890, p. 315-317.
- [153] E.-J.-Ph. Jonquières (Fauque de), Note sur un Mémoire présenté, qui contient, avec le texte complet et revu de l'écrit posthume de Descartes: De solidorum elementis, la traduction et le commentaire de cet Ouvrage, *Comptes rendus des séances de l'Académie des sciences de Paris*, 110, 1890, p. 677-680.
- [154] C. Jordan, Sur la déformation des surfaces, *Journal de mathématiques pures et appliquées*, ser. 2, t. XI (1866).
- [155] C. Jordan, Résumé de recherches sur la symétrie des polyèdres non eulériens, *Journal für die reine und angewandte Mathematik (Journal de Crelle)*, Berlin, 66, 1866, p. 86-91 ; *Œuvres de Camille Jordan*, vol. IV, p. 79-84.
- [156] C. Jordan, Recherches sur les polyèdres, *Journal für die reine und angewandte Mathematik (Journal de Crelle)*, Berlin, 66, 1866, p. 22-85 ; *Œuvres de Camille Jordan*, vol. IV, p. 15-78.
- [157] C. Jordan, Recherches sur les polyèdres. Comptes Rendus des Séances de l'Académie des Sciences. Paris, 62 (1866) 1339–1341.
- [158] C. Jordan, Recherches sur les polyèdres (second Mémoire), *Journal für die reine und angewandte Mathematik (Journal de Crelle)*, Berlin, 68, 1868, p. 297-349 ; *Œuvres de Camille Jordan*, vol. IV, p. 119-172.
- [159] C. Jordan, Note sur la symétrie inverse des polyèdres non eulériens, *Journal für die reine und angewandte Mathematik (Journal de Crelle)*, Berlin, 68, 1868, p. 350-353 ; *Œuvres de Camille Jordan*, vol. IV, p. 173-178.
- [160] I. Kant, De Mundi Sensibilis atque Intelligibilis Forma et Principiis (Inaugural dissertation, On the form and principles of the sensible and the intelligible world), 1770, translated by William J. Eckoff, New York, Columbia College, 1894.
- [161] A. B. Kempe, On a General Method of describing Plane Curves of the nth degree by linkwork. *Proc. London Math. Soc.* 7 (1876) 213–216.
- [162] F. Klein, *Über Riemanns Theorie der algebraischen Funktionen und ihrer Integrale*, Teubner, Leipzig, 1882. English translation : *On Riemann's theory of algebraic functions and their integrals; a supplement to the usual treatises*. Translated from the German by Frances Hardcastle, Macmillan and Bowes, Cambridge, 1893. Reprint : New York, Dover Publications, 1963.
- [163] F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. 2 volumes*, Julius Springer, Berlin, 1926. English translation by M. Ackermann: *Development of mathematics in the 19th century*, Volume IX of the Series *Lie Groups: History, Frontiers and Applications*, edited by R. Hermann, Math. Sci. Press, Brookline, Brookline, MA, 1979.
- [164] A. W. Knap, André Weil: A Prologue, *Notices of the AMS*, Vol. 46 (1999) No. 4, 434–439.
- [165] J.-L. de Lagrange *Théorie des fonctions analytiques contenant les principes du calcul différentiel dégagés de toute considération d'infiniment petits ou d'évanouissans de limites ou de fluxions*, Paris, Imprimerie de la République, 1797.
- [166] J.-L. de Lagrange, Sur la construction des cartes géographiques, *Nouveaux mémoires de l'Académie Royale des Sciences et Belles-lettres de Berlin*, année 1779, Premier

- mémoire, *Œuvres complètes*, tome 4, 637-664. Second mémoire *Œuvres complètes*, tome 4, 664-692.
- [167] J.-L. Lagrange, *Œuvres*, 14 volumes, published by J.-A. Serret (t. I–X and XIII) and G. Darboux, Paris, Gauthier-Villars, 1867–1892.
- [168] J.-L. de Lagrange and J. Le Rond d'Alembert, Correspondance inédite de Lagrange et d'Alembert, *Œuvres de Lagrange*, publiées par J.-A. Serret, t. 13, Gauthier-Villars, Paris, 1882.
