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SOLUTION BY CONVEX MINIMIZATION OF THE CAUCHY PROBLEM FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS WITH CONVEX ENTROPY

YANN BRENIER

ABSTRACT

We show that, for first-order systems of conservation laws with a strictly convex entropy, in particular for the very simple so-called "inviscid" Burgers equation, it is possible to address the Cauchy problem by a suitable convex minimization problem, quite similar to some problems arising in optimal transport or variational mean-field game theory. In the general case, we show that smooth, shock-free, solutions can be recovered on some sufficiently small interval of time. In the special situation of the Burgers equation, we further show that every "entropy solution" (in the sense of Kruzhkov) including shocks, can be recovered, for arbitrarily long time intervals.

Keywords. Partial differential evolution equations, calculus of variations, conservation laws, entropy solutions, convex optimization, optimal transport, mean-field games.

INTRODUCTION

Solving Cauchy problems by convex minimization techniques is definitely not a new idea, in particular in the framework of linear evolution PDEs, as illustrated by the classical least square method. For instance, in the case of a linear transport equation such as $\partial_t u + \partial_x u = 0$, one can try to minimize

$$\int \int (\partial_t u + \partial_x u)^2 dx dt, \quad u(0, \cdot) = u_0,$$

where u_0 is the initial condition. This typically leads to (degenerate) space-time elliptic problems. In the framework of nonlinear equations, similar strategies can be used but usually lead to non-convex ill-conditioned minimization problems. In this paper, we discuss a somewhat different strategy for the special class of "systems of conservation laws with convex entropy" [9, 15], namely systems of first order evolution PDEs of form

$$(0.1) \quad \partial_t U^\alpha + \partial_i(\mathcal{F}^{i\alpha}(U)) = 0, \quad \alpha = 1, \dots, m,$$

(with implicit summation on repeated indices), where $U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m$, $t \in [0, T]$, $x \in \mathcal{D} = (\mathbb{R}/\mathbb{Z})^d$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$, \mathcal{W} is a smooth convex open set, while the so-called "flux" $\mathcal{F} : \mathcal{W} \rightarrow \mathbb{R}^{d \times m}$ is a smooth function enjoying the symmetry property

$$(0.2) \quad \forall i \in \{1, \dots, n\}, \quad \forall \beta, \gamma \in \{1, \dots, m\}, \quad \partial_{\alpha\beta}^2 \mathcal{E} \partial_\gamma \mathcal{F}^{i\alpha} = \partial_{\alpha\gamma}^2 \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha},$$

for some smooth function, called "entropy", $\mathcal{E} : \mathcal{W} \rightarrow \mathbb{R}$, which is supposed to be convex in the strong sense that $(\partial_{\alpha\beta}^2 \mathcal{E})$ is a positive definite matrix, everywhere on \mathcal{W} .

The symmetry condition (0.2) enforces the conservation of entropy, in the sense that every smooth solution U to (0.1) must satisfy the extra-conservation law

$$(0.3) \quad \partial_t(\mathcal{E}(U)) + \partial_i(\mathcal{Q}^i(U)) = 0,$$

where the "entropy-flux" function $\mathcal{Q} : \mathcal{W} \rightarrow \mathbb{R}^d$ depends on \mathcal{F} and \mathcal{E} . Indeed, (0.2) is equivalent to

$$\partial_\gamma(\partial_\alpha \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha}) = \partial_\beta(\partial_\alpha \mathcal{E} \partial_\gamma \mathcal{F}^{i\alpha})$$

which means that $\partial_\alpha \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha}$ is the gradient of some smooth function $Q^i : \mathcal{W} \rightarrow \mathbb{R}$, namely $\partial_\alpha \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha} = \partial_\beta Q^i$. So, every smooth solution U of system (0.1) satisfies

$$-\partial_t(\mathcal{E}(U)) = \partial_\alpha \mathcal{E}(U) \partial_i(\mathcal{F}^{i\alpha}(U)) = \partial_\alpha \mathcal{E}(U) \partial_\beta \mathcal{F}^{i\alpha}(U) \partial_i U^\beta = \partial_\beta \mathcal{Q}^i(U) \partial_i U^\beta = \partial_i(\mathcal{Q}^i(U)),$$

which exactly is (0.3).

This important class of PDEs contains many classical models in continuum mechanics and material sciences (Euler equations of hydrodynamics, Elastodynamics with polyconvex energy, ideal Magnetohydrodynamics, etc...[9, 12, 15, 17]). The simplest example, is, of course, the celebrated and so-called "inviscid" Burgers equation

$$(0.4) \quad \partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad u \in \mathbb{R}, \quad \mathcal{E}(u) = \frac{u^2}{2},$$

A richer example is the Euler equation of isothermal fluids:

$$(0.5) \quad \partial_t \rho + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left(\frac{q \otimes q}{\rho} \right) + \nabla \rho = 0, \quad \mathcal{E} = \frac{|q|^2}{2\rho} + \rho \log \rho, \quad \rho > 0, \quad q \in \mathbb{R}^d.$$

Under mild additional conditions, PDEs of that class are (locally) well-posed [typically in Sobolev spaces H^s for $s > d/2 + 1$ [9, 15]]. Interestingly enough, in most cases, smooth solutions develop singularities, called "shock waves", in finite time, and they cease to be continuous. Thus, it is interesting to consider weak solutions U , for which the initial value problem with initial condition U_0 means

$$(0.6) \quad \int_{[0,T] \times \mathcal{D}} \partial_t W_\alpha U^\alpha + \partial_i W_\alpha F^{i\alpha}(U) + \int_{\mathcal{D}} W_\alpha(0, \cdot) U_0^\alpha = 0,$$

for all smooth functions $W = W(t, x) \in \mathbb{R}^m$ such that $W(T, \cdot) = 0$. Weak solutions are not expected to satisfy the extra-conservation law (0.3). In particular

