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# Some Results About a Conjecture on Identifying Codes in Complete Suns 

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#### Abstract

Consider a graph $G=(V, E)$ and, for every vertex $v \in V$, denote by $B(v)$ the set $\{v\} \cup\{u: u v \in E\}$. A subset $C \subseteq V$ is an identifying code if the sets $B(v) \cap C, v \in V$, are all nonempty and distinct. It is a locating-dominating code if the sets $B(v) \cap C, v \in V \backslash C$, are all nonempty and distinct.

Let $S_{n}$ be the graph whose vertex set can be partitioned into two sets $U_{n}$ and $V_{n}$, where $U_{n}=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ induces a clique, and $V_{n}=\left\{v_{1,2}, v_{2,3}, \ldots, v_{n-1, n}, v_{n, 1}\right\}$ induces an independent set, with edges $v_{i, i+1} u_{i}$ and $v_{i, i+1} u_{i+1}, 1 \leq i \leq n$; computations are carried modulo $n$. This graph is called a complete sun. We prove the conjecture, stated by Argiroffo, Bianchi and Wagler in 2014, that the smallest identifying code in $S_{n}$ has size equal to $n$. We also characterize and count all the identifying codes with size $n$ in $S_{n}$. Finally, we determine the sizes of the smallest locating-dominating codes in $S_{n}$.


Key Words: Graph Theory, Identifying Codes, Locating-Dominating Codes

## 1 Introduction

This paper was first motivated by a conjecture, stated by Argiroffo, Bianchi and Wagler in [1], about the size of optimal identifying codes in a special class of graphs, the complete suns. We prove below that this conjecture is true, but we also investigate these graphs further: we characterize and count all their optimal identifying codes, and we determine the size of their optimal locating-dominating codes.

Consider a graph $G=(V, E)$; the order of $G$ is its number of vertices. For every vertex $v \in V$, denote by $B(v)$ the ball of radius one and centre $v$, i.e., the set $\{v\} \cup\{u: u v \in E\}$. Two vertices $u \in V, v \in V$ are said to dominate each other if $u \in B(v)$, which is equivalent to $v \in B(u)$. A vertex $z \in V$ is said to separate $u$ and $v$, or $u$ from $v$, if it dominates $u$ but not $v$ (note that in this case $z=u$ is possible), or the other way round. A set of vertices $Y \subseteq V$ is said to dominate a vertex $u \in V$ (respectively, to separate two vertices $u \in V$ and $v \in V)$ if $Y$ contains at least one vertex which dominates $u$ (respectively, which separates $u$ and $v$ ).

A subset $C \subseteq V$ is an identifying code if the sets $B(v) \cap C, v \in V$, are all nonempty and distinct. In other words, every vertex is dominated by $C$, and every pair of vertices is separated by $C$. The vertices in $C$ are called codewords and those in $V \backslash C$ non-codewords. With a slight abuse of language, when the context is clear, we shall often say that "a vertex is dominated", or "a pair of vertices is separated", omitting to add ". . . by the code" or ". . .by such codeword".

When $G$ admits an identifying code, we denote by $\gamma^{I D}(G)$ the minimum cardinality of an identifying code in $G$. Any identifying code $C$ with $|C|=$ $\gamma^{I D}(G)$ is said to be optimal.

Closely related are the locating-dominating codes, which are less demanding: a subset $C \subseteq V$ is a locating-dominating code (LD code for short) if the sets $B(v) \cap C, v \in V \backslash C$, are all nonempty and distinct. In other words, every non-codeword is dominated by $C$, and every pair of non-codewords is separated by $C$. For LD codes, we shall often say that we separate two vertices also when we put one of them in the code. We denote by $\gamma^{L D}(G)$ the minimum cardinality of a LD code in $G$. Any LD code $C$ with $|C|=\gamma^{L D}(G)$ is said to be optimal.

The motivations for identifying and locating-dominating codes may come, for instance, from fault diagnosis in multiprocessor systems. Identifying codes were introduced in the seminal paper [6], which establishes general bounds on $\gamma^{I D}$, then studies particular classes of graphs, such as hypercubes, trees, or the hexagonal and triangular infinite grids. Locating-dominating codes were introduced in [9] and further developed in [3], with a focus on complexity and algorithms. Both families of codes now constitute a topic of their own, as shows the bibliography at [7], which contains approximately 340 references.


Figure 1: The complete sun $S_{4}$.

A complete sun $S_{n}$ is a graph whose vertex set can be partitioned into two sets $U_{n}$ and $V_{n}$, where $U_{n}=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ induces a clique (or complete graph), and $V_{n}=\left\{v_{1,2}, v_{2,3}, \ldots, v_{n-1, n}, v_{n, 1}\right\}$ induces an independent (or stable) set, with edges $v_{i, i+1} u_{i}$ and $v_{i, i+1} u_{i+1}, 1 \leq i \leq n$; computations are carried modulo $n$. See Figure 1 for $n=4$.

Complete suns belong to a larger class of graphs, the split graphs, whose vertex set can be partitioned into a clique and an independent set. Complexity results for the identifying code problem in split graphs can be found in [4].

In [1], the authors investigate the minimum cardinality of an identifying code for different families of suns, and in particular they conjecture that for $n \geq 3, \gamma^{I D}\left(S_{n}\right)=n$ (note that $V_{n}$ is obvioulsy an identifying code in $S_{n}$ ). In Section 2, we prove that the conjecture is true. We then widen our study of complete suns: in Section 3, we characterize all the optimal identifying codes in $S_{n}$, and we count them in Section 4; in Section 5, we prove that $\gamma^{L D}\left(S_{3}\right)=3$ and $\gamma^{L D}\left(S_{n}\right)=\left\lfloor\frac{4 n+2}{5}\right\rfloor$ for $n \geq 4$.

NB: In [1, Fig. 2], the graphs (b) and (c) are erroneously inverted.

## 2 Proof of the Conjecture

In the subsequent figures, black circles will represent codewords, white circles will represent non-codewords, and when necessary there will be squares for the vertices with unknown or immaterial status; we do not represent all the edges in the clique, except in Figure 4.