- [169] H. Lebesgue, Remarques sur les deux premières démonstrations du théorème d'Euler relatif aux polyèdres, *Bulletin de la Société mathématique de France*, tome 52, 1924, p. 315-336.
- [170] H. Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, Paris, Gauthier-Villars, 1904.
- [171] A.-M. Legendre, *Essai sur la théorie des nombres*, Duprat, Paris, 1798.
- [172] A.-M. Legendre, *Exercices de calcul intégral sur divers ordres de transcendentes et sur les quadratures*, 3 volumes, Paris, Courcier, 1811–1816.
- [173] A.-M. Legendre, *Traité des fonctions elliptiques et des intégrales eulériennes, avec des tables pour en faciliter le calcul numérique*, 3 volumes, Paris, Huzard-Courcier, 1825–1828.
- [174] A.-M. Legendre, *Éléments de géométrie, avec des notes*, Paris, Firmin Didot, 1794.
- [175] G. W. Leibniz, *Mathematische Schriften*, 1 Abt., vol. II, C. I. Gerhardt, Berlin, 1850.
- [176] G. W. Leibniz, *La caractéristique géométrique*, text edited and annotated by J. Acheverría, translation, notes and psotface by M. Parmentier, Coll. Mathesis, Vrin, Paris, 1995.
- [177] G. W. Leibniz, In: *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern. Erster Band. Hrsg. von C. I. Gerhardt. Mit Unterstützung der Königl. Preussischen Akademie der Wissenschaften*, Berlin : Mayer & Müller, 1899.
- [178] S.-A.-J. Lhuillier, Démonstrations diverses du théorème d'Euler, *Annales de Gergonne*, 3, 1812, p. 169-189.
- [179] G. Loria, Commemorazione del compianto Socio Prof. Placido Tardy, *rend. Acad. Lincei (V)* 24 (1° semestre) 1915, 521-521.
- [180] C. MacLaurin, *A treatise on fluxions*, two volumes London, Baynes, 1801.
- [181] P. Mengoli, *Novae quadrature arithmeticae, seu de additione fractionum*, Bononiæ, ex Typografia Iacobi Montij, 1650.
- [182] , J.-B. Meusnier, Mémoire sur la courbure des surfaces, *Mém. div. sav. Paris*, t. X, 1785, p. 477–510.
- [183] A. de Moivre, *Miscellanea analytica de seriebus et quadraturis*. London, J. Tonson & J. Watts, 1730.
- [184] G. H. Moore, The evolution of the concept of homeomorphism, *Historia Mathematica* 34, (2007), Issue 3, 333–343.
- [185] G. Monge, Mémoire sur les développées, les rayons de courbure, et les différens genres d'Inflexions des courbes a double courbure, Paris, Imprimerie Royale, 1785.
- [186] G. Monge, Mémoire sur les propriétés de plusieurs genres de surfaces courbes et particulièrement sur celles des surfaces développables avec une application à la théorie générale des ombres et des pénombres, *Mém. Div. savants*, 9 (1780), 382–440.
- [187] C. Neumann, *Vorlesungen über Riemann's Theorie der Abel'schen Integrale*, Leipzig, Teubner, 1865. Second revised edition, 1884.
- [188] I. Newton, In: *Sir Isaac's Newton's mathematical principles of natural philosophy and his system of the world*. Translated by A. Motte (1729), Revision by F. Cajori, Cambridge University Press, 1934.
- [189] A. B. Niewenglowski, Exposition de la méthode de Riemann pour la détermination des surfaces minima de contour donné. Thesis submitted to the Faculté des Sciences, Paris, Gauthier-Villars, 1880.
- [190] A. Papadopoulos, Euler et les débuts de la topologie. In: *Leonhard Euler : Mathématicien, physicien et théoricien de la musique* (X. Hascher and A. Papadopoulos, ed.), CNRS Editions, p. 321–347, 2015.

