Directional dimensions of ergodic currents on $CP(2)$

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Abstract

Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. We estimate the local directional dimensions of closed positive currents S with respect to ergodic dilating measures ν. We infer several applications. The first one shows that the currents S containing a measure of entropy $h_\nu > \log d$ have a directional dimension > 2, which answers a question by de Thélin-Vigny. The second application asserts that the Dujardin's semi-extremal endomorphisms are close to suspensions of one-dimensional Lattès maps. Finally, we obtain an upper bound for the dimension of the equilibrium measure, towards the formula conjectured by Binder-DeMarco.

Key words: dimension theory, positive closed currents, invariant measures, Lyapunov exponents, normal forms.

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1 Introduction

This article concerns the ergodic properties of holomorphic endomorphisms of $\mathbb{P}^2$, see [16]. Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. The Green current T is defined as $T := \lim_n \frac{1}{d^n} f^{n*}\omega$, where ω is the Fubini-Study form of $\mathbb{P}^2$. The equilibrium measure is defined as $\mu := T \wedge T$, this is an ergodic measure of entropy $\log d^2$, its Lyapunov exponents are $\geq \frac{1}{2}\log d$, see [6, 16].

We say that an ergodic measure ν is *dilating* if its Lyapunov exponents are positive. The ergodic measures of entropy $h_\nu > \log d$ are dilating: their exponents are larger than or equal to $\frac{1}{2}(h_\nu - \log d)$, see [11, 20]. It is known that the support of every ergodic measure ν of entropy $h_\nu > \log d$ is contained in the support of μ, see [10, 14]. The article [20] constructs such measures by using coding techniques.

1.1 Directional dimensions

Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. The algebraic subsets of $\mathbb{P}^2$ do not contain any ergodic measure ν of entropy $h_\nu > \log d$: this comes from the Gromov's iterated graph argument and from the relative variational principle, see [7] and [16, Section 1.7]. In this Section, we quantify that property thanks to the Lyapunov exponents

of the measures ν. We shall work in the more general setting of $(1,1)$ closed positive currents S. Those currents are as big as the algebraic subsets, since we have:

$$\forall x \in \mathbb{P}^2 \ , \ \underline{d_S}(x) := \liminf_{r\to 0} \frac{\log S \wedge \omega(B_x(r))}{\log r} \geq 2,$$

which comes from $S \wedge \omega(B_x(r)) \leq c(x) r^2$, see [13, Chapitre 3]. We shall use the notation $\overline{d_S}(x)$ for the lim sup. A drawback of the trace measure $S \wedge \omega$ is that it does not distinguish any specific direction. If Z is a holomorphic coordinate in the neighbourhood of $x \in \mathbb{P}^2$, one defines the lower local directional dimension of S with respect to Z by

$$\underline{d_{S,Z}}(x) := \liminf_{r\to 0} \frac{\log \left[S \wedge (\frac{i}{2} dZ \wedge d\overline{Z})(B_x(r)) \right]}{\log r},$$

we shall denote the lim sup by $\overline{d_{S,Z}}(x)$. Geometrically, the positive measure $S \wedge (\frac{i}{2} dZ \wedge d\overline{Z})$ is the average with respect to Lebesgue measure of the slices of the current S transversaly to the direction Z, see Proposition 9.4. If (Z, W) are holomorphic coordinates near x, the directional dimensions of S are related to the dimension of S by

$$\underline{d_S}(x) = \min \left\{ \underline{d_{S,Z}}(x), \underline{d_{S,W}}(x) \right\} \ , \ \overline{d_S}(x) = \min \left\{ \overline{d_{S,Z}}(x), \overline{d_{S,W}}(x) \right\}, \qquad (1.1)$$

see Proposition 9.2. We start with showing lower and upper bounds for the directional dimensions of the Green current T (Theorems 1.1 and 1.2). Next we display our result concerning the general closed positive currents S.

Directional dimensions of the Green current

The invariance property of the current T allows to obtain estimates on the directional dimensions ν-almost everywhere. In what follows, the functions $O(\epsilon)$ only depend on ϵ, the degree d of the endomorphism, the exponents and the entropy of ν. They tend to 0 when ϵ tends to 0 and are positive for ϵ small enough. We shall say that the exponents of ν do not resonate if $\lambda_1 \neq k\lambda_2$ for every $k \geq 2$.

Theorem 1.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic diltating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. Then there exist functions $O_1(\epsilon), O_2(\epsilon)$ satisfying the following properties. For every $\epsilon > 0$ and for ν-almost every x, there exist holomorphic coordinates (Z, W) in a neighbourhood of x such that*

$$\overline{d_{T,Z}}(x) \geq 2 + \frac{h_\nu - \log d}{\lambda_2} - O_1(\epsilon),$$

$$\overline{d_{T,W}}(x) \geq 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2} - O_2(\epsilon).$$

This result applies to the measure μ and allows to improve a classical lower bound concerning the dimension of the Green current T. This current has local γ-Hölder *psh* potentials for every $\gamma < \gamma_0 := \min\{1, \log d / \log d_\infty\}$, where $d_\infty := \lim_n ||Df^n||_\infty^{1/n}$, see

[16, Proposition 1.18]. This implies that $T \wedge \omega(B_x(r)) \leq c_\gamma(x) r^{2+\gamma}$ for every $x \in \mathbb{P}^2$ and every $\gamma < \gamma_0$, see [24, Théorème 1.7.3]. We deduce that for every $x \in \mathbb{P}^2$ and for every local holomorphic coordinates (Z, W) in a neighbourhood of x:

$$\min\{\overline{d_{T,Z}}(x), \overline{d_{T,W}}(x)\} = \overline{d_T}(x) \geq 2 + \gamma_0. \tag{1.2}$$

Now, since $h_\mu = \log d^2$ and $\lambda_1 \leq \log d_\infty$, we get:

$$\frac{h_\mu - \log d}{\lambda_2} = \frac{\log d}{\lambda_2} > \frac{\log d}{\lambda_1} \geq \frac{\log d}{\log d_\infty} \geq \gamma_0.$$

The lower bound in the direction Z provided by Theorem 1.1 is therefore better than (1.2) when ϵ is small enough.

Now we show directional upper bounds for the current T with respect to every dilating measure ν whose support is contained in the support of μ.

Theorem 1.2. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. We assume that the support of ν is contained in the support of μ. Then there exist functions $O_3(\epsilon), O_4(\epsilon)$ satisfying the following properties. For every $\epsilon > 0$ and for ν-almost every x, there exist holomorphic coordinates (Z, W) in a neighbourhood of x such that*

$$\underline{d_{T,Z}}(x) \leq \frac{\log d}{\lambda_2} + 2\frac{\lambda_1}{\lambda_2} + O_3(\epsilon),$$

$$\underline{d_{T,W}}(x) \leq \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon).$$

By combining these two theorems, we manage to separate coordinates (Z, W) in terms of local dimensions.

Corollary 1.3. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. For every $\epsilon > 0$ and for ν-almost every x, there exist holomorphic coordinates (Z, W) in a neighbourhood of x such that*

$$\underline{d_{T,W}}(x) \leq \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon) \quad , \quad \frac{\log d}{\lambda_2} + 2 - O_1(\epsilon) \leq \overline{d_{T,Z}}(x).$$

In the three preceding results, the coordinates (Z, W) come from a normal form Theorem for the inverse branches of f^n, see Section 2. They depend on $\hat{x}$ in the natural extension $(\hat{f}, \hat{\nu})$, we shall denote them by $(Z^\epsilon_{\hat{x}}, W^\epsilon_{\hat{x}})$. The coordinates (Z, W) at a point x are coordinates of type $(Z^\epsilon_{\hat{x}}, W^\epsilon_{\hat{x}})$ where $\pi_0(\hat{x}) = x$. The coordinate $W^\epsilon_{\hat{x}}$ is always invariant by the shift $\hat{f}$, $Z^\epsilon_{\hat{x}}$ is invariant when the exponents do not resonate. Since the current T is f-invariant, the functions $\underline{d_{T,Z}}(\hat{x})$ and $\overline{d_{T,W}}(\hat{x})$ are $\hat{f}$-invariant, hence $\hat{\nu}$-almost everywhere constant, see Proposition 2.4. We shall denote them by $\underline{d_{T,Z}}(\nu)$ and $\underline{d_{T,W}}(\nu)$. We have the same properties for the upper dimensions.

Dimension of ergodic currents S

Theorem 1.4. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let S be a $(1,1)$-closed positive current on $\mathbb{P}^2$. We assume that the support of S contains an ergodic measure ν of entropy $h_\nu > \log d$ whose exponents satisfy $\lambda_1 > \lambda_2$ and do not resonate. Then there exists a function $O_5(\epsilon)$ satisfying the following properties. For every $\epsilon > 0$, there exist $x \in \operatorname{Supp} \nu$ and a holomorphic coordinate Z in the neighbourhood of x such that:*

$$\overline{d_{S,Z}}(x) \geq 2 + \frac{h_\nu - \log d}{\lambda_2} - O_5(\epsilon).$$

In particular, S has a local directional dimension > 2 at some $x \in \operatorname{Supp} \nu$.

This result is localized at a point x because S is not assumed to be f-invariant. For every closed positive current S containing an ergodic dilating measure ν with exponents $\lambda_1 \geq \lambda_2$, de Thélin-Vigny proved in [12] that there exists $x \in supp(\nu)$ such that

$$\overline{d_S}(x) \geq 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2}. \tag{1.3}$$

Theorem 1.4 improves this estimate by replacing λ_2/λ_1 by 1 for a coordinate Z. When $\lambda_1 = \lambda_2$, (1.1) shows that this substitution is valid for every coordinate Z. Our lower bound answers a question of [12] in a directional way. In the framework of invertible and meromorphic mappings, the preprint [9] gives a lower bound > 2 for the dimensions of the Green currents $T^\pm$ by using coding techniques and laminar properties of $T^\pm$.

1.2 Dimension of dilating measures

Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let ν be an ergodic measure. We refer to [23] and [26] for the beginning of this Section. The dimension of ν is defined by

$$\dim_H(\nu) := \inf \left\{ \dim_H(A)\,,\ A \text{ Borel set of } \mathbb{P}^2\,,\ \nu(A) = 1 \right\}.$$

The lower and upper local dimensions of ν are

$$\underline{d_\nu} := \liminf_{r \to 0} \frac{\log(\nu(B_x(r)))}{\log r} \quad , \quad \overline{d_\nu} := \limsup_{r \to 0} \frac{\log(\nu(B_x(r)))}{\log r},$$

these limits are ν-almost everywhere constant, by ergodicity of ν. If $a \leq \underline{d_\nu} \leq \overline{d_\nu} \leq b$, then $a \leq \dim_H(\nu) \leq b$. If ν is dilating, we have the classical inequalities $\frac{h_\nu}{\lambda_1} \leq \underline{d_\nu} \leq \overline{d_\nu} \leq \frac{h_\nu}{\lambda_2}$ where $\lambda_1 \geq \lambda_2$ are the exponents of ν. For the equilibrium measure μ, Binder-DeMarco [4] conjectured the formula:

$$\dim_H(\mu) = \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} \tag{1.4}$$

which generalizes the one-dimensional Mañé's formula [22]. The article [4] proves the following upper bound for polynomial mappings

$$\dim_H(\mu) \leq 4 - \frac{2(\lambda_1 + \lambda_2) - \log d^2}{\lambda_1}, \tag{1.5}$$

the article [15] extends this upper bound in a meromorphic context. For every dilating measure ν, we have at our disposal the following lower bound proved in [19]:

$$\dim_H(\nu) \geq \underline{d_\nu} \geq \frac{\log d}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2}. \tag{1.6}$$

We obtain a new upper bound which comes closer to Binder-DeMarco's conjecture (1.4).

Theorem 1.5. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure, of exponents $\lambda_1 > \lambda_2$ and whose support is contained in the support of μ. Then*

$$\underline{d_\nu} \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\left(1 - \frac{\lambda_2}{\lambda_1}\right).$$

Moreover, if the exponents do not resonate, then:

$$\underline{d_\nu} \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\min\left(1 - \frac{\lambda_2}{\lambda_1}; \frac{\lambda_1}{\lambda_2} - 1\right).$$

The proof extends the arguments of Theorem 1.2.

1.3 Dimension of semi-extremal endomorphisms

We say that f is *extremal* if the exponents $\lambda_1 \geq \lambda_2$ of its equilibrium measure μ satisfy $\lambda_1 = \lambda_2 = \frac{1}{2}\log d$. The articles [1, 3, 15] characterize these endomorphisms by the equivalent properties:

1. $\dim_H(\mu) = 4$.
2. $\mu << Leb_{\mathbb{P}^2} := \omega \wedge \omega$.
3. T is a $(1,1)$ positive smooth form on an open set of $\mathbb{P}^2$.
4. f is a Lattès map: there exist a complex torus $\mathbb{C}^2/\Lambda$, an affine dilation D on this torus and a finite galoisian covering $\sigma : \mathbb{C}^2/\Lambda \to \mathbb{P}^2$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}^2/\Lambda & \xrightarrow{D} & \mathbb{C}^2/\Lambda \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2 \end{array}$$

There exist such applications for every degree $d \geq 2$, see for instance [18].

