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QUANTUM MECHANISM OF CONDENSATION AND HIGH TC SUPERCONDUCTIVITY

TIAN MA AND SHOUHONG WANG

ABSTRACT. The main objective of this paper is to introduce a new quantum mechanism of condensates and for superconductivity based on new interpretation of quantum mechanical wave functions, and on recent developments in quantum physics and statistical physics. First, we postulate that the wave function $\psi = |\psi|e^{i\varphi}$ is the common wave function for all particles in the same class determined by the external potential $V(x)$, $|\psi(x)|^2$ represents the distribution density of the particles, and $\frac{\hbar}{m}\nabla\varphi$ is the velocity field of the particles. Although the new interpretation does not alter the basic theories of quantum mechanics, it is an entirely different interpretation from the classical Bohr interpretation, removes all absurdities, and offers new insights for quantum physics and for condensed matter physics. Second, we show that the key for condensation of bosonic particles is that their interaction is sufficiently weak to ensure that a large collection of boson particles are in a state governed by the same condensation wave function field ψ under the same bounding potential V . For superconductivity, the formation of superconductivity comes down to conditions for the formation of electron-pairs, and for the electron-pairs to share a common wave function. Thanks to the recently developed PID interaction potential of electrons and the average-energy level formula of temperature, these conditions for superconductivity are explicitly derived. Furthermore, we obtain both microscopic and macroscopic formulas for the critical temperature. The field and topological phase transition equations for condensates are also derived.

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1. INTRODUCTION

The main objective of this paper is to attempt to derive the quantum mechanism of condensates and high temperature superconductivity, based on the new interpretation of quantum mechanical wave functions, and on the recent developments of fundamental interactions and statistical physics. The main results of the paper include the following:

- (1) new interpretation of wave functions in quantum mechanics,
- (2) quantum mechanism of condensates and (high T_c) superconductivity,
- (3) new theory of electron-pairs, and
- (4) microscopic and macroscopic formulas of the critical temperature for superconductivity.

Motivations

The study of this paper is strongly motivated by the lack of physical mechanism of condensates and high temperature of superconductivity. The phenomenon of superconductivity was first discovered by Kamerlingh Onnes in 1911, in metallic mercury below 4K. Since then, the main goals of study have been to attempt to observe superconductivity at increasing temperatures, and to derive its physical mechanism.

The classical BCS theory [8], developed by John Bardeen, Leon Cooper, and John Schrieffer for which they shared the 1972 Nobel Prize, is a microscopic theory based on the Cooper electron-pair mechanism, which was used to explain superconducting currents, the Meissner effect, the isotope effect, and to calculate specific heats and penetration

depths. The Cooper pair theory provides the mechanism of electron-phonon interaction in low temperature superconductors, and a physical mechanism form the Bose electron-pair as required in (4.1). However, the BCS theory is applicable only for $T < 35\text{K}$.

In 1986, Bednorz and Müller [10] discovered superconductivity in a lanthanum-based cuprate perovskite material, which had a transition temperature of 35 K (Nobel Prize in Physics, 1987). Bednorz and Müller's work opened the door for high T_c superconductors. The current highest temperature of superconductors reaches $T_c = 130\text{K}$, and $T_c = 160\text{K}$ under high pressure.

The discovery of High T_c superconductors shows the limitation of the BCS theory. There have been other attempts for the understanding of the nature of superconductivity; see among others [4, 5, 86, 25, 97, 123, 63, 47, 117, 96, 9, 32, 136, 120, 20, 36, 110, 137, 40, 116, 104, 108, 82, 77, 99, 51, 111, 27, 73, 35, 46, 33, 18, 78, 2, 50, 118, 41, 23, 93, 43, 44, 129, 3, 61, 103, 17, 64, 55, 91, 134, 60, 94, 70, 49, 127, 125, 100, 62, 31, 26, 84, 28, 79, 38, 65, 98, 106, 76, 21, 109, 113, 80, 7, 119, 57, 131, 112, 52, 22, 75, 132, 13, 69, 6, 34, 58, 68, 72, 30, 42, 122, 133, 92, 24, 121, 107, 135, 114, 105, 19, 126, 102, 128, 74, 66, 53, 48, 67, 16, 29, 85, 138, 115, 37, 59, 95, 101, 45, 11, 15, 56, 130, 1, 71, 124].