$$t \in [0, T] \rightarrow \int_{\mathcal{D}} \mathcal{E}(U(t, \cdot))$$

is not expected to be constant in t . Whenever a weak solution satisfies the "entropy inequality" in the sense of distributions

$$(0.7) \quad \partial_t(\mathcal{E}(U)) + \partial_i(\mathcal{Q}^i(U)) \leq 0,$$

it is called a "weak entropy solutions". The concept of weak solutions is quite faulty since, for a fixed initial condition U_0 , weak solutions may not be unique and the conservation of entropy is generally not true. (This is now well established in the case of the Euler equations, in Hydrodynamics, through the results of Scheffer, Shnirelman, De Lellis-Székelyhidi Jr. [16, 18, 10, 20].) However, a weak solution \tilde{U} cannot differ from a smooth

solution U given on $[0, T] \times \mathcal{D}$ as long as

$$\int_{\mathcal{D}} \mathcal{E}(\tilde{U}(t, \cdot)) \leq \int_{\mathcal{D}} \mathcal{E}(U(0, \cdot)), \quad a.e. t \in [0, T],$$

which is certainly true for weak entropy solutions, because of (0.7).

The main idea of our paper is to look, given an initial condition U_0 , for weak solutions that minimize over $[0, T]$ the time integral of their entropy. As explained above, this problem is not void since weak solutions may not be unique and do not conserve their entropy. Using the trial functions W in (0.6) as Lagrange multipliers, we get the following min-max problem

$$(0.8) \quad I = \inf_U \sup_W \int_{[0, T] \times \mathcal{D}} E(U) - \partial_t W_\alpha U^\alpha - \partial_i W_\alpha F^{i\alpha}(U) - \int_{\mathcal{D}} W_\alpha(0, \cdot) U_0^\alpha,$$

where $W = W(t, x) \in \mathbb{R}^m$ are smooth functions, vanishing at $t = T$. This indeed amounts to looking for a weak solution U of our system of conservation laws with initial condition U_0 that minimizes the time integral of its entropy. Let us now exchange the infimum and the supremum in the definition of I and get the lower bound

$$(0.9) \quad J = \sup_W \inf_U \int_{[0, T] \times \mathcal{D}} E(U) - \partial_t W_\alpha U^\alpha - \partial_i W_\alpha F^{i\alpha}(U) - \int_{\mathcal{D}} W_\alpha(0, \cdot) U_0^\alpha$$

which can be reduced to the concave maximization problem

$$(0.10) \quad J = \sup_W \int_{[0, T] \times \mathcal{D}} -K(\partial_t W, DW) - \int_{\mathcal{D}} W_\alpha(0, \cdot) U_0^\alpha$$

where W is still subject to $W(T, \cdot) = 0$ and K is the *convex* function defined by

$$(0.11) \quad K(A, B) = \sup_{V \in \mathcal{W}} A_\alpha V^\alpha + B_{\alpha i} F^{i\alpha}(V) - \mathcal{E}(V), \quad A \in \mathbb{R}^m, \quad B \in \mathbb{R}^{m \times d}.$$

This concave maximization problem is somewhat similar to the Monge optimal mass transport problem with quadratic cost in its so-called "Benamou-Brenier" formulation [3, 1, 19]. [This is particularly true in the case of the "inviscid" Burgers equation, as will be seen in section 2.]

In the first part of the paper (section 1) we establish that smooth solutions of general systems of conservation laws with convex entropy can be recovered at least for short enough intervals of time. In the special case of the "inviscid" Burgers equation, the shortness condition is sharp in the sense that it exactly corresponds to the formation of a shock. Thus it is tempting to investigate the possibility of recovering solutions beyond the formation of shocks. At first glance, this seems difficult. Indeed, solutions of a concave space-time maximization problem are expected to enjoy some (limited) elliptic regularity, which does not seem compatible with shocks. We leave this question open in the general case. Nevertheless, in the second part of this paper (sections 2 and 3), we are able to prove, in the very elementary case of the "inviscid" Burgers equation, that, indeed, solutions with shocks (more precisely, "entropy solutions", in the sense of Kruzhkov, which are known to be unique for each given initial condition [9, 12, 15, 17]) solve the maximization problem in a suitable sense. This result is obtained by combining convex duality arguments (as in [3]), of optimal transport type and related to mean-field games (for which we refer to [1, 3, 8, 13, 19]), and classical properties of entropy solutions and Hamilton-Jacobi

equations (for which we refer to [2, 9, 6, 12, 15, 17]). Finally, let us mention our previous work [4] where the Euler equations of incompressible fluids [14] have been investigated with similar ideas.

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1. RECOVERY OF SMOOTH SOLUTIONS

Theorem 1.1. *Let U be a smooth solution of the entropic system of conservation laws (0.1). Then, as long as $T > 0$ is not too large, more precisely as long as*

$$(1.12) \quad \partial_{\beta\gamma}^2 \mathcal{E}(V) + (T - t) \partial_i (\partial_\alpha \mathcal{E}(U(t, x))) \partial_{\beta\gamma}^2 \mathcal{F}^{i\alpha}(V)$$

stays a positive definite matrix, for all $t \in [0, T]$, $x \in \mathcal{D}$ and $V \in \mathcal{W}$, then the maximization problem (0.10) admits

$$(1.13) \quad W_\alpha(t, x) = (t - T) \partial_\alpha \mathcal{E}(U(t, x)), \quad \alpha \in \{1, \dots, m\}, \quad (t, x) \in [0, T] \times \mathcal{D},$$

as solution.

Notice that condition (1.12) requires, at $t = 0$,

$$(1.14) \quad \partial_{\beta\gamma}^2 \mathcal{E}(V) + T \partial_i (\partial_\alpha \mathcal{E}(U_0(x))) \partial_{\beta\gamma}^2 \mathcal{F}^{i\alpha}(V)$$

to be a positive definite matrix, for all $x \in \mathcal{D}$ and $V \in \mathcal{W}$, which restricts the choice of T with respect to the initial condition U_0 . Observe, however, that condition (1.12) gets less restrictive as t approaches T and even allows a blow-up of $\partial_i (\partial_\alpha \mathcal{E}(U(t, x)))$ of order $(T - t)^{-1}$.

