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A currently true statement G of the form " $\exists f : \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability . . .", where f is conjecturally uncomputable, G remains unproven when f is computable, and G holds forever when f is uncomputable

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Abstract

We define a function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ which eventually dominates every computable function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$. We present a simple code in MuPAD which for $n \in \mathbb{N}$ prints the sequence $\{\gamma_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $\gamma(n)$. It establishes the most effective proof that there exists a non-computable function from \mathbb{N} to \mathbb{N} . We define a function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation. We present a simple code in MuPAD which for $n \in \mathbb{N}$ prints the sequence $\{\beta_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $\beta(n)$. Let G denote the following statement: $\exists f : \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability such that f eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation and there is a computer program which for $n \in \mathbb{N}$ prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$. We show that G has all properties from the title of the article.

Key words and phrases: computable function, eventual domination, finite-fold Diophantine representation, limit-computable function, predicate \mathcal{K} of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate \mathcal{K} of the current mathematical knowledge.

2020 Mathematics Subject Classification: 03D20.

1 Predicate \mathcal{K} of the current mathematical knowledge

In sections 1–4, \mathcal{K} denotes both the predicate satisfied by every currently known theorem and the set of all currently known theorems. The set \mathcal{K} is time-dependent. Observations 1–5 explain the definition of \mathcal{K} . This definition is intentionally simple, which restricts the set \mathcal{K} . Observations 1–4 hold at any time.

Observation 1. Every written down theorem from past or present belongs to \mathcal{K} .

Observation 2. If $\Delta \in \mathcal{K}$, then every particular case of Δ belongs to \mathcal{K} . For example,

$$\{\emptyset \cup \{n\} = \{n\} : n \in \mathbb{N}\} \subseteq \mathcal{K}$$

Observation 3. Since the Replacement Axiom is a scheme of axioms, only finitely many axioms of ZFC belong to \mathcal{K} .

Observation 4. \mathcal{K} is not deductively closed.

Proof. There exists $n \in \mathbb{N}$ such that $\neg \mathcal{K}(\underbrace{(2 \cdot 2 = 4) \wedge \dots \wedge (2 \cdot 2 = 4)}_{\text{conjunction of } n+3 \text{ statements}})$. □

Observation 5. *Currently,*

$(\neg \mathcal{K}(\text{th digit of } \pi \text{ is even})) \wedge$

$(\neg \mathcal{K}(\text{th digit of } \pi \text{ is odd}))$

Hence, \mathcal{K} is not deductively closed.

2 A limit-computable function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ which is constructively defined and eventually dominates every computable function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$

For $n \in \mathbb{N}$, let

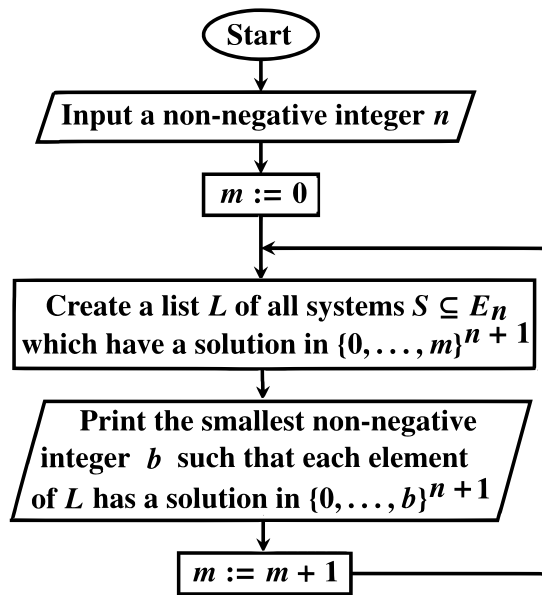
$$E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \dots, n\}\}$$

Theorem 1. ([8, p. 118]). *There exists a limit-computable function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ which eventually dominates every computable function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$.*

We present an alternative proof of Theorem 1. For $n \in \mathbb{N}$, $\gamma(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in \mathbb{N}^{n+1} , then \mathcal{S} has a solution in $\{0, \dots, b\}^{n+1}$. The function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$, see [9].

Corollary 1. *For every $k \in \mathbb{N}$, the function $\mathbb{N} \ni n \rightarrow k + \gamma(n) \in \mathbb{N}$ is uncomputable.*

Flowchart 1 shows a semi-algorithm which computes $\gamma(n)$ in the limit, see [9].



