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# A common approach to the problem of the infinitude of twin primes, primes of the form $n! + 1$ , and primes of the form $n! - 1$

Apoloniusz Tyszka

## Abstract

For a positive integer  $x$ , let  $\Gamma(x)$  denote  $(x - 1)!$ . Let  $\text{fact}^{-1}: \{1, 2, 6, 24, \dots\} \rightarrow \mathbb{N} \setminus \{0\}$  denote the inverse function to the factorial function. For positive integers  $x$  and  $y$ , let  $\text{rem}(x, y)$  denote the remainder from dividing  $x$  by  $y$ . For a positive integer  $n$ , by a computation of length  $n$  we understand any sequence of terms  $x_1, \dots, x_n$  such that  $x_1$  is defined as the variable  $x$ , and for every integer  $i \in \{2, \dots, n\}$ ,  $x_i$  is defined as  $\Gamma(x_{i-1})$ , or  $\text{fact}^{-1}(x_{i-1})$ , or  $\text{rem}(x_{i-1}, x_{i-2})$  (only if  $i \geq 3$  and  $x_{i-1}$  is defined as  $\Gamma(x_{i-2})$ ). Let  $f(4) = 3$ , and let  $f(n + 1) = f(n)!$  for every integer  $n \geq 4$ . For an integer  $n \geq 4$ , let  $\Psi_n$  denote the following statement: if a computation of length  $n$  returns positive integers  $x_1, \dots, x_n$  for at most finitely many positive integers  $x$ , then every such  $x$  does not exceed  $f(n)$ . We prove: (1) the statement  $\Psi_4$  equivalently expresses that there are infinitely many primes of the form  $n! + 1$ ; (2) the statement  $\Psi_6$  implies that for infinitely many primes  $p$  the number  $p! + 1$  is prime; (3) the statement  $\Psi_6$  implies that there are infinitely many primes of the form  $n! - 1$ ; (4) the statement  $\Psi_7$  implies that there are infinitely many twin primes.

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For a positive integer  $x$ , let  $\Gamma(x)$  denote  $(x - 1)!$ . Let  $\text{fact}^{-1}: \{1, 2, 6, 24, \dots\} \rightarrow \mathbb{N} \setminus \{0\}$  denote the inverse function to the factorial function. For positive integers  $x$  and  $y$ , let  $\text{rem}(x, y)$  denote the remainder from dividing  $x$  by  $y$ .

**Definition.** For a positive integer  $n$ , by a computation of length  $n$  we understand any sequence of terms  $x_1, \dots, x_n$  such that  $x_1$  is defined as the variable  $x$ , and for every integer  $i \in \{2, \dots, n\}$ ,  $x_i$  is defined as  $\Gamma(x_{i-1})$ , or  $\text{fact}^{-1}(x_{i-1})$ , or  $\text{rem}(x_{i-1}, x_{i-2})$  (only if  $i \geq 3$  and  $x_{i-1}$  is defined as  $\Gamma(x_{i-2})$ ).

Let  $f(4) = 3$ , and let  $f(n + 1) = f(n)!$  for every integer  $n \geq 4$ . For an integer  $n \geq 4$ , let  $\Psi_n$  denote the following statement: if a computation of length  $n$  returns positive integers  $x_1, \dots, x_n$  for at most finitely many positive integers  $x$ , then every such  $x$  does not exceed  $f(n)$ .

**Lemma 1.** For every positive integer  $n$ , there are only finitely many computations of length  $n$ .

**Theorem 1.** For every integer  $n \geq 4$ , the statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .

*Proof.* It follows from Lemma 1. □

Let  $\mathcal{P}$  denote the set of prime numbers.

**Lemma 2.** ([4, pp. 214–215]). For every positive integer  $x$ ,  $\text{rem}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\}$  if and only if  $x \in \{4\} \cup \mathcal{P}$ .

**Theorem 2.** For every integer  $n \geq 4$  and for every positive integer  $x$ , the following computation  $\mathcal{H}$

$$\left\{ \begin{array}{l} x_1 := x \\ \forall i \in \{2, \dots, n-3\} x_i := \text{fact}^{-1}(x_{i-1}) \\ x_{n-2} := \Gamma(x_{n-3}) \\ x_{n-1} := \Gamma(x_{n-2}) \\ x_n := \text{rem}(x_{n-1}, x_{n-2}) \end{array} \right.$$

returns positive integers  $x_1, \dots, x_n$  if and only if  $x = f(n)$ .

*Proof.* We make three observations.

**Observation 1.** If  $x_{n-3} = 3$ , then  $x_1, \dots, x_{n-3} \in \mathbb{N} \setminus \{0\}$  and  $x = x_1 = f(n)$ .

