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On uniformly continuous functions for some profinite topologies*

Dedicated to Antonio Restivo for his 70th birthday

Jean-Éric Pin², Pedro V. Silva³

June 28, 2016

Abstract

Given a variety of finite monoids \mathbf{V} , a subset of a monoid is a \mathbf{V} -subset if its syntactic monoid belongs to \mathbf{V} . A function between two monoids is \mathbf{V} -preserving if it preserves \mathbf{V} -subsets under preimages and it is hereditary \mathbf{V} -preserving if it is \mathbf{W} -preserving for every subvariety \mathbf{W} of \mathbf{V} . The aim of this paper is to study hereditary \mathbf{V} -preserving functions when \mathbf{V} is one of the following varieties of finite monoids: groups, p -groups, aperiodic monoids, commutative monoids and all monoids.

1 Introduction

This article is a follow-up of [12], where the authors started the study of \mathbf{V} -preserving functions. Let us first remind the definition. Let M be a monoid and let \mathbf{V} be a variety of finite monoids. A recognizable subset S of M is said to be a \mathbf{V} -subset if its syntactic monoid belongs to \mathbf{V} . A function $f : M \rightarrow N$ is called \mathbf{V} -preserving if, for each \mathbf{V} -subset L of N , $f^{-1}(L)$ is a \mathbf{V} -subset of M . A function is *hereditary \mathbf{V} -preserving* if it is \mathbf{W} -preserving for every subvariety \mathbf{W} of \mathbf{V} .

Let us first consider the case where f is a function from A^* to B^* , where A and B are finite alphabets. If \mathbf{V} is the variety \mathbf{M} of all finite monoids, a \mathbf{V} -preserving function is also called *regularity-preserving*, according to the terminology used in [5, 16, 18]. The characterization of regularity-preserving functions is a long-term objective, but in spite of intensive research (see [10] for a detailed bibliography), it is still out of reach. For the variety \mathbf{G}_p of finite p -groups, the situation is more advanced. Indeed, the authors gave in [13] a characterization of \mathbf{G}_p -preserving functions when B is a one-letter alphabet

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and a preliminary step towards a general solution can be found in [10]. For the variety \mathbf{G} of finite groups and for the variety \mathbf{A} of finite aperiodic monoids, the only known contribution to the study of \mathbf{V} -preserving functions seems to be the article of Reutenauer and Schützenberger on rational functions [14].

This paper focuses on hereditary \mathbf{V} -preserving functions when \mathbf{V} is one of the varieties \mathbf{M} , \mathbf{G} , \mathbf{G}_p and \mathbf{A} . We consider functions from a free monoid or a free commutative monoid to \mathbb{N} and, in the case of the varieties \mathbf{G} and \mathbf{G}_p , we also study functions from A^* to \mathbb{Z} or from \mathbb{Z}^k to \mathbb{Z} . The case of a one-letter alphabet was also discussed in [3]. Our results are summarized in the table below.

\mathbf{V}	\mathbf{G}_p	\mathbf{G}	\mathbf{A}	\mathbf{M}
$A^* \rightarrow \mathbb{Z}$	Theorem 3.19	Theorem 4.3	Open	Open
$\mathbb{Z}^k \rightarrow \mathbb{Z}$	Theorem 3.5	Theorem 4.1	Irrelevant	Corollary 7.5
$\mathbb{N}^k \rightarrow \mathbb{Z}$	Theorem 3.12	Theorem 4.2	Irrelevant	Open
$\mathbb{N}^k \rightarrow \mathbb{N}$	Theorem 3.12	Theorem 4.2	Theorem 5.4	Theorem 7.2

Characterization of hereditary \mathbf{V} -preserving functions.

2 Preliminaries

In this section, we review the basic notions used in this paper.

2.1 Varieties

A *variety of finite monoids* is a class of finite monoids closed under taking submonoids, quotients and finite direct products. In the sequel, we shall use freely the term *variety* instead of *variety of finite monoids*.

We denote by \mathbf{M} (respectively \mathbf{Com} , \mathbf{G} , \mathbf{Ab} , \mathbf{A}) the variety of all finite monoids (respectively finite commutative monoids, finite groups, finite abelian groups, finite aperiodic monoids). Given a prime number p , we denote by \mathbf{G}_p the variety of all finite p -groups and by \mathbf{Ab}_p the variety of all finite abelian p -groups. Each finite monoid M generates a variety, denoted by (M) . The join of a family of varieties $(\mathbf{V}_i)_{i \in I}$ is the least variety containing all the varieties \mathbf{V}_i , for $i \in I$.

For $n > 0$, C_n denotes the cyclic group of order n . Throughout the paper, we shall use the well-known structure theorem for finite abelian groups [15], which shows that \mathbf{Ab} is the variety generated by the finite cyclic groups.

Proposition 2.1 *Every finite abelian group is isomorphic to a direct product of finite cyclic groups.*

2.2 Ultrametrics and pseudo-ultrametrics

A *pseudo-ultrametric* on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ satisfying the following properties, for all $x, y, z \in X$:

$$(P_1) \quad d(x, y) \geq 0,$$

- (P₂) $d(x, x) = 0$,
(P₃) $d(x, y) = d(y, x)$,
(P₄) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

An *ultrametric* satisfies a stronger version of (P₂):

- (P₅) $d(x, y) = 0$ if and only if $x = y$.

2.3 Uniformly continuous functions

Given two pseudometric spaces (X_1, d_1) and (X_2, d_2) , a function $f : X_1 \rightarrow X_2$ is *uniformly continuous* if, for every positive real number ε there exists a positive real number $\delta > 0$ such that for all $(x, y) \in X^2$,

$$d_1(x, y) < \delta \text{ implies } d_2(f(x), f(y)) < \varepsilon. \quad (2.1)$$

It follows in particular that if $d_1(x, y) = 0$, then $d_2(f(x), f(y)) = 0$. Moreover this condition is sufficient if 0 is an isolated point in the range of d_1 and d_2 . We shall only need a weaker version of this result.

Proposition 2.2 *If d_1 and d_2 have finite range, a function $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is uniformly continuous if and only if*

$$d_1(x, y) = 0 \text{ implies } d_2(f(x), f(y)) = 0. \quad (2.2)$$

Proof. Since d_2 has finite range, there exists a positive real number ε such that $d_2(u, v) < \varepsilon$ implies $d_2(u, v) = 0$. If f is uniformly continuous, there exists δ such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. By the choice of ε , this actually implies $d_2(f(x), f(y)) = 0$ and thus (2.2) holds.

Since d_1 has finite range, there exists a positive real number δ such that $d_1(u, v) < \delta$ implies $d_1(u, v) = 0$. Suppose that (2.2) holds and let ε be a positive integer. If $d_1(u, v) < \delta$ then $d_1(u, v) = 0$ and by (2.2), $d_2(f(x), f(y)) = 0$. It follows in particular that $d_2(f(x), f(y)) < \varepsilon$ and thus f is uniformly continuous. \square

Nonexpansive functions form an interesting subclass of the class of uniformly continuous functions. A function $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is *nonexpansive* if, for all $(x, y) \in X_1 \times X_1$,

$$d_2(f(x), f(y)) \leq d_1(x, y)$$

We shall use nonexpansive functions in Section 3.

2.4 Pro-V metrics

For the remainder of this section, let \mathbf{V} denote a variety of finite monoids. Let M be a monoid and let $u, v \in M$. We say that a monoid N *separates* u and v if there exists a monoid morphism $\varphi : M \rightarrow N$ such that $\varphi(u) \neq \varphi(v)$. A monoid M is *residually \mathbf{V}* if any two distinct elements of M can be separated by a monoid in \mathbf{V} .

We shall use the conventions $\min \emptyset = \infty$ and $2^{-\infty} = 0$. For all $u, v \in M$, let

$$r_{\mathbf{V}}(u, v) = \min \left\{ |N| \mid N \text{ is in } \mathbf{V} \text{ and separates } u \text{ and } v \right\}$$

and $d_{\mathbf{V}}(u, v) = 2^{-r_{\mathbf{V}}(u, v)}$. Then $d_{\mathbf{V}}$ is a pseudo-ultrametric, called the *pro- \mathbf{V}* metric on M (see [12]). If the monoid is residually \mathbf{V} , then $d_{\mathbf{V}}$ is an ultrametric.

In this paper, we consider free monoids, free commutative monoids and free abelian groups of finite rank: they are all finitely generated and residually \mathbf{V} for the main varieties considered in this paper: monoids, (abelian) groups, abelian p -groups, (commutative) aperiodic monoids.

2.5 \mathbf{V} -uniform continuity and \mathbf{V} -hereditary continuity

Let M and N be monoids. A function $f : M \rightarrow N$ is said to be *\mathbf{V} -uniformly continuous* if it is uniformly continuous for the pro- \mathbf{V} pseudometric on M and N . The following result was proved in [12, Theorem 4.1].

Proposition 2.3 *A function $f : M \rightarrow N$ is \mathbf{V} -preserving if and only if it is \mathbf{V} -uniformly continuous.*

We say that f is *\mathbf{V} -hereditarily continuous* if it is \mathbf{W} -uniformly continuous for each subvariety \mathbf{W} of \mathbf{V} . Closure properties of this notion under various operators are analysed in [12, Subsection 4.3].