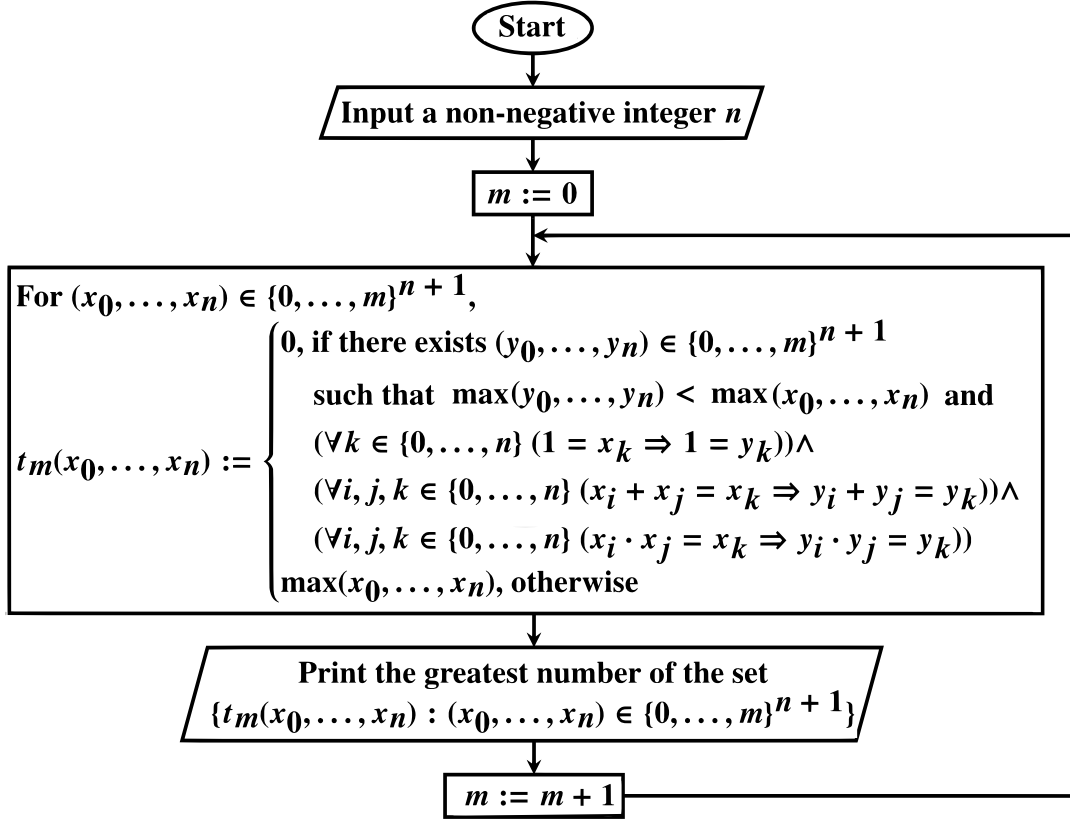
Flowchart 1

A semi-algorithm which computes $\gamma(n)$ in the limit

Proposition 1. *If $k \in \mathbb{N}$, then the statement "the function $\mathbb{N} \ni n \rightarrow k + \gamma(n) \in \mathbb{N}$ is uncomputable" belongs to \mathcal{K} .*

Proof. It follows from Corollary 1 and Observations 1 and 2. □

Flowchart 2 shows a simpler semi-algorithm which computes $\gamma(n)$ in the limit.



Flowchart 2

A simpler semi-algorithm which computes $\gamma(n)$ in the limit

Lemma 1. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$\begin{aligned}
 E_n \supseteq & \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\
 & \{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\
 & \{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}
 \end{aligned}$$

□

Lemma 2. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ and (a_0, \dots, a_n) solves \mathcal{S} , then (a_0, \dots, a_n) solves the system of equations

$$\begin{aligned}
 \tilde{\mathcal{S}} := & \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\
 & \{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\
 & \{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}
 \end{aligned}$$

□

Theorem 2. For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the same number.

Proof. It follows from Lemmas 1 and 2.