- [191] A. Papadopoulos, Physics in Riemann's mathematical papers, In: From Riemann to differential geometry and relativity (L. Ji, A. Papadopoulos and S. Yamada, ed.) Berlin: Springer, 2017.
- [192] A. Papadopoulos, Cauchy and Puiseux: Two precursors of Riemann, In: From Riemann to differential geometry and relativity (L. Ji, A. Papadopoulos and S. Yamada, ed.) Berlin: Springer, 2017.
- [193] A. Papadopoulos, Riemann surfaces: reception by the French school, In: From Riemann to differential geometry and relativity (L. Ji, A. Papadopoulos and S. Yamada, ed.) Berlin: Springer, 2017.
- [194] A. Papadopoulos, Euler and Chebyshev: From the sphere to the plane and backwards, Proceedings in Cybernetics (A volume dedicated to the jubilee of Academician Vladimir Betelin), 2 (2016) p. 55–69.
- [195] A. Papadopoulos, Quasiconformal mappings, from Ptolemy's geography to the work of Teichmüller, to appear in: *Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau Manifolds, and Picard-Fuchs Equations* (ed. L. Ji and S.-T. Yau), International Press and Higher Education Press. To appear in 2017, 65p.
- [196] J. F. Pfaff, Disquisitiones analyticae maxime ad calculum integrelem et doctrinam serierum pertinentes, Helmstadii, leckeisen, 1797.
- [197] É. Picard, L'œuvre mathématique de É. Galois, Revue générale des sciences pures et appliquées, t. VIII (1897) 339–340.
- [198] É. Picard, Traité d'Analyse, 3 volumes, Paris, Gauthier-Villars, 1891, 1893, 1896.
- [199] É. Picard, Sur le Développement de l'Analyse et ses Rapports avec diverses Sciences: Conférences faites en Amérique, Paris, Gauthier-Villars, 1905.
- [200] H. Poincaré, Sur la généralisation d'un théorème d'Euler relatif aux polyèdres, *Comptes rendus des séances de l'Académie des sciences de Paris*, 117, 1893, p. 144–145 ; *Œuvres*, vol. XI, p. 6–7.
- [201] H. Poincaré, Analysis situs, *Journal de l'École Polytechnique*, (2) 1, 1895, p. 1–121 ; *Œuvres*, vol. VI, p. 193–288.
- [202] H. Poincaré, Sciences et méthode, Paris, Flammarion, 1908, several later editions.
- [203] H. Poincaré, Analyse des travaux scientifiques de Henri Poincaré faite par lui-même, *Acta Mathematica*, 38, 1921, pp. 36–135.
- [204] L. Poincot, Note sur la théorie des polyèdres, *Comptes rendus des séances de l'Académie des sciences de Paris*, 46, 1858, p. 65–79.
- [205] L. Poincot, Mémoire sur les polygones et les polyèdres, *Journal de l'École Polytechnique*, Paris, 4, 1810, p. 16–49.
- [206] S.-D. Poisson, Mémoire sur la courbure des surfaces, *J. reine angew. Math*, 8 (1832), 280–297.
- [207] J.-C. Pont, La topologie algébrique des origines à Poincaré, Presses Universitaires de France, Paris, 1974.
- [208] C. A. Possé, Excerpts of a biography of Chebyshev, contained in his *Collected Works*, Edited by A. Markoff and N. Sonin, Vol. II, p. I–VI.
- [209] P.-M.-E. Prouhet, Sur le nombre des diagonales d'un polyèdre, *Nouvelles Annales mathématiques*, Paris, 2, 1863, p. 77–79.
- [210] P.-M.-E. Prouhet, Remarques sur un passage des Œuvres inédites de Descartes, *Comptes rendus des séances de l'Académie des sciences de Paris*, 50, 1860, p. 779–781.
- [211] R. Rashed, Histoire de l'analyse diophantienne classique : D'Abū Kāmil à Fermat, de Gruyter 349 p. (2013).