However, after more than thirty years of intensive research, the origin of high-temperature superconductivity is still not completely understood. Fortunately, the recent developments of quantum physics and statistical physics, together with a new interpretation of quantum wave functions, provide the needed new insights for this challenging problem, which is the main aim of the paper.

New interpretation of wave functions in QM

It is inevitable that a better understanding of the nature of superconductivity should built upon the fundamental level understanding of quantum physics. In classical quantum mechanics, a micro-particle is described by a complex-valued wave function Ψ , satisfying such a wave equation as the Schrödinger equation (2.1) with external interaction potential $V(x)$. The Schrödinger equation (2.1) conserves the energy, and the wave function Ψ can be expressed as $\Psi = e^{-iEt/\hbar}\psi(x)$, where E is the energy, and ψ is the time-independent wave function.

The classical Born statistical interpretation of quantum mechanics amounts to saying that without constraints, the motion of a micro-particle is random and there is no definite trajectory of the motion. Also $|\psi(x)|^2$ stands for the probability density of the particle appearing at the particular point x . The Born interpretation of the wave function is treated as a fundamental postulate of quantum mechanics. This leads

to the classical Einstein-Bohr debates, and is the origin of absurdities associated with the interpretation of quantum mechanics; see among many others [39, 14, 12, 54].

Our new interpretation, presented in detail in Section 2, says that $\psi = |\psi|e^{i\varphi}$ is the common wave function for all particles in the same class determined by the external potential $V(x)$, $|\psi(x)|^2$ represents the distribution density of the particles, and

$$(1.1) \quad \frac{\hbar}{m} \nabla \varphi$$

is the velocity field of the particles. The trajectories of the motion of the particles are then dictated by this velocity field, as shown in (2.10). The observed particles are the particles in the same class described by the same wave function, rather than a specific particle in the sense of classical quantum mechanics.

This is an entirely different interpretation from the classical Bohr interpretation. Also this new interpretation of wave functions does not alter the basic theories of quantum mechanics, and instead offers new understanding of quantum mechanics, and plays a fundamental role for the quantum theory of condensed matter physics and quantum physics.

It is worth mentioning that the Landau school of physics was the first who noticed that the relation between the superfluid velocity v_s and the wave function $\psi = |\psi|e^{i\varphi}$ of the *condensate* is given by (1.1); see [81, (26.12) on page 106]. However they fail to make an important connection between (1.1) and the basic interpretation of quantum mechanics, and consequently they were not able to derive the new quantum mechanism of condensates presented in this paper.

Quantum condensation mechanism and superconductivity

In view of the above new interpretation of quantum mechanic wave functions and the Pauli exclusion principle, under the condition of very weak interactions between particles, a large collection of Bose particles can undergo a motion governed by the same wave function field when the particles are all under the same bounding potential field. This gives rise to the physical picture of a condensate:

a condensate is a large collection of bosonic particles with the same wave function field ψ , called the condensation wave function or the condensation field, determined by the same bounding potential field V .

The physical picture results immediately the quantum condensation mechanism, stated in Condensation Mechanism 3.1. One key ingredient of this mechanism is that the condition for condensation of bosonic

particles is that their interaction is sufficiently weak to ensure that the a large collection of boson particles are in a state governed by the same condensation wave function field ψ under the same external bounding potential V . Another ingredient is that the condensation wave function ψ is the smallest allowed energy state of the wave equation. Hence the motion of particles in a condensate system under the same wave function field resembles the motion of stars in a galaxy, and can last permanently.

With this quantum condensation mechanism applied to superconducting systems, we obtain the following common physical mechanism of superconductivity:

- *there is a critical temperature $T_c > 0$ such that for $T < T_c$, a large number of free electrons in the system form bosonic electron-pairs; and*
- *the interactions between the bosonic electron-pairs and other particles in the system are negligible, so that the electron-pairs share a common interaction potential Φ . This common potential Φ enables the electron-pairs to occupy a single wave function field ψ with the velocity field (1.1), forming the condensate, and Φ is the potential given by the interaction between the lattice of the system and the external field.*

With this mechanism, the formation of superconductivity comes down to two important aspects: 1) the physical mechanism and condition for the formation of electron-pairs, and 2) the condition for the electron-pairs to share a common wave function. Hereafter we address these two questions in turn.

