Error estimation for FETI(DP) and BDD(C)

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Context

Virtual testing aims at replacing (expensive) physical tests on large structures (up to failure) by simulations.

Figure: A380 wing torsion test (supaero)

Figure: Delamination of bolted composite plates

This implies several capabilities:

- (Ensuring the quality of models).
- Obtaining results in a reasonable time, with a correct exploitation of current hardware, even if models are ugly (nonlinearity, softening, instabilities)
- Generating and handling distributed data (pre/post).

$\longrightarrow$ Ensuring the quality of computations, adapting the discretizations

Context

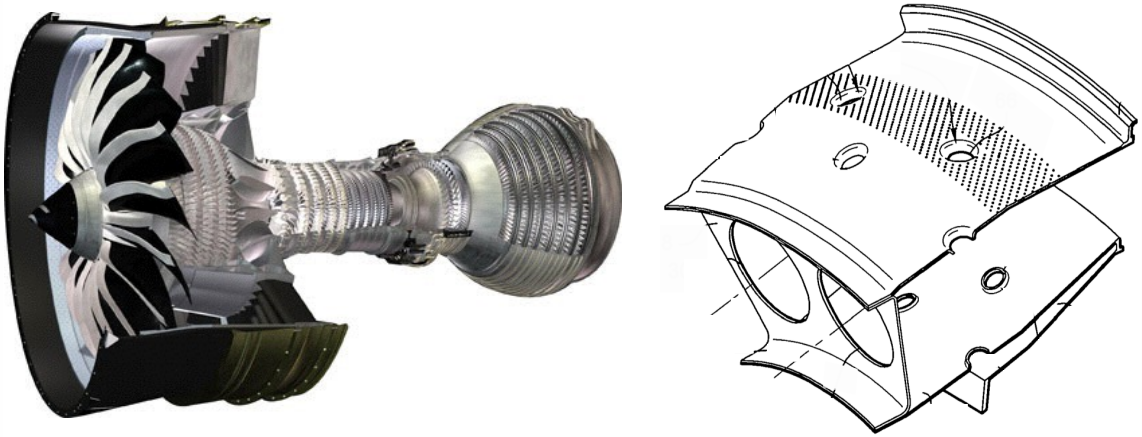

Figure: LEAP engine, combustion chamber (Safran)

Classical design practice

Mesh and model are based on engineer rules, continuous Galerkin finite element, implicit time integration, Newton solver, direct sparse solver for the resulting system.
+ huge safety factor (which accounts for both model and numerical errors).

The new designs and working regimes (e.g. $T \simeq 1500^\circ$, extremely small perforations) are almost beyond the domain of validity of current practices.

Error estimation enables to reduce the safety factor $\longrightarrow$ better performance.

Classical linear elasticity problem

Ω open polyhedral domain of $\mathbb{R}^d$ ($d = 2$ ou 3), $\mathbb{H}$: Hooke's tensor
Load: body force $f \in L^2(\Omega)$,
bcs: Neumann $g \in L^2(\partial_n\Omega)$, Dirichlet $u_d \in H^{1/2}(\partial_d\Omega)$.

Usual variational formulation

Kinematically admissible displacement ($H^1(\Omega)$):
$\mathrm{KA}(\Omega) = \{u \in H^1(\Omega),\ u = u_b \text{ sur } \partial_d\Omega\}$

Find $u \in \mathrm{KA}(\Omega)$ s.t. $\forall v \in \mathrm{KA}_0(\Omega)$,

$$\int_\Omega \varepsilon(u) : \mathbb{H} : \varepsilon(v)\, dx =: a(u,v) = l(v) := \int_\Omega f \cdot v\, dx + \int_{\partial_n\Omega} g \cdot v\, dS$$

with ε the symmetric part of the gradient.

[1] Ladevèze, 1975.

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with ε the symmetric part of the gradient.

Formulation by the **Error in constitutive relation**[1]

Statically admissible stress ($H_{div}(\Omega)$)
$\mathrm{SA}(\Omega) = \{\sigma \in L^2_{sym}(\Omega),\ \forall v \in \mathrm{KA}_0(\Omega), \int_\Omega \sigma : \varepsilon(v)\, dx = \int_\Omega f \cdot v\, dx + \int_{\partial_n\Omega} g \cdot v\, dS\}$

Find $(u,\sigma) \in \mathrm{KA}(\Omega) \times \mathrm{SA}(\Omega)$ s.t. $\mathrm{e_{cr}}(\varepsilon(u),\sigma) := |||\mathbb{H}^{-1} : \sigma - \varepsilon(u)|||_\Omega = 0$

$|||\varepsilon|||_\Omega$ is the energy norm, in particular $|||\varepsilon(u)|||_\Omega = \|u\|_a$.

[1] Ladevèze, 1975.

Classical approximation

continuous Galerkin finite element

Find $u_h \in \mathrm{KA}_h(\Omega)$ subspace of $\mathrm{KA}(\Omega)$ of finite dimension, s.t. $\forall v_h \in \mathrm{KA}_{h0}(\Omega)$

$$r_{u_h}(v_h) := a(u_h, v_h) - l(v_h) = 0$$

and we define $\sigma_h = \mathbb{H} : \varepsilon(u_h)$

Error estimation

Classical approximation

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and we define $\sigma_h = \mathbb{H} : \varepsilon(u_h)$

Error estimation

- σ_h is not smooth $\rightarrow$ ZZ2 estimator[2],

Let $\tilde{\sigma}$ be a smoothing of σ_h,

$$\eta_{ZZ2} = |||\sigma_h - \tilde{\sigma}|||$$

Very often $\|u - u_h\|_a \simeq \eta_{ZZ2}$ but no warranty

[2] Zienkiewicz and Zhu, 1987.

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and we define $\sigma_h = \mathbb{H} : \varepsilon(u_h)$

Error estimation

- σ_h is not smooth $\rightarrow$ ZZ2 estimator,
- $\sigma_h \notin \mathrm{SA}(\Omega) \rightarrow$ explicit residuals,[2]

$$\eta^2 = \sum_T h_T^2 \| \operatorname{div} \sigma_h + f \|_{0,T}^2 + \sum_E h_E \| [\sigma_h]_E \cdot n_E \|_{0,E}^2$$

$$\| u - u_h \|_a^2 \leqslant C_1 \eta^2$$

$$C_2 \eta^2 \leqslant \| u - u_h \|_a^2 + \mathrm{osc}^2$$

Oscillation term: $\mathrm{osc}^2 = \sum_T \| h_T (f - \bar{f}_T) \|_{0,T}^2 \dots$

[2] Verfürth, 1996, review.

Classical approximation

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Error estimation

- σ_h is not smooth $\rightarrow$ ZZ2 estimator,
- $\sigma_h \notin \mathrm{SA}(\Omega) \rightarrow$ explicit residuals,
- Stein's formula[2] $\rightarrow$ constant-free explicit residuals,

$$\eta_{new}^2 = \sum_T h_T^2 (\| \operatorname{div} \sigma_h + f \|_{0,T} + C_T \sum_{E \in \partial T} \frac{\sqrt{h_E}}{\sqrt{|T|}} \| [\sigma_h]_E \cdot n_E \|_{0,E})^2$$

$$\| u - u_h \|_a \leqslant C \eta_{new}$$

all constants are computable

[2] Gerasimov, Stein, and Wriggers, 2015.

