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The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory *ZFC* as it refers to our current knowledge on \mathcal{X}

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form n^2+1 is infinite. Let $\beta = (((24!)!)!)!$, and let Φ denote the implication: $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement Φ without invoking Landau's conjecture. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)-(4). (1) There are a large number of elements of \mathcal{X} and it is conjectured that \mathcal{X} is infinite. (2) No known algorithm decides the finiteness/infiniteness of \mathcal{X} . (3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. (4) There is an explicitly known integer n such that $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. (5) There is an explicitly known integer n such that $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ and some known definition of \mathcal{X} is much simpler than every known definition of $\mathcal{X} \setminus (-\infty, n]$. The following problem is open: *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)-(3) and (5)?* The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)-(3). Let $[\cdot]$ denote the integer part function. For every explicitly given integer $m \geq 1$, the set $\mathcal{X} = \left\{ k \in \mathbb{N} : \left[\frac{k}{m} \right]^2 + 1 \text{ is prime} \right\}$ contains m consecutive integers and satisfies conditions (1)-(3). The statement Φ implies that both sets \mathcal{X} satisfy condition (5).

Key words and phrases: complexity of a mathematical definition, computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on \mathcal{X} , explicitly known integer n bounds \mathcal{X} from above when \mathcal{X} is finite, infiniteness of \mathcal{X} remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, large number of elements of \mathcal{X} , mathematical statement that cannot be formalized in the set theory *ZFC*, no known algorithm decides the finiteness/infiniteness of \mathcal{X} , physical impossibility of machine computations on sufficiently large integers.

1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

Definition 1. *Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.*

(1) *There are a large number of elements of X and it is conjectured that X is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of X .*

(3) *There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in X$.*

(4) *There is an explicitly known integer n such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.*

(5) *There is an explicitly known integer n such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$ and some known definition of X is much simpler than every known definition of $X \setminus (-\infty, n]$.*

Definition 2. *We say that an integer n is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, cf. [6] and [7].*

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of X . If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $[\max(X), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [4] and [5].

Definition 3. *Let Φ denote the implication:*

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)]$$

Landau's conjecture implies the statement Φ . In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *There is no explicitly known threshold number of \mathcal{P}_{n^2+1} . It means that there is no explicitly known integer k such that $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.*

Proving the statement Φ will falsify Statement 1. Statement 1 cannot be formalized in the set theory *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2–4 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $X \subseteq \mathbb{N}$.

2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

Definition 4. Let $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{25.16114896940657}}}$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. \square

Statement 2. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\mathcal{X} \supseteq \{0, \dots, \beta\}$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β , see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

In Statement 2,

$$\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, \beta]$$

and the sets

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

and

$$\mathcal{X} \setminus (-\infty, \beta] = \{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

have definitions of similar complexity. The following problem arises:

Open Problem 1. Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions (1)–(3) and (5)?

Theorem 1. Assume that for every positive integers b and s , at some future day, machine computations will be possible on every integers from the interval $[-b, b]$ and this will be possible with the speed of s FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever.

Proof. The proof goes by contradiction. Since conditions (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X}, n + 2 \notin \mathcal{X}, n + 3 \notin \mathcal{X}, \dots \quad (\text{T})$$

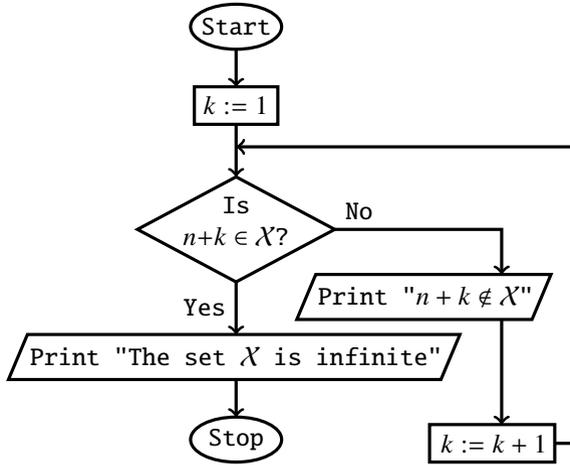


Fig. 1 Algorithm whose execution never terminates if the set \mathcal{X} is finite

The sentences from the sequence (T) and our assumptions alone imply that for every explicitly given integer $m > n$, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set \mathcal{X} is finite, contrary to the conjecture in condition (1). \square

3. Number-theoretic statements Ψ_n

Let $f(1) = 2, f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let \mathcal{U}_1 denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \dots, n - 1\} x_i! = x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system \mathcal{U}_n .