As a matter of fact, in the very special and elementary case of the "inviscid" Burgers equation (0.4) with initial condition u_0 , condition (1.12) reads

$$1 + (T - t) \partial_x u(t, x) > 0, \quad \forall t \in [0, T], x \in \mathbb{R}/\mathbb{Z}$$

and turns out to be equivalent to (1.14), namely

$$1 + T u'_0(x) > 0, \quad \forall x \in \mathbb{R}/\mathbb{Z}.$$

This exactly means that T is smaller than

$$T^* = \inf_{x \in \mathcal{D}} \frac{1}{\max\{-u'_0(x), 0\}} \in]0, +\infty],$$

which is exactly the first time when a shock forms. So, at least in this very elementary case, *all* smooth solutions can be recovered from the maximization problem in just one step.

Proof of Theorem 1.1. Since U is supposed to be a smooth solution of (0.1), we have

$$\partial_t U^\alpha + \partial_\beta \mathcal{F}^{i\alpha}(U) \partial_i U^\beta = 0.$$

Thus W defined by (1.13) satisfies

$$\partial_t W_\gamma - \partial_\gamma \mathcal{E}(U) = (t - T) \partial_{\alpha\gamma}^2 \mathcal{E}(U) \partial_t U^\alpha = -(t - T) \partial_{\alpha\gamma}^2 \mathcal{E}(U) \partial_\beta \mathcal{F}^{i\alpha}(U) \partial_i U^\beta$$

which is equal, thanks to symmetry property (0.2), to

$$-(t - T) \partial_{\alpha\beta}^2 \mathcal{E}(U) \partial_\gamma \mathcal{F}^{i\alpha}(U) \partial_i U^\beta = -(t - T) \partial_i (\partial_\alpha \mathcal{E}(U)) \partial_\gamma \mathcal{F}^{i\alpha}(U) = -\partial_i W_\alpha \partial_\gamma \mathcal{F}^{i\alpha}(U).$$

Thus, we have obtained

$$\partial_t W_\gamma + \partial_i W_\alpha \partial_\gamma \mathcal{F}^{i\alpha}(U) - \partial_\gamma \mathcal{E}(U) = 0,$$

which precisely means that, at each point (t, x) , $V = U(t, x)$ satisfies the first order optimality condition in the definition of $K(\partial_t W(t, x), DW(t, x))$ through (0.11), namely

$$K(\partial_t W(t, x), DW(t, x)) = \sup_{V \in \mathcal{W}} \partial_t W_\gamma(t, x) V^\gamma + \partial_i W_\alpha(t, x) \mathcal{F}^{i\alpha}(V) - \mathcal{E}(V).$$

Meanwhile, condition (1.12) tells us, by definition (1.13) of W , that

$$\partial_{\beta\gamma}^2 \mathcal{E}(V) - \partial_i W_\alpha(t, x) \partial_{\beta\gamma}^2 \mathcal{F}^{i\alpha}(V)$$

is a positive definite matrix for all (t, x, V) , which means that, for each fixed (t, x) ,

$$V \in \mathcal{W} \rightarrow \partial_i W_\alpha(t, x) \mathcal{F}^{i\alpha}(V) - \mathcal{E}(V)$$

is a concave function. So the first order optimality condition we have already obtained for $V = U(t, x)$ is enough to deduce that

$$K(\partial_t W, DW) = \partial_t W_\gamma U^\gamma + \partial_i W_\alpha \mathcal{F}^{i\alpha}(U) - \mathcal{E}(U).$$

Thus, integrating on $[0, T] \times \mathcal{D}$ and using that U is solution of (0.1), we get

$$\begin{aligned} \int_{[0, T] \times \mathcal{D}} K(\partial_t W, DW) + \mathcal{E}(U) &= \int_{[0, T] \times \mathcal{D}} \partial_t W_\gamma U^\gamma + \partial_i W_\alpha \mathcal{F}^{i\alpha}(U) = \\ &= \int_{\mathcal{D}} W_\gamma(T, \cdot) U^\gamma(T, \cdot) - W_\gamma(0, \cdot) U^\gamma(0, \cdot) = \int_{\mathcal{D}} -W_\gamma(0, \cdot) U_0 \end{aligned}$$

since U_0 is the initial condition and, by definition (1.13), $W(T, \cdot) = 0$. By definition (0.10), the optimal value J of the maximization problem is larger than

$$\int_{[0, T] \times \mathcal{D}} -K(\partial_t W, DW) - \int_{\mathcal{D}} -W_\gamma(0, \cdot) U_0.$$

Thus, we have obtained

$$J \geq \int_{[0, T] \times \mathcal{D}} \mathcal{E}(U).$$

But, by definition (0.8), the right-hand side is certainly larger than I which is an upper bound for J . (Indeed $\inf \sup \geq \sup \inf$ is always true.) We conclude that $I = J$ which shows that W must be optimal for the maximization problem (0.10), and completes the proof.

End of proof.