Classical approximation

continuous Galerkin finite element

Find $u_h \in \mathrm{KA}_h(\Omega)$ subspace of $\mathrm{KA}(\Omega)$ of finite dimension, s.t. $\forall v_h \in \mathrm{KA}_{h0}(\Omega)$

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and we define $\sigma_h = \mathbb{H} : \varepsilon(u_h)$

Error estimation

- σ_h is not smooth $\rightarrow$ ZZ2 estimator,
- $\sigma_h \notin \mathrm{SA}(\Omega) \rightarrow$ explicit residuals,
- Stein's formula $\rightarrow$ constant-free explicit residuals,
- Error in constitutive relation[2] (implicit residuals),

Prager-Synge theorem: let $\hat{u} \in \mathrm{KA}(\Omega)$, $\hat{\sigma} \in \mathrm{SA}(\Omega)$

$$\|u - \hat{u}\|_a^2 + |||\sigma - \hat{\sigma}|||^2 = \mathrm{e_{cr}}^2(\varepsilon(\hat{u}), \hat{\sigma})$$

Use $\hat{u} = u_h$ and compute $\hat{\sigma}$ from σ_h (aka. equilibration). Better if $\hat{\sigma} \simeq \sigma$:

$$\|u - u_h\|_a \leqslant \mathrm{e_{cr}}(\varepsilon(u_h), \hat{\sigma})$$

[2] Ladevèze and Leguillon, 1983.

Classical approximation

continuous Galerkin finite element

Find $u_h \in \mathrm{KA}_h(\Omega)$ subspace of $\mathrm{KA}(\Omega)$ of finite dimension, s.t. $\forall v_h \in \mathrm{KA}_{h0}(\Omega)$

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and we define $\sigma_h = \mathbb{H} : \varepsilon(u_h)$

Error estimation

- σ_h is not smooth $\rightarrow$ ZZ2 estimator,
- $\sigma_h \notin \mathrm{SA}(\Omega) \rightarrow$ explicit residuals,
- Stein's formula $\rightarrow$ constant-free explicit residuals,
- Error in constitutive relation (implicit residuals),
- Lower bound by estimation of the residual,

$$\forall \hat{w} \in \mathrm{KA}_0(\Omega), \quad \frac{|r_{u_h}(\hat{w})|}{\|\hat{w}\|_a} \leqslant \|u - u_h\|_a$$

$\hat{w}$ must be of high order. Better if $\hat{w} \simeq (u - u_h)$.

Classical approximation

continuous Galerkin finite element

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$$r_{u_h}(v_h) := a(u_h, v_h) - l(v_h) = 0$$

and we define $\sigma_h = \mathbb{H} : \varepsilon(u_h)$

Error estimation

- σ_h is not smooth $\rightarrow$ ZZ2 estimator, easy to compute, often efficient but not rigorous (in its early versions)
- $\sigma_h \notin \mathrm{SA}(\Omega) \rightarrow$ explicit residuals, perfect for adaptation, not for error measurement
- Stein's formula $\rightarrow$ constant-free explicit residuals, not tested
- Error in constitutive relation (implicit residuals), computationally demanding but constant free $\longleftarrow$
- Lower bound by estimation of the residual, computationally demanding but constant free and by-product of equilibration[2] $\longleftarrow$

[2] Díez, Parés, and Huerta, 2003.

Given an error estimation with local (element) contributions (η_T), there are mainly two strategies:

- The nested discretization method, based on the chain:

$$SOLVE \longrightarrow ESTIMATE \longrightarrow MARK \longrightarrow REFINE$$

  - MARK: select elements most contributing to the error (most famous: Dörfler's marking[3])
  - REFINE: iterative or recursive bisection of elements (even local errors impact lots of elements)

  $\longrightarrow$ lots of interesting properties (error decrease, quasi-optimality)

- The full remeshing technique based on the definition of map of characteristic lengths computed using a priori estimators with the objective to have all elements contributing identically to the error.

[3] Dörfler, 1996.

Linear quantities of interest

Bounds on the energy norm of the error may not be interesting for an engineer who may prefer the error on specific values (average stress in a region, SIF around a crack, displacement of a node...).

Extractor and adjoint problem

We consider continuous linear quantities of interest of the form $\tilde{l}(u)$.
The adjoint problem writes:

$$\text{Find } \tilde{u} \in \mathrm{KA}_0(\Omega) \text{ s.t. } \forall v \in \mathrm{KA}_0(\Omega),\ a(v, \tilde{u}) = \tilde{l}(v)$$

Let $\tilde{u}_h$ be the finite element approximation of $\tilde{u}$, we have:

$$\tilde{l}(u - u_h) = a(u - u_h, \tilde{u} - \tilde{u}_h)$$

Cauchy-Schwarz bound

$$|\tilde{l}(u - u_h)| \leqslant \|u - u_h\|_a \|\tilde{u} - \tilde{u}_h\|_a \leqslant \mathrm{e_{cr}} \tilde{\mathrm{e}}_{\mathrm{cr}}$$

Linear quantities of interest

Parallelogram bound

Let $e_h = u - u_h$ and $\tilde{e}_h = \tilde{u} - \tilde{u_h}$,

$$\tilde{l}(u - u_h) = \frac{1}{4}\left(\underbrace{\|se_h + \frac{1}{s}\tilde{e}_h\|_a}_{S^+} - \underbrace{\|se_h - \frac{1}{s}\tilde{e}_h\|_a}_{S^-}\right)$$

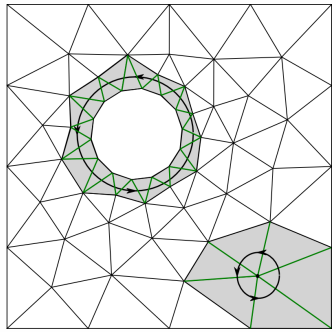
with $s = \|\tilde{e}_h\|_a / \|e_h\|_a$

Let $\beta^{+/-}_{\text{inf}\,/\,\text{sup}}$ be bounds for $S^{+/-}$, we have:

$$\beta^+_{\text{inf}} - \beta^-_{\text{sup}} \leqslant 4\tilde{l}(u - u_h) \leqslant \beta^+_{\text{sup}} - \beta^-_{\text{inf}}$$

Sequential recovery of admissible fields

$\hat{\sigma} \in \mathrm{SA}(\Omega) \subset H_{div}(\Omega)$
$w \in \mathrm{KA}_0(\Omega)$



- Element equilibration techniques
  1. Compute face tractions from σ_h, optimize on closed loops[4] or at least on star-patches[5].
  2. Solve Neumann problems on elements with high order elements[6].
- Flux-free technique[7]
  - Using partition of unity, directly solve the error equation $a(e_h, v) = r_h(v)$ with high order on star-patches.
  - Automatically provides both $\hat{\sigma}$ and $\hat{w}$ (for the lower bound).

The recovery involves many localized operations. Evaluating the error roughly doubles the computational cost.

In the following we assume we have a recovery procedure $(\hat{\sigma}, w) = \mathcal{F}(\sigma_h, f, g)$.
It works for pure Neumann problems if rigid body balance is satisfied by (f, g).

---

[4] V. Rey, Gosselet, and C. Rey, 2014.

[5] Pled, Chamoin, and Ladevèze, 2011.

[6] Babuška et al., 1994.

[7] Parés, Díez, and Huerta, 2006.