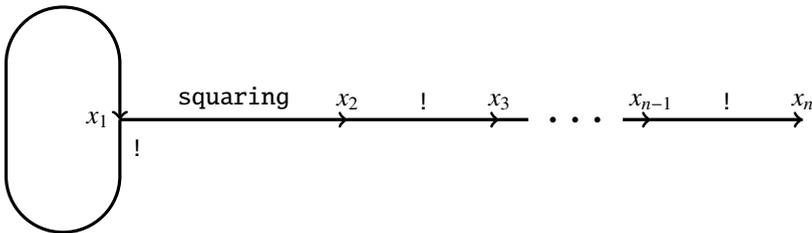


Fig. 2 Construction of the system \mathcal{U}_n

Lemma 2. For every positive integer n , the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(f(1), \dots, f(n))$.

Let

$$B_n = \{x_i! = x_k : i, k \in \{1, \dots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For a positive integer n , let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$* . The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement Ψ_9 , cf. Conjecture 1 in Section 4.

Theorem 2. *For every statement Ψ_n , the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geq 2$, the statement Ψ_{n+1} implies the statement Ψ_n .*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers x_1, \dots, x_n , then for every integer $i \in \{1, \dots, n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers x_1, \dots, x_{n+1} . The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leq f(n+1) = f(n)!$. Hence, $x_i \leq f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n .*

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y , $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 4. (*Wilson's theorem*, [1, p. 89]). *For every integer $x \geq 2$, x is prime if and only if x divides $(x - 1)! + 1$.*

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system \mathcal{A} .

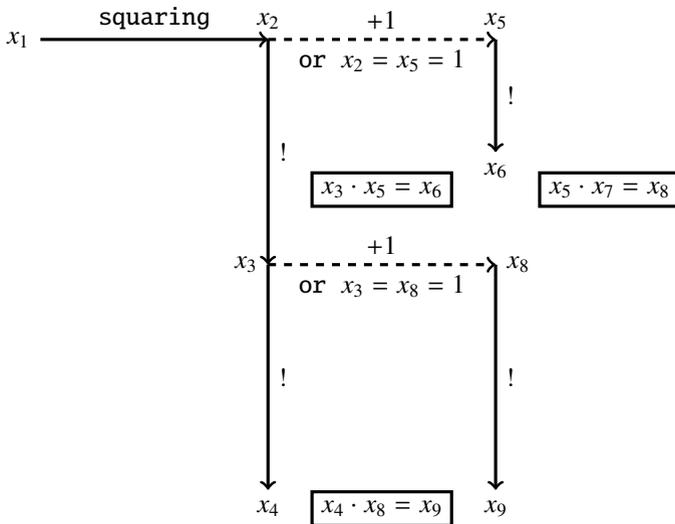


Fig. 3 Construction of the system \mathcal{A}

Lemma 5. *For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:*

$$\begin{aligned}
 x_2 &= x_1^2 \\
 x_3 &= (x_1^2)! \\
 x_4 &= ((x_1^2)!)! \\
 x_5 &= x_1^2 + 1 \\
 x_6 &= (x_1^2 + 1)! \\
 x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
 x_8 &= (x_1^2)! + 1 \\
 x_9 &= ((x_1^2)! + 1)!
 \end{aligned}$$

Proof. By Lemma 3, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Lemma 6. *There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system \mathcal{A} and satisfy $x_1 = 1$. This is true as every such tuple (x_1, \dots, x_9) satisfies $x_1, \dots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system \mathcal{A} .*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set \mathcal{P}_{n^2+1} is infinite. \square

Theorem 6. *Conjecture 1 implies the statement Φ .*

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. \square

Theorem 7. *The statement Φ implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers x_1, \dots, x_9 solve the system \mathcal{A} , then

$$(x_1 \geq 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \dots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system \mathcal{A} has at most finitely many solutions in positive integers x_1, \dots, x_9 . Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. \square

Statement 3. *The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3). The statement Φ implies that the set \mathcal{X} satisfies condition (5).*

Proof. Since the set \mathcal{P}_{n^2+1} is conjecturally infinite, condition (1) holds for \mathcal{X} . Condition (3) holds trivially. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds for \mathcal{X} . Suppose that the statement Φ holds. This implies that β is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. Thus condition (4) holds for \mathcal{X} . The definition of \mathcal{P}_{n^2+1} is much simpler than the definition of $\mathcal{P}_{n^2+1} \setminus (-\infty, \beta]$. The last two sentences imply that condition (5) holds for \mathcal{X} . \square

Let $[\cdot]$ denote the integer part function.

Statement 4. *For every explicitly given integer $m \geq 1$, the set $\mathcal{X} = \{k \in \mathbb{N} : \left[\frac{k}{m}\right]^2 + 1 \text{ is prime}\}$ contains m consecutive integers and satisfies conditions (1)–(3). The statement Φ implies that the set \mathcal{X} satisfies condition (5).*

Proof. The set \mathcal{X} contains m consecutive integers because the number 2 is prime and the equality $\left[\frac{k}{m}\right]^2 + 1 = 2$ holds for every integer $k \in \{m, \dots, 2m - 1\}$. The rest of the proof goes as in the proof of Statement 3, although the statement Φ allows us to compute a threshold number of \mathcal{X} that depends on m . \square

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