A monoid N is called *\mathbf{V} -projective* if the following property holds: if $\alpha : N \rightarrow R$ is a morphism and if $\beta : T \rightarrow R$ is a surjective morphism, where T (and hence R) is a monoid of \mathbf{V} , then there exists a morphism $\gamma : N \rightarrow T$ such that $\alpha = \beta \circ \gamma$.

$$\begin{array}{ccc}
 & & T \\
 & \nearrow \gamma & \downarrow \beta \\
 N & & R \\
 & \searrow \alpha & \\
 & &
 \end{array}$$

For example, any free monoid (in particular \mathbb{N}) is \mathbf{V} -projective for every variety of finite monoids. Similarly, any free group (in particular \mathbb{Z}) is \mathbf{V} -projective for every variety of finite groups. Note that a \mathbf{V} -projective monoid is \mathbf{W} -projective for every subvariety \mathbf{W} of \mathbf{V} .

The following results were proved in [12]:

Proposition 2.4 [12, Proposition 5.7] *Let \mathbf{V} be the join of a family $(\mathbf{V}_i)_{i \in I}$ of varieties of finite commutative monoids. A function from a monoid to a \mathbf{V} -projective monoid is \mathbf{V} -hereditarily continuous if and only if it is \mathbf{V}_i -hereditarily continuous for all $i \in I$.*

Proposition 2.5 [12, Proposition 5.4] *A function from a monoid to a commutative monoid is \mathbf{V} -hereditarily continuous if and only if it is $(\mathbf{V} \cap \mathbf{Com})$ -hereditarily continuous.*

In contrast, note that a \mathbf{V} -uniformly continuous function from a monoid to a commutative monoid is not necessarily $(\mathbf{V} \cap \mathbf{Com})$ -hereditarily continuous. For instance, the function f from $\{a, b\}^*$ to \mathbb{N} defined by $f(ab) = 1$ and $f(u) = 0$ if $u \neq ab$ is \mathbf{M} -uniformly continuous but is not \mathbf{Com} -uniformly continuous.

2.6 p -adic valuations

Let p be a prime number. If n is a non-zero integer, the p -adic valuation of n is the integer

$$v_p(n) = \max \{k \in \mathbb{N} \mid p^k \text{ divides } n\}$$

By convention, $v_p(0) = +\infty$. Note that the equality $v_p(nm) = v_p(n) + v_p(m)$ holds for all integers n, m .

The p -adic norm of n is the real number

$$|n|_p = p^{-v_p(n)}.$$

The p -adic norm satisfies the following properties, for all $n, m \in \mathbb{Z}$:

- (N₁) $|n|_p \geq 0$,
- (N₂) $|n|_p = 0$ if and only if $n = 0$,
- (N₃) $|mn|_p = |m|_p |n|_p$,
- (N₄) $|m + n|_p \leq \max\{|m|_p, |n|_p\}$.

The p -adic valuation and the p -adic norm can be extended to \mathbb{Z}^k as follows. Given $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$, we set

$$v_p(n) = \min_{1 \leq j \leq k} \{v_p(n_j)\} \quad \text{and} \quad |n|_p = p^{-v_p(n)} = \max_{1 \leq j \leq k} \{|n_j|_p\}.$$

The p -adic norm on \mathbb{Z}^k still satisfies (N₁), (N₂) and (N₄), as well as the following weaker version of (N₃):

- (N₅) for all $n, m \in \mathbb{Z}^k$, $|mn|_p \leq |m|_p |n|_p$.

The p -adic norm on \mathbb{Z}^k induces the p -adic ultrametric d_p on \mathbb{Z}^k , defined by $d_p(u, v) = |u - v|_p$. Note that the pro- \mathbf{Ab}_p metric $d_{\mathbf{Ab}_p}$ and d_p are strongly equivalent metrics.

2.7 Binomial coefficients

Let A be a finite alphabet. We denote by A^* the free monoid on A . Note that if $|A| = 1$, then A^* is isomorphic to the additive monoid \mathbb{N} .

Let u and v be two words of A^* . Let $u = a_1 \cdots a_n$, with $a_1, \dots, a_n \in A$. Then u is a *subword* of v if there exist $v_0, \dots, v_n \in A^*$ such that $v = v_0 a_1 v_1 \cdots a_n v_n$. Set

$$\binom{v}{u} = |\{(v_0, \dots, v_n) \mid v = v_0 a_1 v_1 \cdots a_n v_n\}|.$$

Note that if $A = \{a\}$, $u = a^n$ and $v = a^m$, then $\binom{v}{u} = \binom{m}{n}$ and hence these numbers constitute a generalization of the classical binomial coefficients. See [7, Chapter 6] for more details. Sometimes, it will be useful to use the convention $\binom{m}{n} = 0$ for $m \geq 0$ and $n \in \mathbb{Z} \setminus \{0, \dots, m\}$, which is compatible with the usual properties of binomial coefficients.

2.8 Mahler expansions

For a fixed $v \in A^*$, we can view the generalized binomial coefficient $\binom{\cdot}{v}$ as a function from A^* to \mathbb{N} . The functions $\{\binom{\cdot}{v} \mid v \in A^*\}$ constitute a *locally finite* family of functions in the sense that, for each $u \in A^*$, the image of u is 0 for all but finitely many elements of the family.

It is clear that the sum of a locally finite family of functions is well defined. In particular, if $(g_v)_{v \in A^*}$ is a family of elements of an abelian group G , then there is a well-defined function f from A^* into G defined by the formula (in additive notation)

$$f(u) = \sum_{v \in A^*} g_v \binom{u}{v}$$

The generalized binomial coefficients provide a unique decomposition of the functions from A^* into G , which will be referred as *Mahler expansion*:

Proposition 2.6 (Lothaire [7]) *Let G be an abelian group and let $f : A^* \rightarrow G$ be an arbitrary function. Then there exists a unique family $\langle f, v \rangle_{v \in A^*}$ of elements of G such that, for all $u \in A^*$, $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$. This family is given by the inversion formula*

$$\langle f, v \rangle = \sum_{w \in A^*} (-1)^{|v|+|w|} \binom{v}{w} f(w) \quad (2.3)$$

A similar result holds for functions from \mathbb{N}^k to an abelian group G . If r is an element of \mathbb{N}^k (or more generally of \mathbb{Z}^k), we denote by r_i its i -th component, so that $r = (r_1, \dots, r_k)$. First observe that the family

$$\left\{ \binom{-}{r_1} \cdots \binom{-}{r_k} \mid r \in \mathbb{N}^k \right\}$$

is a locally finite family of functions from \mathbb{N}^k into \mathbb{N} . Thus, given a family $(g_r)_{r \in \mathbb{N}^k}$, the formula

$$f(n) = \sum_{r \in \mathbb{N}^k} g_r \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

defines a function $f : \mathbb{N}^k \rightarrow G$. Conversely, each function from \mathbb{N}^k to G admits a unique *Mahler expansion*, a result proved in a more general setting in [2, 1].

Proposition 2.7 *Let G be an abelian group and let $f : \mathbb{N}^k \rightarrow G$ be an arbitrary function. Then there exists a unique family $\langle f, r \rangle_{r \in \mathbb{N}^k}$ of elements of G such that, for all $n \in \mathbb{N}^k$,*

$$f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}.$$

The coefficients $\langle f, r \rangle$ are given by

$$\langle f, r \rangle = \sum_{i_1=0}^{r_1} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\dots+r_k+i_1+\dots+i_k} \binom{r_1}{i_1} \cdots \binom{r_k}{i_k} f(i).$$

3 \mathbf{G}_p -hereditary continuity

Let p be a prime number. We proved in [11, 13] that \mathbf{G}_p -uniformly continuous functions from A^* to \mathbb{Z} can be characterized by properties of their Mahler expansions. The case where A is a one-letter alphabet corresponds to the classical Mahler's Theorem from p -adic number theory [8, 9].

Theorem 3.1 *Let $f : A^* \rightarrow \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then the following conditions are equivalent:*

- (1) f is \mathbf{G}_p -uniformly continuous;
- (2) $\lim_{|v| \rightarrow \infty} |\langle f, v \rangle|_p = 0$.

A similar result (Amice, [2]) holds when A^* is replaced by \mathbb{Z}^k (see also [13, Corollary 6.3] for an alternative proof). In this section, we obtain analogous results for \mathbf{G}_p -hereditary continuity. A first step is to reduce \mathbf{G}_p -hereditary continuity to a simpler property.

Lemma 3.2 *A function from a monoid to a \mathbf{G}_p -projective commutative monoid is \mathbf{G}_p -hereditarily continuous if and only if it is (C_{p^n}) -uniformly continuous for all $n > 0$.*

Proof. By Proposition 2.5, f is \mathbf{G}_p -hereditarily continuous if and only if it is $(\mathbf{G}_p \cap \mathbf{Com})$ -hereditarily continuous. Since

$$\mathbf{G}_p \cap \mathbf{Com} = \mathbf{G}_p \cap \mathbf{Ab} = \mathbf{Ab}_p = \bigvee_{n>0} (C_{p^n})$$

by Proposition 2.1, Proposition 2.4 implies that f is \mathbf{G}_p -hereditarily continuous if and only if f is (C_{p^n}) -hereditarily continuous for every $n \in \mathbb{N}$. Since the only subvarieties of (C_{p^n}) are those of the form (C_{p^i}) with $i \leq n$, the lemma follows. \square

Let \mathbf{V} be a variety of groups. Since any morphism from \mathbb{N}^k to a finite group extends uniquely to a morphism from \mathbb{Z}^k to that same group, the pro- \mathbf{V} pseudo-metric on \mathbb{N}^k is the restriction of the pro- \mathbf{V} pseudo-metric on \mathbb{Z}^k . Therefore the forthcoming results hold for \mathbb{N}^k even though they are stated and proved for \mathbb{Z}^k .