- [212] R. Rashed, Les mathématiques infinitésimales du IXème au XIème siècle. Vol. IV. Ibn al-Haytham. Méthodes géométriques, transformations ponctuelles et philosophie des mathématiques. Al-Furqān Islamic Heritage Foundation Publication. 60. London: Al-Furqān Islamic Heritage Foundation, 1064 p. (2002).
- [213] R. Rashed and A. Papadopoulos, *Menelaus' Spherics: Early Translation and al-Māhānī/ al-Harawī 's Version* (Critical edition of Menelaus' *Spherics* from the Arabic manuscripts, with historical and mathematical commentaries), De Gruyter, Series: Scientia Graeco-Arabica, 21, 2017, 890 pages.

- [214] R. Remmert, *Theory of complex variables*, Springer, New York, English edition, 1989.
- [215] B. Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, (Göttingen, 1851), [226] pp. 3–48.
- [216] B. Riemann, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*. Aus dem dreizehnten Bande der *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 1867.
- [217] B. Riemann, *Commentatio mathematica, qua respondere tentatur quaestioni ab Ill<sup>ma</sup> Academia Parisiensi propositae: “Trouver quel doit être l’état calorifique d’un corps solide homogène indéfini pour qu’un système de courbes isothermes, à un instant donné, restent isothermes après un temps quelconque, de telle sorte que la température d’un point puisse s’exprimer en fonction du temps et de deux autres variables indépendantes.”* In: *Bernhard Riemann’s Gesammelte Mathematische Werke* [226], 2nd Edition, Teubner 1892, pp. 391–404.
- [218] B. Riemann, *Theorie der Abel’schen Functionen*. *Journal für die reine und angewandte Mathematik*, 54 (1857), 115–155. Reprinted in his *Gesammelte mathematische Werke* [226], pp. 88–144.
- [219] B. Riemann, *Lehrsätze aus der analysis situs für die Theorie der Integrale von zweigliedrigen vollständigen Differentialien*, *Journal für die reine und angewandte Mathematik* 54 (1857), 105–110.
- [220] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, *Monatsberichte der Berliner Akademie*, November 1859, 671–680, *Gesammelte mathematische Werke* [226], pp. 145–153.
- [221] B. Riemann, *Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung*, *Abh. Königl. d. Wiss. Göttingen, Mathem. Cl.*, 13 (1867) 3–52.
- [222] B. Riemann, *Ein Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoides*, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, vol. 9, 1861, 3–36.
- [223] B. Riemann, *Beiträge zur Theorie der durch die Gauss’sche Reihe  $F(\alpha, \beta, \gamma, x)$  darstellbaren Functionen*. Aus dem siebenten Band der *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* (1857) 3–32.
- [224] B. Riemann, *Ein Beitrag zur Elektrodynamik*, *Ann. Phys.*, 131 (1867), 237–43. English translation: *A contribution to electrodynamics*, *Phil. Mag.*, ser. 4, 35 (1867), 368–372.
- [225] B. Riemann, *Über das Verschwinden der  $\vartheta$ -Functionen*, *J. für die r. und a. Math.*, 65 (1866), 161–172 = *Ges. Math. Werke*, 212–224.
- [226] B. Riemann, *Gesammelte mathematische Werke*, wissenschaftlicher Nachlass und Nachträge. Nach der Ausgabe von Heinrich Weber und Richard Dedekind neu herausgegeben von Raghavan Narasimhan. (Collected mathematical works, scientific posthumous works and supplements. According to the edition by H. Weber and R. Dedekind newly edited by R. Narasimhan). Teubner Verlagsgesellschaft, Leipzig, 1862; Springer-Verlag, Berlin (1990).
- [227] B. Riemann, *Gleichgewicht der Electricität auf Cylindern mit kreisförmigen Querschnitt und parallelen Axen*, *Conforme Abbildung von durch Kreise bergrenzten Figuren*, (Nachlass XXVI) 1857, In: *Ges. math. Werke*, p. 472–476.
- [228] B. Riemann and K. Hattendorff, *Schwere, Elektrizität und Magnetismus*, Hannover, Carl Rümpler, 1876, Nachdruck VDM, Müller, Saarbrücken, 2006.