We say that f is *semi-extremal* if the exponents $\lambda_1 \geq \lambda_2$ of its equilibrium measure μ satisfy $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$. Using (1.5) and (1.6) one sees that the Binder-DeMarco's conjecture (1.4) holds for these mappings:

$$\dim_H(\mu) = 2 + \frac{\log d}{\lambda_1}. \tag{1.7}$$

Classical examples of semi-extremal endomorphisms are suspensions of one-dimensional Lattès maps, they satisfy $\mu << T \wedge \omega$. More generally,

Theorem 1.6 (Dujardin [17]). *If $\mu << T \wedge \omega$, then f is semi-extremal.*

In [17] Dujardin asked if $\mu << T \wedge \omega$ implies the existence of a one-dimensional Lattès factor for f. Theorem 1.7 below provides one step in that direction. It shows that, from a theoritical dimensional point of view, these endomorphisms look like suspensions of one-dimensional Lattès maps: the dimension of T is maximal equal to 4 in a coordinate Z, and equal to $2 + \log d/\lambda_1$ (the dimension of μ, see (1.7)) in a coordinate W. The functions $O_1(\epsilon), O_2(\epsilon)$ come from Theorem 1.1.

Theorem 1.7. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. We assume that $\mu << T \wedge \omega$ and that $\overline{d_\mu} = \underline{d_\mu}$. We also assume that the exponents $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$ of μ do not resonate. For every $\epsilon > 0$ and for μ-almost every $x \in \mathbb{P}^2$, there exist holomorphic coordinates (Z, W) in a neighbourhood of x such that*

$$4 - O_1(\epsilon) \leq \overline{d_{T,Z}}(x) \quad \text{and} \quad 2 + \frac{\log d}{\lambda_1} - O_2(\epsilon) \leq \overline{d_{T,W}}(x) \leq 2 + \frac{\log d}{\lambda_1}.$$

1.4 Organization of the article

Sections 2, 3 et 4 are devoted to normal forms, the geometry of inverse branches and separated sets. Sections 5 et 6 establish Theorems 1.1, 1.4 and 1.7, the proofs rest on Theorem 5.2 which relies on the lower local dimension $\underline{d_\nu}$ (the lower bound (1.6) is therefore crucial to deduce these results). Theorems 1.2 and 1.5 are proved in Sections 7 and 8. Section 9 brings together technical results.

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2 Normal forms and Oseledec-Poincaré coordinates

2.1 Natural extension and normal forms

Let $\mathcal{C}_f$ be the critical set of f, this is an algebraic subset of $\mathbb{P}^2$. If ν is an ergodic dilating measure, then $x \mapsto \log|Jac\, f(x)| \in L^1(\nu)$, which implies $\nu(\mathcal{C}_f) = 0$. Let X be the f-invariant Borel set $\mathbb{P}^2 \setminus \cup_{n\in\mathbb{Z}} f^n(\mathcal{C}_f)$ and let

$$\hat{X} := \left\{\hat{x} = (x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}, \;\; x_{n+1} = f(x_n)\right\}.$$

Let $\hat{f}$ be the left shift on $\hat{X}$ and $\pi_0(\hat{x}) := x_0$. There exists a unique $\hat{f}$-invariant measure $\hat{\nu}$ on $\hat{X}$ such that $(\pi_0)_*\hat{\nu} = \nu$. We set $\hat{x}_n := \hat{f}^n(\hat{x})$ for $n \in \mathbb{Z}$. A function $\alpha : \hat{X} \to]0, +\infty]$ is *ϵ-tempered* if $\alpha(\hat{f}^{\pm 1}(\hat{x})) \geq e^{-\epsilon}\alpha(\hat{x})$. For every $\hat{x} \in X$ we denote by $f_{\hat{x}}^{-n}$ the inverse branch of f^n defined in a neighbourhood of x_0 with values in a neighbourhood of x_{-n}. The articles [2] and [21] provide normal forms for these mappings.

Theorem 2.1. *([2, Proposition 4.3]) Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure with exponents $\lambda_1 > \lambda_2$. Let $\epsilon > 0$.*

There exists an $\hat{f}$-invariant Borel set $\hat{F} \subset \hat{X}$ such that $\hat{\nu}(\hat{F}) = 1$ and satisfying the following properties. There exist ϵ-tempered functions $\eta_\epsilon, \rho_\epsilon : \hat{F} \to]0,1]$ and $\beta_\epsilon, L_\epsilon, M_\epsilon : \hat{F} \to [1, +\infty[$ and for every $\hat{x} \in \hat{F}$, there exists an injective holomorphic mapping

$$\xi^\epsilon_{\hat{x}} : B_{x_0}(\eta_\epsilon(\hat{x})) \to \mathbb{D}^2(\rho_\epsilon(\hat{x}))$$

such that the following diagram commutes for every $n \geq n_\epsilon(\hat{x})$:

$$\begin{array}{ccc} B_{x_{-n}}(\eta_\epsilon(\hat{x}_{-n})) & \xleftarrow{f^{-n}_{\hat{x}}} & B_{x_0}(\eta_\epsilon(\hat{x})) \\ \downarrow{\scriptstyle \xi^\epsilon_{\hat{x}_{-n}}} & & \downarrow{\scriptstyle \xi^\epsilon_{\hat{x}}} \\ \mathbb{D}^2(\rho_\epsilon(\hat{x}_{-n})) & \xleftarrow[R_{n,\hat{x}}]{} & \mathbb{D}^2(\rho_\epsilon(\hat{x})) \end{array}$$

and such that

1. $\forall (p,q) \in B_{x_0}(\eta_\epsilon(\hat{x})),\ \frac{1}{2} d(p,q) \leq |\xi^\epsilon_{\hat{x}}(p) - \xi^\epsilon_{\hat{x}}(q)| \leq \beta_\epsilon(\hat{x}) d(p,q)$.
2. $\mathrm{Lip}(f^{-n}_{\hat{x}}) \leq L_\epsilon(\hat{x}) e^{-n\lambda_2 + n\epsilon}$ *on* $B_{x_0}(\eta_\epsilon(\hat{x}))$.
3. *if* $\lambda_1 \notin \{k\lambda_2, k \geq 2\}$, $R_{n,\hat{x}}(z,w) = (\alpha_{n,\hat{x}} z, \beta_{n,\hat{x}} w)$,
   if $\lambda_1 = k\lambda_2$ *where* $k \geq 2$, $R_{n,\hat{x}}(z,w) = (\alpha_{n,\hat{x}} z, \beta_{n,\hat{x}} w) + (\gamma_{n,\hat{x}} w^k, 0)$, *with*
   (a) $e^{-n\lambda_1 - n\epsilon} \leq |\alpha_{n,\hat{x}}| \leq e^{-n\lambda_1 + n\epsilon}$ *and* $|\gamma_{n,\hat{x}}| \leq M_\epsilon(\hat{x}) e^{-n\lambda_1 + n\epsilon}$,
   (b) $e^{-n\lambda_2 - n\epsilon} \leq |\beta_{n,\hat{x}}| \leq e^{-n\lambda_2 + n\epsilon}$.

Remark 2.2. *The diagram commutes for every $n \in \{1, \ldots, n_\epsilon(\hat{x})\}$ for the germs of the mappings, see [2]. The integer $n_\epsilon(\hat{x})$ is the smallest integer such that $L_\epsilon(\hat{x}) e^{-n\lambda_2 + n\epsilon} \leq e^{-n\epsilon}$, so that $L_\epsilon(\hat{x}) e^{-n\lambda_2 + n\epsilon} \eta_\epsilon(\hat{x}) \leq e^{-n\epsilon} \eta_\epsilon(\hat{x}) \leq \eta_\epsilon(\hat{x}_{-n})$. Item 2 thus ensures that $f^{-n}_{\hat{x}}(B_{x_0}(\eta_\epsilon(\hat{x})) \subset B_{x_{-n}}(\eta_\epsilon(\hat{x}_{-n}))$.*

We shall need the following Lemma. Let $n_1(L)$ be the smallest integer n satisfying $L/4 \leq e^{n\epsilon}$. The first item uses the upper bound for $\mathrm{Lip}(f^{-n}_{\hat{x}})$ provided by Theorem 2.1. The second item comes from [15, Proposition 3.1].

Lemma 2.3. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \geq \eta$ and $L_\epsilon(\hat{x}) \leq L$. If $n \geq n_1(L)$ and $r \leq \eta$,*

1. $f^{-n}_{\hat{x}_n}(B_{x_n}(r/4)) \subset B_{x_0}(re^{-n\lambda_2 + 3n\epsilon})$ *and* $f^{-n}_{\hat{x}}(B_{x_0}(r/4)) \subset B_{x_{-n}}(re^{-n\lambda_2 + 3n\epsilon})$.
2. $B_{x_0}(re^{-n\lambda_1 - 4n\epsilon}) \subset f^{-n}_{\hat{x}_n}(B_{x_n}(\frac{r}{4} e^{-2n\epsilon}))$.

2.2 Oseledec-Poincaré coordinates

Let ν be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$. We assume that the exponents do not resonate, which means that $\lambda_1 \notin \{k\lambda_2, k \geq 2\}$. Let $\epsilon > 0$, and let us apply Theorem 2.1. For every $\hat{x} \in \hat{F}$ we denote by $(Z^\epsilon_{\hat{x}}, W^\epsilon_{\hat{x}})$ the coordinates of $\xi^\epsilon_{\hat{x}}$. The commutative diagram given by Theorem 2.1 implies:

$$Z^\epsilon_{\hat{x}_{-n}} \circ f^{-n}_{\hat{x}} = \alpha_{n,\hat{x}} \times Z^\epsilon_{\hat{x}}, \quad W^\epsilon_{\hat{x}_{-n}} \circ f^{-n}_{\hat{x}} = \beta_{n,\hat{x}} \times W^\epsilon_{\hat{x}}. \tag{2.1}$$

Hence, $f^{-n}_{\hat{x}}$ multiplies the first coordinate by $e^{-n\lambda_1 \pm n\epsilon}$ and multiplies the second coordinate by $e^{-n\lambda_2 \pm n\epsilon}$. Let us note that the second property holds in the resonant case $\lambda_1 \in \{k\lambda_2, k \geq 2\}$. We shall name the collection of local holomorphic coordinates

$$(Z, W)_\epsilon := (Z^\epsilon_{\hat{x}}, W^\epsilon_{\hat{x}})_{\hat{x} \in \hat{F}}$$

Oseledec-Poincaré coordinates for (f, ν). Using (2.1) and the fact that the Green current is f-invariant, we obtain the following Proposition.

Proposition 2.4. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let T be its Green current. Let ν be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$. Let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). Then there exists a $\hat{f}$-invariant Borel set $\hat{\Lambda}_T \subset \hat{F}$ of $\hat{\nu}$-mesure 1 such that*

1. *$\hat{x} \mapsto \overline{d_{T,W}}(\hat{x})$ and $\hat{x} \mapsto \underline{d_{T,W}}(\hat{x})$ are $\hat{f}$-invariant on $\hat{\Lambda}_T$.*
2. *$\hat{x} \mapsto \overline{d_{T,Z}}(\hat{x})$ and $\hat{x} \mapsto \underline{d_{T,Z}}(\hat{x})$ are $\hat{f}$-invariant on $\hat{\Lambda}_T$ if $\lambda_1 \notin \{k\lambda_2, k \geq 2\}$.*

In particular, if the exponents do not resonate, these functions are constant $\hat{\nu}$-almost everywhere. We shall denote them by

$$\overline{d_{T,Z}}(\nu),\ \underline{d_{T,Z}}(\nu),\ \overline{d_{T,W}}(\nu),\ \underline{d_{T,W}}(\nu).$$

Proof. We prove the invariance of $\overline{d_{T,W}}(\hat{x})$, the same arguments hold for the other functions. For every $z \in \mathbb{P}^2 \setminus \mathcal{C}_f$ we denote

$$a(z) := \frac{1}{2}||(D_z f)^{-1}||^{-1}\ ,\ \gamma(z) := \min\{a(z)\,\|f\|^{-1}_{\mathcal{C}^2,\mathbb{P}^2}\ ,\ 1\}.$$

Then [6, Lemme 2] asserts that f is injective on $B_z(\gamma(z))$ and

$$\forall r \in [0, \gamma(z)]\ ,\ B_{f(z)}\left(a(z)r\right) \subset f(B_z(r)).$$

Let $\hat{x} \in \hat{F}$. Since $x_n \notin \mathcal{C}_f$ for every $n \in \mathbb{Z}$, we obtain for every $r \leq \gamma(x_0)$:

$$T \wedge \left(\frac{i}{2} dW^\epsilon_{\hat{f}(\hat{x})} \wedge d\overline{W^\epsilon_{\hat{f}(\hat{x})}}\right)\left[B_{f(x_0)}\left(a(x_0)r\right)\right] \leq T \wedge \left(\frac{i}{2} dW^\epsilon_{\hat{f}(\hat{x})} \wedge d\overline{W^\epsilon_{\hat{f}(\hat{x})}}\right)\left[f(B_{x_0}(r))\right]. \tag{2.2}$$

Since f is injective on $B_{x_0}(r)$, we can change the variables to get:

$$T \wedge \left(\frac{i}{2} dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right) [f(B_{x_0}(r))] = \int_{B_{x_0}(r)} f^*T \wedge f^* \left(\frac{i}{2} dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right). \quad (2.3)$$