Formation of electron-pairs

The key issue is to reveal in the microscopic level the needed attractive potential/force between two electrons to counteract the repulsive Coulomb potential/force. In the classical BCS theory [8], the Cooper pair theory provides the mechanism of electron-phonon interaction in low temperature superconductors, and a physical mechanism form the Bose electron-pair as required. However, the BCS theory is applicable only for $T < 35\text{K}$. In fact, in the BCS theory, the Cooper electron-pair is explained as the electron-phonon interaction. This interaction is reminiscent to the Yukawa interaction mechanism in quantum field theory. The electron-phonon interaction means that by exchanging phonons, electrons induces an attractive interaction. However, the physical picture of such an interaction is not clear. The reason is that phonon is a hypothetical pseudo-particle, and is introduced in the quantum field

theory as a quantized energy to compute the binding energy Δ of the Cooper pair. In other words, the phonon is essentially a hypothesis, introduced for computing Δ , that in addition to the repulsive Coulomb potential between electrons, the deformation of lattices induces an attractive interaction potential.

Recently, in [88], using the principle of interaction dynamics (PID) and the $SU(N)$ gauge theory for multi-particle systems, the authors have obtained a layered electromagnetic interaction potential formula, which can be used here to derive conditions to ensure the validity of the mechanism (4.1) and (4.2) for superconductivity. For an electron pair, this new interaction potential, modifying the classical Coulomb potential, takes the form:

$$(1.2) \quad \Phi_e = \frac{e_0}{r} - e_0 A_0 (1 + kr) e^{-kr}.$$

Here e_0 is the electric charge of an electron, the first term in the right hand side is the Coulomb potential, and the second term represents the effect of the dual field ϕ of the electromagnetic potential induced from PID.

Notice also that there is also spin magnetic moment interaction between two electrons, which can lead to attractive potential between the electrons. We can show that the only stable spin arrangement is the anti-parallel electron-pair, with attracting spin magnetic moment potential given by $-\alpha\mu_0^2/e_0r$. Then we derive the interaction force formula of an anti-parallel electron-pair is given by

$$(1.3) \quad F = \text{Coulomb} + \text{magnetic-moment} + \text{effect of PID dual field} \\ = \frac{e_0^2}{r^2} \left(1 - \frac{\alpha\hbar^2}{4m^2c^2} \right) - k^2 A_0 r e^{-kr} e_0^2.$$

The interval, on which $F < 0$, defines the attracting shell region of two electrons: $\bar{r}_0 < r < \bar{r}_1$, whose existence is ensured by the condition:

$$(1.4) \quad A_0 > \left(\frac{e}{3} \right)^3 k \left(1 - \frac{\alpha\hbar^2}{4m^2c^2} \right).$$

A similar condition (4.30) is also derived to ensure that an anti-parallel electron-pair possesses attracting negative potential.

Binding energy of electron-pairs

One key ingredient for a theory of superconductivity is, as in the BCS theory, to examine the binding energy $2\Delta(T)$ of electron pairs. In a recent paper [90] on statistical theory of heat, we have derived an average energy level formula of temperature T . This, together with

potential associated with the force (1.3), leads to the following new formula for the binding energy:

$$(1.5) \quad \Delta = \frac{e_0^2}{2\bar{r}_0} \left[A(1 + k\bar{r}_0)e^{-k\bar{r}_0} + \frac{\alpha\hbar^2}{4m^2c^2} - 1 \right] - \varepsilon(T),$$

where $A = r_0A_0$ is a parameter as in (1.2), and $\varepsilon(T)$ is the average kinetic energy of the two-electrons.

One consequence of (1.5) is the equation for critical temperature T_c , i.e. $\Delta(T_c) = 0$:

$$(1.6) \quad \varepsilon(T_c) = \frac{e_0^2}{2\bar{r}_0} \left[A(1 + k\bar{r}_0)e^{-k\bar{r}_0} + \frac{\alpha\hbar^2}{4m^2c^2} - 1 \right].$$

With the above formulas, one can then easily verify the two conditions on the formation of electron pairs and on the existence of a global wave function ψ for the condensate, as required in the physical mechanism of superconductivity stated earlier.