- [229] B. Riemann, *Collected papers*, English translation by R. Baker, Ch. Christenson and H. Orde, Kendrick Press, Heber City, UT, 2004.
- [230] B. Riemann, *Collected works translated into French: Œuvres mathématiques de Riemann*, traduites par L. Laugel, avec une préface de C. Hermite, Paris, Gauthier-Villars, 1898.
- [231] B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, published by R. Dedekind, after Riemann’s death, in *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 13, 1867. French translation in [230] pp. 280–299.



- [232] B. Riemann, On the Hypotheses which lie at the Bases of Geometry. Translation of [231] by W. K. Clifford, *Nature*, Vol. VIII. Nos. 183, 184, pp. 14–17, 36, 37.
- [233] B. Riemann and H. Weber, *Die partiellen Differential-Gleichungen der mathematischen Physik nach Riemanns Vorlesungen*, 2 vols, Vieweg, Braunschweig, 1912.
- [234] The Riemann letters at the Prussian cultural archive, translated by R. Gallagher and M. Weissbach, mimeographed notes, 1981.
- [235] V. de Risi, *Geometry and Monadology: Leibniz’s Analysis Situs and Philosophy of Space*, Birkhäuser, 2007.
- [236] O. Rodrigues, Recherches sur la théorie des lignes et des rayons de courbure des surfaces, et sur la transformation d’une classe d’intégrales doubles qui ont un rapport direct avec les formules de cette théorie, *Correspondance de l’Ecole Polytechnique*, 3 (1815), 162–183.
- [237] B. Russell, *Principles of Mathematics*, Cambridge, 1903
- [238] M. Schmitz, The life of Gotthold Ferdinand Eisenstein, *Res. Lett. Inf. Math. Sci.*, 6 (2004) Vol. 6–13.
- [239] A. Sossinsky, Configuration spaces of planar linkages, In: *Handbook of Teichmüller theory* (ed. A. Papadopoulos) Vol. VI, p. 335–373, European Mathematical Society, Zurich, 2016.
- [240] J. Steiner, Leichter Beweis eines stereometrischen Satzes von Euler nebst einem Zusatz X, S. 48, tome 1, *Journal für die reine und angewandte Mathematik (Journal de Crelle)*, Berlin, 1, 1826, p. 364-367.
- [241] J. Tannery, Pensées, ed. É. Boutoux and É. Borel, *Revue du mois*, 10 mars 1911 and 10 avril 1911. Re-edition: En souvenir de Jules Tannery, brochure, Imprimerie Crété, Corbeil, 1912.
- [242] P. Tannery, Le concept scientifique du continu: Zénon d’Elée et Georg Cantor, *Revue philosophique de la France et de l’Étranger*, Xe année, t. XX (1885), 385–410.
- [243] B. Taylor, De motu nervi tensi, *Philosophical Transactions of the Royal Society of London*, 28 (1713, published in 1714) 26–32.
- [244] B. Taylor, *Methodus incrementorum directa et inversa*, Impensis Gulielmi Innys, London, 1715; 2nd ed. 1717.
- [245] J.-M. Thiel, Démonstration nouvelle du théorème d’Euler pour des polyèdres convexes, *Nieuw Archief voor wiskunde uitgegeven door bet Wiskundig Genootschap*. Amsterdam, 19 (1892), 98–99.
- [246] R. Thiele, The Rise of the Function Concept in Analysis, In *Euler Reconsidered: Tercentenary Essays*, Kendrick Press, Heber City, UT. 422-461,
- [247] R. Thom, Les intuitions topologiques primordiales de l’aristotélisme, *Revue thomiste*, juillet-septembre 1988, XCVI<sup>e</sup> année, 88 (3), 1988, p. 393-409.
- [248] R. Thom, Aristote topologue, *Revue de synthèse*, 120 (1), 1999, p. 39-47.
- [249] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University, Princeton, 1976.
- [250] J.-P.-F. Valat, Nouvelles remarques sur l’interprétation d’un théorème de Descartes, *Comptes rendus des séances de l’Académie des sciences de Paris*, 51, 1860, 1031-1033.