2. RECOVERY OF ENTROPY SOLUTIONS FOR THE "INVISCID" BURGERS EQUATION

As already mentioned, the simplest example of entropic conservation law is the so-called "inviscid" Burgers equation (0.4) on the torus $\mathcal{D} = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, namely

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad u = u(t, x) \in \mathbb{R}, \quad x \in \mathbb{T}, \quad t \in [0, T].$$

Let us recall some of its properties. This equation is Galilean invariant. [Indeed, for each constant c , $u(t, x - ct) + c$ is a solution whenever u is itself a solution.] Thus, it is not a restriction to assume that u_0 has zero mean on \mathbb{T} . This allows us to introduce a unique periodic anti-derivative ϕ_0 with zero mean on \mathbb{T} , so that $u_0 = \phi_0'$. Given $u_0 = \phi_0'$, this equation admits a unique so-called entropy solution [9, 12, 15, 17] explicitly given by the Hopf-Cole formula

$$(2.15) \quad \begin{aligned} u(t, x) &= \partial_x \phi(t, x), \quad \phi(t, x) = \inf_{a \in \mathbb{R}} \phi_0(a) + \frac{|a - x|^2}{2t} \\ &= \inf_{a \in [0, 1], k \in \mathbb{Z}} \phi_0(a) + \frac{|a + k - x|^2}{2t}. \end{aligned}$$

(since ϕ_0 is \mathbb{Z} -periodic). The entropy solution stays smooth as long as

$$a \in \mathbb{R} \rightarrow \phi_0(a) + \frac{|a - x|^2}{2t}$$

admits a single non-degenerate minimum for each (t, x) , which is equivalent to $t\phi_0'' + 1 > 0$ and ceases to be true as soon as $t > T^*$ where

$$(2.16) \quad T^* = \inf_{a \in \mathbb{T}} \frac{1}{\max\{-\phi_0''(a), 0\}} \in]0, +\infty].$$

Beyond T^* , the entropy solution admits discontinuities in x , which are called "shocks". A striking property of entropy solutions, on top of their forward uniqueness, is their lack of backward uniqueness after shocks form. Indeed, from formula (2.15), we easily deduce that, for each $T > T^*$, there is another entropy solution u^T which is shock-free before $t = T$ and coincide with u at time $t = T$. More precisely:

Proposition 2.1. *Let u be the unique entropy solution with initial condition $u_0 = \phi_0'$. For any fixed $T > 0$, let us introduce ϕ_0^T such that $a \in \mathbb{R} \rightarrow 2T\phi_0^T(a) + a^2$ is the convex hull of $a \in \mathbb{R} \rightarrow 2T\phi_0(a) + a^2$.*

Let u^T be the unique entropy solution with initial condition $u_0^T = \phi_0^T$. Then, $u^T(t, \cdot)$ is shock-free for $0 \leq t < T$ and coincides with u at time $t = T$. We call u^T the "shock-free substitute" of u on $[0, T]$.

This concept will be crucially used later on.

Let us now move back to the maximization problem 0.10. In the case of the Burgers equation, with initial condition u_0 , we get

$$(2.17) \quad J = \sup \left\{ \int_{[0, T] \times \mathbb{T}} -\frac{\partial_t W^2}{2(1 - \partial_x W)} - \int_{\mathbb{T}} W(0, \cdot) u_0, \quad W(T, \cdot) = 0 \right\}$$

where $W = W(t, x) \in \mathbb{R}$ is a smooth function subject to $\partial_x W \leq 1$ (and $\partial_t W = 0$ whenever $\partial_x W = 1$). Indeed, according to (0.11), the value of K in (0.10) is given by

$$(2.18) \quad K(A, B) = \sup_{v \in \mathbb{R}} Av + B \frac{v^2}{2} - \frac{v^2}{2} = \frac{A^2}{2(1-B)},$$

whenever $B > 1$, $K = 0$ for $(A, B) = (0, 1)$ and $K = +\infty$ otherwise. Since

$$- \int_{\mathbb{T}} W(0, \cdot) u_0 = \int_{\mathbb{T}} (W(T, \cdot) - W(0, \cdot)) u_0 = \int_{[0, T] \times \mathbb{T}} \partial_t W u_0$$

(because $W(T, \cdot) = 0$ and u_0 does not depend on t), the maximization problem equivalently reads

$$(2.19) \quad J = \sup \left\{ \int_{[0, T] \times \mathbb{T}} - \frac{\partial_t W^2}{2(1 - \partial_x W)} + \partial_t W u_0, \quad W(T, \cdot) = 0 \right\}.$$

Our goal is to prove that we may recover all entropy solutions of the "inviscid" Burgers equation from this maximization problem, in the sense:

Theorem 2.2. *Let $u_0 = \phi'_0$ where ϕ_0 is a smooth function on \mathbb{T} and both u_0 and ϕ_0 have zero mean. Let u be the unique entropy solution of the "inviscid" Burgers equation (0.4) with initial condition u_0 and let u^T be its "shock-free substitute" u^T on $[0, T]$ (as in Proposition 2.1).*

Then, for any fixed time $T > 0$, the maximization problem (2.19) admits a generalized solution W for which $q = \partial_t W$ and $\rho = 1 - \partial_x W \geq 0$ are bounded Borel measures on $[0, T] \times \mathbb{T}$, where $\rho(T, \cdot) = 1$ and q is absolutely continuous with respect to ρ , with density $v = u^T$. In particular $v(T, \cdot) = u(T, \cdot)$.

Remark. Notice that $v = u^T$ is a shock-free entropy solution on $[0, T[$. This is not so surprising, since v is obtained from a concave maximization problem in both space and time, which is, in some vague sense, a (degenerate) space-time elliptic problem so that some partial regularity of its solution should be expected.

Remark: an analogy with... mountain climbing! Through Theorem 2.2, we have a rather unusual way of solving the Cauchy problem, somewhat reminiscent of some well-known techniques in mountaineering. Indeed, we may interpret pursuing the entropy solution beyond T^* after shocks have formed as walking along a sharp crest. Through the concave maximization problem, we prefer accessing to the point of the crest (namely $u(T, \cdot)$) by climbing from a different initial point (namely $u^T(0, \cdot)$) following a less dangerous way up to destination (through the shock-free substitute u^T)! Of course, this strategy is rather cumbersome if our goal is to explore each point of the crest, but actually safer (*).

(*) For instance, so far, nobody has succeeded in crossing, without supplemental oxygen, the long ridge linking mount Everest to mount Lhotse at about 8000 meters above sea-level! Many thanks to Thomas Gallouët for this information.