Since $V_{n}$ is an identifying code, all we have to prove is that $\gamma^{I D}\left(S_{n}\right) \geq n$. Let $C_{n}$ be an identifying code in $S_{n}$, optimal or not. When we speak of consecutive vertices in $U_{n}$ or $V_{n}$, "consecutive" refers to the subscripts. A vertex $v_{i, i+1}$ such that $v_{i-1, i} \notin C_{n}$ and $v_{i+1, i+2} \notin C_{n}$ is called a single. A sequence of two consecutive non-codewords $v_{i-1, i}, v_{i, i+1}$ is called a duet; two duets with no vertices in common form a quartet.

Lemma 1 Let $C_{n}$ be an identifying code in $S_{n}$. Then:
(a) There is no single in $V_{n}$;
(b) There is at most one duet in $V_{n}$; equivalently, there is no quartet in $V_{n}$.

(b)


Figure 2: Codewords and non-codewords in $V_{n}$.

Proof. (a) If $v_{i, i+1}$ is a single, then $u_{i}$ and $u_{i+1}$ are not separated by the code, since $v_{i-1, i}$ and $v_{i+1, i+2}$ are the only vertices separating $u_{i}$ and $u_{i+1}$.
(b) If $v_{i-1, i}, v_{i, i+1}$ and $v_{j-1, j}, v_{j, j+1}$ form a quartet, then $u_{i}$ and $u_{j}$ are not separated, with $j \in\{1, \ldots, n\} \backslash\{i-1, i, i+1\}$.

Proposition 2 For all integers $n \geq 3$, we have: $\gamma^{I D}\left(S_{n}\right) \geq n$.
Proof. Using Lemma 1, we study the possible sequences of codewords and non-codewords in $V_{n}$. In particular, no sequence of three consecutive noncodewords exists in $V_{n}$.

There is at least one codeword in $V_{n}$, because the vertices in $U_{n}$ are separated, and without loss of generality (wlog), we assume that $v_{1,2} \in C_{n}$. Then by Lemma 1 (a) $v_{n, 1} \in C_{n}$ or $v_{2,3} \in C_{n}$ : wlog, we take $v_{2,3} \in C_{n}$. We can then alternate the codewords and the non-codewords in $V_{n}$ as in Figure 2; in Figure 2(a), we have:

- Sequences of consecutive codewords of various lengths, with at least one sequence, and the lengths at least two by Lemma 1(a).
- One duet; by Lemma $1(\mathrm{~b})$, there is at most one duet.
- Sequences of isolated non-codewords; their number can range between zero and $\left\lfloor\frac{n-4}{3}\right\rfloor$.

In Figure 2(b), we have only isolated non-codewords in $V_{n}$; their number can range between zero and $\left\lfloor\frac{n}{3}\right\rfloor$ (but as we shall see later on, exactly one [isolated] non-codeword in $V_{n}$ is impossible).

We denote by $q$ the number of codewords in $V_{n}$; then there are $n-q$ non-codewords in $V_{n}$, distributed according to Figure 2(a) or (b). These non-codewords must be dominated and separated by codewords in $U_{n}$. It is straightforward to observe that each isolated non-codeword requires (at least) one codeword in $U_{n}$ (in order to be dominated), that the duet requires (at least) two codewords in $U_{n}$ (so that its two vertices are dominated and separated), and that these tasks do not overlap. So the $n-q$ non-codewords


Figure 3: (a) How to treat isolated non-codewords in $V_{n}$; (b) and (c) How to treat a duet.
in $V_{n}$ require (at least) $n-q$ codewords in $U_{n}$, and the total number of codewords is (at least) $n$.

Thus, the conjecture is proved:
Corollary 3 For all integers $n \geq 3$, we have: $\gamma^{I D}\left(S_{n}\right)=n$.

## 3 Characterization of the Optimal Codes

We construct all the optimal identifying codes in $S_{n}$ by first choosing the configuration of codewords/non-codewords in $V_{n}$; this configuration is one of the two given in Figure 2. Then we complete the code by choosing the remaining codewords in $U_{n}$ : for each isolated non-codeword in $V_{n}$, we arbitrarily choose one of the two vertices in $U_{n}$ that dominates it, see Figure 3(a); for each duet in $V_{n}$, we arbitrarily choose two vertices among the three vertices in $U_{n}$ that dominate at least one vertex of the duet, see Figure 3(b)-(c). We claim that, except in some cases that will be described below, the code $C_{n}$ thus constructed is identifying. Since it contains $n$ elements, it is optimal.
(1) Obviously, all the vertices are dominated by $C_{n}$.
(2) The vertices in $U_{n}$ are separated between themselves:
two consecutive vertices are separated because there is no single in $C_{n}$; two non-consecutive vertices are separated because there is no quartet in $C_{n}$.
(3) The vertices in $U_{n}$ are separated from the vertices in $V_{n}$ : indeed, unless
(a) there is exactly one codeword $u_{i}$ in $U_{n}$, or
(b) there are exactly two codewords $u_{i}, u_{i+1}$ in $U_{n}$,
the codewords in $U_{n}$ perform this separating task. In Case (a) however, $u_{i}$ and the codeword in $V_{n}$ that $u_{i}$ dominates are not separated (cf. Figure 3(a)); this case can be avoided only if we do not take exactly $n-1$ codewords in $V_{n}$. In Case (b), if $u_{i}$ and $u_{i+1}$ have been put in the code to dominate and separate $v_{i, i+1}$ and $v_{i+1, i+2}$, then $u_{i+1}$ and $v_{i, i+1}$ are not separated, (cf. Figure 3(b)), but by taking rather $u_{i}$ and $u_{i+2}$, like in Figure 3(c), we can overcome this difficulty.


Figure 4: All the nonisomorphic optimal identifying codes.