We denote by e_1, \dots, e_k the canonical generators of both \mathbb{N}^k and \mathbb{Z}^k . Thus $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 occurs in position j .

Lemma 3.3 *Let $n \in \mathbb{N}$ and let d be the pro- (C_{p^n}) pseudo-metric on \mathbb{Z}^k . For $r, s \in \mathbb{Z}^k$, one has $d(r, s) = 2^{-p^m}$ where*

$$m = \min \{ i \leq n \mid \text{there exists } j \in \{1, \dots, k\} \text{ such that } r_j \not\equiv s_j \pmod{p^i} \}.$$

Proof. Suppose that $r_j \not\equiv s_j \pmod{p^i}$ for some $i \leq n$ and $j \in \{1, \dots, k\}$. Let $f : \mathbb{Z}^k \rightarrow C_{p^i}$ be defined by $f(n) = n_j$. Clearly, $C_{p^i} \in (C_{p^n})$ and f separates r and s , hence $d(r, s) \geq 2^{-p^i}$ and so $d(r, s) \geq 2^{-p^m}$. Note that this last inequality holds trivially if $m = \infty$.

If $d(r, s) = 0$, equality follows. Otherwise, we may assume that $f : \mathbb{Z}^k \rightarrow G \in (C_{p^n})$ is a morphism that separates r and s with $|G|$ minimum. By Proposition 2.1, G is a direct product of cyclic groups. Since their order must divide $|G|$ which is a power of p , each one of these factor groups is of the form C_{p^i} . Since any group in (C_{p^n}) must satisfy the identity $x^{p^n} = 1$, we conclude that $i \leq n$ in each case. If G were a nontrivial direct product, we could decompose f into its components and contradict the minimality of G , thus $G = C_{p^i}$ with $i \leq n$.

Suppose that $r_j \equiv s_j \pmod{p^i}$ for every $j \in \{1, \dots, k\}$. Then $r_j = s_j$ in C_{p^i} for every j and so

$$f(r) = \sum_{j=1}^k r_j f(e_j) = \sum_{j=1}^k s_j f(e_j) = f(s),$$

a contradiction. Thus $r_j \not\equiv s_j \pmod{p^i}$ for some $j \in \{1, \dots, k\}$ and so $i \geq m$. It follows that $d(r, s) = 2^{-p^i} \leq 2^{-p^m}$ and so $d(r, s) = 2^{-p^m}$ as required. \square

The next corollary shows how the pro- (C_{p^n}) pseudo-metric relates to the p -adic norm:

Corollary 3.4 *Let $n \in \mathbb{N}$ and let d denote the pro- (C_{p^n}) pseudo-metric on \mathbb{Z}^k . For all $r, s \in \mathbb{Z}^k$, we have*

$$d(r, s) = \begin{cases} 2^{-\frac{p}{|r-s|_p}} & \text{if } |r-s|_p > p^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let

$$m = \min \{i \leq n \mid \text{exists } j \in \{1, \dots, k\} \text{ such that } r_j \not\equiv s_j \pmod{p^i}\}.$$

It is easy to check that

$$m = \begin{cases} v_p(r-s) + 1 & \text{if } v_p(r-s) < n \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, $v_p(r-s) < n$ if and only if $|r-s|_p > p^{-n}$. In this case,

$$p^m = p^{v_p(r-s)+1} = \frac{p}{|r-s|_p}$$

and the claim follows from Lemma 3.3. \square

We arrive to our characterization of \mathbf{G}_p -hereditarily continuous functions.

Theorem 3.5 *A function from \mathbb{Z}^k to \mathbb{Z} is \mathbf{G}_p -hereditarily continuous if and only if it is nonexpansive for the p -adic norm.*

Proof. Let d_n denote the pro- (C_{p^n}) pseudo-metric. By Lemma 3.2, f is hereditarily \mathbf{G}_p -uniformly continuous if and only if, for all $n > 0$, it is uniformly continuous for d_n . By Proposition 2.2, this holds if and only if, for all $r, s \in \mathbb{Z}^k$,

$$d_n(r, s) = 0 \text{ implies } d_n(f(r), f(s)) = 0. \quad (3.4)$$

By Corollary 3.4, $d_n(r, s) = 0$ if and only if $|r-s|_p \leq p^{-n}$, thus (3.4) is equivalent to stating that for all $r, s \in \mathbb{Z}^k$,

$$|r-s|_p \leq p^{-n} \text{ implies } |f(r) - f(s)|_p \leq p^{-n}. \quad (3.5)$$

Clearly, (3.5) holds for every n if and only if $|f(r) - f(s)|_p \leq |r-s|_p$, which proves the result. \square

It follows easily from Theorem 3.5 that all polynomial functions from \mathbb{Z}^k to \mathbb{Z} are \mathbf{G}_p -hereditarily continuous. We shall use the Mahler expansion of functions given by Proposition 2.7 to characterize all the \mathbf{G}_p -hereditarily continuous functions from \mathbb{N}^k to \mathbb{Z} . Polynomial functions will appear then as the finitely generated case. We shall need a few lemmas:

Lemma 3.6 *The sum of a locally finite family of \mathbf{G}_p -hereditarily continuous functions from \mathbb{N}^k to \mathbb{Z} is \mathbf{G}_p -hereditarily continuous.*

Proof. Let $\{f_i : \mathbb{N}^k \rightarrow \mathbb{Z} \mid i \in I\}$ be a locally finite family of \mathbf{G}_p -hereditarily continuous functions and let $f = \sum_{i \in I} f_i$. By Theorem 3.5, each f_i is nonexpansive for the p -adic norm, and since the p -adic norm satisfies (N₄), f is also nonexpansive. \square

The following result is due to Kummer [6]. See also [17, 4].

Proposition 3.7 *Let $n, r \in \mathbb{N}$ with $0 \leq r \leq n$. Then $v_p\left(\binom{n}{r}\right)$ is equal to the number of carries it takes to add r and $n - r$ in base p .*

Taking $n = p^s$ yields the following corollary

Lemma 3.8 *Let $r, s \in \mathbb{N}$ with $0 < r \leq p^s$. Then $v_p\left(\binom{p^s}{r}\right) = s - v_p(r)$.*

We also need a result stated in [3, Lemma 2.8], for which we give a shorter proof.

Lemma 3.9 *Let $n, r, s \in \mathbb{N}$. Then*

$$p^s \text{ divides } \left(\operatorname{lcm}_{1 \leq j \leq r} j \right) \left(\binom{n+p^s}{r} - \binom{n}{r} \right) \quad (3.6)$$

or equivalently,

$$s \leq \max_{1 \leq j \leq r} v_p(j) + v_p\left(\binom{n+p^s}{r} - \binom{n}{r}\right). \quad (3.7)$$

Proof. Since $\binom{n+p^s}{r} - \binom{n}{r} = \sum_{j=1}^r \binom{p^s}{j} \binom{n}{r-j}$, one gets by Lemma 3.8 the relation

$$\left| \binom{n+p^s}{r} - \binom{n}{r} \right|_p \leq \max_{1 \leq j \leq r} \left| \binom{p^s}{j} \right|_p \left| \binom{n}{r-j} \right|_p \leq \max_{1 \leq j \leq r} \left| \binom{p^s}{j} \right|_p = \max_{1 \leq j \leq r} p^{v_p(j)-s}$$

or equivalently,

$$v_p\left(\binom{n+p^s}{r} - \binom{n}{r}\right) \geq \min_{1 \leq j \leq r} (s - v_p(j)) = s - \max_{1 \leq j \leq r} v_p(j)$$

which gives (3.7). \square

We shall need two elementary results on nonexpansive functions.

Lemma 3.10 *Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ be a nonexpansive function for the p -adic norm and let $s \in \mathbb{N}$. Then for $0 \leq i \leq p^s$, p^s divides $\binom{p^s}{i} (f(i) - f(0))$, or equivalently, $s \leq v_p\left(\binom{p^s}{i}\right) + v_p(f(i) - f(0))$.*

Proof. Since f is nonexpansive, one has $|f(i) - f(0)|_p \leq |i - 0|_p$ and thus $v_p(f(i) - f(0)) \geq v_p(i)$. Since $v_p\left(\binom{p^s}{i}\right) = s - v_p(i)$ by Lemma 3.8, the relation $s \leq v_p\left(\binom{p^s}{i}\right) + v_p(f(i) - f(0))$ follows immediately. \square

Corollary 3.11 *Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ be a nonexpansive function for the p -adic norm and let $s \in \mathbb{N}$. Then p^s divides $\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i)$.*

Proof. Newton's binomial formula yields

$$0 = (1 - 1)^{p^s} = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i},$$

hence

$$\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i) = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} (f(i) - f(0)).$$

The result now follows from Lemma 3.10. \square

Theorem 3.12 *Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$ be the Mahler expansion of a function $f : \mathbb{N}^k \rightarrow \mathbb{Z}$. Then the following conditions are equivalent:*

- (1) f is \mathbf{G}_p -hereditarily continuous,
- (2) $v_p(j) \leq v_p(\langle f, r \rangle)$ holds for all j, r such that $1 \leq j \leq \max\{r_1, \dots, r_k\}$.