3. PROOF OF THEOREM 2.2

The first step of the proof consists in establishing a suitable generalized framework to solve the maximization problem (2.19), relying on generalized solutions W of bounded

variation on $[0, T] \times \mathbb{T}$. Next, we use standard convex analysis to find the corresponding dual (or rather pre-dual) minimization problem, which is very similar to the most standard optimal transportation problem, as in [3], and actually corresponds to the most elementary first-order mean-field game à la Lasry-Lions [13, 8] (**). This problem can be solved almost explicitly, thanks to the properties of the "inviscid" Burgers equations such as Proposition 2.1, which completes the proof.

(**) At a computational level, using the augmented Lagrangian method of [3], these two problems differ just by...two lines of code!

3.1. A priori estimates and generalized solutions for problem 2.19. Let us get few estimates. Trivially, we get $J \geq 0$ (just take $W = 0$). Next, for each $r > 0$

$$\left| \int_{[0, T] \times \mathbb{T}} \partial_t W u_0 \right| \leq \frac{r \sup u_0^2}{2} \int_{[0, T] \times \mathbb{T}} (1 - \partial_x W) + \int_{[0, T] \times \mathbb{T}} \frac{\partial_t W^2}{2r(1 - \partial_x W)}$$

(by Cauchy-Schwarz-Young's inequality, using that $1 - \partial_x W \geq 0$)

$$= \frac{rT \sup u_0^2}{2} + \int_{[0, T] \times \mathbb{T}} \frac{\partial_t W^2}{2r(1 - \partial_x W)} \quad (\text{since } \int_{\mathbb{T}} \partial_x W = 0).$$

Thus

$$(3.20) \quad \left| \int_{[0, T] \times \mathbb{T}} \partial_t W u_0 \right| \leq \frac{rT \sup u_0^2}{2} + \int_{[0, T] \times \mathbb{T}} \frac{\partial_t W^2}{2r(1 - \partial_x W)}$$

which already shows (taking $r = 1$) that

$$J \leq \frac{T \sup u_0^2}{2}.$$

So, we can consider a maximizing sequence W_n , $n \geq 1$, such that

$$\int_{[0, T] \times \mathbb{T}} \frac{\partial_t W_n^2}{2(1 - \partial_x W_n)} - \partial_t W_n u_0 \leq -J + n^{-1}.$$

From estimate (3.20), taking $r = 2$, we deduce

$$\int_{[0, T] \times \mathbb{T}} \frac{\partial_t W_n^2}{4(1 - \partial_x W_n)} \leq -J + n^{-1} + T \sup u_0^2 \leq T \sup u_0^2 + n^{-1}$$

(since $J \geq 0$). Thus (again by Cauchy-Schwarz' inequality)

$$\begin{aligned} \left(\int_{[0, T] \times \mathbb{T}} |\partial_t W_n| \right)^2 &\leq \int_{[0, T] \times \mathbb{T}} (1 - \partial_x W_n) \int_{[0, T] \times \mathbb{T}} \frac{\partial_t W_n^2}{1 - \partial_x W_n} \\ &= T \int_{[0, T] \times \mathbb{T}} \frac{\partial_t W_n^2}{1 - \partial_x W_n} \leq 4T^2 \sup u_0^2 + 4Tn^{-1}. \end{aligned}$$

Since $\partial_x W_n \leq 1$, $\int_{\mathbb{T}} \partial_x W_n = 0$ and $W_n(T, \cdot) = 0$, this is already enough to deduce that W_n , up to a subsequence, strongly converges in $L^1([0, T] \times \mathbb{T})$ to some limit W which is necessarily a function of bounded-variation on $[0, T] \times \mathbb{T}$ (which exactly means that $\partial_t W_n$,

$\partial_x W_n$ are bounded Borel measures). Since K , defined by (2.18), is convex, we further have, by upper semi-continuity, that

$$(3.21) \quad \tilde{J} = \int_{[0,T] \times \mathbb{T}} -\frac{\partial_t W^2}{2(1 - \partial_x W)} + \partial_t W u_0 \geq J,$$

but, a priori, $\tilde{J} > J$ is possible. As a function of bounded variation, $W(t, \cdot)$ has a limit at $t \uparrow T$, namely $W(T-, \cdot)$, in the $L^1(\mathbb{T})$ sense. However, it is not clear, a priori, that this limit is zero, in spite of the fact that $W_n(T, \cdot) = 0$. We need a more precise estimate to prove it. For each smooth function $\psi(x)$ and (s, t) such that $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} \int_{\mathbb{T}} (W_n(t, \cdot) - W_n(s, \cdot)) \psi &= \int_{[s,t] \times \mathbb{T}} \partial_t W_n \psi \leq \sup |\psi| \sqrt{\int_{[0,T] \times \mathbb{T}} \frac{\partial_t W_n^2}{1 - \partial_x W_n} \int_{[s,t] \times \mathbb{T}} 1 - \partial_x W_n} \\ &= \sqrt{|t - s|} \sup |\psi| \sqrt{\int_{[0,T] \times \mathbb{T}} \frac{\partial_t W_n^2}{1 - \partial_x W_n}} \quad (\text{since } \int_{\mathbb{T}} \partial_x W_n = 0) \\ &\leq 2\sqrt{|t - s|} \sup |\psi| \sqrt{T \sup u_0^2 + n^{-1}} \end{aligned}$$

(as already established). All these estimates show that the $W_n = W_n(t, x)$ are actually uniformly bounded in the space \mathcal{W} of all bounded-variation functions that are uniformly Hölder continuous (of exponent 1/2) functions of t valued in the space of distributions on \mathbb{T} , which is enough to guarantee that they converge to W in \mathcal{W} (in the weak-* sense, since the closed bounded convex subset of \mathcal{W} are weak-* compact). Hence, $W(T, \cdot) = 0$ just follows from $W_n(T, \cdot) = 0$. Thus, it is natural to consider the maximization problem (2.19) in the generalized class of bounded-variation functions $W(t, x)$ that are uniformly continuous in t with values in distributions in x such that $W(T, \cdot) = 0$. In this large class, the supremum is certainly attained (as can be seen by using the same a priori estimates as above). However, the supremum may a priori exceed the value J . Let us now show that this is not the case.