- [251] V. Volterra, Betti, Brioschi, Casorati : Trois analystes italiens et trois manières d’envisager les questions d’analyse, *Comptes Rendus du deuxième congrès international des mathématiciens tenu à Paris, 1900*; Gauthier-Villars, 1902, 43–57.
- [252] J. Wallis, *Arithmetica infinitorum*, 1656. *Opera Mathematica*, Vol. 1, Oxford, 1695.
- [253] J. Wallis, A treatise on algebra, both historical and practical, shewing, the original, progress, and advancement thereof, from time to time, and by what steps it hath attained to the height at which now it is. With some additional treatises, I. Of the cono-cuneus; being a body representing in part a conus, in part a cuneus. II. Of angular sections; and other things relating there-unto, and to trigonometry. III. Of the angle of contact; with other things appertaining to the composition of magnitudes, the inceptives of magnitudes, and the composition of motions, with the Results thereof. IV. Of combinations, alternations, and aliquot parts. London, John Playford for Richard Davis, 1685.

- [254] A. Weil, Essais historiques sur la théorie des nombres, Monographie N0 22 de L'Enseignement Mathématique, Université de Genève, Genève, Imprimerie Kundig, 1975.
- [255] A. Weil, History of Mathematics: Why and How. Proceedings of the International Congress of Mathematicians. Helsinki, 1978.
- [256] A. Weil, Number Theory: An approach through history from Hammurapi to Legendre, Birkhäuser, Boston-Basel-Berlin, First edition 1984.
- [257] A. Weil, Prehistory of the zeta function. In: Number theory, trace formulas and discrete groups: Symposium in Honor of Atle Selberg, Oslo, Norway, July 14–21, 1987, (K. E. Aubert, E. Bombieri and D. Goldfeld, ed.) Acad. Press, Boston, 1989, p. 1–9.
- [258] A. Weil, Riemann, Betti and the birth of topology, Archive for History of Exact Sciences 20 (1979) Issue 2, 91-96.
- [259] H. Weyl, Philosophy of Mathematics and Natural Science, Princeton University Press, New edition, 2009, With a new introduction by Frank Wilczek. Translation of *Philosophie der Mathematik and Naturwissenschaft*, Munich, R. Oldenburg, 1927.
- [260] S. Yamada, Riemann on minimal surfaces. In: From Riemann to differential geometry and relativity (L. Ji, A. Papadopoulos and S. Yamada, ed.) Berlin: Springer, 2017.
- [261] A. P. Youschkevitch, *The Concept of Function up to the Middle of the 19th Century*, Archive for History of Exact Sciences 16 (1976) 37-85.
- [262] A. P. Youschkevitch and R. Taton, Introduction to Volume V of Series IV A of Euler's *Opera omnia* (The correspondence of Euler with Clairaut, d'Alembert and Lagrange).