1. Context

2. Verification in a nutshell
   - Error estimators
   - Adaptation
   - Linear quantities of interest
   - Sequential recovery of admissible fields

3. Verification applied to FETI(DP) and BDD(C)
   - Distributed admissible fields
   - FETI
   - First bounds
   - Bounds with separated contributions
   - Adaptation and recycling

Distributed admissible fields

Let $\omega \subset \Omega$

- Subspace of kinematically admissible fields (KA)

$$\mathrm{KA}(\omega) = \left\{ \mathbf{u} \in \left(\mathrm{H}^1(\omega)\right)^d,\ \mathbf{u} = \mathbf{u}_d \text{ on } \partial\omega \bigcap \partial_u\Omega \right\}$$

- Subspace of statically admissible fields(SA)

$$\mathrm{SA}(\omega) = \left\{ \tau \in \left(\mathrm{L}^2(\omega)\right)^{d\times d}_{\mathrm{sym}};\ \forall v \in \mathrm{KA}_{00}(\omega), \right.$$
$$\left. \int_\omega \tau : \varepsilon(v)\, dx = \int_\omega f \cdot v\, dx + \int_{\partial_g\Omega \bigcap \partial\omega} g \cdot v\, dS \right\}$$

Global admissibility

$$u \in \mathrm{KA}(\Omega) \Leftrightarrow \begin{vmatrix} u^s \in \mathrm{KA}(\Omega^s),\ \forall s \\ \mathrm{tr}(u^s) = \mathrm{tr}(u^{s'}) \text{ on } \Gamma^{(s,s')} \end{vmatrix}$$

$$\sigma \in \mathrm{SA}(\Omega) \Leftrightarrow \begin{vmatrix} \sigma^s \in \mathrm{SA}(\Omega^s),\ \forall s \\ \sigma^s \cdot n^s + \sigma^{(s')} \cdot n^{(s')} = 0 \text{ on } \Gamma^{(s,s')} \end{vmatrix}$$

FETI

[Farhat et Roux 91, 94]

Global system on domain Ω

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

$\mathbf{K}$ is SPD

Substructured formulation

N non-overlapping subdomains, conforming mesh)

$$\mathbf{K}^s\mathbf{u}^s = \mathbf{f}^s + {\mathbf{t}^s}^T{\mathbf{B}^s}^T\boldsymbol{\lambda}$$

$$\sum_s \mathbf{B}^s\mathbf{t}^s\mathbf{u}^s = 0$$

$\boldsymbol{\lambda}$ = Lagrange multipliers that connect subdomains

Classical FETI system

$$\begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{d} \\ \mathbf{e} \end{pmatrix}$$

Topology

$\mathbf{t}^s$ = trace operators
$\mathbf{B}^s$ = signed Boolean assembly operators

Local operators

$\mathbf{S}^s = \mathbf{K}^s_{bb} - \mathbf{K}^s_{bi}{\mathbf{K}^s_{ii}}^{-1}\mathbf{K}^s_{ib}$ Schur complement
$\mathbf{F}^s = (\mathbf{S}^s)^+ = \mathbf{t}^s{\mathbf{K}^s}^+{\mathbf{t}^s}^T$ Dual Schur
$\mathbf{R}^s = \ker(\mathbf{K}^s)$ basis of rigid body modes

Global operators

$$\mathbf{e} = -\left(\ldots, {\mathbf{f}^s}^T\mathbf{R}^s, \ldots\right)^T$$

$$\mathbf{G} = \left(\ldots, \mathbf{B}^s\mathbf{t}^s\mathbf{R}^s, \ldots\right)$$

$$\mathbf{F} = \sum_s \mathbf{B}^s\mathbf{F}^s{\mathbf{B}^s}^T$$

$$\mathbf{d} = -\sum_s \mathbf{B}^s\mathbf{t}^s{\mathbf{K}^s}^+\mathbf{f}^s$$

Rigid body constraint

$$\boldsymbol{\lambda}_0 = \mathbf{AG}(\mathbf{G}^T\mathbf{AG})^{-1}\mathbf{e}$$
$$\mathbf{P} = \mathbf{I} - \mathbf{AG}(\mathbf{G}^T\mathbf{AG})^{-1}\mathbf{G}^T$$

Matrix $\mathbf{A}$ is a SPD matrix, $\mathbf{A} \simeq \tilde{\mathbf{S}}$ approximates the preconditioner.

$\boldsymbol{\lambda}$ is sought as $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \mathbf{P}\tilde{\boldsymbol{\lambda}}$ where $\tilde{\boldsymbol{\lambda}}$ is a solution of:

$$\mathbf{P}^T\mathbf{FP}\tilde{\boldsymbol{\lambda}} = \mathbf{P}^T\left(\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0\right) = \mathbf{P}^T\left(\sum_s \mathbf{B}^s\mathbf{K}^{s^+}\left(\mathbf{f}^s - \mathbf{B}^{s^T}\boldsymbol{\lambda}_0\right)\right)$$

Dirichlet preconditioner

This system is solved by an iterative solver, the preconditioner $\tilde{\mathbf{S}}$ being

$$\tilde{\mathbf{S}} = \sum_s \tilde{\mathbf{B}}^s\mathbf{S}^s\tilde{\mathbf{B}}^{s^T}$$

$\tilde{\mathbf{B}}^s$ are scaled[8] assembling operators / $[\ldots\tilde{\mathbf{B}}^s\ldots]^T = [\ldots\mathbf{B}^s\ldots]^+$
$\mathbf{S}^s$ are the Schur complements

[8] Klawonn and Widlund, 2001; Rixen and Charbel Farhat, 1999.

Illustration of local equilibrium

Dirichlet and Neumann bcs

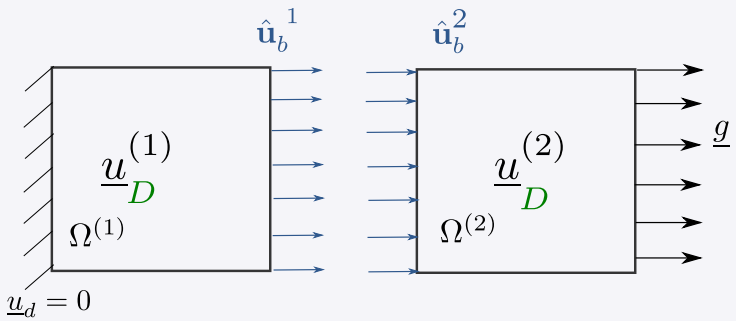
Continuous displacement at the interface

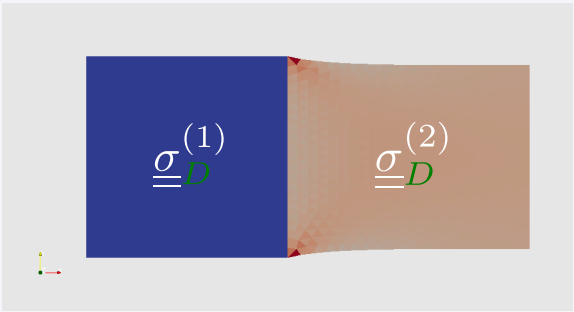
$$\hat{\mathbf{u}}_b^2 - \hat{\mathbf{u}}_b^1 = 0 \Rightarrow \hat{\mathbf{u}}_b^2 = \hat{\mathbf{u}}_b^1$$

Balanced reactions at the interface

$$\hat{\boldsymbol{\lambda}}_N^2 + \hat{\boldsymbol{\lambda}}_N^1 = 0 \Rightarrow \hat{\boldsymbol{\lambda}}_N^2 = -\hat{\boldsymbol{\lambda}}_N^1$$