Proposition 3.1. *In definition (2.17) (equivalently (2.19)), the optimal value J is achieved in the generalized class \mathcal{W} of all bounded-variation functions $W = W(t, x)$ defined on $[0, T] \times \mathbb{T}$ which are uniformly continuous in t with values in distributions in x and satisfy $\partial_x W \leq 1$ and $W(T, \cdot) = 0$.*

Proof of Proposition 3.1. It is enough to show that, given a maximizer W in the large class \mathcal{W} , there is a corresponding smooth function W_ϵ such that $\partial_x W_\epsilon < 1$, $W_\epsilon(T, \cdot) = 0$ and

$$(3.22) \quad \liminf_{\epsilon \downarrow 0} \int_{[0,T] \times \mathbb{T}} \frac{\partial_t W_\epsilon^2}{2(1 - \partial_x W_\epsilon)} - \partial_t W_\epsilon u_0 \leq \int_{[0,T] \times \mathbb{T}} \frac{\partial_t W^2}{2(1 - \partial_x W)} - \partial_t W u_0.$$

Let us introduce two nonnegative mollifiers on \mathbb{R} , ζ and γ with respective supports in $[0, 1]$ and $[-1, 1]$. Then we define

$$W_\epsilon(t, x) = \int_0^1 \left(\int_{-1}^1 W(t + \epsilon\tau, x + \epsilon\xi) \gamma(\xi) d\xi \right) \zeta(\tau) d\tau,$$

where $W(t, x)$ has been extended by 0 beyond $t = T$. By construction $W_\epsilon(T, \cdot) = 0$, $\partial_x W_\epsilon < 1$, and, by Jensen's inequality,

$$\int_{[0, T] \times \mathbb{T}} \frac{\partial_t W_\epsilon^2}{2(1 - \partial_x W_\epsilon)} \leq \int_{[0, T] \times \mathbb{T}} \frac{\partial_t W^2}{2(1 - \partial_x W)},$$

since W_ϵ has been defined by local average of W . Furthermore,

$$\int_{[0, T] \times \mathbb{T}} \partial_t W_\epsilon u_0 = - \int_{\mathbb{T}} W_\epsilon(0, \cdot) u_0 = - \int_{\mathbb{T}} \int_0^1 \int_{-1}^1 u_0(x) W(\epsilon\tau, x + \epsilon\xi) \zeta(\tau) \gamma(\xi) d\xi d\tau dx$$

has the same limit (because $W(t, x)$ is uniformly continuous in t with values in distributions in x) as

$$- \int_{\mathbb{T}} \int_{-1}^1 u_0(x) W(0, x + \epsilon\xi) \gamma(\xi) d\xi dx = - \int_{\mathbb{T}} \int_{-1}^1 u_0(x - \epsilon\xi) W(0, x) \gamma(\xi) d\xi dx$$

which, itself, converges to

$$- \int_{\mathbb{T}} u_0(x) W(0, x) dx,$$

since u_0 has been supposed to be smooth. This is enough to deduce (3.22) and conclude the proof of Proposition 3.1.

3.2. Primal and dual formulations of Problem 2.19. In this next step, we get primal and dual formulations for Problem 2.19, after introducing

$$\rho = 1 - \partial_x W \geq 0, \quad q = \partial_t W \in \mathbb{R}.$$

We see that, from definition (2.19) J can be now written

$$(3.23) \quad J = \sup \left\{ \int_{[0, T] \times \mathbb{T}} -\frac{q^2}{2\rho} + qu_0, \quad (\rho, q) \text{ s.t. } \rho \geq 0, \quad \partial_t \rho + \partial_x q = 0, \quad \rho(T, \cdot) = 1 \right\}$$

where the supremum is performed over all bounded Borel measures (ρ, q) over $[0, T] \times \mathbb{T}$. This is very close to an optimal transport problem with quadratic cost, in its "Benamou-Brenier" formulation [3]. At this point, with the help of the Fenchel-Rockafellar duality theorem [5], following [3], we may assert

Proposition 3.2. *In the maximization problem (2.19), rewritten as (3.23), the supremum is achieved by some $(\rho \geq 0, q \in \mathbb{R})$ enjoying the following properties: i) ρ is a non-negative measure, ii) q is absolutely continuous with respect to ρ with a square-integrable density $v = v(t, x)$ so that*

$$J = \int_{[0, T] \times \mathbb{T}} -\frac{q^2}{2\rho} + qu_0 = \int_{[0, T] \times \mathbb{T}} -\frac{1}{2}\rho v^2 + \rho v u_0.$$

In addition, we have the duality result

$$(3.24) \quad J = \inf \left\{ \int_{\mathbb{T}} -\theta(T, \cdot); \quad \theta \text{ s.t. } \partial_t \theta + \frac{1}{2}(\partial_x \theta)^2 \leq 0, \quad \theta(0, \cdot) = \phi_0 \right\}$$

where the supremum is performed over all smooth functions θ and ϕ_0 is the unique function with zero mean on \mathbb{T} such that $\phi'_0 = u_0$.

The proof of this rather standard result (see also [7] for similar proofs) is postponed to Appendix 1.