□

MuPAD is a part of the Symbolic Math Toolbox in MATLAB R2019b. The following code in MuPAD implements the semi-algorithm shown in Flowchart 2. It establishes the most effective proof that there exists a non-computable function from \mathbb{N} to \mathbb{N} .

```

input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do
for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do
for j from i to n+1 do
if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if max(op(X[q]))<max(op(X[p])) and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
end_while:

```

3 A limit-computable function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability which is constructively defined and eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation

The Davis-Putnam-Robinson-Matiyasevich theorem states that every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ ($n \in \mathbb{N} \setminus \{0\}$) has a Diophantine representation, that is

$$(a_1, \dots, a_n) \in \mathcal{M} \iff \exists x_1, \dots, x_m \in \mathbb{N} \ W(a_1, \dots, a_n, x_1, \dots, x_m) = 0 \quad (\text{R})$$

for some polynomial W with integer coefficients, see [5]. The representation (R) is said to be single-fold, if for any $a_1, \dots, a_n \in \mathbb{N}$ the equation $W(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has at most one solution $(x_1, \dots, x_m) \in \mathbb{N}^m$. The representation (R) is said to be finite-fold, if for any $a_1, \dots, a_n \in \mathbb{N}$ the equation $W(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has only finitely many solutions $(x_1, \dots, x_m) \in \mathbb{N}^m$.

Conjecture 1. ([1], [2], [3], [4, pp. 341–342], [6, p. 42], [7, p. 745]). *Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ ($k \in \mathbb{N} \setminus \{0\}$) has a single-fold Diophantine representation.*

Conjecture 2. ([1], [2], [3], [4, pp. 341–342], [6, p. 42], [7, p. 745]). *Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ ($k \in \mathbb{N} \setminus \{0\}$) has a finite-fold Diophantine representation.*

Let Φ denote the following statement: *the function $\mathbb{N} \ni n \rightarrow 2^n \in \mathbb{N}$ eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.* For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \gamma(n), & \text{otherwise} \end{cases}$$

The function $g : \mathbb{N} \rightarrow \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge (\neg\mathcal{K}(g \text{ is computable})) \wedge (\neg\mathcal{K}(g \text{ is uncomputable}))$$

Let Ψ denote the following statement: *the function $\mathbb{N} \ni n \rightarrow 2^n \in \mathbb{N}$ eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a finite-fold Diophantine representation.* For $n \in \mathbb{N}$, let

$$h(n) = \begin{cases} 2^n, & \text{if } \Psi \text{ holds} \\ \gamma(n), & \text{otherwise} \end{cases}$$

The function $h : \mathbb{N} \rightarrow \mathbb{N}$ is computable if and only if Ψ holds. Currently,

$$(\neg\mathcal{K}(\Psi)) \wedge (\neg\mathcal{K}(\neg\Psi)) \wedge (\neg\mathcal{K}(h \text{ is computable})) \wedge (\neg\mathcal{K}(h \text{ is uncomputable}))$$

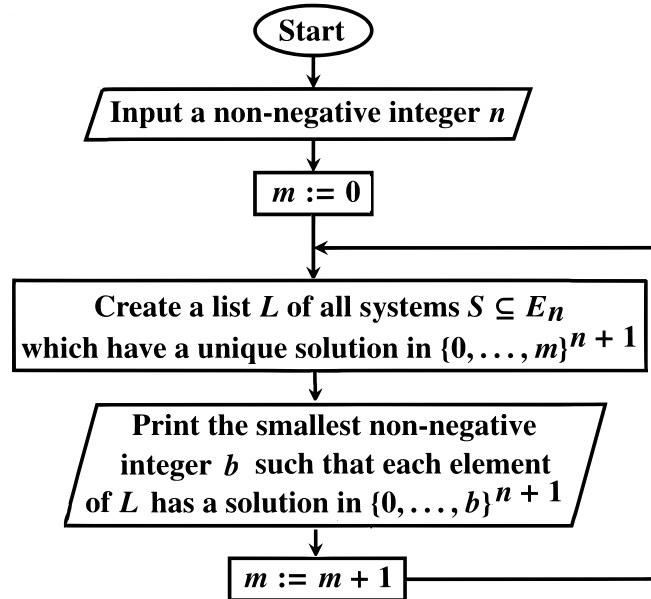
Lemma 3. *The function g is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation. The function h is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a finite-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in \mathbb{N}^{n+1} , then this solution belongs to $\{0, \dots, b\}^{n+1}$.