Proof. (1) \Rightarrow (2). For all $r, t \in \mathbb{N}^k$, let us set

$$m_r(t) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\dots+r_k+i_1+\dots+i_k} \binom{r_1}{i_1} \cdots \binom{r_k}{i_k} f(i+t).$$

By Proposition 2.7, we have $m_r(0, \dots, 0) = \langle f, r \rangle$. We next show that

$$\min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\} \leq \min_{t \in \mathbb{N}^k} \{v_p(m_{r+s}(t))\} \quad (3.8)$$

for all $r, s \in \mathbb{N}^k$. By transitivity, we may assume that $s_1 + \dots + s_k = 1$. By symmetry, we may assume that $s = (1, 0, \dots, 0)$. Let $\ell = \min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\}$. For all $t \in \mathbb{N}^k$, we have

$$\begin{aligned} m_{r+s}(t) &= \sum_{i_1=0}^{r_1+1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\dots+r_k+i_1+\dots+i_k} \binom{1+r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i+t) \\ &= \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\dots+r_k+i_1+\dots+i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i+t) \\ &\quad + \sum_{i_1=1}^{r_1+1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\dots+r_k+i_1+\dots+i_k} \binom{r_1}{i_1-1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i+t) \\ &= - \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\dots+r_k+i_1+\dots+i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i+t) \\ &\quad + \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\dots+r_k+i_1+\dots+i_k} \binom{r_1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_k}{i_k} f(i+t+s) \\ &= -m_r(t) + m_r(t+s). \end{aligned}$$

Since $p^\ell \mid m_r(t)$ and $p^\ell \mid m_r(t+s)$, it follows that $p^\ell \mid m_{r+s}(t)$ and so (3.8) holds.

Now we show that

$$s \leq v_p(m_{p^s e_j}(t)) \quad (3.9)$$

for all $s \in \mathbb{N}$, $t \in \mathbb{N}^k$ and $j = 1, \dots, k$.

By symmetry, we may assume that $j = 1$, so that (3.9) becomes

$$p^s \mid \sum_{i=0}^{p^s} (-1)^{p^s+i} \binom{p^s}{i} f(i + t_1, t_2, \dots, t_k). \quad (3.10)$$

Fix $t \in \mathbb{N}^k$ and let $g : \mathbb{N} \rightarrow \mathbb{Z}$ be the function defined by

$$g(n) = f(n + t_1, t_2, \dots, t_k).$$

By Theorem 3.5, g is \mathbf{G}_p -hereditarily continuous and thus (3.10) follows from Corollary 3.11. Therefore (3.10) holds and so does (3.9).

We now show that

$$1 \leq j \leq \max\{r_1, \dots, r_k\} \Rightarrow v_p(j) \leq v_p(m_r(t)) \quad (3.11)$$

holds for all $j \in \mathbb{N}$ and $r, t \in \mathbb{N}^k$.

We use induction on $q = r_1 + \dots + r_k$. The claim holds trivially for $q = 0$, hence we assume that $q > 0$ and (3.11) holds for smaller values of q . By symmetry, we may assume that $r_1 > 0$.

Assume first that $1 \leq j \leq \max\{r_1 - 1, \dots, r_k\}$. By the induction hypothesis on q , we have $v_p(j) \leq v_p(m_{r_1-1, r_2, \dots, r_k}(t))$ for all $t \in \mathbb{N}^k$. Thus $v_p(j) \leq v_p(m_r(t))$ by (3.8).

The remaining case corresponds to $j = r_1 > \max\{r_1 - 1, \dots, r_k\}$. If j is not a power of p , then we may write $j = j_1 j_2$ with $j_1 < j$ and $v_p(j_1) = v_p(j)$, falling into the previous case. Thus we may assume that $j = p^i$ for some $i \in \mathbb{N}$. By (3.9), we have $i \leq v_p(m_{p^i, 0, \dots, 0}(t))$ for all $t \in \mathbb{N}^k$. Since $r_1 = j = p^i$, it follows from (3.8) that $v_p(j) = i \leq v_p(m_r(t))$ and (3.11) holds.

Considering now the particular case $t = 0$, we obtain Condition (2).

(2) \Rightarrow (1). By Lemma 3.6, it is enough to show that the function

$$g(n) = \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

is \mathbf{G}_p -hereditarily continuous for a fixed $r \in \mathbb{N}^k$. Write $m = \langle f, r \rangle$. Let $x, y \in \mathbb{N}^k$ and assume that $p^s \mid x - y$. By Theorem 3.5, it suffices to show that

$$p^s \mid m \left(\binom{x_1}{r_1} \cdots \binom{x_k}{r_k} - \binom{y_1}{r_1} \cdots \binom{y_k}{r_k} \right). \quad (3.12)$$

We have $p^s \mid x - y$ if and only if $y = x + p^s z$ for some $z \in \mathbb{Z}^k$. Clearly, we can obtain y from x by successively adding or subtracting $p^s e_i$ ($i = 1, \dots, k$). Since $p^s \mid \ell$ and $p^s \mid \ell'$ together imply $p^s \mid \ell - \ell'$, we may assume without loss of generality that $x = y + p^s e_i$. By symmetry, we may also assume that $i = 1$. Therefore (3.12) will follow from

$$p^s \mid m \left(\binom{y_1 + p^s}{r_1} - \binom{y_1}{r_1} \right). \quad (3.13)$$

By condition (2), we have $v_p(j) \leq v_p(m)$ if $1 \leq j \leq r_1$, hence Lemma 3.9 yields

$$s \leq \max_{1 \leq j \leq r_1} v_p(j) + \nu_p \left(\binom{y_1 + p^s}{r_1} - \binom{y_1}{r_1} \right) \leq v_p(m) + \nu_p \left(\binom{y_1 + p^s}{r_1} - \binom{y_1}{r_1} \right)$$

and (3.13) holds as required. \square

It followed from Theorem 3.5 that all polynomial functions $f : \mathbb{N}^k \rightarrow \mathbb{Z}$ with integer coefficients are \mathbf{G}_p -hereditarily continuous. There are of course only countably many such functions. Theorem 3.12 implies the existence of uncountably many \mathbf{G}_p -hereditarily continuous functions:

Corollary 3.13 *There are uncountably many \mathbf{G}_p -hereditarily continuous functions $f : \mathbb{N}^k \rightarrow \mathbb{Z}$.*

Proof. For every $r \in \mathbb{N}^k$, let

$$\ell_r = \max\{v_p(j) \mid 1 \leq j \leq \max\{r_1, \dots, r_k\}\}.$$

By Theorem 3.12 and Proposition 2.7, the map

$$(n_r)_{r \in \mathbb{N}^k} \mapsto \sum_{r \in \mathbb{N}^k} p^{\ell_r} n_r \binom{-}{r_1} \cdots \binom{-}{r_k}$$

is a bijection between $\mathbb{Z}^{(\mathbb{N}^k)}$ and the set of all \mathbf{G}_p -hereditarily continuous functions from \mathbb{N}^k to \mathbb{Z} . \square

We now consider functions from a free monoid A^* to \mathbb{Z} . Let $h : A^* \rightarrow \mathbb{N}^A$ be the canonical morphism defined by $h(u) = (|u|_a)_{a \in A}$, where $|u|_a$ denotes as usual the number of occurrences of the letter a in u . Let \sim be the *commutative equivalence*, formally defined by $u \sim v$ if and only if $h(u) = h(v)$.