Now let us recall that

$$f^*T = dT \text{ and } f^* \left(\frac{i}{2} dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right) = |c(\hat{x})|^2 \frac{i}{2} dW^{\epsilon}_{\hat{x}} \wedge d\overline{W^{\epsilon}_{\hat{x}}}, \quad (2.4)$$

where the second equality comes from (2.1) by setting $c(\hat{x})^{-1} := \beta_{1,\hat{f}(\hat{x})}$, it is valid near x_0 according to Remark 2.2. By combining (2.2), (2.3) and (2.4) we deduce:

$$T \wedge \left(\frac{i}{2} dW^{\epsilon}_{\hat{f}(\hat{x})} \wedge d\overline{W^{\epsilon}_{\hat{f}(\hat{x})}}\right) \left[B_{f(x_0)}\left(a(x_0)r\right)\right] \leq d|c(\hat{x})|^2 \; T \wedge \left(\frac{i}{2} dW^{\epsilon}_{\hat{x}} \wedge d\overline{W^{\epsilon}_{\hat{x}}}\right) [B_{x_0}(r)]$$

for every r small enough. Taking the logarithm and dividing by $\log(a(x_0)r) < 0$, we get $\overline{d_{T,W}}(\hat{f}(\hat{x})) \geq \overline{d_{T,W}}(\hat{x})$ by taking limits. Since $\hat{\nu}$ is ergodic, the function $\overline{d_{T,W}}(\hat{x})$ is constant on a Borel set $\hat{\Lambda}_T$ of $\hat{\nu}$-measure 1 (see [25, Chapter 1.5]). One can replace it by $\bigcap_{n\in\mathbb{Z}} \hat{f}^n(\hat{\Lambda}_T)$ to obtain an invariant set. □

Proposition 2.5. *Let f be an endomorphism of $\mathbb{P}^2$ and let T be its Green current. Let ν be an ergodic measure. Then the functions $x \mapsto \underline{d_T}(x)$ and $x \mapsto \overline{d_T}(x)$ are invariant, hence ν-almost everywhere constant. We denote them by $\underline{d_T}(\nu)$ and $\overline{d_T}(\nu)$.*

Proof. The arguments follow the proof of Proposition 2.4. In this case we study the measure $T \wedge \omega$, and we replace the second equality in (2.4) by $f^*\omega \leq \rho(x_0)\omega$ on $B_{x_0}(\gamma(x_0))$, where $\rho(x_0) > 0$ is a large enough positive constant. □

3 Geometry of the inverse branches and uniformizations

Let ν be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). Our aim is to construct, for every $\delta > 0$, a Borel set $\hat{\Lambda}_\epsilon \subset \hat{X}$ satisfying $\hat{\nu}(\hat{\Lambda}_\epsilon) \geq 1 - \delta/2$ which provides convenient uniformizations.

3.1 Dynamical balls

The dynamical distance is defined by $d_n(x, y) := \max_{0\leq k\leq n} d(f^k(x), f^k(y))$. We denote by $B_n(x, r)$ the ball centered at x and of radius r for the distance d_n.

Lemma 3.1. *There exist $r_0 > 0$, $n_2 \geq 1$ and $C \subset \mathbb{P}^2$ such that $\nu(C) \geq 1 - \delta/8$ and satisfying the following properties: for every $x \in C$ and every $n \geq n_2$:*

$$\nu(B_n(x, r_0/8)) \geq e^{-nh_\nu - \epsilon n}.$$

$$\forall r \leq r_0 \;,\; \nu(B_n(x, 5r)) \leq \nu(B_n(x, 5r_0)) \leq e^{-nh_\nu + \epsilon n}.$$

Proof. Brin-Katok Theorem [8] ensures that there exists $C_1 \subset \mathbb{P}^2$ of full ν-measure such that for every $x \in C_1$:

$$\lim_{r\to 0}\left(\liminf_{n\to+\infty}\frac{-1}{n}\log\nu(B_n(x,r))\right) = \lim_{r\to 0}\left(\limsup_{n\to+\infty}\frac{-1}{n}\log\nu(B_n(x,r))\right) = h_\nu.$$

Hence, for every $x \in C_1$ there exists $r_0(x)$ such that $r \le r_0(x)$ implies

$$\liminf_{n\to+\infty}\frac{-1}{n}\log(\nu(B_n(x,5r))) \ge h_\nu - \epsilon/2 \text{ and } \limsup_{n\to+\infty}\frac{-1}{n}\log(\nu(B_n(x,r/8))) \le h_\nu + \epsilon/2.$$

Let r_0 such that $C_2 := \{x \in C_1\,,\ r_0(x) \ge r_0\}$ satisfies $\nu(C_2) \ge 1-\delta/16$. For every $x \in C_2$, there exists $n_2(x)$ such that $n \ge n_2(x)$ implies

$$\nu(B_n(x, r_0/8)) \ge e^{-nh_\nu - \epsilon n}.$$

$$\forall r \le r_0\ ,\ \nu(B_n(x,5r)) \le \nu(B_n(x,5r_0)) \le e^{-nh_\nu+\epsilon n}.$$

Let $n_2 \ge 1$ such that $C := C_2 \cap \{x \in C_1\,,\ n_2(x) \le n_2\}$ satisfies $\nu(C) \ge 1-\delta/8$. □

For every $L > 0$, let m_L be the smallest integer m such that $Le^{-m(\lambda_2+\epsilon)} \le 1$ and let $n_3(L)$ be the smallest integer larger than m_L such that $e^{-n\epsilon} \le M^{-m_L}$, where $M := \max\{\|Df\|_{\infty,\mathbb{P}^2}\,,1\}$.

Lemma 3.2. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \ge \eta$ and $L_\epsilon(\hat{x}) \le L$. For every $n \ge n_3(L)$ and every $r \le \eta$,*

$$f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-2n\epsilon})) \subset B_n(x_0, r).$$

Proof. Let us observe that for every $0 \le k \le n$, f^k is injective on $f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-2n\epsilon}))$ and that $f^k f_{\hat{x}_n}^{-n} = f_{\hat{x}_n}^{-n+k}$. By setting $p = n-k$, it suffices to show that

$$\forall p \in [\![0,n]\!] \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset B_{x_{n-p}}(r). \tag{3.1}$$

To simplify let us set $m := m_L$ et $n_3 := n_3(L)$. We immediately have

$$\forall n \ge n_3,\ \forall p \in [\![0,n]\!]\,, \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset f_{\hat{x}_n}^{-p}(B_{x_n}(\frac{r}{M^m}e^{-n\epsilon})). \tag{3.2}$$

To verify (3.1), we shall consider separately the cases $p \le m$ and $p > m$. We know that for every p, $\operatorname{Lip} f_{\hat{x}_n}^{-p} \le L(\hat{x}_n)e^{-p\lambda_2+p\epsilon} \le Le^{n\epsilon}e^{-p\lambda_2+p\epsilon}$ on $B_{x_n}(\eta_\epsilon(\hat{x}_n))$, which contains $B_{x_n}(\eta e^{-n\epsilon})$. Hence for every $n \ge n_3 \ge m$, $p \in [\![m,n]\!]$ and $r \le \eta$:

$$f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-n\epsilon})) \subset B_{x_{n-p}}(re^{-n\epsilon}Le^{n\epsilon}e^{-p\lambda_2+p\epsilon}) = B_{x_{n-p}}(rLe^{-p\lambda_2+p\epsilon}) \subset B_{x_{n-p}}(r).$$

Since $M^m \ge 1$ this implies for every $n \ge n_3 \ge m$, $p \in [\![m,n]\!]$ and $r \le \eta$:

$$f_{\hat{x}_n}^{-p}(B_{x_n}(\frac{r}{M^m}e^{-n\epsilon})) \subset B_{x_{n-p}}(\frac{r}{M^m}). \tag{3.3}$$

Thus, by using (3.2) and $M^m \geq 1$:

$$\forall p \in [\![m,n]\!], \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset B_{x_{n-p}}(r). \tag{3.4}$$

We have proved (3.1) for $p \in [\![m,n]\!]$. Let us show this inclusion for $p \in [\![0,m]\!]$. For every $p \in [\![0,m]\!]$, let us set $p = m - p'$ where $p' \in [\![0,m]\!]$. Then

$$f_{\hat{x}_n}^{-p}(B_{x_n}(\frac{r}{M^m}e^{-n\epsilon})) = f^{p'}(f_{\hat{x}_n}^{-m}(B_{x_n}(\frac{r}{M^m}e^{-n\epsilon})) \subset f^{p'}(B_{x_{n-m}}(\frac{r}{M^m})),$$

where the inclusion comes from (3.3) with $p = m$. We deduce:

$$\forall p \in [\![0,m]\!], \quad f_{\hat{x}_n}^{-p}(B_{x_n}(re^{-2n\epsilon})) \subset B_{x_{n-m+p'}}(\frac{r}{M^m}M^p) \subset B_{x_{n-p}}(r). \tag{3.5}$$

We finally obtain (3.1) by combining (3.4) and (3.5). □

3.2 Pullback of the Fubini-Study form ω

Let ν be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$. Let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for (f,ν). Let $n_4(\beta)$ be the smallest integer n such that $e^{-n\epsilon} \leq \beta^{-1}$.

Proposition 3.3. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \geq \eta$ and $\beta_\epsilon(\hat{x}) \leq \beta$. If $n \geq \max\{n_4(\beta), n_\epsilon(\hat{x}_n)\}$ and if $r \leq \eta$, then we have on $f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-n\epsilon}))$:*

1. $(f^n)^*\omega \geq e^{-4n\epsilon+2n\lambda_1}\left(\frac{i}{2}dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\right)$ *if the exponents do not resonate.*
2. $(f^n)^*\omega \geq e^{-4n\epsilon+2n\lambda_2}\left(\frac{i}{2}dW_{\hat{x}}^\epsilon \wedge d\overline{W_{\hat{x}}^\epsilon}\right)$.

The remainder of this Section is devoted to the proof. Theorem 2.1 gives

$$(f^n)^*\omega = (\xi_{\hat{x}}^\epsilon)^*((R_{n,\hat{x_n}})^{-1})^*((\xi_{\hat{x}_n})^{-1})^*\omega$$

on $f_{\hat{x}_n}^{-n}(B_{x_n}(re^{-n\epsilon}))$. Let $\omega_0 := \frac{i}{2}dz \wedge d\overline{z} + \frac{i}{2}dw \wedge d\overline{w}$ be the standard form on $\mathbb{D}^2$.

Lemma 3.4. *Let $\hat{x} \in \hat{F}$ such that $\eta_\epsilon(\hat{x}) \geq \eta$ and $\beta_\epsilon(\hat{x}) \leq \beta$. For every $n \geq n_4(\beta)$ and $r \leq \eta$, we have on $\xi_{\hat{x}_n}^\epsilon(B_{x_n}(re^{-n\epsilon}))$:*

$$e^{-2n\epsilon}\omega_0 \leq ((\xi_{\hat{x}_n}^\epsilon)^{-1})^*\omega \leq 2\omega_0.$$

Proof. For every $p = (z,w)$ and $p' = (z',w')$ in $\xi_{\hat{x}_n}^\epsilon(B_{x_n}(re^{-n\epsilon}))$, we have

$$e^{-n\epsilon}\beta^{-1}d(p,p') \leq \left|(\xi_{\hat{x}_n}^\epsilon)^{-1}(p) - (\xi_{\hat{x}_n}^\epsilon)^{-1}(p')\right| \leq 2d(p,p').$$

This implies for every $n \geq n_4(\beta)$ and $(z,w) \in \xi_{\hat{x}_n}^\epsilon(B_{x_n}(re^{-n\epsilon}))$:

$$\forall u \in \mathbb{C}^2, \quad e^{-2n\epsilon}\left|u\right| \leq \left|D_{(z,w)}(\xi_{\hat{x}_n}^\epsilon)^{-1}(u)\right| \leq 2\left|u\right|.$$

This provides the desired estimates. □

Lemma 3.5. *Let $\hat{x} \in \hat{F}$. If $n \geq n_\epsilon(\hat{x}_n)$, then*

1. $((R_{n,\hat{x}_n})^{-1})^*\omega_0 \geq e^{2(n\lambda_1 - n\epsilon)} \frac{i}{2} dz \wedge d\overline{z}$ *if the exponents do not resonate.*
2. $((R_{n,\hat{x}_n})^{-1})^*\omega_0 \geq e^{2(n\lambda_2 - n\epsilon)} \frac{i}{2} dw \wedge d\overline{w}$.

Proof. We use the fact that the linear part of $R_{n,\hat{x}_n}$ is diagonal with coefficients $e^{-n\epsilon - n\lambda_1} \leq |\alpha_{n,\hat{x}_n}| \leq e^{n\epsilon - n\lambda_1}$ and $e^{-n\epsilon - n\lambda_2} \leq |\beta_{n,\hat{x}_n}| \leq e^{n\epsilon - n\lambda_2}$ (see Theorem 2.1) and the fact that the $(1,1)$-forms $\frac{i}{2} dz \wedge d\overline{z}$ and $\frac{i}{2} dw \wedge d\overline{w}$ are positive. □

To end the proof of Proposition 3.3, we observe that for every $\hat{x} \in \hat{F}$:

$$(\xi^\epsilon_{\hat{x}})^*(\frac{i}{2} dz \wedge d\overline{z}) = (\frac{i}{2} dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}) \qquad (\xi^\epsilon_{\hat{x}})^*(\frac{i}{2} dw \wedge d\overline{w}) = (\frac{i}{2} dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}})$$

which follows from the definitions of $Z^\epsilon_{\hat{x}}$ and $W^\epsilon_{\hat{x}}$.