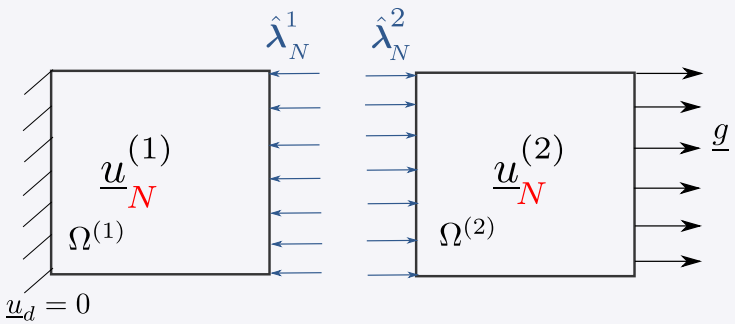
Dirichlet problem on Ω^s

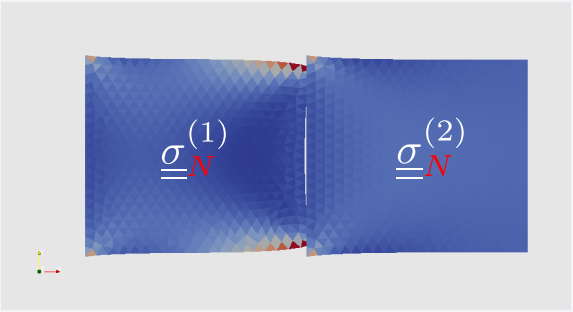




out of balance reactions

Neumann problem on Ω^s

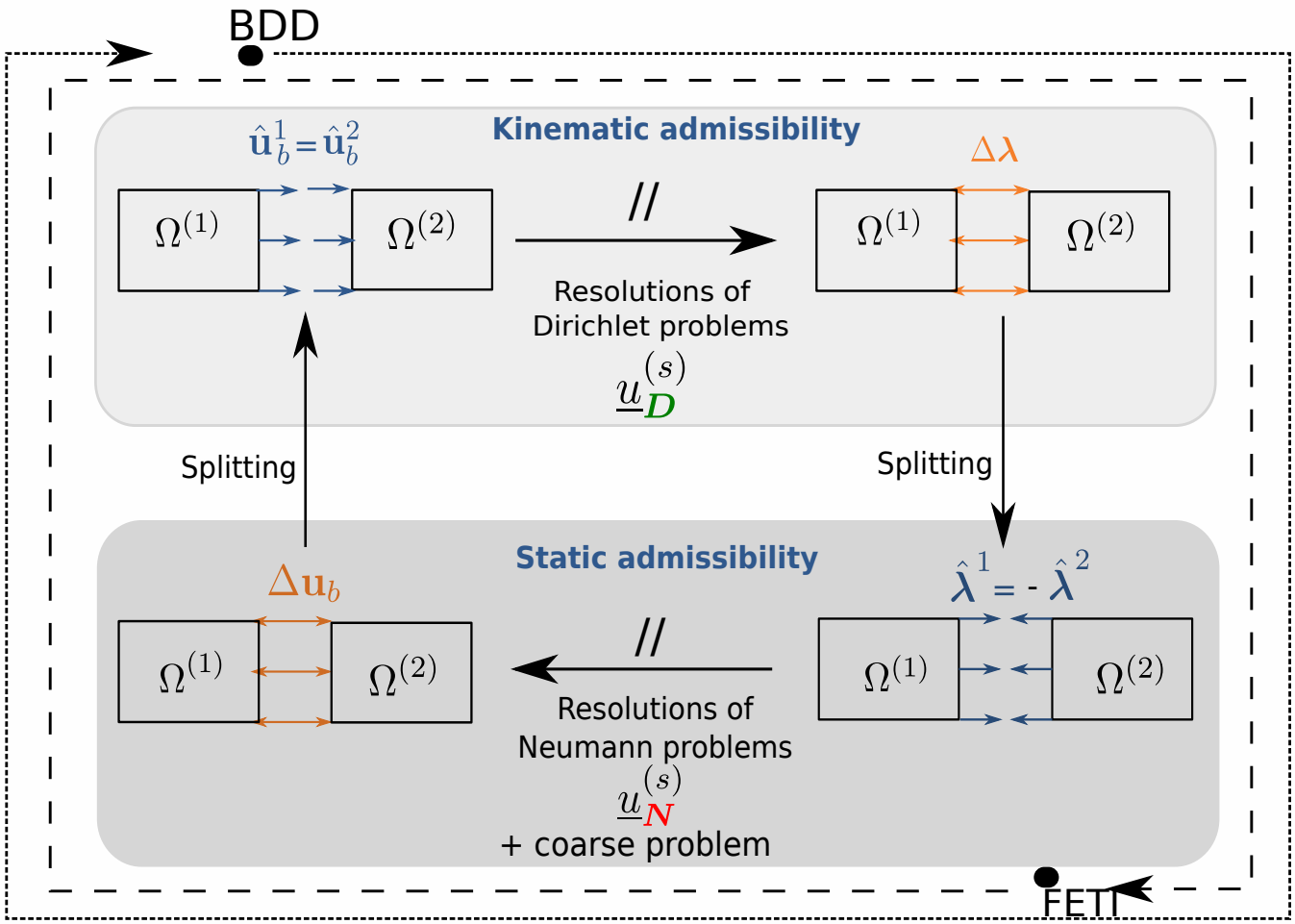




discontinuous displacements

Interface iterations

Dual (FETI)[9] and primal (BDD)[10] approaches



+ no stationarity, Krylov solver is mandatory

[9] C. Farhat and Roux, 1994.

[10] Mandel, 1993.

Algorithm 1: FETI: main unknown $\boldsymbol{\Lambda}$

$\boldsymbol{\Lambda} = \texttt{Initialize}(\mathbf{f}^s)$

Local reactions $\boldsymbol{\lambda}_N^s = {\mathbf{B}^s}^T \boldsymbol{\Lambda}$

$(\mathbf{u}_N^s) = \texttt{Solve}_N(\boldsymbol{\lambda}_N^s, \mathbf{f}^s)$

Compute residual $\mathbf{r} = \mathbf{P}_2^T(\sum_s \mathbf{B}^s \mathbf{t}^s \mathbf{u}_N^s)$

Define local displacement $\tilde{\mathbf{u}}_b^s = {\widetilde{\mathbf{B}}^s}^T \mathbf{r}$;

$(\tilde{\boldsymbol{\lambda}}^s, \tilde{\mathbf{u}}^s) = \texttt{Solve}_D(\tilde{\mathbf{u}}_b^s, 0)$

// $\mathbf{u}_D^s = \mathbf{u}_N^s - \tilde{\mathbf{u}}^s$

// $\boldsymbol{\lambda}_D^s = \boldsymbol{\lambda}_N^s - \tilde{\boldsymbol{\lambda}}^s$

Preconditioned residual $\mathbf{z} = \mathbf{P}_2(\sum_s \widetilde{\mathbf{B}}^s \tilde{\boldsymbol{\lambda}}^s)$

Search direction $\mathbf{w} = \mathbf{z}$

while $\sqrt{(\mathbf{r}, \mathbf{z})} > \epsilon\sqrt{(\mathbf{r}_0, \mathbf{z}_0)}$ **do**

$(\delta\mathbf{u}_N^s) = \texttt{Solve}_N({\mathbf{B}^s}^T \mathbf{w}, 0)$

$\mathbf{p} = \mathbf{P}_2^T(\sum_s \mathbf{B}^s \mathbf{t}^s \delta\mathbf{u}_N^s)$

$\alpha = (\mathbf{r}, \mathbf{z})/(\mathbf{p}, \mathbf{w})$

$\boldsymbol{\Lambda} \leftarrow \boldsymbol{\Lambda} + \alpha\mathbf{w}$

// $\mathbf{u}_N^s \leftarrow \mathbf{u}_N^s + \alpha\delta\mathbf{u}_N^s$

// $\boldsymbol{\lambda}_N^s = {\mathbf{B}^s}^T \boldsymbol{\Lambda}$

$\mathbf{r} \leftarrow \mathbf{r} - \alpha\mathbf{p}$

$\tilde{\mathbf{u}}_b^s = {\widetilde{\mathbf{B}}^s}^T \mathbf{r}$

$(\tilde{\boldsymbol{\lambda}}^s, \tilde{\mathbf{u}}^s) = \texttt{Solve}_D(\tilde{\mathbf{u}}_b^s, 0)$

// $\mathbf{u}_D^s = \mathbf{u}_N^s - \tilde{\mathbf{u}}^s$

// $\boldsymbol{\lambda}_D^s = \boldsymbol{\lambda}_N^s - \tilde{\boldsymbol{\lambda}}^s$

$\mathbf{z} = \mathbf{P}_2(\sum_s \widetilde{\mathbf{B}}^s \tilde{\boldsymbol{\lambda}}^s)$

$\mathbf{w} \leftarrow \mathbf{z} - (\mathbf{p}, \mathbf{z})/(\mathbf{p}, \mathbf{w})\mathbf{w}$

end

Construction of global admissible fields

At each iteration of FETI(DP)/BDD(C)[11]

- $u_D \in H^1(\Omega)$ continuous everywhere and in equilibrium inside subdomains,
- $u_N \in H^1(\cup\Omega^s)$ in equilibrium inside subdomains, with associated interface nodal reactions $\boldsymbol{\lambda}_N^s$ in balance between subdomains $\boldsymbol{\lambda}_N^{s^T} = \mathbf{B}^{s^T}\boldsymbol{\Lambda}$ and in balance wrt rigid body motions and external load. Let σ_N be the associated FE stress field.