If  $x = f(n)$ , then  $x_1, \dots, x_{n-3} \in \mathbb{N} \setminus \{0\}$  and  $x_{n-3} = 3$ .

Hence,  $x_{n-2} = \Gamma(x_{n-3}) = 2$  and  $x_{n-1} = \Gamma(x_{n-2}) = 1$ . Therefore,  $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 1$ .

**Observation 2.** If  $x_{n-3} = 2$ , then  $x = x_1 = \dots = x_{n-3} = 2$ .

If  $x = 2$ , then  $x_1 = \dots = x_{n-3} = 2$ . Hence,  $x_{n-2} = \Gamma(x_{n-3}) = 1$  and  $x_{n-1} = \Gamma(x_{n-2}) = 1$ .

Therefore,  $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$ .

**Observation 3.** If  $x_{n-3} = 1$ , then  $x_{n-2} = \Gamma(x_{n-3}) = 1$ . Hence,  $x_{n-1} = \Gamma(x_{n-2}) = 1$ .

Therefore,  $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$ .

Observations 1–3 cover the case when  $x_{n-3} \in \{1, 2, 3\}$ . If  $x_{n-3} \geq 4$ , then  $x_{n-2} = \Gamma(x_{n-3})$  is greater than 4 and composite. By Lemma 2,  $x_n = \text{rem}(x_{n-1}, x_{n-2}) = \text{rem}(\Gamma(x_{n-2}), x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$ .  $\square$

**Corollary 1.** For every integer  $n \geq 4$ , the bound  $f(n)$  in the statement  $\Psi_n$  cannot be decreased.

**Lemma 3.** (Wilson's theorem, [2, p. 89]). For every positive integer  $x$ ,  $x$  divides  $\Gamma(x) + 1$  if and only if  $x \in \{1\} \cup \mathcal{P}$ .

**Corollary 2.** If  $x \in \mathcal{P}$ , then  $\text{rem}(\Gamma(x), x) = x - 1$ .

**Lemma 4.** For every positive integer  $x$ , the following computation  $\mathcal{A}$

$$\left\{ \begin{array}{l} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \text{rem}(x_2, x_1) \\ x_4 := \text{fact}^{-1}(x_3) \end{array} \right.$$

returns positive integers  $x_1, \dots, x_4$  if and only if  $x = 4$  or  $x$  is a prime number of the form  $n! + 1$ .

*Proof.* For an integer  $i \in \{1, \dots, 4\}$ , let  $A_i$  denote the set of positive integers  $x$  such that the first  $i$  instructions of the computation  $\mathcal{A}$  returns positive integers  $x_1, \dots, x_i$ . We show that

$$A_4 = \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P}) \quad (1)$$

For every positive integer  $x$ , the terms  $x_1$  and  $x_2$  belong to  $\mathbb{N} \setminus \{0\}$ . By Lemma 2, the term  $x_3$  (which equals  $\text{rem}(\Gamma(x), x)$ ) belongs to  $\mathbb{N} \setminus \{0\}$  if and only if  $x \in \{4\} \cup \mathcal{P}$ . Hence,  $A_3 = \{4\} \cup \mathcal{P}$ . If  $x = 4$ , then  $x_1, \dots, x_4 \in \mathbb{N} \setminus \{0\}$ . Hence,  $4 \in A_4$ . If  $x \in \mathcal{P}$ , then Corollary 2 implies that  $x_3 = \text{rem}(\Gamma(x), x) = x - 1 \in \mathbb{N} \setminus \{0\}$ . Therefore, for every  $x \in \mathcal{P}$ , the term  $x_4 = \text{fact}^{-1}(x_3)$  belongs to  $\mathbb{N} \setminus \{0\}$  if and only if  $x \in \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\}$ . This proves equality (1).  $\square$

It is conjectured that there are infinitely many primes of the form  $n! + 1$ , see [1, p. 443] and [5].

**Theorem 3.** *The statement  $\Psi_4$  implies that the set of primes of the form  $n! + 1$  is infinite.*

*Proof.* The number  $3! + 1 = 7$  is prime. By Lemma 4, for  $x = 7$  the computation  $\mathcal{A}$  returns positive integers  $x_1, \dots, x_4$ . Since  $x = 7 > 3 = f(4)$ , the statement  $\Psi_4$  guarantees that the computation  $\mathcal{A}$  returns positive integers  $x_1, \dots, x_4$  for infinitely many positive integers  $x$ . By Lemma 4, there are infinitely many primes of the form  $n! + 1$ .  $\square$