Lemma 3.14 *Let $f : A^* \rightarrow \mathbb{Z}$ be a \mathbf{G}_p -hereditarily continuous function and let $u, v \in A^*$ be commutatively equivalent. Then $f(u) = f(v)$.*

Proof. Let us choose s such that $p^s > |f(u) - f(v)|$ and let d (respectively d') be the pro- (C_{p^s}) pseudo-metric on A^* (respectively \mathbb{Z}). Since f is hereditarily \mathbf{G}_p -uniformly continuous, it is in particular (C_{p^s}) -uniformly continuous. Now, if u and v are commutatively equivalent, then $d(x, y) = 0$ and hence $d'(f(x), f(y)) = 0$, which means that $f(x) \equiv f(y) \pmod{p^s}$. Since $p^s > |f(u) - f(v)|$, this finally implies that $f(u) = f(v)$. \square

Lemma 3.15 *Let $u \in A^*$ and $r = (r_a)_{a \in A} \in \mathbb{N}^A$. Then $\sum_{v \in h^{-1}(r)} \binom{u}{v} = \prod_{a \in A} \binom{|u|_a}{r_a}$.*

Proof. Let $\mathbb{Z}\langle A \rangle$ be the ring of polynomials in noncommutative variables in A with integer coefficients. The monoid morphism μ from A^* to the multiplicative monoid $\mathbb{Z}\langle A \rangle$ defined, for each letter $a \in A$, by $\mu(a) = 1 + a$, is called the *Magnus transformation*. By [7, Proposition 6.3.6], the following formula holds for all $u \in A^*$:

$$\mu(u) = \sum_{v \in A^*} \binom{u}{v} v \tag{3.14}$$

Let $\mathbb{Z}[A]$ be the ring of polynomials in commutative variables in A with integer coefficients. The commutative version of the Magnus transformation is the monoid morphism $\underline{\mu}$ from A^* to the multiplicative monoid $\mathbb{Z}[A]$ defined, for

each letter $a \in A$, by $\underline{\mu}(a) = 1 + a$. Thus by definition, one has, for each word $v \in A^*$,

$$\underline{\mu}(u) = \prod_{a \in A} (1+a)^{|u|_a} = \prod_{a \in A} \left(\sum_{0 \leq r_a \leq |u|_a} \binom{|u|_a}{r_a} a^{r_a} \right) = \sum_{0 \leq r_a \leq |u|_a} \left(\prod_{a \in A} \binom{|u|_a}{r_a} \right) \prod_{a \in A} a^{r_a} \quad (3.15)$$

and on the other hand, (3.14) shows that

$$\underline{\mu}(u) = \sum_{v \in A^*} \binom{u}{v} \prod_{a \in A} a^{|v|_a} = \sum_{r \in \mathbb{N}^A} \left(\sum_{v \in h^{-1}(r)} \binom{u}{v} \right) \prod_{a \in A} a^{r_a} \quad (3.16)$$

Comparing (3.15) and (3.16) now gives the formula $\sum_{v \in h^{-1}(r)} \binom{u}{v} = \prod_{a \in A} \binom{|u|_a}{r_a}$. \square

Lemma 3.16 *Let $f : A^* \rightarrow G$ be a function from A^* to some abelian group with Mahler expansion $f(-) = \sum_{w \in A^*} \langle f, w \rangle \binom{-}{w}$. Then the following conditions are equivalent:*

- (1) for any two commutatively equivalent words u and v , $\langle f, u \rangle = \langle f, v \rangle$,
- (2) for any two commutatively equivalent words u and v , $f(u) = f(v)$.

Proof. (1) implies (2). Suppose that (2) holds. For each $r \in \mathbb{N}^k$, let $\langle k, r \rangle$ be the common value of $\langle f, v \rangle$ for all $v \in h^{-1}(r)$. With the help of Lemma 3.15, we now obtain

$$f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \sum_{v \in h^{-1}(r)} \langle k, r \rangle \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \langle k, r \rangle \sum_{v \in h^{-1}(r)} \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \langle k, r \rangle \prod_{a \in A} \binom{|u|_a}{r_a}$$

It follows immediately that if u and v are commutatively equivalent, then $f(u) = f(v)$.

(2) implies (1). Let $g : A^* \rightarrow G$ be the function defined by $g(u) = (-1)^{|u|} \langle f, u \rangle$. It follows from the inversion formula (2.3) that $\langle g, x \rangle = (-1)^{|x|} f(x)$. Thus if (2) holds, then for any two commutatively equivalent words u and v , $\langle g, u \rangle = \langle g, v \rangle$. By the first part of the proof applied to g , it follows that $g(u) = g(v)$ and thus $\langle f, u \rangle = \langle f, v \rangle$. \square

Lemma 3.17 *Let $g : \mathbb{N}^k \rightarrow \mathbb{Z}$ be a function and let \mathbf{V} be a variety of finite groups. Then g is \mathbf{V} -hereditarily continuous if and only if $g \circ h$ is \mathbf{V} -hereditarily continuous.*

Proof. By Proposition 2.5, g or $g \circ h$ are \mathbf{V} -hereditarily continuous if and only if they are $(\mathbf{V} \cap \mathbf{Ab})$ -hereditarily continuous. Let \mathbf{W} be a subvariety of $\mathbf{V} \cap \mathbf{Ab}$ and let d denote the pro- \mathbf{W} pseudo-metric. Since h is surjective, every element of \mathbb{N}^k can be written in the form $h(u)$ for some $u \in A^*$. Therefore g is \mathbf{W} -uniformly continuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $u, v \in A^*$,

$$d(h(u), h(v)) < \delta \text{ implies } d(g \circ h(u), g \circ h(v)) < \varepsilon \quad (3.17)$$

Since any morphism from A^* to an abelian group factors through \mathbb{N}^k , one has $d(u, v) = d(h(u), h(v))$ for all $u, v \in A^*$. Therefore (3.18) can be rewritten as

$$d(u, v) < \delta \text{ implies } d(g \circ h(u), g \circ h(v)) < \varepsilon \quad (3.18)$$

and thus g is \mathbf{W} -uniformly continuous if and only if $g \circ h$ is \mathbf{W} -uniformly continuous. \square

Lemma 3.18 *Let $g : \mathbb{N}^k \rightarrow \mathbb{Z}$ be a function and let*

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle_{\binom{n_1}{r_1}} \cdots \binom{n_k}{r_k} \text{ and } g \circ h(u) = \sum_{v \in A^*} \langle g \circ h, v \rangle_{\binom{u}{v}}$$

be the Mahler expansions of g and $g \circ h$. Then $\langle g, r \rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle$ for every $r \in \mathbb{N}^k$.

Proof. We have

$$\begin{aligned} g(n) &= g \circ h(a_1^{n_1} \cdots a_k^{n_k}) = \sum_{v \in A^*} \langle g \circ h, v \rangle_{\binom{a_1^{n_1} \cdots a_k^{n_k}}{v}} \\ &= \sum_{v \in a_1^* \cdots a_k^*} \langle g \circ h, v \rangle_{\binom{a_1^{n_1} \cdots a_k^{n_k}}{v}} = \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle_{\binom{n_1}{r_1}} \cdots \binom{n_k}{r_k}. \end{aligned}$$

By the uniqueness of the Mahler expansion in Proposition 2.7, we conclude that $\langle g, r \rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle$ for every $r \in \mathbb{N}^k$. \square

Theorem 3.19 *Let $f : A^* \rightarrow \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle_{\binom{u}{v}}$ be its Mahler expansion. Then f is \mathbf{G}_p -hereditarily continuous if and only if it satisfies the following conditions:*

- (1) *for any two commutatively equivalent words u and v , $\langle f, u \rangle = \langle f, v \rangle$,*
- (2) *$v_p(j) \leq v_p(\langle f, v \rangle)$ holds for all $v \in A^*$ and $1 \leq j \leq \max_{a \in A} |v|_a$.*

Proof. Assume that f is \mathbf{G}_p -hereditarily continuous. By Lemmas 3.14 and 3.16, condition (1) holds. Moreover, by Lemma 3.14, we may write $f = g \circ h$, where $h : A^* \rightarrow \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \rightarrow \mathbb{Z}$ is defined by

$$g(n) = f(a_1^{n_1} \cdots a_k^{n_k}).$$

By Lemma 3.18, the Mahler expansion

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle_{\binom{n_1}{r_1}} \cdots \binom{n_k}{r_k}$$

of g is defined by $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$.

Assume that $v \in A^*$ and $j \in \mathbb{N}$ are such that $1 \leq j \leq |v|_{a_i}$ for every $i \in \{1, \dots, k\}$. Let $r = (|v|_{a_1}, \dots, |v|_{a_k})$. By Lemma 3.17, g is \mathbf{G}_p -hereditarily continuous and so we get $v_p(j) \leq v_p(\langle g, r \rangle) = v_p(\langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle)$ by Theorem 3.12. Since $v \sim a_1^{r_1} \cdots a_k^{r_k}$, we get $\langle f, v \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$ by Lemma 3.16 and so $v_p(j) \leq v_p(\langle f, v \rangle)$. Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, $f(u) = f(v)$ whenever $u \sim v$ and so there exists a function $g : \mathbb{N}^k \rightarrow \mathbb{Z}$ such that $f = g \circ h$. By Lemma 3.17, it suffices to show that g is \mathbf{G}_p -hereditarily continuous. Let

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

be the Mahler expansion of g and suppose that $1 \leq j \leq \max\{r_1, \dots, r_k\}$. By Theorem 3.12, we only need to show that

$$v_p(j) \leq v_p(\langle g, r \rangle). \quad (3.19)$$

By Lemma 3.18, we have $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$. Since

$$1 \leq j \leq \max\{r_1, \dots, r_k\} = \max\{|a_1^{r_1} \cdots a_k^{r_k}|_{a_1}, \dots, |a_1^{r_1} \cdots a_k^{r_k}|_{a_k}\},$$

it follows from condition (2) that $v_p(j) \leq v_p(\langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle)$ and so (3.19) holds as required. \square

4 \mathbf{G} -hereditary continuity

Let \mathbb{P} denote the set of all positive primes.