Figure 5: Distributions of codewords in $V_{7}$.
(4) The vertices in $V_{n}$ are separated between themselves:
(a) The codewords in $V_{n}$ are separated from all the other vertices in $V_{n}$ by themselves.
(b) The non-codewords in $V_{n}$ are separated between themselves by the codewords in $U_{n}$ that dominate them.
Summarizing: in order to construct all the optimal identifying codes in $S_{n}$, we choose a configuration of codewords and non-codewords in $V_{n}$ such as given by Figure 2, with the exception of the configuration with exactly $n-1$ codewords in $V_{n}$. We complete the code with vertices in $U_{n}$, in the way described in the first paragraph of this Section, being careful, in the case when $v_{i, i+1}$ and $v_{i+1, i+2}$ are the only non-codewords in $V_{n}$, to take $u_{i}$ and $u_{i+2}$ as codewords, not $u_{i}$ and $u_{i+1}$ nor $u_{i+1}$ and $u_{i+2}$.

Finally, Figure 4 gives all the nonisomorphic optimal identifying codes in $S_{n}$, for $n=3, n=4, n=5$ and $n=6$, with decreasing number of codewords in $V_{n}$. Figure 5 gives, for $n=7$, the four possible configurations for $V_{n}$, giving respectively one, one, three and one nonisomorphic optimal identifying codes in $S_{7}$.

An open question is to determine the number of nonisomorphic optimal identifying codes in $S_{n}$ for $n>7$, but in the following Section, we manage
(a) $V_{n}$

(b) $V_{n}$

(c) $V_{n}$


Figure 6: Examples of partitions of $V_{n}$ : (a) no duet, $v_{1,2}$ not in a brick; (b) no duet, $v_{1,2}$ in the brick $B$; (c) one duet.
to count all the optimal identifying codes in $S_{n}$, isomorphic or not.

## 4 Counting the Optimal Codes

### 4.1 The case when there is no duet in $V_{n}$

Consider the set $V_{n}=\left\{v_{1,2}, v_{2,3}, \ldots, v_{n-1, n}, v_{n, 1}\right\}$ and an optimal identifying code $C_{n}$ such that there is no duet in $V_{n}$; for $i \in\{1, \ldots, n\}$, the set $\left\{v_{i-1, i}, v_{i, i+1}, v_{i+1, i+2}\right\}$ is called a brick if $v_{i-1, i} \in C_{n}, v_{i, i+1} \notin C_{n}$, and $v_{i+1, i+2} \in C_{n}$. Let $k$ be the number of bricks is $V_{n} ; k$ is also the number of (isolated) non-codewords in $V_{n}$, and we have seen previously that $k \in K_{n}=\{0\} \cup\left\{2,3, \ldots,\left\lfloor\frac{n}{3}\right\rfloor\right\}$. Two sets of vertices of $V_{n}$ are said to be consecutive if both consist of consecutive vertices in $V_{n}$, and if the last vertex of one set and the first vertex of the other set are consecutive.

Assume first that $v_{1,2}$ does not belong to any brick; in particular, $v_{1,2} \in$ $C_{n}$. Going from $v_{2,3}$ to $v_{n, 1}$, we can partition $V_{n} \backslash\left\{v_{1,2}\right\}$ into $2 k+1$ consecutive sets in the following way (see Figure 6(a)):

$$
\begin{equation*}
V_{n} \backslash\left\{v_{1,2}\right\}=H_{0} \cup B_{1} \cup H_{1} \cup \ldots \cup H_{k-1} \cup B_{k} \cup H_{k} \tag{1}
\end{equation*}
$$

where the sets $H_{i}$ contain only codewords and can have size zero, and the sets $B_{i}$ are bricks (of size 3). Let $h_{i}=\left|H_{i}\right|$. The partitions of type (1) are characterized by the sizes of the sets $H_{i}$, and their number is given by the number of solutions of

$$
\begin{equation*}
\sum_{i=0}^{k} h_{i}=n-1-3 k, 0 \leq h_{i} \tag{2}
\end{equation*}
$$

which is known to be $\binom{n-2 k-1}{k}$, see, e.g., [8]. We have seen that for each isolated non-codeword $v_{j, j+1}$ in $V_{n}$, we can choose as codeword in $U_{n}$ either $u_{j}$ or $u_{j+1}$. So, under these conditions (no duet, no brick containing $v_{1,2}$ ), the number of optimal identifying codes is

$$
\begin{equation*}
\alpha_{n}=\sum_{k \in K_{n-1}} 2^{k}\binom{n-2 k-1}{k}=\sum_{k=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor} 2^{k}\binom{n-2 k-1}{k}-2(n-3) . \tag{3}
\end{equation*}
$$

Assume next that $v_{1,2}$ belongs to one brick (hence $k \neq 0$ ), and let $k^{*}=k-1$. There are three possibilities according to the position of $v_{1,2}$ in the brick (leftmost, middle or rightmost). Then we can apply the previous argument, with the following differences nonetheless: $k^{*} \in\left\{1,2, \ldots,\left\lfloor\frac{n}{3}\right\rfloor-1\right\}$, and the partition is done on the $n-3$ vertices surrounding the brick containing $v_{1,2}$, with $k^{*}$ bricks and $k^{*}+1$ sets $H_{i}$; see Figure 6(b). So (2) becomes

$$
\sum_{i=0}^{k^{*}} h_{i}=n-3-3 k^{*}, 0 \leq h_{i},
$$

which admits $\binom{n-2 k^{*}-3}{k^{*}}$ solutions. Keeping in mind that we have three positions for $v_{1,2}$ inside its brick, and two possibilities in the choice of the codeword in $U_{n}$ which dominates $v_{1,2}$, we can see that the number of optimal identifying codes is in this case is given by

$$
\begin{equation*}
\beta_{n}=3 \times 2 \times \sum_{k^{*}=1}^{\left\lfloor\frac{n-3}{3}\right\rfloor} 2^{k^{*}}\binom{n-2 k^{*}-3}{k^{*}} . \tag{4}
\end{equation*}
$$

So, in the absence of duet, the number of optimal identifying codes in $S_{n}$ is equal to

$$
\begin{equation*}
\delta_{n}=\alpha_{n}+\beta_{n}, \tag{5}
\end{equation*}
$$

where $\alpha_{n}$ and $\beta_{n}$ are given by (3) and (4), respectively.