[11] A. Parret-Fréaud et al., 2010

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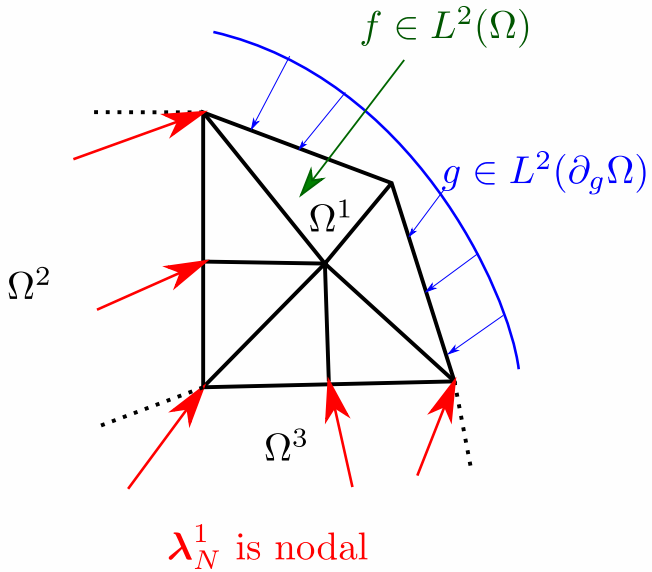

Figure: Subdomain and its surrounding

[11] A. Parret-Fréaud et al., 2010

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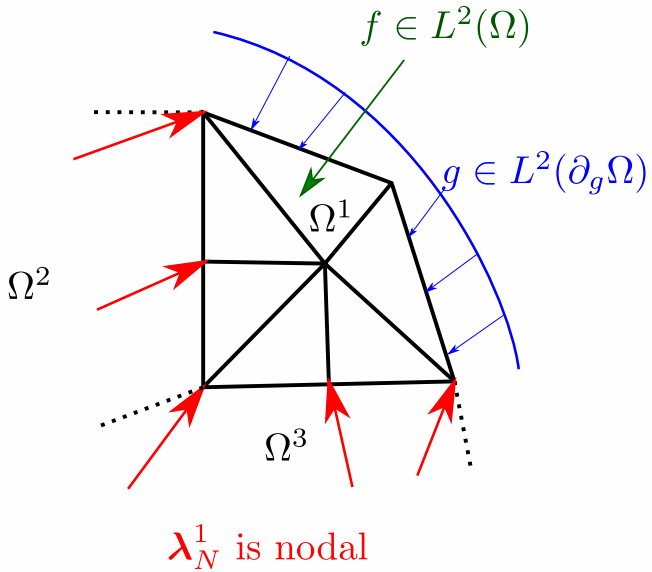


Figure: Subdomain and its surrounding

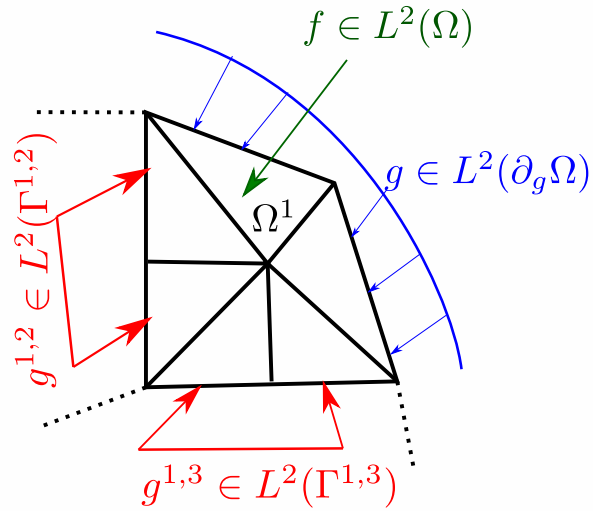


Figure: Preprocessing before recovery

Once we have a L^2 representation of interface traction, we can compute in parallel:

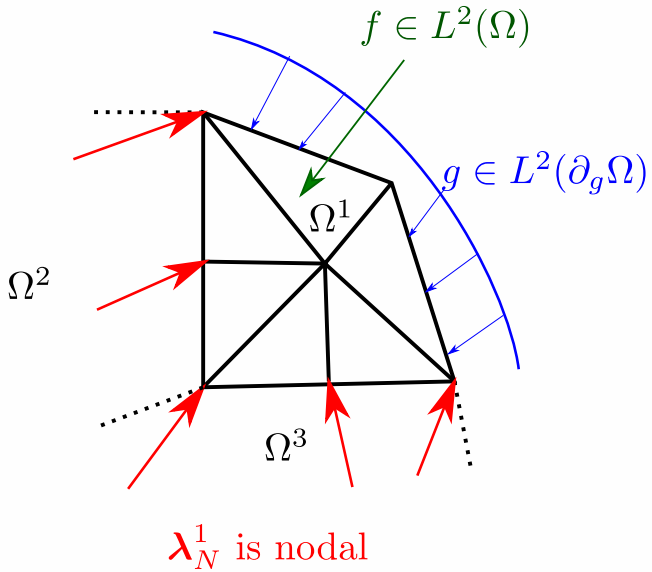
$$\hat{\sigma}_N^s = \mathcal{F}(\sigma_N, f, g, (g^{s,i})_i) \text{ with } \hat{\sigma}_N = (\hat{\sigma}_N^s)_s \in \mathrm{SA}(\Omega)$$

[11] P. Gosselet, A. Parret-Fréaud et al., 2010

Construction of global admissible fields

At each iteration of FETI(DP)/BDD(C)[11]

- $u_D \in H^1(\Omega)$ continuous everywhere and in equilibrium inside subdomains,
- $u_N \in H^1(\cup\Omega^s)$ in equilibrium inside subdomains, with associated interface nodal reactions $\boldsymbol{\lambda}_N^s$ in balance between subdomains $\boldsymbol{\lambda}_N^{s^T} = \mathbf{B}^{s^T}\boldsymbol{\Lambda}$ and in balance wrt rigid body motions and external load. Let σ_N be the associated FE stress field.



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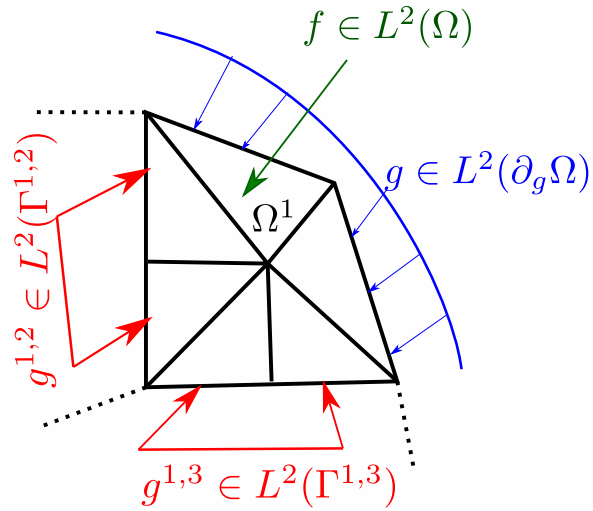

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For the construction of $w \in \mathrm{KA}_0(\Omega)$ we use 0-Dirichlet bc on the interface.

[11] A. Parret-Fréaud et al., 2010

The little difficulty of multiple points[12]

For Ω^s, $g^{s,i}$ is typically developed on the FE basis.
Its nodal components are adjusted in order to develop the same work as $\boldsymbol{\lambda}_N^{s,i}$.