3.3 Uniformizations

Let ν be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$ and let $\delta > 0$. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν).

Measure of dynamical balls

We apply Lemma 3.1. There exist $r_0 > 0$, $n_2 \geq 1$ and $C \subset \mathbb{P}^2$ such that $\nu(C) \geq 1 - \delta/8$ and for every $x \in C$ and $n \geq n_2$:

$$\nu(B_n(x, r_0/8)) \geq e^{-nh_\nu - \epsilon n},$$

$$\forall r \leq r_0 \ , \ \nu(B_n(x, 5r)) \leq \nu(B_n(x, 5r_0)) \leq e^{-nh_\nu + \epsilon n}.$$

We denote $\Lambda^{(1)} := \pi_0^{-1}(C) \cap \hat{F}$.

Control of the functions $n_\epsilon, \rho_\epsilon, L_\epsilon, \eta_\epsilon, \beta_\epsilon$ of Theorem 2.1

Let n_0, $\rho_0 > 0$, $L_0 > 0$, $\eta_0 > 0$ and $\beta_0 > 0$ such that

$$\Lambda^{(2)} := \left\{ \hat{x} \in \hat{F},\ n_\epsilon(\hat{x}) \leq n_0,\ \rho_\epsilon(\hat{x}) \geq \rho_0,\ L_\epsilon(\hat{x}) \leq L_0,\ \eta_\epsilon(\hat{x}) \geq \eta_0,\ \beta_\epsilon(\hat{x}) \leq \beta_0 \right\} \quad (3.6)$$

satisfies $\hat{\nu}(\Lambda^{(2)}) \geq 1 - \delta/8$.

Uniformization of the dimension of the current.

Let S be a positive closed current on $\mathbb{P}^2$ whose support contains the support of ν. Let $r_1 > 0$ such that

$$\Lambda^{(3)} := \left\{ \hat{x} \in \hat{F},\ \ \forall r \leq r_1, \quad \begin{array}{c} r^{\overline{d_{S,Z}}(\hat{x})+\epsilon} \leq (S \wedge (\frac{i}{2} dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}))(B_{x_0}(r)) \leq r^{\underline{d_{S,Z}}(\hat{x})-\epsilon} \\ (S \wedge (\frac{i}{2} dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}))(B_{x_0}(r)) \leq r^{\underline{d_{S,W}}(\hat{x})-\epsilon} \end{array} \right\}$$

satisfies $\hat{\nu}(\Lambda^{(3)}) \geq 1 - \delta/8$. In the case of the Green current T, the functions $\overline{d_{T,Z}}$, $\underline{d_{T,Z}}$ and $\underline{d_{T,W}}$ are $\hat{\nu}$-almost everywhere constant and denoted $\overline{d_{T,Z}}(\nu)$, $\underline{d_{T,Z}}(\nu)$ and $\underline{d_{T,W}}(\nu)$.

Uniformization of the dimension of the measure

The lower dimension $\underline{d_\nu}$ is defined in Section 1.4. Let $r_2 > 0$ such that

$$D := \{x \in \mathbb{P}^2\,,\, \forall r \le r_2,\ \nu(B_x(r)) \le r^{\underline{d_\nu}-\epsilon}\}$$

satisfies $\nu(D) \ge 1 - \delta/8$. We set $\Lambda^{(4)} := \pi_0^{-1}(D) \cap \hat{F}$.

Definition of $\hat{\Lambda}_\epsilon, \eta_1$ and N_ϵ.

The integers $n_1(L), n_3(L)$ and $n_4(\beta)$ were defined before Lemma 2.3, 3.2 and Proposition 3.3. Let n_5 be the smallest integer n such that $e^{-n\epsilon} \le 1/2$ and $2e^{-n(\lambda_1+\epsilon)} < 1$.

$$\hat{\Lambda}_\epsilon := \Lambda^{(1)} \cap \Lambda^{(2)} \cap \Lambda^{(3)} \cap \Lambda^{(4)},$$

$$\eta_1 := \min\{\eta_0, r_0, r_1, r_2\},$$

$$N_\epsilon := \max\{n_0, n_1(L_0), n_2, n_3(L_0), n_4(\beta_0), n_5\}.$$

We have $\hat{\nu}(\hat{\Lambda}_\epsilon) \ge 1 - \delta/2$.

Definition of $\hat{\Delta}_\epsilon^n$.

We set

$$\forall n \ge N_\epsilon \ \ , \ \ \hat{\Delta}_\epsilon^n := \hat{F} \cap \hat{f}^{-n}\{n_\epsilon(\hat{x}) \le n\} = \{\hat{x} \in \hat{F}\ ,\ n_\epsilon(\hat{x}_n) \le n\}.$$

Since $\hat{\nu}$ is $\hat{f}$-invariant and $\Lambda^{(3)} \subset \{n_\epsilon(\hat{x}) \le n\}$, we have $\hat{\nu}(\hat{\Delta}_\epsilon^n) \ge \hat{\nu}(\Lambda^{(3)}) \ge 1 - \delta/8$. Hence

$$\forall n \ge N_\epsilon \ \ , \ \ \hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n) \ge 1 - \delta.$$

4 Separated sets

A subset $\{x_1, \ldots, x_N\} \subset \mathbb{P}^2$ is r-separated if $d(x_i, x_j) \ge r$ for every $i \ne j$. For $A \subset \mathbb{P}^2$, a subset $\{x_1, \ldots, x_N\} \subset A$ is maximal r-separated with respect to A if it is r-separated and if for every $y \in A$, there exists $i \in \{1, \ldots N\}$ such that $d(y, x_i) < r$. We use similar definitions for the distance d_n, in which case we say that the subsets are (n, r) separated.

4.1 Elementary separation

Lemma 4.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \ge 2$ and let ν be an ergodic measure. Let $A \subset \pi_0(\hat{\Lambda}_\epsilon)$ such that $\nu(A) > 0$ and let $c \in]0, 1]$. Let $n \ge N_\epsilon$ and let $\{x_1, \ldots x_{N_n}\} \subset A$ be maximal $(n, c\,\eta_1)$-separated with respect to A. Then*

1. *for every $i \ne j$, $B_n(x_i, c\,\eta_1/2) \cap B_n(x_j, c\,\eta_1/2) = \emptyset$.*
2. *$A \subset \cup_{i=1}^{N_n} B_n(x_i, c\,\eta_1)$.*
3. *$\nu(B_n(x_i, c\,\eta_1)) \le e^{-nh_\nu + n\epsilon}$.*

4. $e^{-nh_\nu - n\epsilon} \leq \nu(B_n(x_i, c\,\eta_1))$ *si* $c \geq 1/8$.

5. $N_n \geq \nu(A) e^{nh_\nu - n\epsilon}$.

Proof. Item 1 comes from separation, Item 2 from the maximal property, Items 3 and 4 from Section 3.3, because $n \geq N_\epsilon$, $c\,\eta_1 \leq \eta_1$ and $x_i \in C$. Items 2 and 3 then imply $\nu(A) \leq \sum_{i=1}^{N_n} \nu(B_n(x_i, c\,\eta_1)) \leq N_n e^{-nh_\nu + n\epsilon}$, which gives Item 5. □

4.2 Concentrated separation

Lemma 4.1 applied with $c = 1/4$ gives $\nu(B_n(x_i, \eta_1/4)) \geq e^{-nh_\nu - n\epsilon}$ for every x_i in a maximal $(n, \eta_1/4)$-separated subset of A. We shall see that it is possible to select a large number of x_i such that

$$\nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_\nu - 2n\epsilon}.$$

We take the arguments of de Thélin-Vigny in [12, Section 6]. Let n_δ be the smallest integer n such that $e^{-n\epsilon} \leq \delta/2$.

Lemma 4.2. *Let $A \subset \pi_0(\hat{\Lambda}_\epsilon)$ such that $\nu(A) \geq \delta$. For every $n \geq \max\{N_\epsilon, n_\delta\}$, there exists a $(n, \eta_1/4)$-separated subset $\{x_1, \ldots, x_{N_{n,2}}\}$ of A such that*

1. *for every $i \neq j$, $B_n(x_i, \eta_1/8) \cap B_n(x_j, \eta_1/8) = \emptyset$.*

2. *for every $1 \leq i \leq N_{n,2}$, $\nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_\nu - 2\epsilon n}$.*

3. *$N_{n,2} \geq \nu(A) e^{nh_\nu - 2n\epsilon}$.*

4. *$e^{-nh_\nu - n\epsilon} \leq \nu(B_n(x_i, c\eta_1))$ si $c \geq 1/8$.*

Proof. Let us apply Lemma 4.1 with $c = 1/4$ and $n \geq \max\{N_\epsilon, n_\delta\}$. There exists a maximal $(n, \eta_1/4)$-separated subset $\{x_1, \ldots, x_{N_{n,1}}\}$ of A satisfying:
- for every $i \neq j$, $B_n(x_i, \eta_1/8) \cap B_n(x_j, \eta_1/8) = \emptyset$,
- $A \subset \cup_{i=1}^{N_{n,1}} B_n(x_i, \eta_1/4)$,
- $e^{-nh_\nu - n\epsilon} \leq \nu(B_n(x_i, \eta_1/8))$,
- $N_{n,1} \geq \nu(A) e^{nh_\nu - n\epsilon}$.

Let us set $I := \left\{1 \leq i \leq N_{n,1}\ ,\ \nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_\nu - 2\epsilon n}\right\}$. Let $N_{n,2}$ be the cardinality of I, and assume that $I = [\![1, N_{n,2}]\!]$ (we may adapt the sums below if $N_{n,2} = 0$). We want to bound $N_{n,2}$ from below. We know that $A \subset \cup_{i=1}^{N_{n,1}} B_n(x_i, \eta_1/4)$, hence

$$\nu(A) \leq \sum_{i=1}^{N_{n,2}} \nu(B_n(x_i, \eta_1/4) \cap A) + \sum_{i=N_{n,2}+1}^{N_{n,1}} \nu(B_n(x_i, \eta_1/4) \cap A).$$

If $i \notin [\![1, N_{n,2}]\!]$, we have $\nu(B_n(x_i, \eta_1/4) \cap A) < e^{-nh_\nu - 2\epsilon n}$ by definition of I. Otherwise, $\nu(B_n(x_i, \eta_1/4)) \leq e^{-nh_\nu + \epsilon n}$ since $x_i \in C$. This implies

$$\nu(A) \leq N_{n,2} e^{-nh_\nu + \epsilon n} + (N_{n,1} - N_{n,2}) e^{-nh_\nu - 2\epsilon n}. \tag{4.1}$$

Let us give an upper bound for $N_{n,1}$. Since the balls $B_n(x_i, \eta_1/8)$ are pairwise disjoint and since $\nu(B_n(x_i, \eta_1/8)) \geq e^{-nh_\nu - \epsilon n}$, we get

$$e^{nh_\nu + \epsilon n} \geq N_{n,1} \geq N_{n,1} - N_{n,2}.$$

Combining this and (4.1), we obtain

$$\nu(A) \leq N_{n,2} e^{-nh_\nu + \epsilon n} + e^{-\epsilon n}.$$

Since $n \geq n_\delta$, we have $e^{-n\epsilon} \leq \delta/2 \leq \nu(A)/2$, and hence $N_{n,2} \geq \nu(A) e^{nh_\nu - \epsilon n}/2$. Finally $N_{n,2} \geq \nu(A) e^{nh_\nu - 2\epsilon n}$ since $n \geq N_\epsilon \geq n_5$. □

Now we put in $B_n(x, \eta_1/2)$ a lot of balls whose centers are in $B_n(x, \eta_1/4) \cap A$.

Lemma 4.3. *Let $A \subset \pi_0(\hat{\Lambda}_\epsilon)$ such that $\nu(A) > 0$. Let $x \in A$ and let $n \geq N_\epsilon$ such that*

$$\nu(B_n(x, \eta_1/4) \cap A) \geq e^{-nh_\nu - 2n\epsilon}.$$

Let $\{y_1, \ldots, y_{M_n}\}$ be a maximal $2\eta_1 e^{-n\lambda_1 - 4n\epsilon}$-separated subset in $B_n(x, \eta_1/4) \cap A$.