### 4.2 The case when there is one duet in $V_{n}$

Now we assume that $C_{n}$ is an optimal identifying code such that there is one duet in $V_{n}$ (we have seen previously that there can be at most one duet).

First we choose the position of the duet; we have $n$ possibilities. The duet, together with its left and right neighbouring codewords, occupies four vertices. Then we proceed as before, with $k$ bricks, $k \in\left\{0,1,2, \ldots,\left\lfloor\frac{n-4}{3}\right\rfloor\right\}$ (no restriction on the number of bricks when there is a duet): we have to choose a partition on the $n-4$ vertices surrounding the duet and its left and right neighbours (see Figure 6(c)), so we have to consider the number of solutions of

$$
\sum_{i=0}^{k} h_{i}=n-4-3 k, 0 \leq h_{i},
$$

which is $\binom{n-2 k-4}{k}$. We have seen that there is no choice for the two codewords in $U_{n}$ taking care of the duet, and we can conclude that the number of optimal identifying codes is given by

$$
\begin{equation*}
\omega_{n}=n \times \sum_{k=0}^{\left\lfloor\frac{n-4}{3}\right\rfloor} 2^{k}\binom{n-2 k-4}{k} \tag{6}
\end{equation*}
$$

in case of a duet.

### 4.3 Conclusion

Gathering (3), (4), (5) and (6) gives the total number of optimal identifying codes, which is

$$
\begin{array}{r}
\Phi_{n}=\sum_{k=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor} 2^{k}\binom{n-2 k-1}{k}+6 \cdot \sum_{k=1}^{\left\lfloor\frac{n-3}{3}\right\rfloor} 2^{k}\binom{n-2 k-3}{k} \\
+n \cdot \sum_{k=0}^{\left\lfloor\frac{n-4}{3}\right\rfloor} 2^{k}\binom{n-2 k-4}{k}-2(n-3) . \tag{7}
\end{array}
$$

The first values given by (7) are $\Phi_{3}=1, \Phi_{4}=5, \Phi_{5}=6, \Phi_{6}=19, \Phi_{7}=50$, $\Phi_{8}=89$ and $\Phi_{9}=160$.

In order to obtain an asymptotic estimation of $\Phi_{n}$, we need to approximate $M_{n}=\max _{k \in\{0, \ldots,\lfloor n / 3\rfloor\}} 2^{k}\binom{n-2 k}{k}$. Setting $N=n-2 k$ and $\varepsilon=k / N$ leads to study $\max _{\varepsilon \in] 0,1[ } 2^{N \varepsilon}\binom{N}{N \varepsilon}$, and, using a standard approximation on the binomial coefficient (see, e.g., (2.4.3) in [2]), to study $\max _{\varepsilon \in] 0,1[ } N(\varepsilon+H(\varepsilon))$, where $H(x)=-x \log _{2}(x)-(1-x) \log _{2}(1-x)$ is the (binary) entropy function. Going back to $n$, we have to search for $\max _{\varepsilon \in] 0,1}\left[n \frac{\varepsilon+H(\varepsilon)}{1+2 \varepsilon}\right.$, that is, to search for $\max _{\varepsilon \in] 0,1[ } \frac{\varepsilon+H(\varepsilon)}{1+2 \varepsilon}$.

This maximum is reached for $\varepsilon_{0} \approx 0.41025$ and $k_{0} \approx 0.22535 n$, and is approximately 0.76181 , which leads finally to $M_{n} \gtrsim 2^{0.76181 n}$. Since $n^{*}=2 n$ is the order of the complete sun $S_{n}$, we have, in terms of order, about $2^{0.38091 n^{*}}$ different optimal identifying codes in the complete sun.

Let us now consider briefly the problem of finding graphs of order $n^{*}$ with a large number of optimal identifying codes, see [5]. The complete suns are graphs for which the optimal identifying codes have size equal to half their order, and as such could be candidates, since $\binom{n^{*}}{\gamma^{I D}(G)}$, the number of all subsets of vertices of size $\gamma^{I D}(G)$ in any graph $G$ of order $n^{*}$, is maximum when $\gamma^{I D}(G)=n^{*} / 2$. However, the above result, $2^{0.38091 n^{*}}$, is modest when compared to the best known result, namely $2^{0.77003 n^{*}}([5$, Th. 11]).


Figure 7: Constructions of LD codes in $S_{n}$.

## 5 Determining $\gamma^{L D}\left(S_{n}\right)$

The first two values of $n$ are easy to study, and we have: $\gamma^{L D}\left(S_{3}\right)=3$, $\gamma^{L D}\left(S_{4}\right)=3$. From now on, we assume that $n \geq 5$.

We first give five constructions, according to the congruences of $n$ modulo 5 , proving that for $n \geq 5$, we have:

$$
\begin{equation*}
\gamma^{L D}\left(S_{n}\right) \leq\left\lfloor\frac{4 n+2}{5}\right\rfloor . \tag{8}
\end{equation*}
$$

These constructions are given by Figure 7 and are straightforward to check (remember that only the non-codewords have to be tested).

We next turn to the lower bound which we want to prove:

$$
\gamma^{L D}\left(S_{n}\right) \geq\left\lfloor\frac{4 n+2}{5}\right\rfloor .
$$

The following lemma, where (a) and (b) match Lemma 1 for identifying codes, will prove useful.

Lemma 4 Let $C_{n}$ be a $L D$ code in $S_{n}$, and $i \in\{1, \ldots, n\}$. Then:
(a) $C_{n} \cap\left\{v_{i-1, i}, u_{i}, u_{i+1}, v_{i+1, i+2}\right\}=\emptyset$ is impossible;
(b) $C_{n} \cap\left\{v_{i-1, i}, u_{i}, v_{i, i+1}\right\}=\emptyset$ is possible for at most one value of $i$;
(c) $C_{n} \cap\left\{u_{i}, v_{i, i+1}, u_{i+1}\right\}=\emptyset$ is impossible;
(d) $C_{n} \cap\left\{u_{i}, v_{i, i+1}, v_{i+1, i+2}, u_{i+2}\right\}$ is impossible.