Theorem 3. *The function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [9]. The term "dominated" in the title of [9] means "eventually dominated". Flowchart 3 shows a semi-algorithm which computes $\beta(n)$ in the limit, see [9].

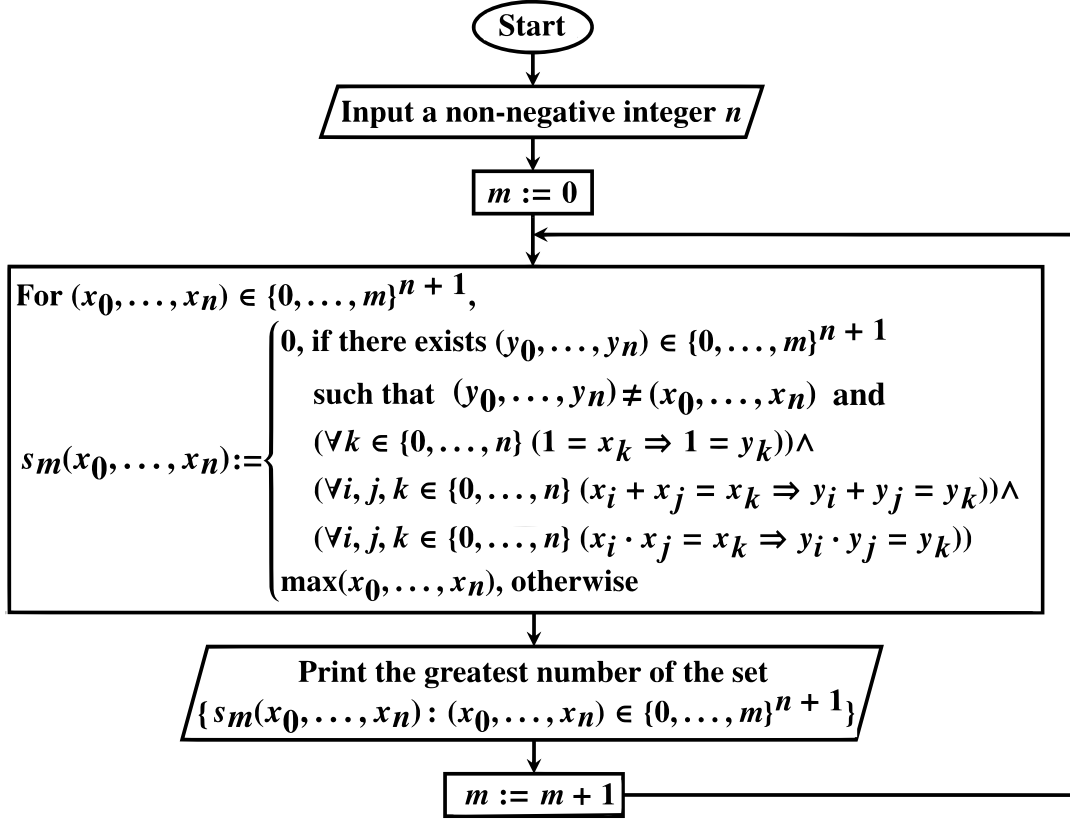


Flowchart 3

A semi-algorithm which computes $\beta(n)$ in the limit

□

Flowchart 4 shows a simpler semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 4

A simpler semi-algorithm which computes $\beta(n)$ in the limit

Lemma 4. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 4 does not exceed the number printed by Flowchart 3.

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$\begin{aligned}
 E_n \supseteq & \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\
 & \{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\
 & \{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}
 \end{aligned}$$

□

Lemma 5. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 3 does not exceed the number printed by Flowchart 4.

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ is a unique solution of \mathcal{S} in $\{0, \dots, m\}^{n+1}$, then (a_0, \dots, a_n) solves the system of equations

$$\begin{aligned}
 \widehat{\mathcal{S}} := & \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\
 & \{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\
 & \{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}
 \end{aligned}$$

By this and the inclusion $\widehat{\mathcal{S}} \supseteq \mathcal{S}$, $\widehat{\mathcal{S}}$ has exactly one solution in $\{0, \dots, m\}^{n+1}$, namely (a_0, \dots, a_n) . □

Theorem 4. For every $n, m \in \mathbb{N}$, Flowcharts 3 and 4 print the same number.