1. *for every $i \neq j$, $B(y_i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \cap B(y_j, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) = \emptyset$.*
2. *$B_n(x, \eta_1/4) \cap A \subset \cup_{i=1}^{M_n} B(y_i, 2\eta_1 e^{-n\lambda_1 - 4n\epsilon})$.*
3. *$B(y_i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset B_n(x, \eta_1/2)$.*
4. *$M_n \geq e^{-nh_\nu - 2n\epsilon} \left(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon}\right)^{\underline{d_\nu} - \epsilon}$.*

Proof. Item 1 comes from separation, Item 2 from the maximal property. Lemmas 2.3 then 3.2 give

$$B(y_i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset f^{-n}_{\hat{y}_{i,n}}(B_{y_{i,n}}(\eta_1 e^{-2n\epsilon}/4) \subset B_n(y_i, \eta_1/4).$$

Since $y_i \in B_n(x, \eta_1/4)$, we have $B_n(y_i, \eta_1/4) \subset B_n(x, \eta_1/2)$, which yields Item 3. Item 2 implies

$$\nu(B_n(x, \eta_1/4) \cap A) \leq \sum_{i=1}^{M_n} \nu(B(y_i, 2\eta_1 e^{-n\lambda_1 - 4n\epsilon})).$$

By assumption, the left hand side is larger than $e^{-nh_\nu - 2n\epsilon}$. For the right hand side, since $n \geq N_\epsilon \geq n_5$, we have $2\eta_1 e^{-n\lambda_1 - n\epsilon} < \eta_1 \leq r_2$ and thus

$$\nu(B(y_i, 2\eta_1 e^{-n\lambda_1 - 4n\epsilon})) \leq (2\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\underline{d_\nu} - \epsilon}$$

by using $y_i \in A \subset \pi_0(\hat{\Lambda}_\epsilon) \subset D$. This shows $e^{-nh_\nu - 2n\epsilon} \leq M_n (2\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\underline{d_\nu} - \epsilon}$. □

5 Lower bounds for the directional dimensions of T

Let ν be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). We have $\overline{d_{T,Z}}(\nu) := \overline{d_{T,Z}}(\hat{x})$ and $\overline{d_{T,W}}(\nu) = \overline{d_{T,W}}(\hat{x})$ for $\hat{\nu}$-almost every $\hat{x}$ (see Proposition 2.4). In this Section we prove Theorems 5.1 and 5.2. These results specify in a directional way Theorems A and B of de Thélin-Vigny [12] concerning the dimension of T. We use below arguments of [12] by replacing the lower bound (obtained in [12] by slicing arguments)

$$(f^n)^*\omega \geq e^{2n\lambda_2}e^{-n\epsilon}\omega$$

by the two lower bounds (obtained by normal forms Theorem 2.1)

$$(f^n)^*\omega \geq e^{2n\lambda_1}e^{-n\epsilon}dZ \wedge d\overline{Z} \text{ and } (f^n)^*\omega \geq e^{2n\lambda_2}e^{-n\epsilon}dW \wedge d\overline{W}.$$

Theorem 5.1 uses elementary separation (Lemma 4.1), Theorem 5.2 uses concentrated separation (Lemmas 4.2 and 4.3). Theorem 5.2 implies Theorem 1.1 (via the lower bound (1.6) for $\underline{d_\nu}$) and Theorem 1.7.

5.1 First lower bounds for the upper dimensions $d_{T,Z}$ et $d_{T,W}$

Theorem 5.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. There exist functions $O_5(\epsilon), O_6(\epsilon)$ satisfying the following properties. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). Then*

$$\overline{d_{T,Z}}(\nu) \geq 2 + \frac{h_\nu - \log d}{\lambda_1} - O_5(\epsilon)$$

$$\overline{d_{T,W}}(\nu) \geq 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_1} - O_6(\epsilon).$$

Proof. Let us denote $\overline{d_{T,Z}} := \overline{d_{T,Z}}(\nu)$. For first estimate, we are going to show

$$(\lambda_1 + 4\epsilon)(\overline{d_{T,Z}} - 2 + \epsilon) + 13\epsilon \geq h_\nu - \log d \tag{5.1}$$

which provides

$$\overline{d_{T,Z}} \geq 2 + \frac{h_\nu - \log d - 13\epsilon}{\lambda_1 + 4\epsilon} - \epsilon =: 2 + \frac{h_\nu - \log d}{\lambda_1} - O_5(\epsilon).$$

Let $\delta > 0$. Let $\hat{\Lambda}_\epsilon$ and N_ϵ be given by Section 3.3. For every $n \geq N_\epsilon$ we set $A_n := \pi_0(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n)$, it satisfies $\nu(A_n) \geq \hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n) \geq 1 - \delta > 0$. Lemma 4.1 applied with $c = 1$ yields a maximal (n, η_1)-separated subset $\{x_1, \ldots, x_{N_n}\}$ of A_n with

$$N_n \geq \nu(A_n)e^{nh_\nu - n\epsilon} \geq (1 - \delta)e^{nh_\nu - n\epsilon}. \tag{5.2}$$

For every i, let us choose $\hat{x}_i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n$ such that $\pi_0(\hat{x}_i) = x_i$. From Proposition 9.3, we get $d^n = \int_{\mathbb{P}^2}((f^n)_* T) \wedge \omega$. Therefore

$$d^n \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (f^n)_* (1_{B_n(x_i, \eta_1/2)} T) \wedge \omega \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (1_{B_n(x_i, \eta_1/2)} T) \wedge (f^n)^* \omega.$$

By using Lemmas 2.3 and 3.2 with $\hat{x}_i \in \hat{\Lambda}_\epsilon$, we get

$$B_{x_i}(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon}) \subset f^{-n}_{\hat{x}_{i,n}}(B_{x_{i,n}}(\frac{\eta_1}{8} e^{-2n\epsilon})) \subset B_n(x_i, \frac{\eta_1}{8}). \tag{5.3}$$

Since $T \wedge (f^n)^* \omega$ is a positive measure, we deduce

$$d^n \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (1_{B_{x_i}(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon})} T) \wedge (f^n)^* \omega.$$

Thanks to the first inclusion of (5.3) and $\hat{x}_i \in \hat{\Delta}_\epsilon^n$ (which implies $n \geq n_\epsilon(\hat{x}_{i,n})$), we can use Proposition 3.3 to bound $(f^n)^* \omega$ from below. We obtain

$$d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left(T \wedge \frac{i}{2} dZ^\epsilon_{\hat{x}_i} \wedge d\overline{Z^\epsilon_{\hat{x}_i}} \right) (B_{x_i}(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon})).$$

Since $\hat{x_i} \in \hat{\Lambda}_\epsilon \subset \Lambda^{(3)}$ and $\eta_1 \leq r_1$, we deduce

$$d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon} \right)^{\overline{d_{T,Z}} + \epsilon}.$$

Finally, we use the estimate (5.2):

$$d^n \geq \nu(A_n) \left(\frac{\eta_1}{2} \right)^{\overline{d^\epsilon_{T,Z}} + \epsilon} e^{n(h_\nu - (\lambda_1 + 4\epsilon)(\overline{d^\epsilon_{T,Z}} - 2 + \epsilon) - 13\epsilon)},$$

where $\nu(A_n) \geq (1 - \delta)$. By taking logarithm and dividing by n, we get (5.1) when $n \to \infty$. The second estimate concerning the coordinate W is proved in a similar way, by using Proposition 3.3 to bound $(f^n)^* \omega$ from below. We precisely get

$$(\lambda_1 + 4\epsilon)(\overline{d_{T,W}} + \epsilon) + 5\epsilon \geq h_\nu - \log d + 2\lambda_2 \tag{5.4}$$

which yields

$$\overline{d_{T,W}} \geq \frac{2\lambda_2}{\lambda_1 + 4\epsilon} + \frac{h_\nu - \log d - 5\epsilon}{\lambda_1 + 4\epsilon} - \epsilon =: 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_1} - O_6(\epsilon).$$

This completes the proof of Theorem 5.1. □

5.2 Proof of Theorem 1.1

Theorem 1.1 is a consequence of Theorem 5.2 below and of the lower bound (1.6).

Theorem 5.2. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. There exist functions $O_1(\epsilon), O_2(\epsilon)$ satisfying the following properties. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). Then*

$$\overline{d_{T,Z}}(\nu) \geq 2 + \underline{d_\nu} - \frac{\log d}{\lambda_1} - O_1(\epsilon),$$

$$\overline{d_{T,W}}(\nu) \geq 2\frac{\lambda_2}{\lambda_1} + \underline{d_\nu} - \frac{\log d}{\lambda_1} - O_2(\epsilon).$$

Proof. Let us set $\overline{d_{T,Z}} := \overline{d_{T,Z}}(\nu)$. We are going to show

$$(\lambda_1 + 4\epsilon)(\overline{d_{T,Z}} - \underline{d_\nu} + 2\epsilon) + 8\epsilon \geq 2\lambda_1 - \log d. \tag{5.5}$$

This yields as desired

$$\overline{d_{T,Z}} - \underline{d_\nu} \geq \frac{2\lambda_1}{\lambda_1 + 4\epsilon} - \frac{\log d - 8\epsilon}{\lambda_1 + 4\epsilon} - 2\epsilon =: 2 - \frac{\log d}{\lambda_1} - O_1(\epsilon). \tag{5.6}$$

Let $\delta > 0$. Let $\hat{\Lambda}_\epsilon$ and N_ϵ be given by Section 3.3. For every $n \geq N_\epsilon$ we set $A_n := \pi_0(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n)$, it satisfies $\nu(A_n) \geq 1 - \delta$. Let $\{x_1, \dots, x_{N_{n,2}}\}$ be a $(n, \eta_1/4)$-separated subset of A_n provided by Lemma 4.2. Then for every x_i, we set a $2\eta_1 e^{-n\lambda_1 - 4n\epsilon}$-separated subset $\{y_1^i, \dots, y_{M_n}^i\}$ given by Lemma 4.3. For every i we choose $\hat{x}_i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n$ such that $\pi_0(\hat{x}_i) = x_i$, and for every j we choose $\hat{y}_j^i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n$ such that $\pi_0(\hat{y}_j^i) = y_j^i$. According to Proposition 9.3, we have $d^n = \int_{\mathbb{P}^2} (f^n)_* T \wedge \omega$, thus

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} \int_{\mathbb{P}^2} (f^n)_* (1_{B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon})} T) \wedge \omega = \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} \int_{B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon})} T \wedge (f^n)^* \omega.$$

Lemma 2.3 with $\hat{y}_j^i \in \hat{\Lambda}_\epsilon$ implies

$$B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset f^{-n}_{\hat{y}_{j,n}^i}(B(y_{j,n}^i, \frac{\eta_1}{4} e^{-2n\epsilon})). \tag{5.7}$$

Since $\hat{y}_j^i \in \hat{\Delta}_\epsilon^n$, we can apply Proposition 3.3 to bound $(f^n)^*\omega$ from below:

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 4n\epsilon} \left(T \wedge \frac{i}{2} dZ^\epsilon_{\hat{y}_j^i} \wedge d\overline{Z^\epsilon_{\hat{y}_j^i}} \right) \left(B_{y_j^i}(\eta_1 e^{-n\lambda_1 - 4n\epsilon}) \right). \tag{5.8}$$

Now $\hat{y}_j^i \in \hat{\Lambda}_\epsilon \subset \Lambda^{(3)}$ and $n \geq N_\epsilon$, hence

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 4n\epsilon} (\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\overline{d^\epsilon_{T,Z}} + \epsilon}.$$

Finally, we use the lower bounds for M_n (Lemma 4.3) and for $N_{n,2}$ (Lemma 4.2). We obtain for every $n \geq \max\{N_\epsilon, n_{1-\delta}\}$:

$$d^n \geq (1-\delta)e^{nh_\nu - 2n\epsilon} \cdot e^{-nh_\nu - 2n\epsilon} \left(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon}\right)^{\underline{d_\nu} - \epsilon} \cdot e^{2n\lambda_1 - 4n\epsilon} (\eta_1 e^{-n\lambda_1 - 4n\epsilon})^{\overline{d^\epsilon_{T,Z}} + \epsilon}.$$

Let us note that the entropy h_ν disappear for the benefit of $\underline{d_\nu}$, and we get

$$d^n \geq c\, e^{-8n\epsilon} \left(e^{n\lambda_1 + 4n\epsilon}\right)^{\underline{d_\nu} - \overline{d^\epsilon_{T,Z}} - 2\epsilon} e^{2n\lambda_1}, \tag{5.9}$$

where $c := (1-\delta)\eta_1^{\overline{d_{T,Z}}+\epsilon}/(2\eta_1)^{\underline{d_\nu}-\epsilon}$. Taking logarithm and then dividing by n, we obtain (5.5) when $n \to +\infty$. Similarly, we can prove

$$(\lambda_1 + 4\epsilon)(\overline{d_{T,W}} - \underline{d_\nu} + 2\epsilon) + 8\epsilon \geq 2\lambda_2 - \log d$$

by using again Proposition 3.3 to bound $(f^n)^*\omega$ from below. □

6 Currents S and semi-extremal endomorphisms

6.1 Proof of Theorem 1.4

Let S be a $(1,1)$ closed positive current of $\mathbb{P}^2$. If S does not satisfy $f^*S = dS$, the directional dimensions may be not $\hat{\nu}$-almost everywhere constant (see Proposition 2.4). In this case, in the manner of de Thélin-Vigny [12], we take on an adapted definition and obtain the following result. The functions $O_5(\epsilon), O_6(\epsilon)$ were defined in Theorem 5.1.