Proof. (a) Because $u_{i}$ and $u_{i+1}$ would not be separated.
(b) Assume on the contrary that there exists $j \in\{1, \ldots, n\}, j \neq i$, such that $C_{n} \cap\left\{v_{j-1, j}, u_{j}, v_{j, j+1}\right\}=\emptyset$; then $u_{i}$ and $u_{j}$ are not separated.
(c) Because $v_{i, i+1}$ must be dominated.
(d) Because $v_{i, i+1}$ and $v_{i+1, i+2}$ must be separated.

In the following, we shall say that the sets $\left\{v_{i-1, i}, u_{i}, v_{i, i+1}\right\}$ and $\left\{u_{i}, v_{i, i+1}\right.$, $\left.u_{i+1}\right\}$ are triangles, denoted by $\nabla$ and $\triangle$, respectively. We shall say that


Figure 8: The 10-slice $T_{i, 10}$ with three codewords, one in $V_{n}$ and two in $U_{n}$.
a triangle is empty if it contains no codeword. Now, with this terminology, Lemma 4(b) states that there is at most one empty triangle $\nabla$, and Lemma 4(c) states that there is no empty triangle $\triangle$.

Amidst all the LD codes in $S_{n}$ with size $\gamma^{L D}\left(S_{n}\right)$, we take one, and we denote it by $C_{n}$. Assume that $V_{n} \cap C_{n}=\emptyset$; then the vertices in $U_{n}$ cannot be separated, unless all of them but one are codewords. This implies $\left|C_{n}\right| \geq n-1 \geq\left\lfloor\frac{4 n+2}{5}\right\rfloor$, for $n \geq 5$. So from now on, we can take for granted that $V_{n}$ contains at least one codeword.

Step 1. Any set $\left\{v_{i, i+1}, v_{i+1, i+2}, \ldots, v_{i+k-1, i+k}, u_{i+1}, u_{i+2}, \ldots, u_{i+k}\right\}, i \in$ $\{1, \ldots, n\}, k \in\{1, \ldots, n\}$, is denoted by $T_{i, 2 k}$ and is called a $2 k$-slice. In this first step, our goal is to show that for all $i \in\{1, \ldots, n\}$,

$$
\begin{equation*}
\left|T_{i, 10} \cap C_{n}\right| \geq 4, \tag{9}
\end{equation*}
$$

with the exception of three cases which will be specifically treated. We are going to study how many codewords can be in $V_{n} \cap T_{i, 10}$, for any $i$.
(1) If there are four or five codewords in $V_{n} \cap T_{i, 10}$, then (9) is satisfied.
(2) Next, assume that there are three codewords in $V_{n} \cap T_{i, 10}$; then, because there are four triangles $\triangle$ in $T_{i, 10}$, at least one more codeword is necessary in $U_{n} \cap T_{i, 10}$, and (9) is satisfied.
(3) We now turn to the case when there are two codewords (among five vertices) in $V_{n} \cap T_{i, 10}$; all the $\binom{5}{2}=10$ configurations require at least two more codewords in $U_{n} \cap T_{i, 10}$, and again (9) is satisfied.
(4) If $\left|V_{n} \cap C_{n} \cap T_{i, 10}\right|=1$, then, among the five possibilities, one can see, in particular using Lemma 4, that only one is possible with only two codewords in $U_{n} \cap T_{i, 10}$, see Figure 8, where one can see an empty triangle $\nabla$.
(5) If $V_{n} \cap C_{n} \cap T_{i, 10}=\emptyset$, then, among the $\binom{5}{3}=10$ configurations for $U_{n} \cap T_{i, 10}$, one can see, again using Lemma 4, that only two of them can possibly lead to LD codes with three codewords in $U_{n}$; they are given in Figure 9. Note that both contain an empty triangle $\nabla$.

So far, the above five cases show that for all $i \in\{1, \ldots, n\}$, the set $C_{n} \cap T_{i, 10}$ has size either (a) at least 4 , or (b) equal to 3 ; in the latter case, only three configurations are possible, given by Figures 8 and 9 , and at most


Figure 9: The 10-slice $T_{i, 10}$ with three codewords, all of them in $U_{n}$.


Figure 10: Two 10-slices containing each three codewords.
one of them can appear, at most once, because they all contain an empty triangle $\nabla$; more precisely, this is true only if no intersection is allowed: see for instance Figure 10, where the configurations of Figures 8 and 9(a) coexist, but share eight vertices. Since our goal is to split $S_{n}$ into disjoint 10 -slices, we shall go on saying, with a slight abuse of language, that at most one of them appears at most once.

Moreover, in Figure 8, $u_{i} \in C_{n}$, by Lemma $4(\mathrm{c})$, and $v_{i+5, i+6} \in C_{n}$, in order to avoid the pattern of Lemma 4(a). The same is true in Figure 9(a), by Lemma 4(d)-(b), as well as in Figure 9(b), by Lemma 4(c)-(b). We shall say that in these three cases, the left outside and right outside neighbours of $T_{i, 10}$ must be codewords.

We can now proceed to Step 2, in which we show how, according to the congruences of $n$ modulo 5 , we can split $S_{n}$ into 10 -slices and one smaller slice, and we prove the lower bound.

Step 2. Given $C_{n}$, for $k \in\{1, \ldots, n\}$ and $j \in\{0, \ldots, 2 k\}$, we say that a $2 k$-slice containing exactly (respectively, at least) $j$ codewords is a $(2 k, j)$ slice (respectively, a $(2 k, \geq j)$-slice). With this notation, what we have just proved in Step 1 is that our graph, when arbitrarily split into disjoint 10 -slices (plus one $r$-slice, $r \in\{0,2,4,6,8\}$ ), can only contain at most one $(10,3)$-slice, and $(10, \geq 4)$-slices. Moreover, any $(10,3)$-slice has its left and right outside neighbours in the code, which in particular helps to show that


Figure 11: Illustration of Lemma 5, the (10, 4)-slice $T_{i, 10}$.
a ( 10,3 )-slice cannot give by itself a LD code in $S_{5}$, and proves that

$$
\begin{equation*}
\gamma^{L D}\left(S_{5}\right) \geq 4 \tag{10}
\end{equation*}
$$

The following lemma gives information on the ( 10,4 )-slices when there is a $(10,3)$-slice in $S_{n}$.