The mixed problem (3.23,3.24) can be interpreted as a *relaxed variational* formulation of a "mean-field game" à la Lasry-Lions [13, 8], namely

$$\partial_t \rho + \partial_x(\rho \partial_x \theta) = 0, \quad \partial_t \theta + \frac{1}{2}(\partial_x \theta)^2 = 0, \quad \rho(T, \cdot) = 1, \quad \theta(0, \cdot) = \phi_0,$$

where the Hamilton-Jacobi equation $\partial_t \theta + \frac{1}{2}(\partial_x \theta)^2 = 0$ is relaxed as an inequality. As a matter of fact, solving (3.24) is very easy, even simpler than solving a standard optimal transport problem! Indeed, from standard Hamilton-Jacobi theory (cf. [2] and [6] as a recent reference)

$$\partial_t \theta + \frac{1}{2}(\partial_x \theta)^2 \leq 0, \quad \theta(0, \cdot) = \phi_0$$

implies

$$\theta(t, x) \leq \inf_{a \in \mathbb{R}} \phi_0(a) + \frac{|a - x|^2}{2t}, \quad \forall t \in]0, T]$$

(or, equivalently,

$$\theta(t, x) \leq \inf_{a \in [0, 1], k \in \mathbb{Z}} \phi_0(a) + \frac{|a + k - x|^2}{2t}, \quad \forall t \in]0, T],$$

since ϕ_0 is \mathbb{Z} -periodic). Thus, by saturating this inequality, we immediately obtain the optimal value for (3.24), namely

$$J = - \int_{\mathbb{T}} \inf_{a \in \mathbb{R}} \left\{ \phi_0(a) + \frac{|a - x|^2}{2T} \right\} dx,$$

i.e.

$$(3.25) \quad J = - \int_{\mathbb{T}} \phi(T, x) dx, \quad \phi(t, x) = \inf_{a \in \mathbb{R}} \phi_0(a) + \frac{|a - x|^2}{2t},$$

where we recognize in $\phi(t, x)$ the anti-derivative of the unique entropy solution with initial condition $u_0 = \phi'_0$, i.e. $\partial_x \phi(t, x) = u(t, x)$. As explained in Proposition 2.1, we also have

$$\phi(T, x) = \phi^T(T, x), \quad \phi^T(t, x) = \inf_{a \in \mathbb{R}} \phi_0^T(a) + \frac{|a - x|^2}{2t},$$

where $a^2 + T\phi_0^T(a)$ is the convex hull of $a^2 + 2T\phi_0(a)$. Denoting by Ω the (closed) set of all points $a \in [0, 1]$ where $a^2 + 2T\phi_0(a)$ coincides with its convex hull (and, therefore, $1 + T\phi_0''(a) \geq 0$), we observe that, for each fixed $t \in [0, T]$,

$$a \in \Omega \rightarrow x = a + t\phi'_0(a)$$

defines a one-to-one change of variable (with range going from Ω at time $t = 0$ to the full torus at time T), under which

$$a + k = \operatorname{Argmin}_{\tilde{a} \in \mathbb{R}} \phi_0(\tilde{a}) + \frac{|\tilde{a} - x|^2}{2t}, \quad \phi^T(t, x) = \phi_0(a) + \frac{|a + k - x|^2}{2t},$$

for a suitable integer $k \in \mathbb{Z}$ (that may depend on (t, x)). In addition,

$$\partial_x \phi^T(t, x) = \phi'_0(a) = -\frac{a + k - x}{t},$$

so that

$$(3.26) \quad \phi^T(t, x) = \phi_0(a) + \frac{t}{2}\phi_0'(a)^2, \quad u^T(t, x) = \phi_0'(a).$$

Let us now introduce, for each $t \in [0, T]$, the probability measure $\rho(t, \cdot)$ defined on \mathbb{T} by

$$(3.27) \quad \forall f \in C^0(\mathbb{T}), \quad \int_{\mathbb{T}} f \rho(t, \cdot) = \int_{\Omega} f(a + t\phi_0'(a))\rho_0(a)da, \quad \rho_0(a) = 1 + T\phi_0''(a),$$

which is just the Lebesgue measure at $t = T$ since

$$\int_{\mathbb{T}} f \rho(T, \cdot) = \int_{\Omega} f(a + T\phi_0'(a))(1 + T\phi_0''(a))da = \int_{\mathbb{T}} f(x)dx.$$

Thus, we have obtained

$$\int_{\mathbb{T}} \phi^T(T, x)dx = \int_{\Omega} (\phi_0(a) + \frac{T}{2}\phi_0'(a)^2)\rho_0(a)da = \int_{\Omega} (\phi_0(a) + \frac{T}{2}\phi_0'(a)^2)(1 + T\phi_0''(a))da,$$

which provides an explicit value for the optimum J of Problem 3.24, namely

$$(3.28) \quad J = - \int_{\Omega} (\phi_0(a) + \frac{T}{2}\phi_0'(a)^2)(1 + T\phi_0''(a))da.$$

In the last step of the proof of Theorem 2.2, we introduce, for each $t \in [0, T]$, a companion measure $q(t, \cdot)$ for $\rho(t, \cdot)$ (already defined by (3.27)):

$$(3.29) \quad \forall f \in C^0(\mathbb{T}), \quad \int_{\mathbb{T}} f q(t, \cdot) = \int_{\Omega} \phi_0'(a)f(a + t\phi_0'(a))\rho_0(a)da, \quad \rho_0(a) = 1 + T\phi_0''(a)$$

and want to show that (ρ, q) is optimal for the dual problem (3.23).

First, we immediately check (from (3.27,3.29))

$$\partial_t \rho + \partial_x q = 0, \quad \rho(T, \cdot) = 1,$$

which already show that (ρ, q) is an admissible solution for problem (3.23). Next, we see that q is absolutely continuous with respect to $\rho(t, \cdot)$ and observe that its Radon-Nikodym derivative $v(t, \cdot)$ is nothing but $u^T(t, \cdot)$, the "shock-free substitute" for u on $[0, T]$. Indeed, through the change of variable

$$a \in \Omega \rightarrow x = a + t\phi_0'(a),$$

we have already used, $\rho(t, \cdot)$ is the image of $\rho_0(a)da$ (restricted to Ω) while $v(t, x)$ is just equal to $\phi_0'(a)$ (through its very definition (3.29)), which is also $u^T(t, x)$, as seen above in (3.26).