- Abel, Nils Henrik (1802–1829), 25–28, 30, 32, 80
- Abelian integral, 30–32
- Adam, Charles (1857–1896), 53
- d’Alembert, Jean le Rond (1717–1783), 2, 19, 24, 25, 68–74
- algebraic equation, 15
- algebraic function, 13, 15
- analysis situs*, 41, 51, 56, 57
- analytic function (in the sense of Euler), 13, 69
- Apollonius of Perga (3d–2d century B.C.), 15
- Arbogast, Louis-François-Antoine (1759–1803), 70
- Archimedes (c. 287 B.C.–c. 212 B.C.), 2, 16, 76, 77
- Archytas of Tarentum (428?347 B.C.), 16, 41
- Aristotle (384–322 B.C.), 15, 41, 47
- Averroes (Ibn Rushd of Cordoba) (1126–1198), 41
- Bernoulli, Daniel (1700–1782), 2, 71
- Bernoulli, Jakob (1654–1705), 35, 36
- Bernoulli, Johann (1667–1748), 12, 21, 36, 62
- Bertrand postulate, 39
- Bessel, Friedrich (1784–1846), 46
- Betti, Enrico (1923–1892), 10, 47, 57
- Bolyai, Wolfgang (1775–1856), 46
- Bonnet, Pierre-Ossian (1819–1892), 9, 66
- Brioschi, Francesco (1824–1897), 10
- Casorati, Felice (1835–1890), 10
- Categories (Aristotle), 41
- Cauchy integral, 79
- Cauchy, Augustin-Louis (1789–1857), 15, 17, 54, 79
- Cauchy–Riemann equations, 19
- characteristic (Leibniz), 48–50
- Chebyshev, Pafnouti Lvovitch (1821–1894), 39, 80
- continuous function (in the sense of Euler), 17
- curvature, 63
- Darboux, Gaston (1842–1917), 75
- Descartes, René (1596–1650), 2, 12, 48, 52, 53, 56, 57, 77
- Diophantus of Alexandria (3d century B.C.), 15
- Dirichlet  $L$ -function, 39
- Dirichlet integral, 79
- Dirichlet series, 39
- Dirichlet, Johann Peter Gustav Lejeune (1805–1859), 2, 6, 20, 30, 33, 38–40, 72, 78
- Dirichlet, Peter Gustav Lejeune (1805–1859), 77
- discontinuous function (in the sense of Euler), 17
- doctoral dissertation
- Gauss, 32
- Kant, 44
- Riemann, 10, 16, 19
- Dupin, Charles (1784–1873), 67
- Ehler, Carl Leonhard Gottlieb (1685–1753), 51
- Einstein, Albert (1879–1955), 61
- Eisenstein, Ferdinand Gotthold Max (1823–1852), 29, 30, 38, 80
- elliptic integral, 15, 30
- Encke, Johann Franz (1791–1865), 39
- equation
- algebraic, 15
- vibrating string, 69
- wave, 71
- equations
- Cauchy–Riemann, 19
- Euclid
- Elements, 12, 34
- Euler
- Introductio in analysin infinitorum, 7, 11, 12, 15, 18
- Euler characteristic, 51, 53, 54
- Euler formula, 55
- Euler, Leonhard (1707–1783), 2, 3, 5, 7, 11–17, 20–25, 28, 29, 36–38, 43, 53–57, 62, 63, 68–70, 72, 74, 75, 78
- Fagnano, Giulio Carlo de’ Toschi di (1682–1766), 21, 25
- Fauque de Jonquières, E.-J.-Ph. (1820–1901), 53
- Foucher de Careil, Louis-Alexandre (1826–1891), 52
- Fourier
- Théorie analytique de la chaleur, 2, 75, 76
- Fourier series, 78, 79
- Fourier, Joseph (1768–1830), 2, 72, 75, 76
- function
- algebraic, 13, 15
- analytic (in the sense of Euler), 13, 69
- continuous (in the sense of Euler), 17
- discontinuous (in the sense of Euler), 17
- multi-valued, 7, 14

- transcendental, 13, 14, 16  
 Fuss, Nicolaus (1755–1826), 74  
 Fuss, Paul Heinrich (1755–1826), 4
- Galilei, Galileo (1564–1642), 2, 76, 77  
 Gauss  
   *Theorema egregium*, 64  
   Disquisitiones arithmeticae, 10, 28  
   Disquisitiones generales circa superficies curvas, 64  
   doctoral dissertation, 32  
 Gauss, Carl Friedrich (1777–1855), 2, 5, 6, 10, 12, 18–20, 25–28, 39  