Critical temperature formulas

There are two approaches to find the critical temperature formula. One is to use the standard model of thermodynamics based on the thermodynamic potential (free energy) under no external magnetic field; see [83]:

$$(1.7) \quad F = -g_0|\psi|^2 + \frac{1}{2}g_1|\psi|^4 - \frac{\alpha_0}{2}TS^2 - \alpha_1TS|\psi|^2 - ST,$$

where $g_0 > 0$ is the binding potential of condensates, g_1 is the coupling constant of particles, α_0 and α_1 are entropy coupling constants. Then the equilibrium state solves the Euler-Lagrangian equation of the potential functional and transition from the normal state to the superconducting state occurs at the critical temperature T_c , which makes the first eigenvalue of the linearized equilibrium state equation vanishing. We derive then the following macroscopic formula for T_c :

$$(1.8) \quad T_c = \frac{g_0^2}{g_1\Delta C_V}(1 - \alpha_0\Delta C_V).$$

The microscopic critical temperature formula is derived using on (1.6) and the average energy level formula of temperature [90]:

$$(1.9) \quad k_B T_c = \frac{N_1 N_2 (1 + \theta_0)}{N^2} E_0,$$

where $E_0 = \varepsilon(T_c)$ is the intrinsic energy of electron-pairs given explicitly by (1.6), $N = N_1 + N_2$ is the total number of particles in the

system, N_1 , N_2 are respectively the numbers of free electrons and lattice particles, and θ_0 is the following ratio at the critical temperature:

$$(1.10) \quad \theta = \frac{\text{average energy level of lattice particles}}{\text{average energy level of free electrons}}.$$

The paper is organized as follows. Section 2 introduces the new interpretation of quantum mechanical wave functions, Section 3 derives 1) the quantum mechanism of condensates and superconductivity, and 2) the field and pattern formation equations of condensates. In Section 4, we derive 1) the physical mechanism of superconductivity, 2) the conditions on formation of electron-pairs, 3) binding energy, and 4) the microscopic and macroscopic formulas for the critical temperature.

2. NEW INTERPRETATION OF QUANTUM WAVE FUNCTIONS

1). *Classical interpretations of wave functions*

The classical quantum mechanics tells us that a micro-particle can be described by a complex-valued wave function $\Psi : \Omega \rightarrow \mathbb{C}$, which satisfies a wave equation such as the Schrödinger equation:

$$(2.1) \quad i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(x)\Psi, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^3$ is the region that the particle occupies, m is the mass of the particle, and $V(x)$ is the external interacting potential. The system (2.1) conserves the energy, and hence the wave function Ψ can be expressed as

$$(2.2) \quad \Psi = e^{-iEt/\hbar} \psi(x),$$

where E is the energy, and ψ is the time-independent wave function. Putting (2.2) into (2.1), we derive that

$$(2.3) \quad -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi = E\psi.$$

The solution of (2.3) can be expressed as

$$(2.4) \quad \psi = |\psi| e^{i\varphi},$$

where φ is the phase of ψ .

The Born interpretation of quantum mechanics amounts to saying that

$$(2.5) \quad |\psi(x)|^2 \text{ represents the probability density of finding the particle at } x.$$

Also, basic quantum mechanics shows that

$$(2.6) \quad -i\hbar\nabla = \text{momentum operator.}$$

Hence

$$(2.7) \quad \text{momentum density of } \psi = \text{imaginary part of } \hbar\nabla\psi \cdot \psi^*.$$

Let the momentum density of ψ be expressed as

$$\text{momentum density of the particle} = m|\psi|^2v,$$

where v is the velocity of the particle. Then we infer from (2.4) and (2.7) that

$$(2.8) \quad v = \frac{\hbar}{m}\nabla\varphi.$$

Namely the velocity is the gradient of the phase of the wave function (2.4). (2.8) is consistent with the classical physical conclusion that the current density J is given by

$$(2.9) \quad J = \frac{ie\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi).$$

2). *New interpretation of the wave function*

Notice that (2.8) is the velocity *field* of particles, rather than the velocity of a single particle. This leads to the new interpretation of the wave function ψ as follows:

