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A conjecture which implies that any twin prime greater than
 (((((((((((((24!)!)!)!)!)!)!)!)!)!)!)! + 3 guarantees that the set of twin primes is infinite

Apoloniusz Tyszka

Abstract

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. For a positive integer n , let Γ_n denote the following statement: if a system $\mathcal{S} \subseteq \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ has at most finitely many solutions in integers x_1, \dots, x_n greater than 1, then each such solution (x_1, \dots, x_n) satisfies $\min(x_1, \dots, x_n) \leq f(n)$. We conjecture that the statements $\Gamma_1, \dots, \Gamma_{16}$ are true. We prove: (1) if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers, then the statement Γ_6 guarantees that each such solution (x, y) satisfies $x \leq f(6)$; (2) the statement Γ_9 proves the implication: if there exists an integer $x > f(9)$ such that $x^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$; (3) the statement Γ_{16} proves the implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.

Key words and phrases: Brocard’s problem, Brocard-Ramanujan equation, prime numbers of the form $n^2 + 1$, single query to an oracle for the halting problem, twin prime conjecture.

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1. Introduction and basic lemmas

In this note, we propose a conjecture which provides a common approach to Brocard’s problem, the problem of the infinitude of primes of the form $n^2 + 1$, and the twin prime problem. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{V}_1 denote the system of equations $\{x_1! = x_1\}$, and let \mathcal{V}_2 denote the system of equations $\{x_1! = x_1, x_1 \cdot x_1 = x_2\}$. For an integer $n \geq 3$, let \mathcal{V}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n - 1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{V}_n .

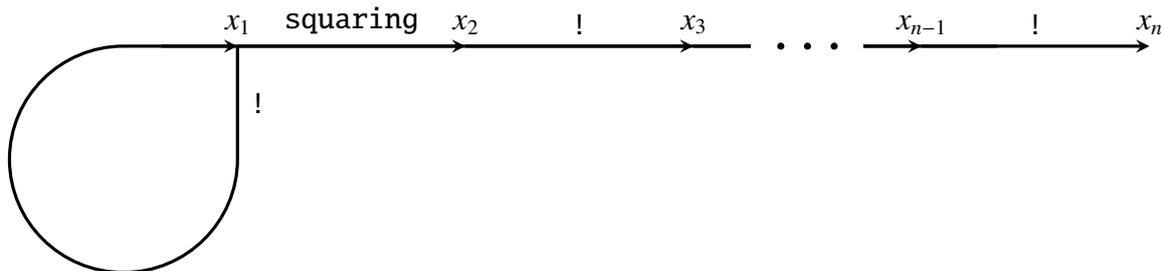


Fig. 1 Construction of the system \mathcal{V}_n

Lemma 1. For every positive integer n , the system \mathcal{V}_n has exactly one solution in integers greater than 1, namely $(f(1), \dots, f(n))$.

Let

$$H_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Γ_n denote the following statement: if a system $\mathcal{S} \subseteq H_n$ has at most finitely many solutions in integers x_1, \dots, x_n greater than 1, then each such solution (x_1, \dots, x_n) satisfies $\min(x_1, \dots, x_n) \leq f(n)$. We conjecture that the statements $\Gamma_1, \dots, \Gamma_{16}$ are true. For every positive integer n , the system H_n has a finite number of subsystems. Therefore, every statement Γ_n is true with an unknown integer bound that depends on n .

Lemma 2. For every integers x and y greater than 1, $x! \cdot y = y!$ if and only if $x + 1 = y$.

Lemma 3. If $x \geq 4$, then $\frac{(x-1)! + 1}{x} > 1$.

Lemma 4. (Wilson's theorem, [2, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x-1)! + 1$.

2. Brocard's problem

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the Brocard-Ramanujan equation $x! + 1 = y^2$, see [4]. It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [5, p. 297].

Let \mathcal{A} denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

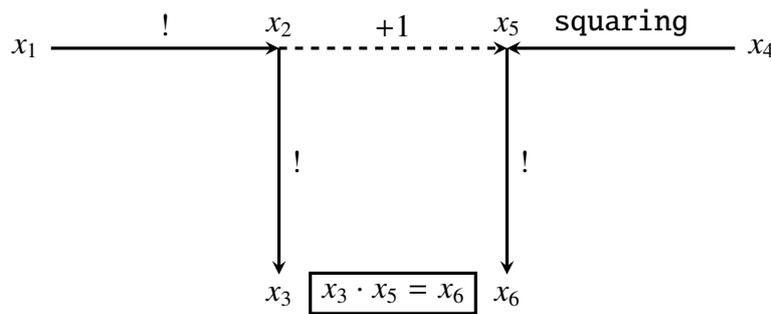


Fig. 2 Construction of the system \mathcal{A}

Lemma 5. For every integers x_1 and x_4 greater than 1, the system \mathcal{A} is solvable in integers x_2, x_3, x_5, x_6 greater than 1 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

and $x_1 = \min(x_1, \dots, x_6)$.