 Goldbach, Christian (1690–1764), 5  
 Grothendieck  
   Récoltes et semailles, 81  
 Grothendieck, Alexandre (1928–2014), 40, 61, 81
- habilitation lecture  
   Riemann, 1, 8, 46, 58, 77  
 habilitation text  
   Riemann, 68  
 Hadamard, Jacques (1865–1963), 38  
 Huygens, Christiaan (1629–1695), 41  
 hypergeometric series, 33
- Ibn al-Haytham, Abū ‘Alī al-Ḥasan ibn al-Ḥasan (965–1039), 41  
 infinity, 15  
 integral  
   Cauchy, 79  
   Dirichlet, 79  
   Riemann, 78, 79
- Jacobi Carl Gustav Jacob (1804–1851), 24–27, 29  
 Jacobi inversion problem, 7, 20, 30, 31  
 Jordan, Camille (1838–1922), 55
- Kant  
   doctoral dissertation, 44  
 Kant, Immanuel (1724–1804), 41, 44, 46  
 Kummer, Ernst (1810–1893), 8
- L*-function, 39  
 Lagrange, Joseph-Louis (1736–1813), 2, 17, 25, 28, 69, 75  
 Lebesgue, Henri (1875–1941), 56  
 Legendre, Adrien-Marie (1752–1833), 24–28, 31, 33, 38, 39, 54–57  
 Leibniz, Gottfried Wilhelm (1646–1716), 5, 12, 15, 35, 42, 48, 49, 51, 52, 56  
 Lorgna, Antonio Maria (1735–1796), 10
- Maclaurin, Colin (1698–1764), 25  
 Marioni, Giovanni Jacopo (1676–1755), 52
- mechanical linkage, 16  
 Menelaus of Alexandria (c. 170–c. 140), 12  
 Mengoli, Pietro (1625–1686), 35  
 Meusnier de La Place, Jean-Baptiste Marie (1754–1793), 67  
 de Moivre, Abraham (1667–1754), 18, 35  
 Monge, Gaspard (1764–1818), 67  
 multi-valued function, 7, 14
- Newton  
   Principia, 2, 12  
 Newton, Isaac (1643–1727), 2, 12, 15, 35, 76, 77
- Pfaff, Johann Friedrich (1765–1825), 8, 33, 58  
 Plato (5th–4th c. B.C.), 41, 47  
 Poincaré, Henri (1854–1912), 53, 59  
 Poinson, Louis (1777–1859), 55  
 Poisson, Siméon Denis (1781–1840), 67  
 Poisson, Siméon-Denis (1781–1840), 67  
 problem  
   Jacobi inversion, 7, 20, 30, 31  
 Puiseux, Victor-Alexandre (1820–1883), 15
- Riemann  
   habilitation lecture, 1, 8, 46, 58, 77  
   habilitation text, 68  
   doctoral dissertation, 10, 16, 19  
   zeta function, 2, 15, 34, 35, 37, 38, 40  
 Riemann hypothesis, 38  
 Riemann integral, 78, 79  
 Riemann, Bernhard (1826–1866), 3, 6, 15, 16, 19, 20  
 Rodrigues, Olinde (1795–1851), 67  
 Russell, Bertrand (1872–1970), 16
- Schrödinger, Erwin (1887–1961), 61  
 Siegel, Carl Ludwig (1896–1981), 40  
 Simplicius (6th c. A.D.), 48  
 sound  
   theory of, 72, 74  
 space, 41  
*speculum musicum*, 51  
 string vibration, 69, 71, 72
- Tardy, Placido (1816–1914), 47  
 theorem  
   Euler characteristic, 51, 53, 54  
   *Theorema egregium* (Gauss), 64  
 Thom, René (1923–2002), 40, 47  
 transcendental function, 13, 14, 16  
 trigonometric series, 2, 68, 71, 72, 79
- de la Vallée Poussin, Charles Jean (1866–1962), 38

- vibrating string, 69, 71, 72
  - equation, 69
- Volterra, Vito (1860–1940), 10
- Wallis, John (1616–1703), 32, 33, 35, 36
- wave equation, 71
- Weil, André (1906–1998), 3, 11, 22, 29,  
30, 36–38, 40, 80
- Weyl, Hermann (1885–1955), 16, 40, 44
- Zeno of Elea (5th c. B.C.), 15, 47
- zeta function, 2, 15, 34, 35, 37, 38, 40

A. PAPADOPOULOS, INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG AND CNRS, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE, AND BROWN UNIVERSITY, MATHEMATICS DEPARTMENT, 151 THAYER STREET PROVIDENCE, RI 02912, USA.