At multiple points, we first need to define $(\boldsymbol{\lambda}_N^{s,i})_i$ from $\boldsymbol{\lambda}_N^s$.
$(\boldsymbol{\lambda}_N^{s,i})_i$ is defined up to an effect-less "cyclic" stress in $\ker(B^{s^T})$.
Optimization is necessary, it must take heterogeneity into account.
One neighbor communication is required.
This is equivalent to what is encountered in the Element Equilibration Technique.

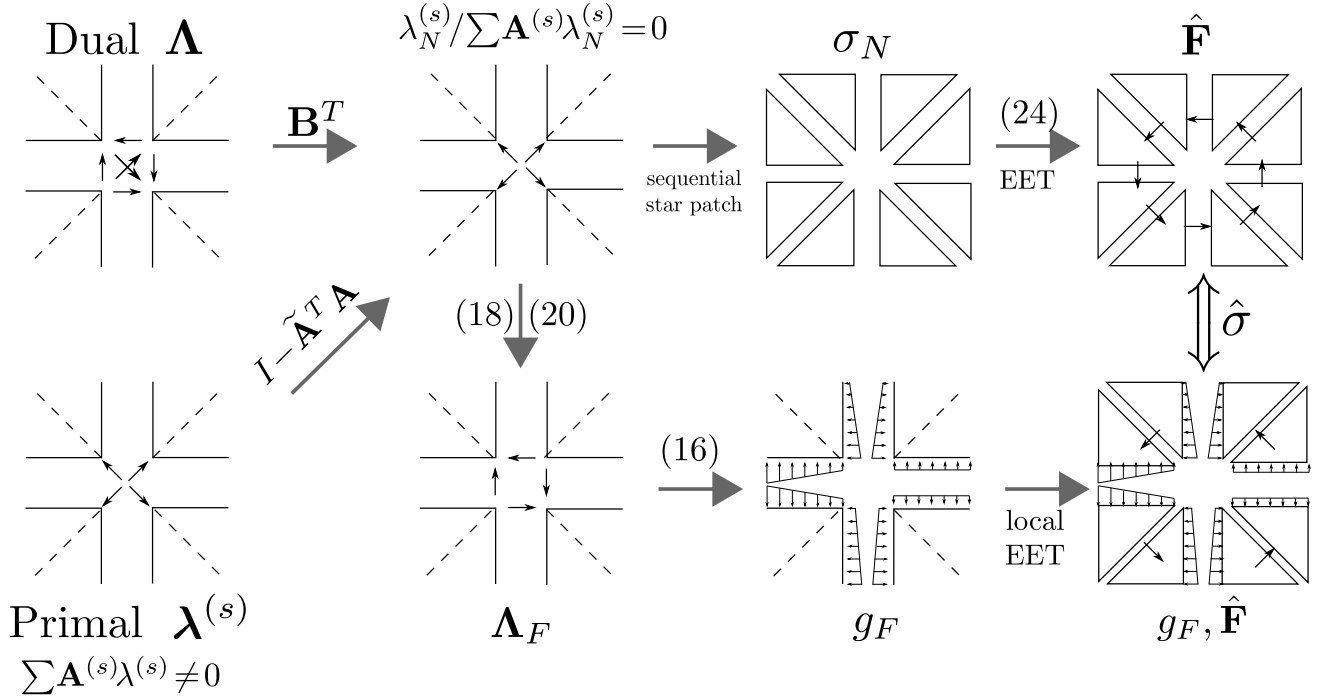


Figure: Methodology for parallel stress recovery / equivalent sequential star-patch

[12] Augustin Parret-Fréaud et al., 2016.

First bounds

Direct transcription of the chosen bounds

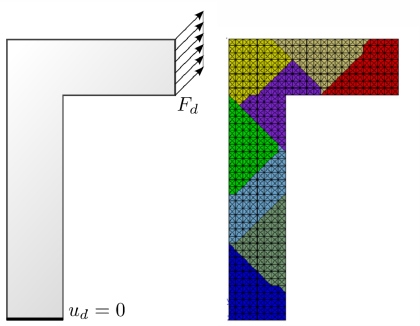
$$\frac{\sum_s R_D^s(\hat{w}^s)}{\sqrt{\sum_s \|\hat{w}^s\|_a^2}} \leqslant \|u - u_D\|_a \leqslant \sqrt{\sum_s \mathrm{e}_{\mathrm{cr}}^2(u_D^s, \hat{\sigma}_N^s)}$$

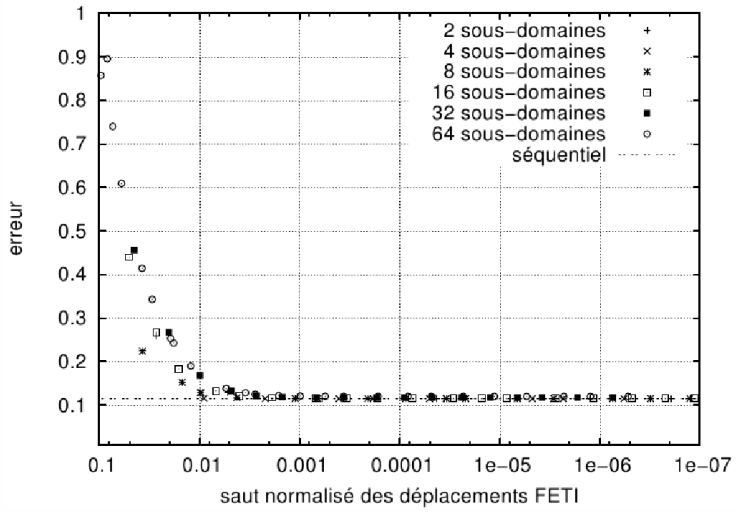
with

$$R_D^s(\hat{w}^s) := \left(\int_{\Omega^s} f \cdot \hat{w}^s d\Omega + \int_{\partial_g \Omega^s} g \cdot \hat{w}^s dS - \int_{\Omega^s} \varepsilon(u_D^s) : \mathbb{H} : \varepsilon(\hat{w}^s) d\Omega^s \right)$$

we recall that $\hat{w}^s$ is 0 on the interface.

Γ-shaped structure — first upper bound[13]

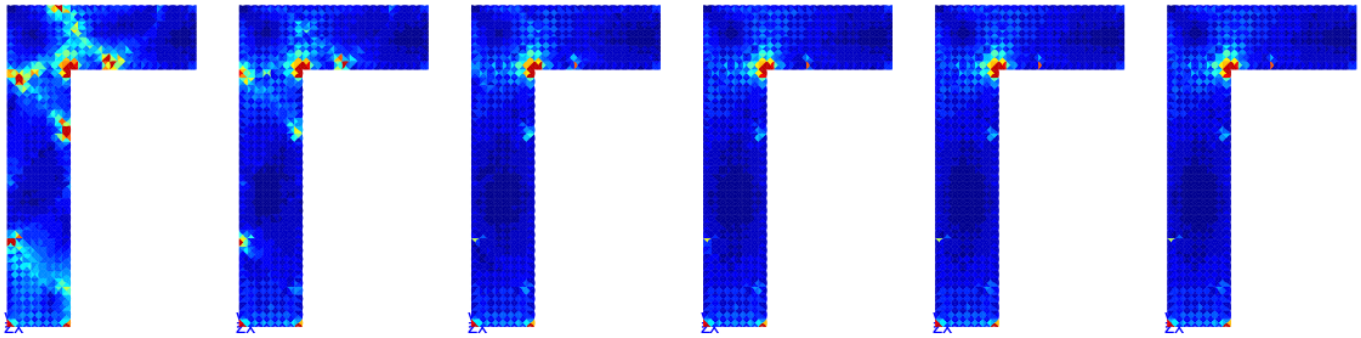




Parallel error estimator:

$$\|u - u_D\|_a \leqslant \sqrt{\sum_s e_{\mathrm{CR}}(\mathbf{u}_D^s, \hat{\sigma}_N^s)_{\Omega^s}}$$

Fast convergence of the estimator



Fast convergence of error maps

The solver converges but the error stagnates

[13] A. Parret-Fréaud et al., 2010.

