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The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory *ZFC* as it refers to our current knowledge on \mathcal{X}

Sławomir Kurpaska, Apoloniusz Tyszką

Abstract. Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form n^2+1 is infinite. Let $\beta = (((24!)!)!)!$, and let Φ denote the implication: $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement Φ without invoking Landau's conjecture. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4). (1) There are a large number of elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite. (2) No known algorithm decides the finiteness/infiniteness of \mathcal{X} . (3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. (4) There is an explicitly known integer n such that $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. (5) There is an explicitly known integer n such that $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ and some known definition of \mathcal{X} is much simpler than every known definition of $\mathcal{X} \setminus (-\infty, n]$. The following problem is open: *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(3) and (5)?* The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3). The set $\mathcal{X} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$ contains $10^{10^{450}}$ consecutive integers and satisfies conditions (1)–(3). The statement Φ implies that both sets \mathcal{X} satisfy condition (5).

Key words and phrases: complexity of a mathematical definition, computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on \mathcal{X} , explicitly known integer n bounds \mathcal{X} from above when \mathcal{X} is finite, infiniteness of \mathcal{X} remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, large number of elements of \mathcal{X} , mathematical statement that cannot be formalized in the set theory *ZFC*, no known algorithm decides the finiteness/infiniteness of \mathcal{X} , physical impossibility of machine computations on sufficiently large integers.

1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

Definition 1. *Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.*

(1) *There are a large number of elements of X and it is conjectured that X is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of X .*

(3) *There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in X$.*

(4) *There is an explicitly known integer n such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.*

(5) *There is an explicitly known integer n such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$ and some known definition of X is much simpler than every known definition of $X \setminus (-\infty, n]$.*

Definition 2. *We say that an integer n is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, cf. [8] and [9].*

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of X . If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $[\max(X), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [5] and [6].

Definition 3. *Let Φ denote the implication:*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement Φ . In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *There is no explicitly known threshold number of \mathcal{P}_{n^2+1} . It means that there is no explicitly known integer k such that $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.*

Proving the statement Φ will falsify Statement 1. Statement 1 cannot be formalized in the set theory *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2 and 3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $X \subseteq \mathbb{N}$.

2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

Definition 4. *Let $\beta = (((24!)!)!)!$.*

Lemma 1. $\beta \approx 10^{10^{10^{25.16114896940657}}}$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. \square

Statement 2. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\mathcal{X} \supseteq \{0, \dots, \beta\}$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

In Statement 2,

$$\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, \beta]$$

and the sets

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

and

$$\mathcal{X} \setminus (-\infty, \beta] = \{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

have definitions of similar complexity. The following problem arises:

Open Problem 1. Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(3) and (5)?

3. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n-1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{U}_n .

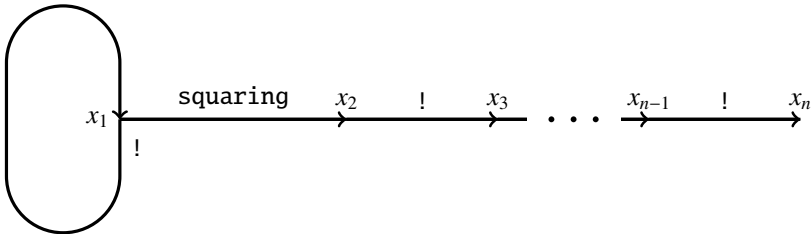


Fig. 1 Construction of the system \mathcal{U}_n

Lemma 2. For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.

Let

$$B_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement Ψ_9 , cf. Conjecture 1 in Section 4.

Theorem 1. *For every statement Ψ_n , the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 2. *For every integer $n \geq 2$, the statement Ψ_{n+1} implies the statement Ψ_n .*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then for every integer $i \in \{1, \dots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers x_1, \dots, x_{n+1} . The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)!$. Hence, $x_i \leq f(n)$. □

Theorem 3. *Every statement Ψ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y , $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. (*Wilson's theorem*, [1, p. 89]). *For every integer $x \geq 2$, x is prime if and only if x divides $(x - 1)! + 1$.*