Proof. It follows from Lemma 2. □

Theorem 1. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Γ_6 implies that each such solution (x_1, x_4) satisfies $x_1 \leq f(6)$.*

Proof. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 5, there exists a unique tuple $(x_2, x_3, x_5, x_6) \in (\mathbb{N} \setminus \{0, 1\})^4$ such that the tuple (x_1, \dots, x_6) solves the system \mathcal{A} . Lemma 5 guarantees that $x_1 = \min(x_1, \dots, x_6)$. By the antecedent and Lemma 5, the system \mathcal{A} has only finitely many solutions in integers x_1, \dots, x_6 greater than 1. Therefore, the statement Γ_6 implies that $x_1 = \min(x_1, \dots, x_6) \leq f(6)$. □

3. Are there infinitely many prime numbers of the form $n^2 + 1$?

Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [3, pp. 37–38].

Let \mathcal{B} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system \mathcal{B} .

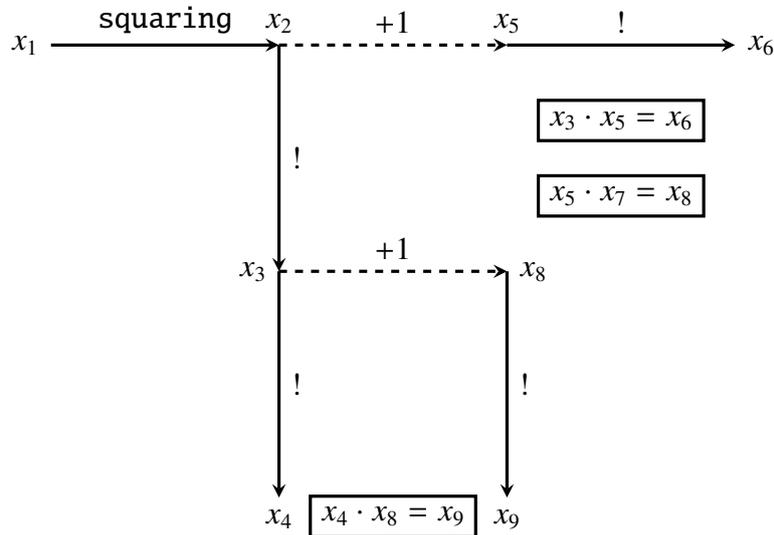


Fig. 3 Construction of the system \mathcal{B}

Lemma 6. *For every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in integers x_2, \dots, x_9 greater than 1 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by*

the following equalities:

$$\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}$$

and $\min(x_1, \dots, x_9) = x_1$.

Proof. By Lemmas 2 and 3, for every integer $x_1 \geq 2$, the system \mathcal{B} is solvable in integers x_2, \dots, x_9 greater than 1 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 6 follows from Lemma 4. \square

Theorem 2. *The statement Γ_9 proves the implication: if there exists an integer $x_1 > f(9)$ such that $x_1^2 + 1$ is prime, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Assume that an integer x_1 is greater than $f(9)$ and $x_1^2 + 1$ is prime. By Lemma 6, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{B} . Lemma 6 guarantees that $\min(x_1, \dots, x_9) = x_1$. Since $\mathcal{B} \subseteq H_9$, the statement Γ_9 and the inequality $\min(x_1, \dots, x_9) = x_1 > f(9)$ imply that the system \mathcal{B} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0, 1\})^9$. According to Lemma 6, there are infinitely many primes of the form $n^2 + 1$. \square

Corollary 1. *Assuming the statement Γ_9 , a single query to an oracle for the halting problem decides the problem of the infinitude of primes of the form $n^2 + 1$.*

4. The twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [3, p. 39].

Let C denote the following system of equations:

$$\left\{ \begin{array}{l}
x_1! = x_2 \\
x_2! = x_3 \\
x_4! = x_5 \\
x_6! = x_7 \\
x_7! = x_8 \\
x_9! = x_{10} \\
x_{12}! = x_{13} \\
x_{15}! = x_{16} \\
x_2 \cdot x_4 = x_5 \\
x_5 \cdot x_6 = x_7 \\
x_7 \cdot x_9 = x_{10} \\
x_4 \cdot x_{11} = x_{12} \\
x_3 \cdot x_{12} = x_{13} \\
x_9 \cdot x_{14} = x_{15} \\
x_8 \cdot x_{15} = x_{16}
\end{array} \right.$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system C .