**Lemma 5.** *If  $x \in \mathbb{N} \setminus \{0, 1\}$ , then  $\text{fact}^{-1}(\Gamma(x)) = x - 1$ .*

**Theorem 4.** *If the set of primes of the form  $n! + 1$  is infinite, then the statement  $\Psi_4$  is true.*

*Proof.* There exist 10 computations of length 4 that differ from  $\mathcal{H}$  and  $\mathcal{A}$ . We claim that every such computation  $\mathcal{F}$  returns positive integers  $x_1, \dots, x_4$  for infinitely many positive integers  $x$ . We omit the easy proof (which uses Lemmas 2 and 5) as it is based on case analysis.  $\square$

**Hypothesis.** *The statements  $\Psi_4, \dots, \Psi_7$  are true.*

**Lemma 6.** *For every positive integer  $x$ , the following computation  $\mathcal{B}$*

$$\left\{ \begin{array}{l} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \text{rem}(x_2, x_1) \\ x_4 := \text{fact}^{-1}(x_3) \\ x_5 := \Gamma(x_4) \\ x_6 := \text{rem}(x_5, x_4) \end{array} \right.$$

*returns positive integers  $x_1, \dots, x_6$  if and only if  $x \in \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}$*

*Proof.* For an integer  $i \in \{1, \dots, 6\}$ , let  $B_i$  denote the set of positive integers  $x$  such that the first  $i$  instructions of the computation  $\mathcal{B}$  returns positive integers  $x_1, \dots, x_i$ . Since the computations  $\mathcal{A}$  and  $\mathcal{B}$  have the same first four instructions, the equality  $B_i = A_i$  holds for every  $i \in \{1, \dots, 4\}$ . In particular,

$$B_4 = \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})$$

We show that

$$B_6 = \{4\} \cup (\{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}) \tag{2}$$

If  $x = 4$ , then  $x_1, \dots, x_6 \in \mathbb{N} \setminus \{0\}$ . Hence,  $4 \in B_6$ . Let  $x \in \mathcal{P}$ , and let  $x = n! + 1$ , where  $n \in \mathbb{N} \setminus \{0\}$ . Hence,  $n \neq 4$ . Corollary 2 implies that  $x_3 = \text{rem}(\Gamma(x), x) = x - 1 = n!$ . Hence,  $x_4 = \text{fact}^{-1}(x_3) = n$  and  $x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\}$ . By Lemma 2, the term  $x_6$  (which equals  $\text{rem}(\Gamma(n), n)$ ) belongs to  $\mathbb{N} \setminus \{0\}$  if and only if  $n \in \{4\} \cup \mathcal{P}$ . This proves equality (2) as  $n \neq 4$ .  $\square$

**Theorem 5.** *The statement  $\Psi_6$  implies that for infinitely many primes  $p$  the number  $p! + 1$  is prime.*

*Proof.* The numbers 11 and  $11! + 1$  are prime, see [1, p. 441] and [7]. By Lemma 6, for  $x = 11! + 1$  the computation  $\mathcal{B}$  returns positive integers  $x_1, \dots, x_6$ . Since  $x = 11! + 1 > 6! = f(6)$ , the statement  $\Psi_6$  guarantees that the computation  $\mathcal{B}$  returns positive integers  $x_1, \dots, x_6$  for infinitely many positive integers  $x$ . By Lemma 6, for infinitely many primes  $p$  the number  $p! + 1$  is prime.  $\square$

**Lemma 7.** For every positive integer  $x$ , the following computation  $C$

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \Gamma(x_2) \\ x_4 & := & \text{fact}^{-1}(x_3) \\ x_5 & := & \Gamma(x_4) \\ x_6 & := & \text{rem}(x_5, x_4) \end{cases}$$

returns positive integers  $x_1, \dots, x_6$  if and only if  $(x-1)! - 1$  is prime.

*Proof.* For an integer  $i \in \{1, \dots, 6\}$ , let  $C_i$  denote the set of positive integers  $x$  such that the first  $i$  instructions of the computation  $C$  returns positive integers  $x_1, \dots, x_i$ . If  $x \in \{1, 2, 3\}$ , then  $x_6 = 0$ . Therefore,  $C_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\}$ . By Lemma 5, for every integer  $x \geq 4$ ,  $x_4 = (x-1)! - 1$ ,  $x_5 = \Gamma((x-1)! - 1)$ , and  $x_1, \dots, x_5 \in \mathbb{N} \setminus \{0\}$ . By Lemma 2, for every integer  $x \geq 4$ ,

$$x_6 = \text{rem}(\Gamma((x-1)! - 1), (x-1)! - 1)$$

belongs to  $\mathbb{N} \setminus \{0\}$  if and only if  $(x-1)! - 1 \in \{4\} \cup \mathcal{P}$ . The last condition equivalently expresses that  $(x-1)! - 1$  is prime as  $(x-1)! - 1 \geq 5$  for every integer  $x \geq 4$ . Hence,

$$C_6 = (\mathbb{N} \setminus \{0, 1, 2, 3\}) \cap \{x \in \mathbb{N} \setminus \{0, 1, 2, 3\} : (x-1)! - 1 \in \mathcal{P}\} = \{x \in \mathbb{N} \setminus \{0\} : (x-1)! - 1 \in \mathcal{P}\}$$

□

It is conjectured that there are infinitely many primes of the form  $n! - 1$ , see [1, p. 443] and [6].