Bounds with separated contributions

Lemma[14]

For FETI(DP) and BDD(C), we have:

$$\|u_D - u_N\|_a = \|\mathbf{r}\|_{\tilde{\mathbf{S}}} =: |r|$$

The energy norm of the gap between u_N and u_D equals the preconditioner-norm of the residual which is naturally computed at each iteration. It can be chosen to control the iterative solver.
Usually iterations stop when $\|\mathbf{r}_i\|_{\tilde{\mathbf{S}}} \leqslant \epsilon \|\mathbf{r}_0\|_{\tilde{\mathbf{S}}}$ and typically $\epsilon = 10^{-6}$.

Bounds

$$\|u - u_N\|_a \leqslant \sqrt{\sum_s \mathrm{e}_{\mathrm{cr}}^2(u_N^s, \hat{\sigma}_N^s)} + |r|$$

$$\frac{\sum_s R_N^s(\hat{w}^s)}{\sqrt{\sum_s \|\hat{w}^s\|_a}} - |r| \leqslant \|u - u_D\|_a \leqslant \sqrt{\sum_s \mathrm{e}_{\mathrm{cr}}^2(u_N^s, \hat{\sigma}_N^s)} + |r|$$

The first bound is adapted from Vohralik. The second bounds simply uses triangular inequality.

$|r|$ only depends on iterations, other quantities depend on the discretization and (very) weakly on the iterations.

[14] Augustin Parret-Fréaud et al., 2016; V. Rey, C. Rey, and Gosselet, 2014.

Unbiased stopping criterion

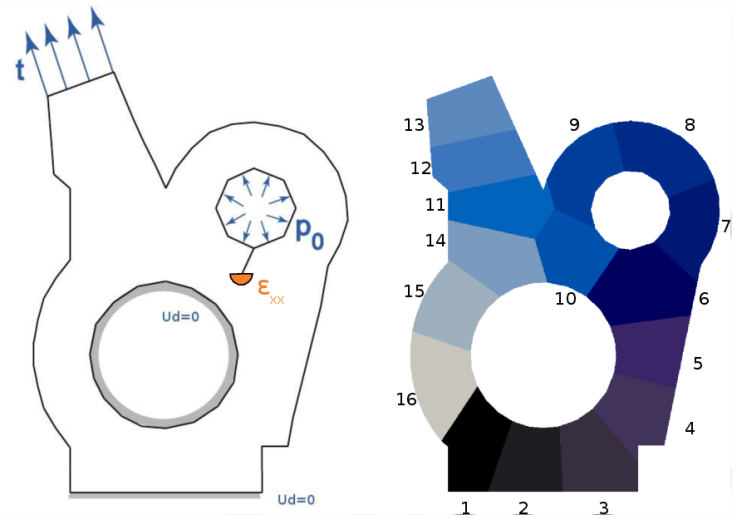


Figure: Pre-cracked structure: forward and adjoint load, decomposition into 16 sd

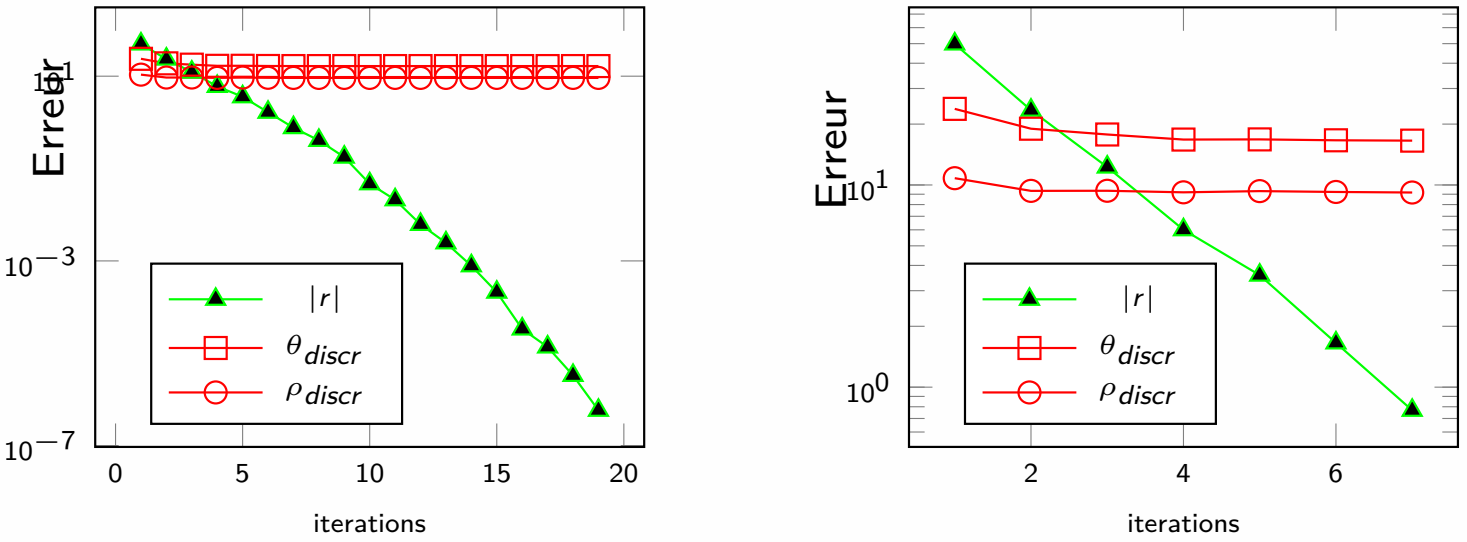


Figure: Envelop of the error due to the discretization and evolution of the residual

Practical considerations

When to compute bounds

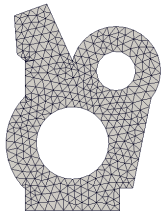
- Evaluate discretization bounds at iteration 1 (avoid 0).
- Iterate until residual is smaller than the inf bound.
- Reevaluate the discretization bounds if they changed to much go on with iterations.
- If needed prepare for adaptation (compute remeshing map)

Quantities of interest

- Solve forward and adjoint problem at the same time using a block solver.
- Also use block computations for the recovery of admissible fields.
- Stop when both residuals are less than their inf bounds.

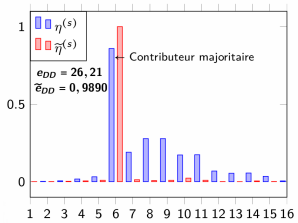
Strategy for adaptation with recycling

Maillage initial



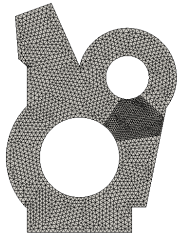
Résolution $\mathrm{S}\underline{x} = \underline{b}$