Lemma 5 Assume that we have one $(10,3)$-slice $T_{j, 10}$ in $S_{n}$. Let $T_{i, 10}$ be any $(10,4)$-slice with no intersection with $T_{j, 10}$. Then:
(a) It is impossible to have $v_{i, i+1} \in C_{n}$ and $u_{i+5} \in C_{n}$ simultaneously;
(b) If $v_{i, i+1} \in C_{n}$, then $v_{i+5, i+6} \in C_{n}$;
(c) If $u_{i+5} \in C_{n}$, then $u_{i} \in C_{n}$.

Proof. The fact that we have a ( 10,3 )-slice forbids that an empty triangle $\nabla$ appears in $T_{i, 10}$. So in Figure 11, there is at least one codeword in each of the triangles $\nabla$. Also, by Lemma 4(c), there is at least one codeword in the triangles $\triangle$. We represent only the triangles which are used in the proof.
(a) Assume that $v_{i, i+1} \in C_{n}$ and $u_{i+5} \in C_{n}$, see Figure 11(a). If $v_{i+1, i+2} \in C_{n}$ or $u_{i+1} \in C_{n}$, then $v_{i+2, i+3}$ and $v_{i+3, i+4}$ are still to be dominated and separated, which is impossible with only one additional codeword. So $v_{i+1, i+2} \notin C_{n}, u_{i+1} \notin C_{n}$, which implies that $u_{i+2} \in C_{n}$ (triangle $\triangle$ ). Then the remaining codeword is at the intersection of the two triangles $\nabla$, $v_{i+3, i+4}$, but we have the forbidden pattern of Lemma 4(a).
(b) Assume that $v_{i, i+1} \in C_{n}$ and $v_{i+5, i+6} \notin C_{n}$, see Figure 11(b). By (a) just above, we have $u_{i+5} \notin C_{n}$. Now necessarily $v_{i+4, i+5} \in C_{n}$ (triangle $\nabla$ ). Then by Lemma $4(\mathrm{a}), u_{i+4} \in C_{n}$ or $v_{i+3, i+4} \in C_{n}$, and there remains only one codeword, which cannot dominate and separate $v_{i+1, i+2}$ and $v_{i+2, i+3}$.
(c) Assume that $u_{i+5} \in C_{n}$ and $u_{i} \notin C_{n}$, see Figure 11(c). By (a), we have $v_{i, i+1} \notin C_{n}$, which implies $u_{i+1} \in C_{n}$ (leftmost triangle $\triangle$ ). But $v_{i, i+1}$ and $v_{i+1, i+2}$ are not separated yet, so $v_{i+1, i+2} \in C_{n}$ or $u_{i+2} \in C_{n}$. If it is $v_{i+1, i+2}$, there remain two empty disjoint triangles, one $\triangle$ and one $\nabla$, with only one more codeword available. If it is $u_{i+2}$, then the fourth codeword is the vertex at the intersection of the two triangles $\nabla, v_{i+3, i+4}$, but we have the forbidden pattern of Lemma 4(a).

Remark 6 The previous lemma is still true if we weaken its condition, by assuming now that there is in $S_{n}$ an empty triangle $\nabla$ with no intersection with $T_{i, 10}$.

We are now going to consider the different congruences of $n$ modulo 5 .
Proposition 7 For every integer $k \geq 1$, we have $\gamma^{L D}\left(S_{5 k}\right)=4 k$.
Proof. The upper bound (8) gives $\gamma^{L D}\left(S_{5 k}\right) \leq 4 k$. By (10), we can assume that $k>1$. The only way to have fewer than $4 k$ codewords in a LD code is to have one $(10,3)$-slice and $k-1(10,4)$-slices, in which case we have $4 k-1$ codewords. Assume wlog that the $(10,3)$-slice is $T_{1,10}$, and consider the $k-1(10,4)$-slices $T_{6,10}, T_{11,10}, \ldots, T_{5 k-4,10}$. We have observed just before inequality (10) that $T_{1,10}$ has its right outside neighbour in the code, i.e., $v_{6,7} \in C_{n}$. Then, by Lemma $5(\mathrm{~b}), v_{11,12} \in C_{n}, \ldots, v_{5 k-4,5 k-3}=v_{n-4, n-3} \in$ $C_{n}$, which implies, by Lemma $5(\mathrm{a})$ applied with $i=n-4$, that $u_{1} \notin C_{n}$, contradicting the fact that $T_{1,10}$ has its left outside neighbour in the code. Note that we could have gone the other way round, using Lemma 5(c): $T_{1,10}$ has its left outside neighbour in the code, so $u_{1} \in C_{n}, u_{5 k-4} \in C_{n}, \ldots$, $u_{11} \in C_{n}$, which implies that $v_{6,7} \notin C_{n}$, which contradicts the fact that $T_{1,10}$ has its right outside neighbour in the code.

Anyway, we have proved that the lower and upper bounds coincide. $\diamond$
Proposition 8 For every integer $k \geq 1$, we have $\gamma^{L D}\left(S_{5 k+1}\right)=4 k+1$.
Proof. The upper bound (8) gives $\gamma^{L D}\left(S_{5 k+1}\right) \leq 4 k+1$. Assume first that $k=1$. Is $\left|C_{6}\right|=4$ possible?

If there is a $(10,3)$-slice in $S_{6}$, then, since its left and right outside neighbours are codewords, we have five codewords. So we can exclude a ( 10,3 )-slice.