Theorem 4.1 *A function from \mathbb{Z}^k to \mathbb{Z} is \mathbf{G} -hereditarily continuous if and only if, for each prime p , it is nonexpansive for the p -adic norm.*

Proof. Since $\mathbf{G} \cap \mathbf{Com} = \bigvee_{p \in \mathbb{P}} (\mathbf{G}_p \cap \mathbf{Com})$, it follows from Propositions 2.4 and 2.5 that a function from \mathbb{Z}^k to \mathbb{Z} is \mathbf{G} -hereditarily continuous if and only if it is \mathbf{G}_p -hereditarily continuous for every $p \in \mathbb{P}$. It now remains to apply Theorem 3.5 to conclude. \square

Theorem 3.12 yields:

Theorem 4.2 *Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$ be the Mahler expansion of a function $f : \mathbb{N}^k \rightarrow \mathbb{Z}$. Then the following conditions are equivalent:*

- (1) f is \mathbf{G} -hereditarily continuous;
- (2) j divides $\langle f, r \rangle$ for all $j \in \mathbb{N}$ and $r \in \mathbb{N}^k$ such that $1 \leq j \leq \max\{r_1, \dots, r_k\}$.

We present now the analogue of Theorem 3.19 through an adaptation of its proof. We keep the notation introduced in Section 3.

Theorem 4.3 *Let $f : A^* \rightarrow \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then f is \mathbf{G} -hereditarily continuous if and only if it satisfies the following conditions:*

- (1) if u and v are commutatively equivalent, then $\langle f, u \rangle = \langle f, v \rangle$,
- (2) j divides $\langle f, v \rangle$ for all $v \in A^*$ and $1 \leq j \leq \max_{a \in A} |v|_a$.

Proof. Assume that f is \mathbf{G} -hereditarily continuous. Since \mathbf{G} -hereditarily continuous implies \mathbf{G}_p -hereditarily continuous, Lemma 3.14 remains valid for \mathbf{G} . Together with Lemma 3.16, this yields condition (1). Moreover, by Lemma 3.14, we may write $f = gh$, where $h : A^* \rightarrow \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \rightarrow \mathbb{Z}$ is defined by

$$g(n) = f(a_1^{n_1} \cdots a_k^{n_k})$$

By Lemma 3.18, the Mahler expansion

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

of g is defined by $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$.

Assume that $1 \leq j \leq |v|_{a_i}$ for some $v \in A^*$ and $i \in \{1, \dots, k\}$. Let $r = (|v|_{a_1}, \dots, |v|_{a_k})$. By Lemma 3.17, g is \mathbf{G} -hereditarily continuous and so we get $j \mid \langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$ by Theorem 4.2. Since $v \sim a_1^{r_1} \cdots a_k^{r_k}$, we get $\langle f, v \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$ by Lemma 3.16 and so $j \mid \langle f, v \rangle$. Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, $f(u) = f(v)$ whenever $u \sim v$ and so there exists a function $g : \mathbb{N}^k \rightarrow \mathbb{Z}$ such that $f = gh$. By Lemma 3.17, it suffices to show that g is \mathbf{G} -hereditarily continuous. Let

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

be the Mahler expansion of g and suppose that $1 \leq j \leq \max\{r_1, \dots, r_k\}$. By Theorem 4.2, we only need to show that

$$j \mid \langle g, r \rangle. \quad (4.20)$$

By Lemma 3.18, we have $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$. Since

$$1 \leq j \leq \max\{r_1, \dots, r_k\} = \max\{|a_1^{r_1} \cdots a_k^{r_k}|_{a_1}, \dots, |a_1^{r_1} \cdots a_k^{r_k}|_{a_k}\},$$

it follows from condition (2) that $j \mid \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$ and so (4.20) holds as required. \square

5 A-uniform continuity

Given a variety \mathbf{V} , let $\mathbf{CV} = \mathbf{Com} \cap \mathbf{V}$. In particular \mathbf{CA} is the variety of commutative and aperiodic monoids. For each $t \in \mathbb{N}$, let $\mathbf{A}_t = \llbracket x^{t+1} = x^t \rrbracket$ and $\mathbf{CA}_t = \mathbf{Com} \cap \mathbf{A}_t$ be the variety of commutative aperiodic monoids of exponent t .

Let also N_t denote the monogenic monoid presented by $\langle x \mid x^t = x^{t+1} \rangle$. We usually view N_t as a quotient of \mathbb{N} in order to represent its elements by natural numbers. The following results are folklore.

Proposition 5.1 *Every variety of commutative monoids is generated by its monogenic monoids. In particular $\mathbf{CA}_t = (N_t)$ for every $t \in \mathbb{N}$. Moreover, if $\mathbf{V} \subseteq \mathbf{CA}$, then $\mathbf{V} = \mathbf{CA}_t$ for some $t \in \mathbb{N}$.*

Given $m, n \in \mathbb{N}$, let us set

$$(m \wedge n) = \begin{cases} \min\{m, n\} & \text{if } m \neq n \\ \infty & \text{if } m = n \end{cases}$$

More generally, for $u, v \in \mathbb{N}^k$, we set write

$$(u \wedge v) = \min\{u_1 \wedge v_1, \dots, u_k \wedge v_k\}.$$

Lemma 5.2 *Let $u, v \in \mathbb{N}^k$ and $t \in \mathbb{N}$. Then:*

- (1) $r_{\mathbf{A}}(u, v) = r_{\mathbf{CA}}(u, v) = (u \wedge v) + 2$;
- (2) $r_{\mathbf{A}_t}(u, v) = r_{\mathbf{CA}_t}(u, v) = \begin{cases} (u \wedge v) + 2 & \text{if } (u \wedge v) < t \\ \infty & \text{otherwise} \end{cases}$

Proof. We may assume that $u \neq v$. Let $\mathbf{V} \subseteq \mathbf{A}$. Since $\mathbf{CV} \subseteq \mathbf{V}$ and every quotient of \mathbb{N}^k in \mathbf{V} is necessarily in \mathbf{CV} , we have $r_{\mathbf{V}}(u, v) = r_{\mathbf{CV}}(u, v)$. We show next that

$$r_{\mathbf{CV}}(u, v) = \min\{|N_t| \mid N_t \in \mathbf{CV} \text{ and separates } u \text{ and } v\}. \quad (5.21)$$

Indeed, if $M \in \mathbf{CV}$ separates u and v through $\psi : \mathbb{N}^k \rightarrow M$, it follows from the proof of Proposition 5.1 that there exists an onto homomorphism $\varphi : N_{t_1} \times \dots \times N_{t_n} \rightarrow M$, where each N_{t_i} may be assumed to be a submonoid of M . Since \mathbb{N}^k is a free commutative monoid, we may factor ψ through θ :

$$\begin{array}{ccc} \mathbb{N}^k & \xrightarrow{\theta} & N_{t_1} \times \dots \times N_{t_n} \\ & \searrow \psi & \nearrow \varphi \\ & & M \end{array}$$

Since $\psi(u) \neq \psi(v)$, one of the component morphisms $\theta_i : \mathbb{N}^k \rightarrow N_{t_i}$ must separate u and v . Therefore the smallest $M \in \mathbf{CV}$ separating u and v must be of the form N_t and so (5.21) holds.

(1) By (5.21), we have

$$r_{\mathbf{CA}}(u, v) = \min\{|N_t| \mid N_t \text{ separates } u \text{ and } v\}. \quad (5.22)$$

If $u \wedge v = u_i \wedge v_i$, it is immediate that the projection on the i -th component induces a morphism from \mathbb{N}^k to $N_{(u \wedge v) + 1}$ separating u and v .

Suppose now that $\eta : \mathbb{N}^k \rightarrow N_t$ separates u and v with $t \leq (u \wedge v)$. Since

$$\sum_{i=1}^k \eta(u_i e_i) = \eta(u) \neq \eta(v) = \sum_{i=1}^k \eta(v_i e_i),$$

we have $\eta(u_i e_i) \neq \eta(v_i e_i)$ for some $i \in \{1, \dots, k\}$. Hence $\eta(e_i) \geq 1$. Since $u_i, v_i \geq t$, it follows that $\eta(u_i e_i) = t = \eta(v_i e_i)$, a contradiction.

Thus $N_{(u \wedge v) + 1}$ is the smallest N_t separating u and v . In view of (5.21), it follows that

$$r_{\mathbf{CA}}(u, v) = |N_{(u \wedge v) + 1}| = (u \wedge v) + 2.$$

(2) By (5.21) and Proposition 5.1, we have

$$r_{\mathbf{CA}_t}(u, v) = \min\{|N_s| \mid s \leq t \text{ and } N_s \text{ separates } u \text{ and } v\}.$$

In view of (5.22), it follows that

$$r_{\mathbf{CA}_t}(u, v) = \begin{cases} r_{\mathbf{CA}}(u, v) & \text{if } r_{\mathbf{CA}}(u, v) \leq t + 1 = |N_t| \\ \infty & \text{otherwise} \end{cases}$$

By (1), $r_{\mathbf{CA}}(u, v) \leq t + 1$ is equivalent to $(u \wedge v) < t$ and the claim follows. \square

Theorem 5.3 *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

- (1) f is \mathbf{A} -uniformly continuous,
- (2) for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u, v \in \mathbb{N}$, $u \wedge v \geq s$ implies $f(u) \wedge f(v) \geq n$,
- (3) for every $n \in \mathbb{N}$, $f^{-1}(n)$ is either finite or cofinite.

Proof. (1) \Leftrightarrow (2). It follows from the definition that f is \mathbf{A} -uniformly continuous if and only if for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u, v \in \mathbb{N}$,

$$r_{\mathbf{A}}(u, v) \geq s \text{ implies } r_{\mathbf{A}}(f(u), f(v)) \geq n,$$

that is equivalent to (2) by Lemma 5.2.