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

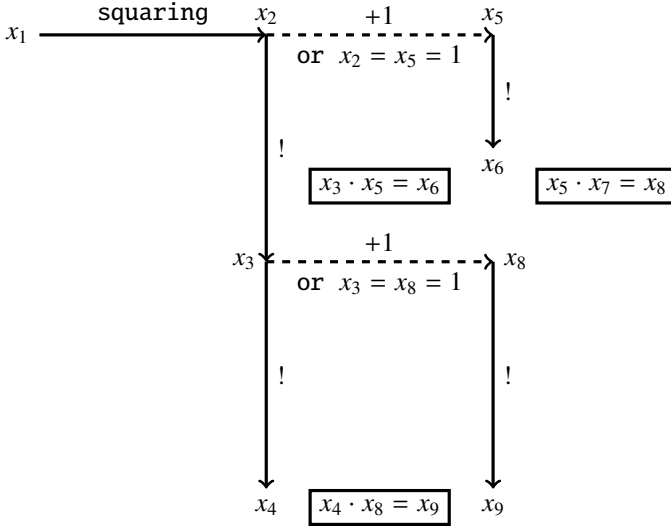


Fig. 2 Construction of the system \mathcal{A}

Lemma 5. For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned}
 x_2 &= x_1^2 \\
 x_3 &= (x_1^2)! \\
 x_4 &= ((x_1^2)!)! \\
 x_5 &= x_1^2 + 1 \\
 x_6 &= (x_1^2 + 1)! \\
 x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
 x_8 &= (x_1^2)! + 1 \\
 x_9 &= ((x_1^2)! + 1)!
 \end{aligned}$$

Proof. By Lemma 3, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \square

Lemma 6. There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$. This is true as every such tuple (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \in \{1, 2\}$.

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. \square

Conjecture 1. The statement Ψ_9 is true when is restricted to the system \mathcal{A} .

Theorem 4. Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set \mathcal{P}_{n^2+1} is infinite. \square

Theorem 5. *Conjecture 1 implies the statement Φ .*

Proof. It follows from Theorem 4 and the equality $f(7) = (((24!)!)!)!$. \square

Theorem 6. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers x_1, \dots, x_9 solve the system \mathcal{A} , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 . Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. \square

Definition 5. *Let $\mathcal{K} = \{k \in \mathbb{N} : \text{the number of digits of } k \text{ belongs to } \mathcal{P}_{n^2+1}\}$.*

Lemma 7. $\text{card}(\mathcal{K}) \geq 9 \cdot 10^9 \cdot 4^{747} \approx 10^{10450.6930560314272}$.

Proof. The following PARI/GP ([4]) command

```
isprime(1+9*4^747, {flag=2})
```

returns %1 = 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([7, p. 226]). It rigorously shows that the number $(3 \cdot 2^{747})^2 + 1$ is prime. Since $9 \cdot 10^9 \cdot 4^{747}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. To establish the approximate equality, we ask Wolfram Alpha about $9 * (10^{(9 * 4^{747})})$. \square

Statement 3. *The sets $\mathcal{X} = \mathcal{P}_{n^2+1}$ and $\mathcal{X} = \mathcal{K}$ satisfy conditions (1) - (3). The statement Φ implies that both sets \mathcal{X} satisfy condition (5).*

Proof. Since the set \mathcal{P}_{n^2+1} is conjecturally infinite, Lemma 7 implies condition (1) for both sets \mathcal{X} . Condition (3) holds trivially for both sets \mathcal{X} . By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds for both sets \mathcal{X} . Suppose that the statement Φ holds. This implies two facts:

$$\beta \text{ is a threshold number of } \mathcal{X} = \mathcal{P}_{n^2+1} \quad (6)$$

and

$$\underbrace{9 \dots 9}_{\beta \text{ digits}} \text{ is a threshold number of } \mathcal{X} = \mathcal{K} \quad (7)$$

Thus condition (4) holds for both sets \mathcal{X} . The definition of \mathcal{P}_{n^2+1} is much simpler than the definition of $\mathcal{P}_{n^2+1} \setminus (-\infty, \beta]$. The definition of \mathcal{K} is much simpler than the definition of $\mathcal{K} \setminus (-\infty, \underbrace{9 \dots 9}_{\beta \text{ digits}}]$. The last three sentences imply that condition (5)

holds for both sets \mathcal{X} . □

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