Then, we compute

$$\begin{aligned} \int_{[0, T] \times \mathbb{T}} -\frac{q^2}{2\rho} + qu_0 &= \int_{[0, T] \times \mathbb{T}} \left(-\frac{v^2}{2} + vu_0\right)\rho = \int_{[0, T] \times \mathbb{T}} \left(-\frac{v^2}{2} + v\phi_0'\right)\rho \\ &= \int_{\Omega} \int_0^T \left(-\frac{1}{2}\phi_0'(a)^2 + \phi_0'(a)\phi_0'(a + t\phi_0'(a))\right) dt \rho_0(a)da \\ &= \int_{\Omega} \left(-\frac{T}{2}\phi_0'(a)^2 + \phi_0(a + T\phi_0'(a)) - \phi_0(a)\right) \rho_0(a)da \\ &= \int_{\Omega} \left(-\frac{T}{2}\phi_0'(a)^2 - \phi_0(a)\right) \rho_0(a)da. \end{aligned}$$

(Indeed $\int_{\Omega} \phi_0(a + T\phi_0'(a))\rho_0(a)da = \int_{\Omega} \phi_0(a + T\phi_0'(a))(1 + T\phi_0''(a))da = \int_{\mathbb{T}} \phi_0(x)dx = 0$.)

This is exactly the value (3.28) just obtained for the optimal value J of both (3.24) and (3.23). We conclude that (ρ, q) defined by (3.27, 3.29) is indeed optimal for (3.23) while q is absolutely continuous with respect to ρ with density $v = u^T$. This completes the proof of Theorem 2.2.

4. APPENDIX 1: PROOF OF PROPOSITION 3.2

It is convenient to introduce (more or less as in [3], see also [7]) the Banach space

$$E = \{(a, b) \in C^0([0, T] \times \mathbb{T}) \times C^0([0, T] \times \mathbb{T})\}$$

equipped with the sup-norm and the convex functions valued in $] -\infty, +\infty]$ respectively defined by

$$\Phi(a, b) = 0, \text{ whenever } a + \frac{b^2}{2} \leq 0, \quad +\infty \text{ otherwise, } \forall (a, b) \in E$$

$$\Psi(a, b) = - \int_{\mathbb{T}} \theta(T, \cdot), \text{ whenever } a = \partial_t \theta, \quad b = \partial_x \theta, \quad \theta(0, \cdot) = \phi_0,$$

for some smooth function $\theta : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$, and $+\infty$ otherwise. [Notice that θ is then defined without ambiguity.] Observe that there is at least one point $(a, b) \in E$ at which Φ is continuous while Ψ is finite. [Take, for instance, $a = -1, b = 0$.] Since both Φ and Ψ are convex functions, the Fenchel-Rockafellar theorem (as stated in [5], chap. I) asserts that

$$\inf\{\Phi(a, b) + \Psi(a, b), (a, b) \in E\} = \max\{-\Phi^*(\rho, q) - \Psi^*(-\rho, -q), (\rho, q) \in E'\}$$

where E' is the dual of E , namely the space of all pairs of real valued bounded Borel measures (ρ, q) on $[0, T] \times \mathbb{T}$, and Φ^*, Ψ^* , are the Legendre transforms of Φ and Ψ . Let us emphasize that notation "max" is used to express that the supremum is always achieved on the right-hand side (while the infimum may not be achieved on the left-hand side). As in [3], we first get $\Phi^*(\rho, q) = +\infty$, unless $\rho \geq 0$, q is absolutely continuous with respect to ρ , with a square integrable derivative v , in which case

$$\Phi^*(\rho, q) = \int_{[0, T] \times \mathbb{T}} \frac{1}{2} \rho v^2.$$

Next, we find $\Psi^*(-\rho, -q) = +\infty$, unless $\partial_t \rho + \partial_x q = 0$, $\rho(T, \cdot) = 1$, in which case

$$\Psi^*(-\rho, -q) = - \int_{[0, T] \times \mathbb{T}} q u_0.$$

[Indeed, by definition of Ψ

$$\Psi^*(-\rho, -q) = \sup_{\theta} \int_{[0, T] \times \mathbb{T}} -\partial_t \theta \rho - \partial_x \theta q + \int_{\mathbb{T}} \theta(T, \cdot)$$

(where the supremum is performed over all smooth functions θ s.t. $\theta(0, \cdot) = \phi_0$)

$$= \sup_{\tilde{\theta}} \int_{[0, T] \times \mathbb{T}} -\partial_t \tilde{\theta} \rho - \partial_x \tilde{\theta} q - \phi_0' q + \int_{\mathbb{T}} \tilde{\theta}(T, \cdot),$$

(where we have set $\tilde{\theta}(t, x) = \theta(t, x) - \phi_0(x)$, which vanishes at $t = 0$, and used that ϕ_0 is a smooth function with zero mean over \mathbb{T}). Thus, whenever (ρ, q) weakly satisfies $\partial_t \rho + \partial_x q = 0$, $\rho(T, \cdot) = 1$,

$$\Psi^*(-\rho, -q) = - \int_{[0, T] \times \mathbb{T}} \phi_0' q = - \int_{[0, T] \times \mathbb{T}} u_0 q,$$

and $+\infty$ otherwise, as announced.]

So, we have, on one side

$$\inf\{\Phi(a, b) + \Psi(a, b), (a, b) \in E\} = \inf\left\{\int_{\mathbb{T}} -\theta(T, \cdot); \theta \text{ s.t. } \partial_t \theta + \frac{1}{2}(\partial_x \theta)^2 \leq 0, \theta(0, \cdot) = \phi_0\right\}$$

where the infimum is taken over all smooth functions θ , which is exactly the optimal value of problem (3.24), while, on the other side

$$\max\{-\Phi^*(\rho, q) - \Psi^*(-\rho, -q), (\rho, q) \in E'\}$$

exactly corresponds to (3.23).

So we have proven that (3.24) and (3.23) are dual to each other, provided (3.23) is performed over the class of bounded Borel measures. and $\rho_e(T, \cdot) = 1$, which concludes the proof of Theorem 3.2.

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