(2) \Rightarrow (3). Suppose that $f^{-1}(m)$ is neither finite nor cofinite. Let $s \in \mathbb{N}$ be arbitrary. Take $u_s \in f^{-1}(m)$ and $v_s \in \mathbb{N} \setminus f^{-1}(m)$ such that $u_s, v_s \geq s$. Thus the relation

$$u_s \wedge v_s \geq s \text{ and } f(u_s) \wedge f(v_s) \leq m$$

holds for all $s \in \mathbb{N}$, and so (2) fails.

(3) \Rightarrow (2). Let $n \in \mathbb{N}$. Suppose first that $f^{-1}(m)$ is cofinite for some $m \in \mathbb{N}$. Let $s = \max(\mathbb{N} \setminus f^{-1}(m))$. If $u \wedge v \geq s + 1$, then $u \neq v$ implies $u, v \geq s + 1$ and so $f(u) = m = f(v)$, hence we have $f(u) = f(v)$ in any case and $f(u) \wedge f(v) > n$ trivially.

Assume now that $f^{-1}(i)$ is finite for every $i \in \mathbb{N}$. Let $s = \max \cup_{i=0}^{n-1} f^{-1}(i)$. If $u \wedge v \geq s + 1$ and $u \neq v$, then $u, v \geq s + 1$ and so $u, v \notin \cup_{i=0}^{n-1} f^{-1}(i)$. Hence $f(u), f(v) \geq n$ and so $f(u) \wedge f(v) \geq n$. Therefore (2) holds. \square

Similarly, we get

Theorem 5.4 *Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

- (1) f is \mathbf{A} -uniformly continuous;
- (2) for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that for all $u, v \in \mathbb{N}^k$, $u \wedge v \geq s$ implies $f(u) \wedge f(v) \geq n$.

However, there is no analogue of condition (3) of Theorem 5.3 in this case: if we define $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ by $f(m, n) = m$, it is immediate that f is \mathbf{A} -uniformly continuous and $f^{-1}(m)$ is infinite for every $m \in \mathbb{N}$.

Theorem 5.5 *Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a mapping and $t \in \mathbb{N}$. Then the following conditions are equivalent:*

- (1) *f is \mathbf{A}_t -uniformly continuous,*
- (2) *for all $u, v \in \mathbb{N}^k$, $u \wedge v \geq t$ implies $f(u) \wedge f(v) \geq t$.*

Proof. Since \mathbb{N}^k and \mathbb{N} are commutative, the pseudo-metrics $d_{\mathbf{A}_t}$ and $d_{\mathbf{CA}_t}$ coincide in both monoids. Hence f is \mathbf{A}_t -uniformly continuous if and only if it is \mathbf{CA}_t -uniformly continuous.

Since $\text{Im } r_{\mathbf{CA}_t} = \{2, \dots, t+1, \infty\}$ by Lemma 5.2 (2), it follows from Proposition 2.2 that f is \mathbf{CA}_t -uniformly continuous if and only if for all $u, v \in \mathbb{N}^k$, $r_{\mathbf{CA}_t}(u, v) = \infty$ implies $r_{\mathbf{CA}_t}(f(u), f(v)) = \infty$. Now the claim follows from the same Lemma 5.2 (2). \square

6 \mathbf{A} -hereditary continuity

Lemma 6.1 *A function from a monoid M to \mathbb{N} is \mathbf{A} -hereditarily continuous if and only if it is \mathbf{CA}_t -uniformly continuous for every $t \in \mathbb{N}$.*

Proof. By Proposition 2.5, a function is \mathbf{A} -hereditarily continuous if and only if it is \mathbf{CA} -hereditarily continuous. The lemma now follows from [12, Proposition 5.9]. \square

Theorem 6.2 *Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

- (1) *f is \mathbf{A} -hereditarily continuous,*
- (2) *for all $u, v \in \mathbb{N}^k$, $u \wedge v \leq f(u) \wedge f(v)$,*
- (3) *f is \mathbf{A} -nonexpansive.*

Proof. (1) is equivalent to (2). By Lemma 6.1, f is \mathbf{A} -hereditarily continuous if and only if it is \mathbf{CA}_t -uniformly continuous for every $t \in \mathbb{N}$. In view of Lemma 5.2, this amounts to stating that, for all $t \in \mathbb{N}$ and for all $u, v \in \mathbb{N}^k$, $u \wedge v \geq t$ implies $f(u) \wedge f(v) \geq t$.

(2) is equivalent to (3). By Lemma (5.2) (1), an equivalent formulation of (2) is that, for all $u, v \in \mathbb{N}^k$, $r_{\mathbf{A}}(u, v) \leq r_{\mathbf{A}}(f(u), f(v))$, which is equivalent to (3). \square

We now look for a more explicit characterization of \mathbf{A} -hereditary continuity. Given a function $f : \mathbb{N}^k \rightarrow \mathbb{N}$, we say that $g : \mathbb{N} \rightarrow \mathbb{N}$ is a *slice function* of f if there exists some $j \in \{1, \dots, k\}$ and $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_k \in \mathbb{N}$ such that $g(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k)$ for every $x \in \mathbb{N}$.

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be *extensive* if $x \leq f(x)$ for every $x \in \mathbb{N}$ and *truncated* if there exists some $m \in \mathbb{N}$ such that $x \leq f(x)$ for $x \leq m$ and $f(x) = m$ for $x > m$. Functions that are either extensive or truncated can be described by the following single property:

- (C) if $b = \min\{x \in \mathbb{N} \mid f(x) < x\}$, then $f(x) = b - 1$ for every $x \geq b$.

Indeed, the case $b = \infty$ corresponds to extensive functions and the case b finite corresponds to truncated functions.

Lemma 6.3 Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a mapping satisfying condition (C) and assume that $f(a_1, \dots, a_k) < \min\{a_1, \dots, a_k\}$. Then:

- (1) $f(x_1, \dots, x_k) = f(a_1, \dots, a_k)$ for all $x_1 \geq a_1, \dots, x_k \geq a_k$;
- (2) there exists some $c \leq \min\{a_1, \dots, a_k\}$ such that $f(a_1, \dots, a_k) = f(c, \dots, c) = c - 1$.

Proof. (1) We use induction on k . For $k = 1$, assume that $f(a) < a$ and $x \geq a$. Then there exists $b = \min\{y \in \mathbb{N} \mid f(y) < y\}$ and so, by condition (C), $x \geq a \geq b$ implies $f(x) = f(a) = b - 1$.

Assume now that $k > 1$ and (1) holds for smaller values of k . Let $x_1 \geq a_1, \dots, x_k \geq a_k$. By condition (C), we have $f(a_1, \dots, a_{k-1}, x_k) = f(a_1, \dots, a_k)$: indeed, if we take $b = \min\{x \in \mathbb{N} \mid f(a_1, \dots, a_{k-1}, x) < x\}$, then $b \leq a_k \leq x_k$ and so $f(a_1, \dots, a_{k-1}, x) = b - 1 = f(a_1, \dots, a_k)$.

Define now $g : \mathbb{N}^{k-1} \rightarrow \mathbb{N}$ by $g(y_1, \dots, y_{k-1}) = f(y_1, \dots, y_{k-1}, x_k)$. Since f satisfies (C), so does g . Moreover,

$$g(a_1, \dots, a_{k-1}) = f(a_1, \dots, a_{k-1}, x) = b - 1 = f(a_1, \dots, a_k) < \min\{a_1, \dots, a_{k-1}\}.$$

By the induction hypothesis, we get $g(x_1, \dots, x_{k-1}) = g(a_1, \dots, a_{k-1})$ since $x_1 \geq a_1, \dots, x_{k-1} \geq a_{k-1}$. Thus

$$\begin{aligned} f(x_1, \dots, x_k) &= g(x_1, \dots, x_{k-1}) = g(a_1, \dots, a_{k-1}) \\ &= f(a_1, \dots, a_{k-1}, x_k) = f(a_1, \dots, a_k) \end{aligned}$$

as required.

(2) We use induction on k . For $k = 1$, assume that $f(a) < a$. Then there exists $c = \min\{y \in \mathbb{N} \mid f(y) < y\}$ and so, by condition (C), $a \geq c$ implies $f(a) = f(c) = c - 1$.

Assume now that $k > 1$ and (ii) holds for smaller values of k . Let

$$b = \min\{y \in \mathbb{N} \mid f(a_1, \dots, a_{k-1}, y) < y\}.$$

Then $b \leq a_k$. Define g as above. We have

$$g(a_1, \dots, a_{k-1}) = f(a_1, \dots, a_{k-1}, b) = b - 1 = f(a_1, \dots, a_k)$$

by condition (C). By the induction hypothesis, there exists $c \leq a_1, \dots, a_{k-1}$ such that $g(a_1, \dots, a_{k-1}) = g(c, \dots, c) = c - 1$. Thus $c - 1 = g(a_1, \dots, a_{k-1}) = b - 1$ and so $b = c$. Since $b \leq a_k$, we get $c \leq a_1, \dots, a_k$. Thus $f(a_1, \dots, a_k) = c - 1 = g(c, \dots, c) = f(c, \dots, c)$ as required. \square

Theorem 6.4 Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a mapping. Then the following conditions are equivalent:

- (1) f is **A**-hereditarily continuous;
- (2) every slice function of f is either extensive or truncated.