Consider a $(2, \geq 1)$-slice (there must be one!). Wlog, it is $T_{1,2}$. Then the 10 -slice $T_{2,10}$ is a $(10, \geq 4)$-slice, and we have at least five codewords. So $\gamma^{L D}\left(S_{6}\right)=5$.

From now on, $k>1$. Assume first that there is a ( 10,3 )-slice, say $T_{2,10}$, and consider the 2 -slice $T_{1,2}$. Because $u_{2}$, the left outside neighbour of $T_{2,10}$, is a codeword, $T_{1,2}$ is a $(2, \geq 1)$-slice, and because there can be only one ( 10,3 )-slice, the $k-110$-slices $T_{7,10}, \ldots, T_{5 k-3,10}$ are ( $10, \geq 4$ )-slices. This implies, if we want to get at most $4 k$ codewords, that $T_{1,2}$ is a $(2,1)$-slice,
that the $k-110$-slices other than $T_{2,10}$ are $(10,4)$-slices, and that we have exactly $4 k$ codewords. Because the right outside neighbour of $T_{2,10}, v_{7,8}$, is a codeword, and using repeatedly Lemma $5(\mathrm{~b})$, we have $v_{7,8} \in C_{n}, v_{12,13} \in C_{n}$, $\ldots, v_{5 k+2,5 k+3}=v_{1,2} \in C_{n}$, i.e., $T_{1,2}$ has its two elements, $u_{2}$ and $v_{1,2}$, in $C_{n}$, a contradiction. So there is no $(10,3)$-slice in $S_{5 k+1}$.

As in the case $k=1$, consider a $(2, \geq 1)$-slice, say $T_{1,2}$. Then the $k$ 10-slices $T_{2,10}, T_{7,10}, \ldots, T_{5 k-3,10}$, which are ( $10, \geq 4$ )-slices, lead to at least $4 k+1$ codewords.

So no LD code with $4 k$ elements exists in $S_{5 k+1}$.
Proposition 9 For every integer $k \geq 1$, we have $\gamma^{L D}\left(S_{5 k+2}\right)=4 k+2$.
Proof. The upper bound (8) gives $\gamma^{L D}\left(S_{5 k+2}\right) \leq 4 k+2$. Assume first that $k=1$. Is $\left|C_{7}\right|=5$ possible? If there is a $(10,3)$-slice in $S_{7}$, say $T_{1,10}$, then, since its two outside neighbours, $u_{1}$ and $v_{6,7}$, are codewords, we have already five codewords. So $u_{7} \notin C_{7}$ and $v_{7,1} \notin C_{7}$. Then, taking the configurations of the $(10,3)$-slice as given by Figures 8 or $9(\mathrm{~b})$, where neither $v_{i, i+1}$ (here, $v_{1,2}$ ) nor $u_{i+1}$ (here, $u_{2}$ ) are codewords, we can see that $v_{7,1}$ and $v_{1,2}$ are not separated, i.e., at least six codewords are necessary. If we use the configuration of Figure $9(\mathrm{a})$, where $v_{i+4, i+5}$ and $u_{i+5}$ (here, $v_{5,6}$ and $u_{6}$ ) are not codewords, we have the pattern forbidden by Lemma 4(a). So no $(10,3)$-slice is possible. If we have a $(10,4)$-slice, see the general case with $k(10,4)$-slices. If we have a $(10,5)$-slice, and since the 4 -slice must be a $(4, \geq 1)$-slice (existence of a $\triangle$ ), we have also at least six codewords. So $\gamma^{L D}\left(S_{7}\right)=6$.

From now on, $k>1$. Can we have $4 k+1$, or fewer, codewords? Assume first that there is a $(10,3)$-slice, say $T_{1,10}$. Its outside neighbours, $u_{1}$ and $v_{6,7}$, are codewords.

Consider the 4 -slice $T_{6,4}$. Because $v_{6,7} \in C_{n}$, and because there is a triangle $\triangle$ in $T_{6,4}, T_{6,4}$ is a $(4, \geq 2)$-slice. The $k-110$-slices $T_{8,10}, \ldots$, $T_{5 k-2,10}$ are $(10, \geq 4)$-slices, so that necessarily we have: one $(10,3)$-slice, one (4, 2)-slice, $k-1$ (10, 4)-slices, and exactly $4 k+1$ codewords.

Assume first that $T_{1,10}$ is given by Figure 9 (a), so that $v_{5,6} \notin C_{n}, u_{6} \notin C_{n}$. The fact that $u_{1} \in C_{n} \cap T_{5 k-2,10}$ implies, by Lemma 5 (c), that $u_{5 k-2} \in C_{n}$, $\ldots, u_{8} \in C_{n}$, so that the two non-codewords in $T_{6,4}$ must be $u_{7}$ and $v_{7,8}$. Now the non-codewords $v_{5,6}, u_{6}, u_{7}$ and $v_{7,8}$ form a forbidden pattern.

Assume next that we are in one of the configurations of Figures 8 or $9(\mathrm{~b})$ : in particular, $v_{1,2} \notin C_{n}$ and $u_{2} \notin C_{n}$. Now we consider the 4 -slice $T_{n-1,4}$, which contains the codeword $u_{1}$, and a second codeword because it contains a triangle $\nabla$, and there is already an empty $\nabla$ in $T_{1,10}$. Since $v_{6,7} \in C_{n} \cap T_{6,10}$, we have, as previously seen several times, $v_{11,12} \in C_{n}, \ldots, v_{5 k+1,5 k+2}=$ $v_{n-1, n} \in C_{n}$, i.e., $C_{n} \cap T_{n-1,4}=\left\{u_{1}, v_{5 k+1,5 k+2}\right\}$. But now the non-codewords $u_{5 k+2}, v_{5 k+2,1}, v_{1,2}$ and $u_{2}$ form a forbidden pattern.

Both cases lead to a contradiction, showing that there is no $(10,3)$-slice in $S_{5 k+2}$.