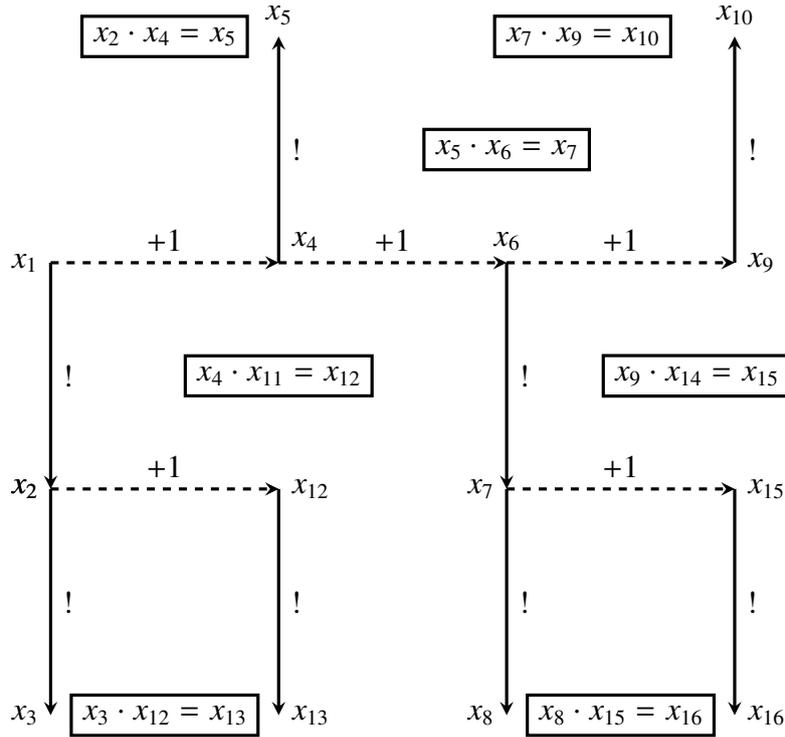


Fig. 4 Construction of the system C

Lemma 7. *If $x_4 = 2$, then the system C has no solutions in integers x_1, \dots, x_{16} greater than 1.*

Proof. The equality $x_2 \cdot x_4 = x_5 = x_4!$ and the equality $x_4 = 2$ imply that $x_2 = 1$. \square

Lemma 8. *If $x_4 = 3$, then the system C has no solutions in integers x_1, \dots, x_{16} greater than 1.*

Proof. The equality $x_4 \cdot x_{11} = x_{12} = (x_4 - 1)! + 1$ and the equality $x_4 = 3$ imply that $x_{11} = 1$. \square

Lemma 9. *For every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ greater than 1 if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$\begin{aligned}
 x_1 &= x_4 - 1 \\
 x_2 &= (x_4 - 1)! \\
 x_3 &= ((x_4 - 1)!)! \\
 x_5 &= x_4! \\
 x_6 &= x_9 - 1 \\
 x_7 &= (x_9 - 1)! \\
 x_8 &= ((x_9 - 1)!)! \\
 x_{10} &= x_9! \\
 x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
 x_{12} &= (x_4 - 1)! + 1 \\
 x_{13} &= ((x_4 - 1)! + 1)! \\
 x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
 x_{15} &= (x_9 - 1)! + 1 \\
 x_{16} &= ((x_9 - 1)! + 1)!
 \end{aligned}$$

and $\min(x_1, \dots, x_{16}) = x_1 = x_9 - 3$.

Proof. By Lemmas 2 and 3, for every $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ and for every $x_9 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ greater than 1 if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | (x_4 - 1)! + 1) \wedge (x_9 | (x_9 - 1)! + 1)$$

Hence, the claim of Lemma 9 follows from Lemma 4. \square

Theorem 3. *The statement Γ_{16} proves the implication: if there exists a twin prime greater than $f(16) + 3$, then there are infinitely many twin primes.*

Proof. Assume the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > f(16) + 3$. Hence, $x_4 \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. By Lemma 9, there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0, 1\})^{14}$ such that the tuple (x_1, \dots, x_{16}) solves the system C . Lemma 9 guarantees that $\min(x_1, \dots, x_{16}) = x_1 = x_9 - 3 > f(16)$. Since $C \subseteq H_{16}$, the statement Γ_{16} and the inequality $\min(x_1, \dots, x_{16}) > f(16)$ imply that the system C has infinitely many solutions in integers x_1, \dots, x_{16} greater than 1. According to Lemmas 7–9, there are infinitely many twin primes. \square

Corollary 2. (cf. [1]). *Assuming the statement Γ_{16} , a single query to an oracle for the halting problem decides the twin prime problem.*

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