Theorem 6.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let S be a $(1,1)$ closed positive current of $\mathbb{P}^2$, of mass 1. Let ν be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. We assume that $\mathrm{Supp}(\nu) \subset \mathrm{Supp}\, S$. Let $\epsilon > 0$ and let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for (f,ν). For every $\hat{\Lambda} \subset \hat{F}$ such that $\hat{\nu}(\hat{\Lambda}) > 0$, we set*

$$\overline{d_{S,Z}}(\hat{\Lambda}) := \sup_{\hat{x}\in\hat{\Lambda}} \overline{d_{S,Z}}(\hat{x}), \qquad \overline{d_{S,W}}(\hat{\Lambda}) := \sup_{\hat{x}\in\hat{\Lambda}} \overline{d_{S,W}}(\hat{x}).$$

Then

$$\overline{d_{S,Z}}(\hat{\Lambda}) \geq 2 + \frac{h_\nu - \log d}{\lambda_1} - O_5(\epsilon),$$

$$\overline{d_{S,W}}(\hat{\Lambda}) \geq 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_1} - O_6(\epsilon).$$

Proof. Let $2\delta := \hat{\nu}(\hat{\Lambda})$. We construct $\hat{\Lambda}_\epsilon$ et N_ϵ for the current S as in Section 3.3. We have $\hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon) \geq 1-\delta$ for every $n \geq N_\epsilon$, thus $\hat{\nu}(\hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon) \geq \delta > 0$. We follow the arguments of Theorem 5.1. Lemma 4.1 applied to $A_n := \pi_0(\hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}^n_\epsilon)$ and $c = 1$ provides a maximal (n,η_1)-separated subset $\{x_1, \ldots, x_{N_n}\}$ of A_n. For every i, let us

choose $\hat{x}_i \in \hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n$ such that $\pi_0(\hat{x}_i) = x_i$. According to Proposition 9.3 we have $d^n = \int_{\mathbb{P}^2} (f^n)_* S \wedge \omega$, hence

$$d^n = \int_{\mathbb{P}^2} ((f^n)_* S) \wedge \omega \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (1_{B_n(x_i, \eta_1/2)} S) \wedge (f^n)^* \omega.$$

Then we use the inclusions and the lower bound for $(f^n)^*\omega$ to obtain:

$$d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left(S \wedge \frac{i}{2} dZ^\epsilon_{\hat{x}_i} \wedge d\overline{Z^\epsilon_{\hat{x}_i}} \right) (B_{x_i}(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon})).$$

Since $\hat{x}_i \in \hat{\Lambda}_\epsilon$ and $n \geq N_\epsilon$, we get

$$d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon} \right)^{\overline{d_{S,Z}}(\hat{x}_i) + \epsilon}.$$

Now we use the adapted definition of $\overline{d_{S,Z}}(\hat{\Lambda})$ and the lower estimate (5.2) to obtain

$$d^n \geq \nu(A_n) \left(\frac{\eta_1}{2} \right)^{\overline{d_{S,Z}}(\hat{\Lambda}) + \epsilon} e^{nh_\nu - 13n\epsilon - n(\lambda_1 + 4\epsilon)(\overline{d_{S,Z}}(\hat{\Lambda}) - 2 + \epsilon)}, \tag{6.1}$$

where $\nu(A_n) \geq \delta$. The lower bound concerning W is proved in a similar way. □

Remark 6.2. *Theorem 6.1 is the counterpart of Theorem 5.1 for currents S. Similarly, the counterpart of Theorem 5.2 can be proved with $n \geq \max\{N_\epsilon, n_\delta\}$ in the proof.*

6.2 Proof of Theorem 1.7

Since T is f-invariant and ν is ergodic, we have $\overline{d_T}(x) = \overline{d_T}(\mu)$ for μ-almost every x, see Proposition 2.5. According to Proposition 9.1, $\mu << \sigma_T$ implies

$$\overline{d_T}(\mu) \leq \overline{d_\mu}. \tag{6.2}$$

Let us analyse these quantities. On the one hand, Proposition 9.2 yields $\overline{d_T}(\mu) = \min\left\{\overline{d_{T,Z}}(x), \overline{d_{T,W}}(x)\right\}$ for μ-almost every $x \in \mathbb{P}^2$ and for every holomorphic coordinates (Z, W) near x. On the other hand, since $\underline{d_\mu} = \overline{d_\mu}$, then $\underline{d_\mu} = \overline{d_\mu} = \dim_H(\mu)$, which is equal to $2 + \frac{\log d}{\lambda_1}$ by (1.7). One deduces from (6.2) that if $\mu << \sigma_T$, then

$$\min\left\{\overline{d_{T,Z}}(x), \overline{d_{T,W}}(x)\right\} \leq 2 + \frac{\log d}{\lambda_1}. \tag{6.3}$$

Now we use Theorem 5.2. Let $\epsilon > 0$ such that $4 - O_1(\epsilon) > 2 + \frac{\log d}{\lambda_1}$, where the function $O_1(\epsilon)$ is defined by (5.6). Let $(Z, W)_\epsilon$ Oseledec-Poincaré coordinates for (f, μ).

First we bound $\overline{d_{T,Z}}(\mu)$ from below, then we establish the formula for $\overline{d_{T,W}}(\mu)$ modulo the function $O_2(\epsilon)$. If $\mu << \sigma_T$, then $\lambda_2 = \frac{1}{2}\log d$ by Theorem 1.6. We deduce from (1.6) that $2 + \frac{\log d}{\lambda_1} \leq \underline{d_\mu}$. Theorem 5.2 then provides

$$\overline{d_{T,Z}}(\mu) \geq 4 - O_1(\epsilon) \quad , \quad \overline{d_{T,W}}(\mu) \geq 2 + \frac{\log d}{\lambda_1} - O_2(\epsilon).$$

Finally, using $4 - O_1(\epsilon) > 2 + \frac{\log d}{\lambda_1}$ and (6.3) with Oseledec-Poincaré coordinates $(Z, W)_\epsilon$, we get $\overline{d_{T,W}}(\mu) \leq 2 + \frac{\log d}{\lambda_1}$ as desired.

7 Upper bounds for the directional dimensions of T

In this Section we show Theorem 1.2. Let ν be an ergodic dilating measure such that $\mathrm{Supp}(\nu) \subset \mathrm{Supp}(\mu)$ and whose exponents $\lambda_1 > \lambda_2$ do not resonate. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). We want to prove

$$\underline{d_{T,Z}}(\nu) \leq \frac{\log d}{\lambda_2} + 2\frac{\lambda_1}{\lambda_2} + O_3(\epsilon) \quad \text{and} \quad \underline{d_{T,W}}(\nu) \leq \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon).$$

We shall directly obtain these upper bounds for $\hat{\nu}$-almost every $\hat{x}$, by using the jacobians of $T \wedge dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}$ and $T \wedge dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}$ with respect to f. In particular, we shall not use separated subsets. The Monge-Ampère equation $\mu = T \wedge T$ will be crucial.

7.1 Dimensions of the Green current on the equilibrium measure

The following Proposition is proved in Section 7.3.

Proposition 7.1. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let T be its Green current. Let $x \in \mathrm{Supp}\,\mu$ and let Z be a local holomorphic coordinate (submersion) in a neighbourhood V of x. Then $T \wedge (\frac{i}{2} dZ \wedge d\overline{Z})$ is not the zero measure on V.*

This implies:

Proposition 7.2. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$ and whose support is contained in the support of μ. Let $\epsilon > 0$ and let $(Z, W)_\epsilon$ be Oseledec-Poincaré coordinates for (f, ν). We recall that for every $\hat{x} \in \hat{F}$, $(Z^\epsilon_{\hat{x}}, W^\epsilon_{\hat{x}})$ is defined on $B_{x_0}(\eta_\epsilon(\hat{x}))$. Then, for every $0 < r < \eta_\epsilon(\hat{x})$,*

$$\left[T \wedge \frac{i}{2} dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right](B_x(r)) > 0 \quad \textit{and} \quad \left[T \wedge \frac{i}{2} dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}\right](B_x(r)) > 0.$$

In particular, for every $\delta > 0$, there exist $m_0 \geq 1$, $L_0 \geq 1$ and $q_0 \geq 1$ such that

$$\hat{\Omega}_\epsilon := \left\{ \eta_\epsilon(\hat{x}) \geq \frac{1}{4m_0} \ , \ L(\hat{x}) \leq L_0 \ , \ \begin{matrix} \left[T \wedge \frac{i}{2} dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right](B_x(\frac{1}{4m_0})) \geq \frac{1}{q_0} \\ \left[T \wedge \frac{i}{2} dW^\epsilon_{\hat{x}} \wedge d\overline{W^\epsilon_{\hat{x}}}\right](B_x(\frac{1}{4m_0})) \geq \frac{1}{q_0} \end{matrix} \right\}$$

satisfies $\hat{\nu}(\hat{\Omega}_\epsilon) \geq 1 - \delta$.

Proof. The first part immediately follows from Proposition 7.1. To prove the second part, let $m_0 \geq 1$ and $L_0 \geq 1$ be such that $\hat{\nu}\left\{\eta_\epsilon \geq \frac{1}{4m_0}\right\} \cap \{L \leq L_0\} \geq 1 - \delta/2$. Then we choose q_0 large enough so that $\hat{\nu}(\hat{\Omega}_\epsilon) \geq 1 - \delta$. □

We define for every $n \geq 1$:

$$\hat{\Omega}_\epsilon^n := \hat{\Omega}_\epsilon \cap \hat{f}^{-n}(\hat{\Omega}_\epsilon).$$

Since $\hat{\nu}$ is invariant, we have:

$$\hat{\nu}(\hat{\Omega}_\epsilon^n) \geq 1 - 2\delta. \tag{7.1}$$

The following Proposition will be useful to prove Theorems 1.2 and 1.5. L_0 is defined in Proposition 7.2 and $n_1(L_0) \geq 1$ is defined before Lemma 2.3.

Proposition 7.3. *Let f be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let ν be a ergodic dilating measure of exponents $\lambda_1 > \lambda_2$ and such that $\mathrm{Supp}(\nu) \subset \mathrm{Supp}(\mu)$. For every $n \geq n_1(L_0)$ and $\hat{x} \in \hat{\Omega}_\epsilon^n$, we have:*

$$\left[T \wedge \frac{i}{2} dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\right]\left(B_x\left(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\right)\right) \geq \frac{1}{d^n} e^{-2n\lambda_1 - 2n\epsilon} \frac{1}{q_0} \quad \text{if } \lambda_1 \notin \{k\lambda_2, k \geq 2\},$$

$$\left[T \wedge \frac{i}{2} dW_{\hat{x}}^\epsilon \wedge d\overline{W_{\hat{x}}^\epsilon}\right]\left(B_x\left(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\right)\right) \geq \frac{1}{d^n} e^{-2n\lambda_2 - 2n\epsilon} \frac{1}{q_0} \quad \text{for every } \lambda_1 > \lambda_2.$$

Proof. Let $\hat{x} \in \hat{\Omega}_\epsilon^n$ and let

$$E_n := f_{\hat{x}_n}^{-n}(B_{x_n}(\frac{1}{4m_0})).$$

The inverse branch $f_{\hat{x}_n}^{-n}$ is well defined on $B_{x_n}(\frac{1}{4m_0})$ since $\hat{x}_n \in \hat{\Omega}_\epsilon$. Let g_n be the restriction of f^n on E_n. By using $f_{\hat{x}_n}^{-n} \circ g_n = Id_{E_n}$ and $T = \frac{1}{d^n} g_n^* T$ on E_n, we obtain

$$\begin{aligned} T \wedge \frac{i}{2} dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon} &= \frac{1}{d^n} g_n^* T \wedge g_n^*(f_{\hat{x}_n}^{-n})^*(\frac{i}{2} dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}) \\ &= \frac{1}{d^n} g_n^* \left[T \wedge \frac{i}{2}(dZ_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n})) \wedge d(\overline{Z_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n})})\right] \end{aligned}$$

on the open subset E_n. Now we use (2.1) to write $Z_{\hat{x}}^\epsilon \circ (f_{\hat{x}_n}^{-n}) = \alpha_{n,\hat{x}_n} Z_{\hat{x}_n}$. Since $|\alpha_{n,\hat{x}_n}|^2 \geq e^{-2n\lambda_1 - 2n\epsilon}$, we get on E_n:

$$T \wedge \frac{i}{2} dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon} \geq \frac{1}{d^n} e^{-2n\lambda_1 - 2n\epsilon} g_n^* \left[T \wedge \frac{i}{2} dZ_{\hat{x}_n}^\epsilon \wedge d\overline{Z_{\hat{x}_n}^\epsilon}\right]. \tag{7.2}$$

We are going to bound from above the left hand side and to bound from below the right hand side (applied to E_n). Using Lemma 2.3 with $r = 1/m_0$ and $n \geq L_0$ we obtain $E_n \subset B_x\left(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\right)$, hence

$$T \wedge \frac{i}{2} dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}\left(B_x\left(\frac{1}{m_0} e^{-n\lambda_2 + 3n\epsilon}\right)\right) \geq T \wedge \frac{i}{2} dZ_{\hat{x}}^\epsilon \wedge d\overline{Z_{\hat{x}}^\epsilon}(E_n) \tag{7.3}$$

For the right hand side, since g_n is injective on E_n and $g_n(E_n) = B_{x_n}\left(\frac{1}{4m_0}\right)$, we get

$$g_n^*\left[T \wedge \frac{i}{2}(dZ^\epsilon_{\hat{x}_n} \wedge \overline{dZ^\epsilon_{\hat{x}_n}})\right](E_n) = \left[T \wedge \frac{i}{2}(dZ^\epsilon_{\hat{x}_n} \wedge (\overline{dZ^\epsilon_{\hat{x}_n}})\right](B_{x_n}(\frac{1}{4m_0})) \geq \frac{1}{q_0}, \qquad (7.4)$$

where the inequality comes from $\hat{x}_n \in \hat{\Omega}_\epsilon$. By combining (7.2), (7.3) and (7.4) we obtain

$$\left[T \wedge \frac{i}{2}dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right]\left(B_x\left(\frac{1}{m_0}e^{-n\lambda_2+3n\epsilon}\right)\right) \geq \frac{1}{d^n}e^{-2n\lambda_1-2n\epsilon}\frac{1}{q_0}.$$