Contributions à l'erreur

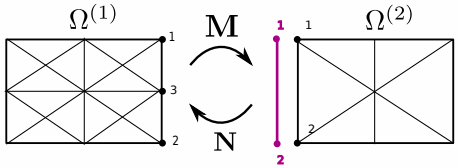


Raffinement local parallèle
discrétisation
de l'interface inchangée

Nouveau maillage



Gestion des incompatibilités aux interfaces



Directions de recherche

V_p base de $K_p(\mathbf{S}, \underline{r}_0)$

Utilisation directe des directions de recherche

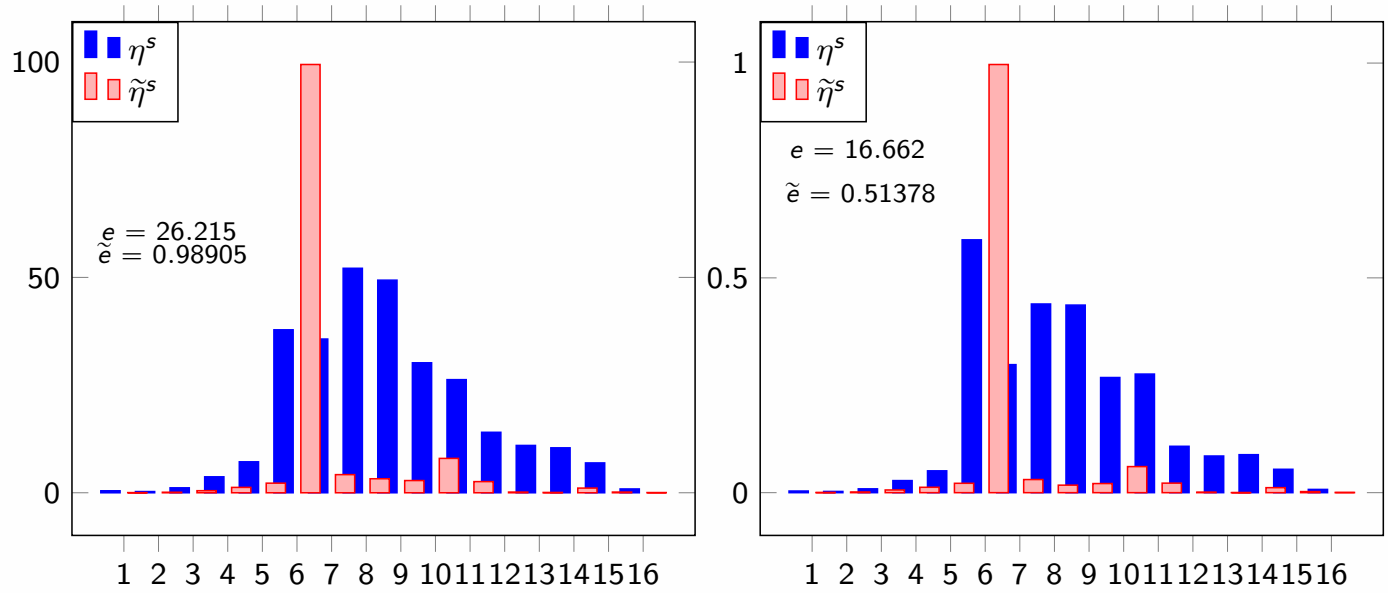
$\mathbf{C}$

Espace de recherche réduit

$$\widetilde{K}_m(\mathbf{S}\underline{r}_0, \mathbf{C}) = \mathrm{Ker}(\mathbf{C^T S}) \oplus \mathrm{Im}(\mathbf{C})$$

Strategy for adaptation with recycling

We select subdomains most contributing to the errors.
We use hierarchical refinement (at least on the interface)so that the building of admissible fields remains easy.



(a) Initial mesh (b) Locally refined mesh

Figure: Distribution of error within subdomains

Mesh	e	$\widetilde{e}$	I_H	$I_{HH,2}$	$\frac{1}{2}e\widetilde{e}$
Uniform	26.215	0.98905	2.4915	3.1935	12.964
Locally refined	16.662	0.51378	3.2165	0.086055	4.2803

Table: Performance of local refined for the cracked structure

Adaptation and recycling

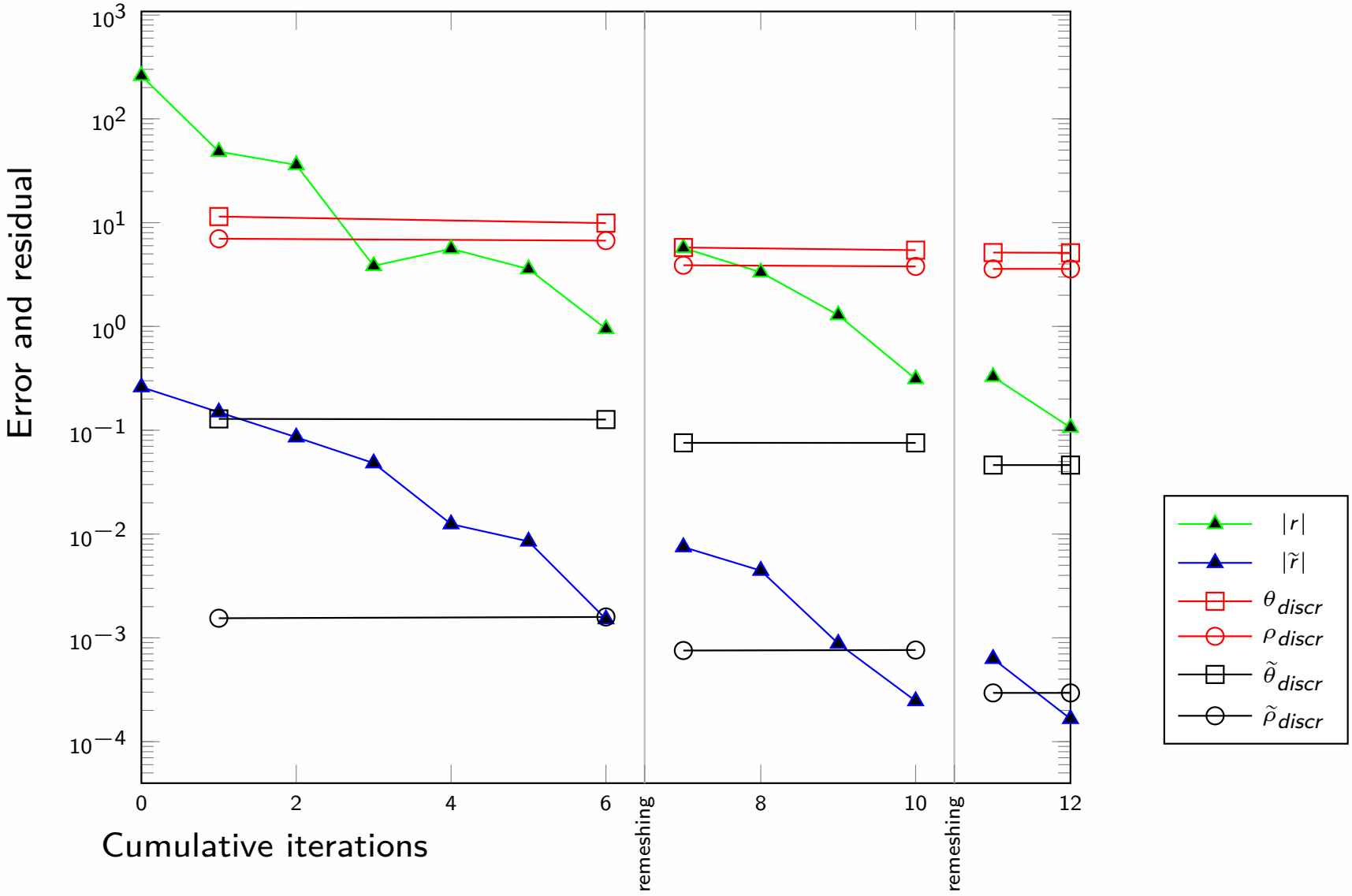


Figure: Erreurs de discrétisation et résidus au cours des itérations

Conclusion

- The recovery of admissible fields can be fully parallel (after just one neighbor communication).
- There are inf and sup bounds of the error which separates the contribution of the solver and of the discretization.
- We can stop the iterative solver based on an unbiased criterion. In practice the discretization error quickly dominates.
- If quantity of interest are wanted, block solvers can be used.
- For the adaptation, hierarchical refinement of interfaces allows to reuse the numerical information.

Ongoing and future work:

- Real implementation for HPC, with improved sequential recovery techniques.
- Error estimation for nonlinear problems[15] with separation of sources[16] with nonlinear DD solvers[17].
- More evolved marking and refinement techniques with good load balancing $\rightarrow$ adaptation of the mesh and of the decomposition.

[15] Ladevèze, 2008.

[16] El Alaoui, Ern, and Vohralik, 2010; Moës, 1996.

[17] Dolean et al., 2015; Klawonn, Lanser, and Rheinbach, 2014; Negrello et al., 2016.