- (1) Under the external potential field $V(x)$, the wave function ψ in (2.3) is the field function for the motion of all particles with the same mass in the same class determined by the external potential $V(x)$. In other words, it is not the wave function of a particular particle in the classical sense;
- (2) When a particle is observed at a particular point $x_0 \in \Omega$, then the motion of the particle is fully determined by the solution of the following motion equation with initial position at x_0 :

$$(2.10) \quad \begin{aligned} \frac{dx}{dt} &= \frac{\hbar}{m}\nabla\varphi(x), \\ x(0) &= x_0, \end{aligned}$$

where φ is the phase of the wave function ψ in (2.4);

- (3) With ψ being the field function,

$$(2.11) \quad |\psi(x)|^2 = \text{distribution density of particles at } x;$$

- (4) The energy E in (2.3) represents the average energy level of the particles and can be written as

$$(2.12) \quad E = \int_{\Omega} \left[\frac{\hbar^2}{2m} |\nabla|\psi||^2 + \frac{\hbar^2}{2m} |\nabla\varphi(x)|^2 + V(x)|\psi|^2 \right] dx,$$

where in integrand on the right-hand side, the first term represents the non-uniform distribution potential of particles, the second term is the average kinetic energy, and the third term is the potential energy of the external field. Here $\nabla|\psi|$ is characteristic of quantum mechanics and there is no such a term in classical mechanics.

3). *Physical meaning of new quantum interpretation*

The above new interpretation of wave functions does not alter the basic theories of quantum mechanics. Instead it offers new understanding of quantum mechanics, and plays a fundamental role for the quantum theory of condensed matter physics below.

The classical Born statistical interpretation of quantum mechanics amounts to saying that without constraints, the motion of a micro-particle is random and there is no definite trajectory of the motion. Also $|\psi(x)|^2$ stands for the probability density of the particle appearing at the particular point x . This leads to the classical Einstein-Bohr debates, and is the main reason for absurdities associated with the interpretation of quantum mechanics.

The new interpretation says that $\psi = |\psi|e^{i\varphi}$ is the common wave function for all particles in the same class determined by the external potential $V(x)$, $|\psi(x)|^2$ represents the distribution density of the particles, and $\frac{\hbar}{m}\nabla\varphi$ is the velocity field of the particles. The trajectories of the motion of the particles are given by (2.10). The observed particles are the particles in the same class described by the same wave function, rather than a specific particle in the classical quantum mechanics. Hence the initial position of a particle is determined by the distribution $|\psi|^2$, and its motion trajectory is fully determined by (2.10). This is an entirely different interpretation from the classical Bohr interpretation, and will lead to impact to quantum physics and condensed matter physics.

In summary, there are field functions ψ_1 , ψ_2 , $|\psi|$ and φ in the wave function

$$\psi = \psi_1 + i\psi_2 = |\psi|e^{i\varphi},$$

whose physical meanings are given as follows:

- (1) ψ_1 and ψ_2 are the conjugate functions of the following Hamilton system

$$\begin{aligned}\hbar \frac{\partial \psi_1}{\partial t} &= k \frac{\delta}{\delta \psi_2} H(\psi), \\ \hbar \frac{\partial \psi_2}{\partial t} &= -k \frac{\delta}{\delta \psi_1} H(\psi);\end{aligned}$$

- (2) $|\psi(x)|^2$ represents the distribution density of particles;
 (3) $\nabla|\psi|$ is the potential describing the non-uniform distribution of particles, and $\frac{\hbar^2}{2m}|\nabla|\psi||^2$ is potential energy for the non-uniform distribution of particles; and
 (4) $\frac{\hbar}{m}\nabla\varphi$ is the velocity field of particles.

3. QUANTUM MECHANISM OF CONDENSATION

3.1. Quantum condensation mechanism. Based on the new interpretation of quantum mechanics presented in the previous section, we now establish the quantum mechanism for the micro-structure and its formation of condensates. This new mechanism serves as a unified physical picture and formation mechanism for superconductivity, superfluidity and Bose-Einstein condensation.

1). *Physical picture of the formation of condensates*

We start with the mechanism of the formation of condensates. The new interpretation of quantum mechanics shows that the wave function is no longer the state function of a specific particle, and is now considered as the field function governing the motion of particles. The motion of particles in the field of a wave function is similar to the motion of macroscopic objects in a gravitational field. This new viewpoint enables us to describe precisely the physical picture of condensates.