We use $W^\epsilon_{\hat{x}} \circ (f^{-n}_{\hat{x}_n}) = \beta_{n,\hat{x}} W_{\hat{x}_n}$ et $|\beta_{n,\hat{x}}|^2 \geq e^{-2n\lambda_2-2n\epsilon}$ to prove the other lower bound. □

7.2 Proof of Theorem 1.2

We take the notations of Section 7.1. Let

$$\hat{\Omega}_\epsilon := \limsup_{n\in\mathbb{N}} \hat{\Omega}_\epsilon \cap \hat{f}^{-n}(\hat{\Omega}_\epsilon) = \limsup_{n\in\mathbb{N}} \hat{\Omega}^n_\epsilon.$$

We have $\hat{\nu}(\hat{\Omega}_\epsilon) \geq 1 - 2\delta$ according to (7.1). Let $\hat{x} \in \hat{\Omega}_\epsilon$. Then there exists an increasing sequence of intergers $(l_p)_p$ such that

$$\hat{x} \in \hat{\Omega}_\epsilon \cap \hat{f}^{-l_p}(\hat{\Omega}_\epsilon) = \hat{\Omega}^{l_p}_\epsilon$$

for every $p \geq 0$. Proposition 7.3 then asserts for p large enough:

$$\left[T \wedge \frac{i}{2}dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right]\left(B_x\left(\frac{1}{m_0}e^{-l_p\lambda_2+3l_p\epsilon}\right)\right) \geq \frac{1}{d^{l_p}}e^{-2l_p\lambda_1-2l_p\epsilon}\frac{1}{q_0}.$$

If p is also large enough so that $e^{l_p\epsilon} \geq \frac{1}{m_0}$ and $\frac{1}{q_0} \geq e^{-l_p\epsilon}$, we obtain with $r_p := e^{-l_p(\lambda_2-4\epsilon)}$:

$$\left[T \wedge \frac{i}{2}dZ^\epsilon_{\hat{x}} \wedge d\overline{Z^\epsilon_{\hat{x}}}\right](B_x(r_p)) \geq e^{-l_p(\log d+2\lambda_1+3\epsilon)} = r_p^{(\log d+2\lambda_1+3\epsilon)/(\lambda_2-4\epsilon)}.$$

Since $(r_p)_p$ tends to 0 and $\hat{\nu}(\hat{\Omega}_\epsilon) > 0$, we get

$$\underline{d_{T,Z}}(\nu) \leq \frac{\log d + 2\lambda_1 + 3\epsilon}{\lambda_2 - 4\epsilon} =: \frac{\log d}{\lambda_2} + 2\frac{\lambda_1}{\lambda_2} + O_3(\epsilon).$$

One can prove

$$\underline{d_{T,W}}(\nu) \leq \frac{\log d + 2\lambda_2 + 3\epsilon}{\lambda_2 - 4\epsilon} =: \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon)$$

in a similar way.

7.3 Monge-Ampère mass

We prove Proposition 7.1. Let $x \in \operatorname{Supp}\mu$, let V be a neighbourhood of x and let $Z : V \to \mathbb{C}$ be a holomorphic coordinate (submersion) on V. We want to prove that the positive measure $T \wedge \frac{i}{2} dZ \wedge d\overline{Z}$ is not the zero measure on V. With no loss of generality, we can assume that $x = (0,0)$, $V = \mathbb{D}(2) \times \mathbb{D}(2)$ and $Z(z,w) = z$. Let also $T = 2i\partial\overline{\partial}G$ on V, where G is a continuous psh function. We denote $\sigma_z(u) := (z,u)$.

Lemma 7.4. *If $(T \wedge \frac{i}{2} dZ \wedge d\overline{Z})(\mathbb{D}(2) \times \mathbb{D}(2)) = 0$, then $G \circ \sigma_z$ is harmonic on $\mathbb{D}$ for every $z \in \mathbb{D}$.*

Proof. Let $z_0 \in \mathbb{D}$ and let $\varphi \in C_0^\infty(\mathbb{D})$ be a test function. Let $\psi \in C_0^\infty(\mathbb{D}^2)$ such that $\psi \circ \sigma_{z_0} = \varphi$ on $\mathbb{D}$. According to Proposition 9.4, we have

$$\left(T \wedge \frac{i}{2} dZ \wedge d\overline{Z}\right)(\psi) = \int_{z\in\mathbb{D}} \left(\int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \, \mathrm{d}\,\mathrm{Leb}(w) \right) \mathrm{d}\,\mathrm{Leb}(z),$$

which is equal to zero by our assumption. Since the measurable function

$$z \mapsto \int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \, \mathrm{d}\,\mathrm{Leb}(w)$$

is non negative, there exists $A \subset \mathbb{D}$ such that $\mathrm{Leb}(A) = \mathrm{Leb}(\mathbb{D})$ and

$$\forall z \in A, \quad \int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \, \mathrm{d}\,\mathrm{Leb}(w) = 0. \tag{7.5}$$

Let us extend this property to every $z \in \mathbb{D}$. Since A is a dense subset of $\mathbb{D}$, there exists a sequence $(z_n)_n$ of points in A which converges to z. Using (7.5), we get

$$\forall n \geq 1, \quad \int_{w\in\mathbb{D}} (G \circ \sigma_{z_n})(w) \times \Delta(\psi \circ \sigma_{z_n})(w) \, \mathrm{d}\,\mathrm{Leb}(w) = 0. \tag{7.6}$$

Since G is continuous on $\overline{\mathbb{D}^2}$ and ψ is smooth on $\overline{\mathbb{D}^2}$, G and $2i\partial\overline{\partial}\psi$ are uniformly continuous on $\overline{\mathbb{D}^2}$. This implies that $G \circ \sigma_{z_n}$ uniformly converges to $G \circ \sigma_z$ on $\mathbb{D}$ and that $\Delta(\psi \circ \sigma_{z_n})$ uniformly converges to $\Delta(\psi \circ \sigma_z)$ on $\mathbb{D}$. Taking the limits in (7.6), we get

$$\forall z \in \mathbb{D}, \quad \int_{w\in\mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \, \mathrm{d}\,\mathrm{Leb}(w) = 0.$$

In particular, we obtain using $\psi \circ \sigma_{z_0} = \varphi$:

$$\int_{w\in\mathbb{D}} (G \circ \sigma_{z_0})(w) \times \Delta\varphi(w) \, \mathrm{d}\,\mathrm{Leb}(w) = 0.$$

This holds for every $\varphi \in C_0^\infty(\mathbb{D})$, hence the function $G \circ \sigma_{z_0}$ is harmonic on $\mathbb{D}$. □

Now we use the following result, see [5, Lemme IV.1.1] and [24, Section A.10].

Theorem 7.5 (Briend)**.** *Let G be a continuous psh function on $\mathbb{D}(2) \times \mathbb{D}(2)$. Let E be the set of points $p \in \mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4})$ such that there exists a holomorphic disc $\sigma_p : \mathbb{D} \to \mathbb{D}(2) \times \mathbb{D}(2)$ satisfying*

1. *the boundary of* σ_p *is outside* $\mathbb{D}(\frac{1}{2}) \times \mathbb{D}(\frac{1}{2})$,

2. $G \circ \sigma_p$ *is harmonic* $\mathbb{D}$.

Then $(2i\partial\overline{\partial}G \wedge 2i\partial\overline{\partial}G)(E) = 0$.

In our situation, $\mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4}) = E$ since one can take for σ_p the discs $\sigma_z : \mathbb{D} \to \mathbb{D} \times \mathbb{D},\ u \mapsto (z, u)$. Indeed, the boundary of σ_z is contained in $\{z\} \times \partial\mathbb{D}$ and $G \circ \sigma_z$ is harmonic on $\mathbb{D}$ according to Lemma 7.4. Theorem 7.5 then gives:

$$(2i\partial\overline{\partial}G \wedge 2i\partial\overline{\partial}G)(\mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4})) = 0, \tag{7.7}$$

which contradicts $x = 0 \in \operatorname{Supp}\mu = \operatorname{Supp}(2i\partial\overline{\partial}G \wedge 2i\partial\overline{\partial}G)$.

8 Upper bound for the dimension of dilating measures

We prove Theorem 1.5. We shall take the proof of Theorem 5.2 and use Proposition 7.3. Let $\epsilon > 0$ and let $\hat{\Lambda}_\epsilon$ and $\hat{\Delta}_\epsilon^n$ be the sets defined in Section 3.3. The set $\hat{\Omega}_\epsilon^n$ has been defined in Section 7.1, it satisfies $\hat{\nu}(\hat{\Omega}_\epsilon^n) \geq 1-2\delta$. Hence we have $\hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n \cap \hat{\Omega}_\epsilon^n) \geq 1-3\delta$ for every $n \geq N_\epsilon$. Now let K_n be the unique integer satisfying

$$\eta_1 e^{-n\lambda_1 - 4n\epsilon} e^{-\lambda_2 + 3\epsilon} \leq \frac{1}{m_0} e^{-K_n\lambda_2 + 3K_n\epsilon} \leq \eta_1 e^{-n\lambda_1 - 4n\epsilon}. \tag{8.1}$$

so that $K_n \simeq n\lambda_1/\lambda_2$. Let $\{x_1, \cdots, x_{N_{n,2}}\}$ be a $(n, \eta_1/4)$-separated subset of $A_n := \pi_0(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n \cap \hat{\Omega}_\epsilon^{K_n})$ provided by Lemma 4.2. We have for every $n \geq \max\{N_\epsilon, n_{1-3\delta}\}$:

$$N_{n,2} \geq \nu(A_n) e^{nh_\nu - 2n\epsilon} \geq (1-3\delta) e^{nh_\nu - 2n\epsilon}. \tag{8.2}$$

Then, for every x_i, let $\{y_1^i, \ldots, y_{M_n}^i\}$ be a $2e^{-n\lambda_1 - 4n\epsilon}$-separated subset of $B_n(x_i, \eta_1/4) \cap A_n$ provided by Lemma 4.3. The cardinality of this set satisfies:

$$M_n \geq e^{-nh_\nu - 2n\epsilon} \left(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon} \right)^{\underline{d_\nu} - \epsilon}. \tag{8.3}$$

For every $j \in \{1, \ldots, M_n\}$, we set $\hat{y}_j^i \in \hat{\Lambda}_\epsilon \cap \hat{\Delta}_\epsilon^n \cap \hat{\Omega}_\epsilon^{K_n}$ such that $y_j^i = \pi_0(\hat{y}_j^i)$. Then we follow the proof of Theorem 5.2 until the inequality (5.8):

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 4n\epsilon} \left[T \wedge (\frac{i}{2} dZ_{\hat{y}_j^i}^\epsilon \wedge d\overline{Z_{\hat{y}_j^i}^\epsilon}) \right] (B_{y_j^i}(\eta_1 e^{-n\lambda_1 - 4n\epsilon})). \tag{8.4}$$

We want to apply Proposition 7.3. According to (8.1),

$$B_{y_j^i}(\eta_1 e^{-n\lambda_1 - 4n\epsilon}) \supset B_{y_j^i}\left(\frac{1}{m_0} e^{-K_n\lambda_2 + 3K_n\epsilon} \right).$$

We apply the positive measure $T \wedge (\frac{i}{2} dZ^{\epsilon}_{\hat{y}^i_j} \wedge d\overline{Z^{\epsilon}_{\hat{y}^i_j}})$ to this inclusion. Since $\hat{y}^i_j \in \hat{\Omega}^{K_n}_{\epsilon}$, we deduce from Proposition 7.3 that for every n satisfying $n \geq N_\epsilon$ and $K_n \geq N_\epsilon$:

$$\left[T \wedge (\frac{i}{2} dZ^{\epsilon}_{\hat{y}_{i,j}} \wedge d\overline{Z^{\epsilon}_{\hat{y}_{i,j}}})\right] (B_{y^i_j}(\eta_1 e^{-n\lambda_1 - 4n\epsilon})) \geq \frac{1}{d^{K_n}} e^{-2K_n\lambda_1 - 2K_n\epsilon} \frac{1}{q_0}.$$

We infer from (8.4) that for every n satisfying $n \geq N_\epsilon$ and $K_n \geq N_\epsilon$,

$$d^n \geq N_{n,2} \cdot M_n \cdot e^{2n\lambda_1 - 4n\epsilon} \frac{1}{d^{K_n}} e^{-2K_n\lambda_1 - 2K_n\epsilon} \frac{1}{q_0}. \tag{8.5}$$

Now we use the upper bounds for $N_{n,2}$ and M_n given by (8.2) and (8.3):

$$d^{n+K_n} \geq (1-3\delta) e^{nh_\nu - 2n\epsilon} \cdot e^{-nh_\nu - 2n\epsilon} \left(\frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon}\right)^{\underline{d_\nu} - \epsilon} \cdot e^{2n\lambda_1 - 4n\epsilon} e^{-2K_n\lambda_1 - 2K_n\epsilon} \frac{1}{q_0}.$$

If $C_1(\epsilon) := (1-3\delta)/q_0 (2\eta_1)^{\underline{d_\nu} - \epsilon}$, we get:

$$\log d + \frac{K_n}{n} \log d \geq \frac{1}{n} \log C_1(\epsilon) - 8\epsilon + (\lambda_1 + 4\epsilon)(\underline{d_\nu} - \epsilon) + 2\lambda_1 - 2\frac{K_n}{n}(\lambda_1 + \epsilon).$$

By using (8.1), we have

$$\log d + \frac{\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon} \log d \geq \frac{1}{n} \log C_2(\epsilon) - 8\epsilon + (\lambda_1 + 4\epsilon)(\underline{d_\nu} - \epsilon) + 2\lambda_1 - 2\frac{\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon}(\lambda_1 + \epsilon),$$

where $C_2(\epsilon)$ is another constant. Letting n tend to $+\infty$ and then ϵ to 0, we get

$$\underline{d_\nu} \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\left(\frac{\lambda_1}{\lambda_2} - 1\right).$$

To obtain the other upper bound, we use the analogue of (8.4) for W. Applying Proposition 7.3 with respect to W, we obtain instead of (8.5):

$$d^n \geq N_{n,2} \cdot M_n \cdot e^{2n\lambda_2 - 4n\epsilon} \frac{1}{d^{K_n}} e^{-2K_n\lambda_2 - 2K_n\epsilon} \frac{1}{q_0}.$$

Then we get

$$\underline{d_\nu} \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2\left(1 - \frac{\lambda_2}{\lambda_1}\right),$$

which completes the proof of Theorem 1.5.