Then, to have $4 k+1$ codewords, we can only have $k(10,4)$-slices and one $(4,1)$-slice. Take one codeword $v_{\ell, \ell+1}$ in $V_{n}$ (we know that there is one), and consider the 4 -slice $T_{\ell, 4}$. Because there is a triangle $\triangle$ in $T_{\ell, 4}$, there are at least two codewords in $T_{\ell, 4}$ and we cannot reach $4 k+1$ codewords.

Proposition 10 For every integer $k \geq 1$, we have $\gamma^{L D}\left(S_{5 k+3}\right)=4 k+2$.
Proof. The upper bound (8) gives $\gamma^{L D}\left(S_{5 k+3}\right) \leq 4 k+2$. Observe that any 6 -slice is necessarily a $(6, \geq 2)$-slice: if there is only one codeword, it must be the middle vertex in $U_{n}$, because it is at the intersection of the two triangles $\triangle$, but then there is the forbidden pattern of Lemma 4(d).

Assume first that $k=1$. Is $\left|C_{8}\right|=5$ possible? If there is a $(10, \geq 4)$-slice, then there are at least $4+2=6$ codewords. So we assume that there is a $(10,3)$-slice. Its two outside neighbours are codewords, but there is still one empty triangle $\triangle$ in the 6 -slice (and also one $\nabla$ ), so at least one additional codeword is required. We can conclude that $\gamma^{L D}\left(S_{8}\right)=6$.

Assume next that $k>1$. The only way to have at most $4 k+1$ codewords is to have one $(10,3)$-slice, $k-1(10,4)$-slices and one $(6,2)$-slice, and we have exactly $4 k+1$ codewords. Wlog, the $(10,3)$-slice is $T_{4,10}$, and we consider the (6,2)-slice $T_{1,6}$. Then $u_{4}$, the left outside neighbour of $T_{4,10}$, is a codeword; the second codeword in $T_{1,6}$ must belong to the triangle $\triangle$ consisting of $u_{2}, v_{2,3}, u_{3}$, but each of the three possibilities leads to a forbidden pattern, either by Lemma $4(\mathrm{a})$ or (b) (since an empty triangle $\nabla$ already appears in $\left.T_{4,10}\right)$. This concludes the case $k>1$.

Proposition 11 For every integer $k \geq 1$, we have $\gamma^{L D}\left(S_{5 k+4}\right)=4 k+3$.
Proof. The upper bound (8) gives $\gamma^{L D}\left(S_{5 k+4}\right) \leq 4 k+3$. If we consider any 8 -slice, say $T_{1,8}$, it is straigthforward to see (for instance by checking the nine possibilities given by the two disjoint triangles $\triangle$ ) that the only possibility with two codewords in $T_{1,8}$ that avoids any forbidden pattern is $C_{n} \cap T_{1,8}=$ $\left\{u_{3}, u_{4}\right\}$, which however contains an empty triangle $\nabla$, incompatible with a (10, 3)-slice.

If $k=1$, to have only six codewords would lead to a $(10,3)$-slice with a $(8,3)$-slice, or a $(10,4)$-slice with a $(8,2)$-slice. In the first case, if the $(10,3)$ slice is $T_{5,10}$, its outside neighbours $u_{5}$ and $v_{1,2}$ are codewords, the third codeword in $T_{1,8}$ is $u_{3}$, because it is at the intersection of two triangles $\triangle$ with no codeword yet, but then the forbidden pattern of Lemma 4(d) appears. In the second case, if the $(10,4)$-slice is $T_{5,10}$, and, remembering that the only two codewords in $T_{1,8}$ are $u_{3}$ and $u_{4}$, we can see immediately that $u_{1} \in C_{9}$ (to avoid an empty triangle $\triangle$ ) and $v_{5,6} \in C_{9}$ (to avoid a second empty triangle $\nabla$ ). This however contradicts, for $n=9$ and $i=5$, Lemma 5(a), used together with Remark 6, since there is an empty triangle $\nabla$ outside $T_{5,10}$. So $\gamma^{L D}\left(S_{9}\right)=7$.

Now $k>1$, and we assume that there is a $(10,3)$-slice, say $T_{5,10}$. Then, to obtain $4 k+2$ codewords, we have one $(8,3)$-slice and $k-1(10,4)$-slices.

Consider the 8 -slice $T_{1,8}$. Then the right outside neighbour of $T_{5,10}, v_{10,11} \in$ $T_{10,10}$, is a codeword, and so are, by Lemma $5(\mathrm{~b}), v_{15,16}, \ldots, v_{5 k+5,5 k+6}=$ $v_{1,2}$. Also, the left outside neighbour of $T_{5,10}, u_{5}$, is a codeword. So we have $\left\{v_{1,2}, u_{5}\right\} \subset C_{n} \cap T_{1,8}$, and, exactly as above, in the first case for $k=1$, we are led to a contradiction. Conclusion: there is no (10, 3)-slice in $S_{5 k+4}$.

Now the only way to have at most $4 k+2$ codewords is to have $k(10,4)$ slices and one ( 8,2 )-slice.

Consider the $(8,2)$-slice $T_{1,8}$; we have, as in the second case for $k=1$, $u_{1} \in C_{n}$ and $v_{5,6} \in C_{n}$. Then again, because we are allowed to use Lemma 5 thanks to the existence of an empty triangle $\nabla$ in the $(8,2)$-slice, we have, by Lemma $5(\mathrm{~b}), v_{10,11} \in C_{n}, \ldots, v_{5 k, 5 k+1}=v_{n-4, n-3} \in C_{n}$, but the fact that $v_{n-4, n-3} \in C_{n}$, together with $u_{1} \in C_{n}$, leads to a contradiction by Lemma 5(a), applied to $i=n-4$.

We gather all our results in the following proposition.
Proposition 12 (a) $\gamma^{L D}\left(S_{3}\right)=3$;
(b) For every integer $n \geq 4, \gamma^{L D}\left(S_{n}\right)=\left\lfloor\frac{4 n+2}{5}\right\rfloor$.

It would be interesting to do the same as for identifying codes and be able to characterize all the optimal locating-dominating codes in $S_{n}$, but from what we have seen above, it seems quite intricate to try to do so, and this remains an open problem.

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