By the Pauli exclusion principle, under the condition of very weak interactions between particles, a large collection of Bose particles can undergo a motion governed by the same wave function field, as long as all the particles are in the same external potential field. Hence the physical picture of a condensate is as follows:

- (3.1) *a condensate is a large collection of Bosons in the same wave function field ψ determined by the same external potential field V .*

We call ψ the condensation wave function or the condensation field.

2). *Microscopic quantum mechanism*

Condensation occurs at low temperatures. The main reason is that at higher temperature, particles possess relatively high kinetic energy,

causing them to collide (scattering) with each other. Collision are interactions. Hence under such collisions, each particle is situated at an independent interaction field with its own wave function, and consequently the condensation picture (3.1) does not happen.

However, at a relative low temperature, the kinetic energy of particles decreases since temperature is the average energy level of particles. Hence the possibility of collision decreases, leading to the formation of a common external potential field. Also, since particles with the same wave function field do not collide, the system energy decreases. By the minimum potential principle, the system has the tendency toward condensation of particles. This is the mechanism of the physical condensation depicted in (3.1).

3). *Collisionless*

We now use the motion law (2.10) to verify that particles of a condensate do not collide with each other.

Since the large collection of particles share a common wave function field $\psi = |\psi|e^{i\varphi}$, their motion is dictated by the velocity field $\frac{\hbar}{m}\nabla\varphi$ as shown in (2.10). For different particles, their initial positions are different so that they will not collide with each other. For example, consider two different particles A and B with initial positions at x_1 and x_2 , i.e.

$$x_A(0, x_1) = x_1, \quad x_B(0, x_2) = x_2,$$

where $x_A(t, x_1)$ and $x_B(t, x_2)$ are the trajectories of particles A and B , which are solutions of

$$\frac{dx}{dt} = \frac{\hbar}{m}\nabla\varphi(x)$$

with initial data x_1 and x_2 . It is then clear that if $x_1 \neq x_2$, then

$$x_A(t, x_1) \neq x_B(t, x_2), \quad \forall t > 0.$$

In other words, particles A and B can never collide with each other. The motion of particles in a condensate resembles the motion of planets around the nucleus of a galaxy.

4). *Minimal energy*

For N particles on a state described by one wave function field ψ_0 , by (2.12), their total energy is given by

$$NE_0 = N \int \left[\frac{\hbar^2}{2m} |\nabla|\psi_0||^2 + \frac{\hbar^2}{2m} |\nabla\varphi_0|^2 + V_0|\psi_0|^2 \right] dx.$$

On the other hand, for N particles such that each particle occupies its own state described by its own wave function ψ_i , their total energy is

$$\sum_{i=1}^N E_i = \sum_{i=1}^N \int \left[\frac{\hbar^2}{2m} |\nabla|\psi_i||^2 + \frac{\hbar^2}{2m} |\nabla\varphi_i|^2 + V_i |\psi_i|^2 \right] dx.$$

When condensation occurs, the particles collectively occupy the minimal energy state, so that

$$E_0 \leq E_i \quad (1 \leq i \leq N).$$

Hence the total energy of the system satisfies:

$$NE_0 < \sum_{i=1}^N E_i \quad (\text{if there is an } E_i \text{ such that } E_i > E_0).$$

Namely, when the condensate particles occupy the minimal energy levels, the total energy of the system is less than the total energy of the particles in the normal states. By the minimal potential principle [89], the system will be in a condensate state.

5). *Condensation mechanism of superconductivity and superfluidity*

In summary, we have derived the following condensation formation mechanism. This is the mechanism of superconductivity, superfluidity and BEC.

Condensation Mechanism 3.1. (1) *The condition for condensation of bosonic particles is that their mutual interaction is sufficiently weak to ensure that they are in a state governed by the same wave function field under the common bounding potential V ;*

- (2) *a large collection of bosonic particles occupy the same wave function ψ , called condensation wave function, that is determined by the wave equation under the same potential V ;*
- (3) *the condensation wave function ψ is the smallest allowed energy state of the wave equation; and*
- (4) *the motion of particles in a condensate system under the same wave function with velocity field (2.8) resembles the motion of stars in a galaxy, and can last permanently.*

3.2. Field equations of condensates. There are two cases of condensates: The first is the case where the system is at near the critical temperature T_c , the condensation is in its early stage and the condensed particle density is small. The second is the case where away from the critical temperature, the system enters a deeper condensate state. The

field equations for these two cases are different and will be addressed separately below.