9 Appendix

9.1 Dimension of measures

Proposition 9.1. *Let ν_1 and ν_2 be two probability measures on $\mathbb{P}^2$ such that $\nu_1 << \nu_2$. Then for ν_1-almost every $x \in \mathbb{P}^2$, we have:*

$$\underline{d_{\nu_1}}(x) \geq \underline{d_{\nu_2}}(x) \quad \text{and} \quad \overline{d_{\nu_1}}(x) \geq \overline{d_{\nu_2}}(x).$$

Proof. Let $\varphi \in L^1(\nu_2)$ such that $\nu_1(A) = \int_A \varphi\nu_2$ for every Borel set A of $\mathbb{P}^2$. Using the dominated convergence Theorem,

$$\lim_{M\to\infty} \int_{\mathbb{P}^2} 1_{\{\varphi\leq M\}}\varphi \, \mathrm{d}\nu_2 = \int_{\mathbb{P}^2} \varphi \, \mathrm{d}\nu_2 = 1.$$

For every $n \geq 1$, we let M_n satisfy $\int_{\mathbb{P}^2} 1_{\{\varphi\leq M_n\}}\varphi d\nu_2 \geq 1 - \frac{1}{n}$. By the Lebesgue density Theorem, for ν_1-almost every x in $\{\varphi \leq M_n\}$, we have

$$\lim_{r\to 0} \frac{\nu_1(B_x(r) \cap \{\varphi \leq M_n\})}{\nu_1(B_x(r))} = 1.$$

Then for every r small enough, we have

$$\frac{1}{2}\nu_1(B_x(r)) \leq \nu_1(B_x(r) \cap \{\varphi \leq M_n\}) = \int_{B_x(r)\cap\{\varphi\leq M_n\}} \varphi \, d\nu_2 \leq M_n \int_{B_x(r)} d\nu_2.$$

And thus $\nu_1(B_x(r)) \leq 2M_n\nu_2(B_x(r))$. We deduce that

$$\underline{d_{\nu_1}}(x) \geq \underline{d_{\nu_2}}(x) \text{ and } \overline{d_{\nu_1}}(x) \geq \overline{d_{\nu_2}}(x)$$

for ν_1-almost every $x \in \{\varphi \leq M_n\}$. We end with $\nu_1(\cup_{n\in\mathbb{N}} \{\varphi \leq M_n\}) = 1$. □

Now we take the notations of Section 1.1.

Proposition 9.2. *Let S be a $(1,1)$-closed positive current on $\mathbb{P}^2$. Let $x \in \mathbb{P}^2$ and let (Z, W) be holomorphic coordinates near x. Then*

$$\underline{d_S}(x) = \min\left\{\underline{d_{S,Z}}(x), \underline{d_{S,W}}(x)\right\}, \quad \overline{d_S}(x) = \min\left\{\overline{d_{S,Z}}(x), \overline{d_{T,W}}(x)\right\}.$$

Proof. Let us set $\sigma_{S,Z} = S \wedge (\frac{i}{2}dZ \wedge d\overline{Z})$ and $\sigma_{S,W} = S \wedge (\frac{i}{2}dW \wedge d\overline{W})$. There exists $c > 0$ such that $\frac{1}{c}(\sigma_{S,Z} + \sigma_{S,W}) \leq \sigma_S \leq c(\sigma_{S,Z} + \sigma_{S,W})$ on a neighbourhood of x, see [13] Chapter III, §3. We deduce for every r small enough

$$\frac{1}{c}\max\left[\sigma_{S,Z}(B_x(r)), \sigma_{S,W}(B_x(r))\right] \leq \sigma_S(B_x(r)) \leq 2c\max\left[\sigma_{S,Z}(B_x(r)), \sigma_{S,W}(B_x(r))\right].$$

We finish by observing that the local dimension of the maximum of two measures is equal to the minimum of these two dimensions, since one divides by $\log r$ which is negative. □

9.2 Cohomology and slices

We refer to Sections 1.2 and A.3 of Dinh-Sibony's book [16].

Proposition 9.3. *Let S be a $(1,1)$-closed positive current of $\mathbb{P}^2$ of mass 1. Let ω be the Fubini-Study form on $\mathbb{P}^2$ and let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be an endomorphism of degree d. Then,*

$$\int_{\mathbb{P}^2} (f^n)_* S \wedge \omega = \int_{\mathbb{P}^2} S \wedge (f^n)^*\omega = d^n.$$

Proof. The first equality comes from the definition of duality. We show the second one. By using $f^*\omega = d\cdot\omega + 2i\partial\overline{\partial}u$, where u is a smooth function on $\mathbb{P}^2$, we obtain by induction

$$(f^n)^*\omega = d^n\omega + 2i\partial\overline{\partial}v_n,$$

where $v_n := (d^{n-1}\cdot u + \cdots + d\cdot u\circ f^{n-2} + u\circ f^{n-1})$. Hence

$$\int_{\mathbb{P}^2} S\wedge(f^n)^*\omega = \int_{\mathbb{P}^2} S\wedge\left(d^n\omega + 2i\partial\overline{\partial}v_n\right).$$

Since S is a closed current of mass 1, we have $\int_{\mathbb{P}^2} S\wedge 2i\partial\overline{\partial}v_n = 0$ and $\int_{\mathbb{P}^2} S\wedge d^n\omega = d^n$. □

Proposition 9.4. *Let G be a continuous psh function on $\mathbb{D}^2$ and let $S = 2i\partial\overline{\partial}G$. Let (Z,W) be the coordinates on $\mathbb{D}^2$ and let $\phi\in C_0^\infty(\mathbb{D}^2)$. Then*

$$S\wedge\frac{i}{2}dZ\wedge d\overline{Z}(\phi) = \int_{z\in\mathbb{D}}\left(\int_{w\in\mathbb{D}} G_z(w)\times\Delta\phi_z(w)\ d\,\mathrm{Leb}(w)\right)\ d\,\mathrm{Leb}(z) = \int_{z\in\mathbb{D}}(\sigma_z^*S)(\phi_z)d\,\mathrm{Leb}(z),$$

where $\sigma_z : u\mapsto(z,u)$, $G_z := G\circ\sigma_z$ and $\phi_z := \phi\circ\sigma_z$.

Proof. By definition,

$$S\wedge\frac{i}{2}dZ\wedge d\overline{Z}(\phi) = 2i\partial\overline{\partial}G(\phi\frac{i}{2}dZ\wedge d\overline{Z}) = \int_{\mathbb{D}^2} G.2i\partial\overline{\partial}(\phi\times\frac{i}{2}dZ\wedge d\overline{Z}).$$

The computation

$$2i\partial\overline{\partial}(\phi\times\frac{i}{2}dZ\wedge d\overline{Z}) = 4(\frac{\partial^2\phi}{\partial w\partial\overline{w}})\frac{i}{2}dW\wedge d\overline{W}\wedge\frac{i}{2}dZ\wedge d\overline{Z} = 4(\frac{\partial^2\phi}{\partial w\partial\overline{w}})\ \mathrm{d}\,\mathrm{Leb}(z,w)$$

allows to write

$$\begin{aligned} S\wedge\frac{i}{2}dZ\wedge d\overline{Z}(\phi) &= \int_{(z,w)\in\mathbb{D}^2} G(z,w)\times 4\frac{\partial^2\phi}{\partial w\partial\overline{w}}(z,w)\ \mathrm{d}\,\mathrm{Leb}(z,w)\\ &= \int_{z\in\mathbb{D}}\left(\int_{w\in\mathbb{D}} G_z(w)\times\Delta\phi_z(w)\ \mathrm{d}\,\mathrm{Leb}(w)\right)\ \mathrm{d}\,\mathrm{Leb}(z). \end{aligned}$$

Finally, the quantity in brackets is equal to $(\Delta G_z)(\phi_z) = (\sigma_z^*S)(\phi_z)$. □

References

[1] F. Berteloot and C. Dupont. Une caractérisation des endomorphismes de Lattès par leur mesure de Green. *Comment. Math. Helv.*, 80(2):433–454, 2005.

[2] F. Berteloot, C. Dupont, and L. Molino. Normalization of bundle holomorphic contractions and applications to dynamics. *Ann. Inst. Fourier (Grenoble)*, 58(6):2137–2168, 2008.

[3] F. Berteloot and J.-J. Loeb. Une caractérisation géométrique des exemples de Lattès de $\mathbb{P}^k$. *Bull. Soc. Math. France*, 129(2):175–188, 2001.

[4] I. Binder and L. DeMarco. Dimension of pluriharmonic measure and polynomial endomorphisms of $\mathbb{C}^n$. *Int. Math. Res. Not.*, 2003(11):613–625, 2003.

[5] J.-Y. Briend. *Exposants de Liapounoff et points périodiques d'endomorphismes holomorphes de CP(k).* PhD thesis, Université Paul Sabatier de Toulouse, 1997.

[6] J.-Y. Briend and J. Duval. Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de $\mathbf{C}\mathrm{P}^k$. *Acta Math.*, 182(2):143–157, 1999.

[7] J.-Y. Briend and J. Duval. Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathrm{P}^k(\mathbf{C})$. *Publ. Math. Inst. Hautes Études Sci.*, 2001(93):145–159, 2001.

[8] M. Brin and A. Katok. On local entropy. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 30–38. Springer, Berlin, 1983.

[9] H. de Thélin. Minoration de la dimension de hausdorff du courant de Green. https://arxiv.org/abs/1709.01356.

[10] H. de Thélin. Un phénomène de concentration de genre. *Math. Ann.*, 332(3):483–498, 2005.

[11] H. de Thélin. Sur les exposants de Lyapounov des applications méromorphes. *Invent. Math.*, 172(1):89–116, 2008.

[12] H. de Thélin and G. Vigny. On the measures of large entropy on a positive closed current. *Math. Z.*, 280(3-4):919–944, 2015.

[13] J.-P. Demailly. *Complex Analytic and Differential Geometry.* En ligne, 2012. disponible à https://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf.

[14] T.-C. Dinh. Attracting current and equilibrium measure for attractors on $\mathbb{P}^k$. *J. Geom. Anal.*, 17(2):227–244, 2007.

[15] T.-C. Dinh and C. Dupont. Dimension de la mesure d'équilibre d'applications méromorphes. *J. Geom. Anal.*, 14(4):613–627, 2004.

[16] T.-C. Dinh and N. Sibony. *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, volume 1998 of *Lecture Notes in Math.*, pages 165–294. Springer, Berlin, 2010.

[17] R. Dujardin. Fatou directions along the Julia set for endomorphisms of $\mathbb{CP}^k$. *J. Math. Pures Appl. (9)*, 98(6):591–615, 2012.

[18] C. Dupont. Exemples de Lattès et domaines faiblement sphériques de $\mathbb{C}^n$. *Manuscripta Math.*, 111(3):357–378, 2003.

[19] C. Dupont. On the dimension of invariant measures of endomorphisms of $\mathbb{CP}$(k). *Math. Ann.*, 349(3):509–528, 2011.

[20] C. Dupont. Large entropy measures for endomorphisms of $\mathbb{CP}^k$. *Israel J. Math.*, 192(2):505–533, 2012.

[21] M. Jonsson and D. Varolin. Stable manifolds of holomorphic diffeomorphisms. *Invent. Math.*, 149:409–430, August 2002.

[22] R. Mañé. The Hausdorff dimension of invariant probabilities of rational maps. In *Dynamical systems, Valparaiso 1986*, volume 1331 of *Lecture Notes in Math.*, pages 86–117. Springer, Berlin, 1988.

[23] Y. B. Pesin. *Dimension theory in dynamical systems.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997. Contemporary views and applications.

[24] N. Sibony. Dynamique des applications rationnelles de $\mathbf{P}^k$. In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.

[25] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[26] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynamical Systems*, 2(1):109–124, 1982.

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