Proof. It follows from Lemmas 4 and 5. □

The following code in MuPAD implements the semi-algorithm shown in Flowchart 4.

```

input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do
for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do
for j from i to n+1 do
if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if q<>p and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
end_while:

```

Statement 1. *There exists a limit-computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 1 follows constructively from Theorem 3 by taking $f = \beta$ because the following conjunction

$$(\neg \mathcal{K}(\beta \text{ is computable})) \wedge (\neg \mathcal{K}(\beta \text{ is uncomputable}))$$

holds. Statement 1 follows non-constructively from Lemma 3 by taking $f = g$ because the following conjunction

$$(\neg \mathcal{K}(g \text{ is computable})) \wedge (\neg \mathcal{K}(g \text{ is uncomputable}))$$

holds. □

Since the function γ in Theorem 1 is not computable, Statement 1 does not follow from Theorem 1.

Proposition 2. *Statement 1 strengthens a mathematical theorem. Statement 1 refers to the current mathematical knowledge and may be false in the future. Statement 1 does not express what is currently unproved in mathematics.*

Proof. Statement 1 strengthens Statement 1 without the epistemic condition. The weakened Statement 1 is a theorem which follows from Theorem 1. Statement 1 claims that some mathematically defined function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfies

$$(f \text{ is computable in the limit}) \wedge (\neg \mathcal{K}(f \text{ is computable})) \wedge (\neg \mathcal{K}(f \text{ is uncomputable})) \wedge$$

$(f \text{ eventually dominates every function } \delta : \mathbb{N} \rightarrow \mathbb{N} \text{ with a single-fold Diophantine representation})$

Conjecture 1 disproves Statement 1. □

Statement 2. *Statement 1 holds for finite-fold Diophantine representations.*

Proof. It follows from Lemma 3 by taking $f = h$ because the following conjunction

$$(\neg\mathcal{K}(h \text{ is computable})) \wedge (\neg\mathcal{K}(h \text{ is uncomputable}))$$

holds. □

Statement 2 strengthens Statement 1. For Statement 2, there is no known computer program that computes f in the limit.

4 The statement \mathbf{G} from the title of the article

Statement 3. *In Statement 1, we can require that there exists a computer program which takes as input a non-negative integer n and prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$.*

Proof. Any computer program that implements the semi-algorithm shown in Flowchart 3 or 4 is correct. □

Let \mathbf{G} denote the following statement: $\exists f : \mathbb{N} \rightarrow \mathbb{N}$ of unknown computability such that f eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation and there is a computer program which for $n \in \mathbb{N}$ prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$.

Proposition 3. *The statement \mathbf{G} has all properties from the title of the article.*

Proposition 4. *The statement \mathbf{G} may hold when $\mathcal{K}(\beta \text{ is computable})$.*

Proof. For $n \in \mathbb{N}$, $\xi(n)$ denotes the smallest $b \in \mathbb{N} \setminus \{0\}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in \mathbb{Q}^{n+1} , then \mathcal{S} has a solution which consists of rationals whose numerators and denominators belong to $\{-b, \dots, b\}$. The function $\xi : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit. We skip the proof, which is similar to the proof that γ is computable in the limit. Currently,

$$(\neg\mathcal{K}(\xi \text{ is computable})) \wedge (\neg\mathcal{K}(\xi \text{ is uncomputable}))$$

At any future time, the conjunction

$$(\neg\mathcal{K}(\xi \text{ is computable})) \wedge (\neg\mathcal{K}(\xi \text{ is uncomputable})) \wedge \mathcal{K}(\beta \text{ is computable})$$

may hold and implies the statement \mathbf{G} with $f = \xi + \beta$. □

Summarizing, the predicate \mathcal{K} can strengthen existential mathematical statements when \mathcal{K} is inserted after \exists and refers to a part of the statement.

5 Predicate \mathcal{K} of the written down mathematical knowledge

In this section, \mathcal{K} denotes both the predicate satisfied by every written down theorem and the finite set of all written down theorems. It changes what is taken as known in mathematics.

Proposition 5. *Since \mathcal{K} is finite, there exists $k \in \mathbb{N}$ such that the computability of the function*

$$\mathbb{N} \ni n \rightarrow k + \gamma(n) \in \mathbb{N}$$

is unknown. For this k , Statements 1 and 2 hold when $f(n) = k + \gamma(n)$.

Proposition 5 contradicts Proposition 1 with the right definition of \mathcal{K} .

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