**Theorem 6.** The statement  $\Psi_6$  implies that there are infinitely many primes of the form  $x! - 1$ .

*Proof.* The number  $(975-1)! - 1$  is prime, see [1, p. 441] and [6]. By Lemma 7, for  $x = 975$  the computation  $C$  returns positive integers  $x_1, \dots, x_6$ . Since  $x = 975 > 720 = f(6)$ , the statement  $\Psi_6$  guarantees that the computation  $C$  returns positive integers  $x_1, \dots, x_6$  for infinitely many positive integers  $x$ . By Lemma 7, the set  $\{x \in \mathbb{N} \setminus \{0\} : (x-1)! - 1 \in \mathcal{P}\}$  is infinite. □

**Lemma 8.** For every positive integer  $x$ , the following computation  $\mathcal{D}$

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \text{rem}(x_2, x_1) \\ x_4 & := & \Gamma(x_3) \\ x_5 & := & \text{fact}^{-1}(x_4) \\ x_6 & := & \Gamma(x_5) \\ x_7 & := & \text{rem}(x_6, x_5) \end{cases}$$

returns positive integers  $x_1, \dots, x_7$  if and only if both  $x$  and  $x-2$  are prime.

*Proof.* For an integer  $i \in \{1, \dots, 7\}$ , let  $D_i$  denote the set of positive integers  $x$  such that the first  $i$  instructions of the computation  $\mathcal{D}$  returns positive integers  $x_1, \dots, x_i$ . If  $x = 1$ , then  $x_3 = 0$ . Hence,  $D_7 \subseteq D_3 \subseteq \mathbb{N} \setminus \{0, 1\}$ . If  $x \in \{2, 3, 4\}$ , then  $x_7 = 0$ . Therefore,

$$D_7 \subseteq (\mathbb{N} \setminus \{0, 1\}) \cap (\mathbb{N} \setminus \{0, 2, 3, 4\}) = \mathbb{N} \setminus \{0, 1, 2, 3, 4\}$$

By Lemma 2, for every integer  $x \geq 5$ , the term  $x_3$  (which equals  $\text{rem}(\Gamma(x), x)$ ) belongs to  $\mathbb{N} \setminus \{0\}$  if and only if  $x \in \mathcal{P} \setminus \{2, 3\}$ . By Corollary 2, for every  $x \in \mathcal{P} \setminus \{2, 3\}$ ,  $x_3 = x - 1 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ . By Lemma 5, for every  $x \in \mathcal{P} \setminus \{2, 3\}$ , the terms  $x_4$  and  $x_5$  belong to  $\mathbb{N} \setminus \{0\}$  and  $x_5 = x_3 - 1 = x - 2$ . By Lemma 2, for every  $x \in \mathcal{P} \setminus \{2, 3\}$ , the term  $x_7$  (which equals  $\text{rem}(\Gamma(x_5), x_5)$ ) belongs to  $\mathbb{N} \setminus \{0\}$  if and only if  $x_5 = x - 2 \in \{4\} \cup \mathcal{P}$ . From these facts, we obtain that

$$D_7 = (\mathbb{N} \setminus \{0, 1, 2, 3, 4\}) \cap (\mathcal{P} \setminus \{2, 3\}) \cap (\{6\} \cup \{p + 2 : p \in \mathcal{P}\}) = \{p \in \mathcal{P} : p - 2 \in \mathcal{P}\}$$

□

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [3, p. 39].

**Theorem 7.** *The statement  $\Psi_7$  implies that there are infinitely many twin primes.*

*Proof.* Harvey Dubner proved that the numbers  $459 \cdot 2^{8529} - 1$  and  $459 \cdot 2^{8529} + 1$  are prime, see [8, p. 87]. By Lemma 8, for  $x = 459 \cdot 2^{8529} + 1$  the computation  $\mathcal{D}$  returns positive integers  $x_1, \dots, x_7$ . Since  $x > 720! = f(7)$ , the statement  $\Psi_7$  guarantees that the computation  $\mathcal{D}$  returns positive integers  $x_1, \dots, x_7$  for infinitely many positive integers  $x$ . By Lemma 8, there are infinitely many twin primes. □

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