Proof. (1) \Rightarrow (2). Let g be the slice function of f defined by $g(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k)$ and let $a_j = \min\{x \in \mathbb{N} \mid g(x) < x\}$. If $x > a_j$, then one gets by Theorem 6.2

$$a_j = (a_1, \dots, a_k) \wedge (a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k) \leq g(a_j) \wedge g(x),$$

and since $g(a_j) < a_j$, it follows that $g(a_j) = g(x)$.

Let $z = g(a_j)$. It remains to prove that $z = a_j - 1$. Since $z < a_j$, we have $z + 1 \leq a_j$. Suppose that $z + 1 < a_j$. By Theorem 6.2, one has

$$z+1 = (a_1, \dots, a_k) \wedge (a_1, \dots, a_{j-1}, z+1, a_{j+1}, \dots, a_k) \leq g(a_j) \wedge g(z+1) = z \wedge g(z+1),$$

hence $g(z+1) = z < z+1$. Since $z+1 < a_j$, this contradicts the minimality of a_j . Thus $z+1 = a_j$ and (C) holds.

(2) \Rightarrow (1). We use induction on k . For $k = 1$, assume that f satisfies condition (C) and let $u, v \in \mathbb{N}$ be distinct. By Theorem 6.2, we must prove that $u \wedge v \leq f(u) \wedge f(v)$. Let $Y = \{y \in \mathbb{N} \mid f(y) < y\}$. If $u, v \notin Y$, the claim follows. Hence we may assume that $Y \neq \emptyset$ and $b = \min Y$. If $u, v \in Y$, then $f(u) = b - 1 = f(v)$ by condition (C) and so $u \wedge v \leq \infty = f(u) \wedge f(v)$. Finally, assume that $u \in Y$ and $v \notin Y$. Then $f(u) = b - 1$. Without loss of generality, we may assume that $f(v) \neq b - 1$. Hence $v < b$ and so $v \leq (f(u) \wedge f(v))$. Thus $u \wedge v \leq f(u) \wedge f(v)$ and the result holds for $k = 1$.

Assume now that $k > 1$ and the theorem holds for smaller values of k . Assume that $f : \mathbb{N}^k \rightarrow \mathbb{N}$ satisfies condition (C) and let $u, v \in \mathbb{N}^k$ be distinct.

Assume first that $u_i = v_i$ for some $i \in \{1, \dots, k\}$. Without loss of generality, we may assume that $i = k$. Define $g : \mathbb{N}^{k-1} \rightarrow \mathbb{N}$ by $g(y_1, \dots, y_{k-1}) = f(y_1, \dots, y_{k-1}, u_k)$. Since f satisfies (C), so does g . By the induction hypothesis and Theorem 6.2, we get

$$\begin{aligned} u \wedge v &= (u_1, \dots, u_{k-1}) \wedge (v_1, \dots, v_{k-1}) \\ &\leq g(u_1, \dots, u_{k-1}) \wedge g(v_1, \dots, v_{k-1}) = f(u) \wedge f(v) \end{aligned}$$

as required.

Hence we may assume that $u_i \neq v_i$ for every $i \in \{1, \dots, k\}$. Without loss of generality, we may also assume that $u \wedge v = u_1$. Suppose first that $f(v) < u_1$. Since $u_1 \leq v_1, \dots, v_k$, we may apply Lemma 6.3(2) and get some $c \leq v_1, \dots, v_k$ such that $f(v) = f(c, \dots, c) = c - 1$. Thus $c - 1 < u_1$ and so $c < u_1 \leq u_2, \dots, u_k$. By Lemma 6.3(1), it follows that $f(u) = f(c, \dots, c) = f(v)$. Therefore $u \wedge v \leq f(u) \wedge f(v)$.

Next we assume that $f(u) < u_1$. Since $u_1 \leq u_2, \dots, u_k$, we may apply Lemma 6.3(2) and get some $c \leq u_1$ such that $f(u) = f(c, \dots, c) = c - 1$. Since $v_1, \dots, v_k \geq u_1 \geq c$, it follows from Lemma 6.3(1) that $f(v) = f(c, \dots, c) = f(u)$. Therefore $u \wedge v \leq f(u) \wedge f(v)$ also in this case.

The final case $f(u), f(v) \geq u_1 = u \wedge v$ is trivial. \square

7 M-hereditary continuity

Proposition 7.1 *Let M be a monoid and $f : M \rightarrow \mathbb{N}$ a mapping. Then the following conditions are equivalent:*

- (1) f is **M**-hereditarily continuous;
- (2) f is both **G**- and **A**-hereditarily continuous;
- (3) f is both **Ab**- and **CA**-hereditarily continuous.

Proof. The equivalence of (1) and (3) follows from [12, Proposition 5.8] and that of (2) and (3) from Proposition 2.5. \square

Theorem 7.2 *Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a mapping. Then f is \mathbf{M} -hereditarily continuous if and only if:*

- (1) $\gcd\{u_i - v_i \mid i = 1, \dots, k\}$ divides $f(u) - f(v)$ for all $u, v \in \mathbb{N}^k$,
- (2) every slice function of f is either extensive or constant.

Proof. By Proposition 7.1, f is \mathbf{M} -hereditarily continuous if and only if it is both \mathbf{G} - and \mathbf{A} -hereditarily continuous. Now condition (1) is equivalent to \mathbf{G} -hereditary continuity by Theorem 4.1. By Theorem 6.4, \mathbf{A} -hereditary continuity is equivalent to every slice function of f being either extensive or truncated. Clearly, every constant function $f : \mathbb{N} \rightarrow \mathbb{N}$ is necessarily truncated. It remains to prove that every truncated slice function must be indeed constant in these circumstances.

Suppose that $g : \mathbb{N} \rightarrow \mathbb{N}$ defined by $g(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k)$ is truncated with $f(x) = m$ for every $x > m$. Let $M = \max(\text{Im } f)$ and take $s < m$ arbitrary. We consider

$$u = (a_1, \dots, a_{j-1}, s, a_{j+1}, \dots, a_k), \quad v = (a_1, \dots, a_{j-1}, M + s + 1, a_{j+1}, \dots, a_k).$$

Since

$$M + 1 = \gcd_{1 \leq i \leq k} (u_i - v_i) \mid f(u) - f(v)$$

and $|f(u) - f(v)| \leq M$, it follows that $f(u) = f(v)$, hence $g(s) = g(M + s + 1) = m$. Therefore g is constant as claimed. \square

Corollary 7.3 *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping. Then f is \mathbf{M} -hereditarily continuous if and only if f is extensive or constant, and $u - v$ divides $f(u) - f(v)$ for all $u, v \in \mathbb{N}$.*

We can now adapt the proof of Corollary 3.13 to strengthen it:

Theorem 7.4 *There are uncountably many \mathbf{M} -hereditarily continuous functions from \mathbb{N} to \mathbb{N} .*

Proof. By the uniqueness of Mahler expansions, the mapping

$$(n_r)_{r \in \mathbb{N}} \mapsto \sum_{r \in \mathbb{N}} n_r \text{lcm}(1, \dots, r) \binom{\cdot}{r}$$

induces an injection θ from $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$. Let $f \in \text{Im } \theta$. By Theorem 4.2, f is \mathbf{G} -hereditarily continuous. Since $n_1 \geq 1$, we have

$$\sum_{r \in \mathbb{N}} n_r \text{lcm}(1, \dots, r) \binom{x}{r} \geq n_1 \binom{x}{1} \geq x$$

for every $x \in \mathbb{N}$ and so f is extensive and thus \mathbf{A} -hereditarily continuous by Theorem 6.4. Therefore f is \mathbf{M} -hereditarily continuous by Proposition 7.1. Since $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ is uncountable and θ is one-to-one, $\text{Im } \theta$ is an uncountable set of \mathbf{M} -hereditarily continuous functions from \mathbb{N} to \mathbb{N} . \square

We can also settle the case of functions from \mathbb{Z}^k to \mathbb{Z} .

Corollary 7.5 *A function from \mathbb{Z}^k to \mathbb{Z} is \mathbf{M} -hereditarily continuous if and only if, for each prime p , it is nonexpansive for the p -adic norm.*

Proof. Let $f : \mathbb{Z}^k \rightarrow \mathbb{Z}$ be a function and let \mathbf{V} denote a subvariety of \mathbf{M} . Since every quotient of \mathbb{Z}^k is necessarily a group, the pseudo-metrics $d_{\mathbf{V}}$ and $d_{\mathbf{V} \cap \mathbf{G}}$ coincide in \mathbb{Z}^k (and in particular in \mathbb{Z}). It follows that f is \mathbf{V} -uniformly continuous if and only if it is $\mathbf{V} \cap \mathbf{G}$ -uniformly continuous. Since $\mathbf{V} \cap \mathbf{G}$ takes all possible values among the subvarieties of \mathbf{G} , it follows that f is \mathbf{M} -hereditarily continuous if and only if it is \mathbf{G} -hereditarily continuous. One can now apply Theorem 4.1 to conclude. \square

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