1). *Condensation field equations near T_c*

At this stage, the field equations are the variational equations of the thermodynamic potential. Let $F(\psi, \Phi)$ be the thermodynamic potential, ψ be the wave function, and $\Phi = (S, H, E)$, where S is the entropy, H is the electromagnetic field, and E is the electric field. Then the field equations are the equilibrium equations of the thermodynamic phase transition:

$$(3.2) \quad \frac{\delta}{\delta\psi^*} F(\psi, \Phi) = 0, \quad \frac{\delta}{\delta\Phi} F(\psi, \Phi) = 0.$$

2). *Fully developed condensation field equations*

For the fully developed condensation away from critical temperature, the system obeys both the principle of Lagrangian dynamics and the principle of Hamiltonian dynamics, leading to two equivalent field equations.

Let H be the Hamiltonian energy functional. Then the Lagrangian dynamical equation is given by

$$(3.3) \quad i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta}{\delta \Psi^*} H(\Psi).$$

Also, the Hamiltonian dynamic equations are

$$(3.4) \quad \begin{aligned} 2\hbar \frac{\partial \Psi_1}{\partial t} &= \frac{\delta}{\delta \Psi_2} H(\Psi), \\ 2\hbar \frac{\partial \Psi_2}{\partial t} &= -\frac{\delta}{\delta \Psi_1} H(\Psi). \end{aligned}$$

where $\Psi = \Psi_1 + i\Psi_2$.

The equations (3.3) and (3.4) are equivalent, and both will be used in describing different properties of quantum systems of condensates.

3.3. Pattern formation equations of condensates. Let the wave function ψ of a condensate system be expressed as

$$(3.5) \quad \psi = \zeta e^{i\varphi}, \quad \zeta = |\psi|.$$

In Section 2, we have derived the physical meaning of ζ and φ :

$$(3.6) \quad \begin{aligned} \rho = \zeta^2 & \text{ distribution density of particles in the condensate,} \\ \frac{\hbar}{m} \nabla \varphi & \text{ the velocity field of the flow of particles.} \end{aligned}$$

Hence ζ and φ describe the structure of condensates in the physical space, and the equations of ζ and φ are called *pattern formation equations*.

The pattern formation equations for ζ and φ can be derived from the field equations (3.2) and (3.3) as follows.

First, near the critical temperature, (3.2) is the steady state equation of (3.5). Away from the criticality, (3.3) are evolution equations. The energy of the Hamiltonian system (3.4) is conserved, i.e. the number of particles is conserved. Hence the solution of (3.3) can be written as:

$$(3.7) \quad \Psi = e^{-i\lambda t/\hbar}\psi(x), \quad \psi \text{ is as (3.5),}$$

where $\lambda \in \mathbb{R}$ represents the chemical potential. By (3.7), we infer from (3.3) that

$$(3.8) \quad \lambda\psi = \frac{\delta}{\delta\psi^*}H(\psi).$$

Hence we reduce (3.3) to the steady state form given by (3.8).

1). *Gaseous and liquid condensate systems*

For gaseous and liquid systems, (3.2) and (3.8) can be written in a unified form as follows:

$$(3.9) \quad -\frac{\hbar^2}{2m}\Delta\psi + f(|\psi|)\psi = 0,$$

where $f(x)$ is a function of $x \in \mathbb{R}$. If ψ is a spinor, then $f(x)$ is a matrix-valued function of $x \in \mathbb{R}^3$, and ψ can be written as

$$\psi = \zeta e^{i\varphi}, \quad \zeta = (\zeta_+, \zeta_0, \zeta_-),$$

as the three types of particles in the system possess the same mass, and consequently share the same velocity field $-\frac{\hbar}{m}\nabla\varphi$.

In view of (3.5), we infer from (3.9) the following pattern formation equation for gaseous and liquid condensate systems:

$$(3.10) \quad \text{div}(\zeta^2\nabla\varphi) = 0,$$

$$(3.11) \quad -\frac{\hbar^2}{2m}[\Delta\zeta - \zeta|\nabla\varphi|^2] + f(\zeta)\zeta = 0.$$

2). *Superconducting systems*

For superconducting systems, (3.2) and (3.8) can be written in the following unified form:

$$(3.12) \quad \frac{1}{2m_s} \left(i\hbar\nabla + \frac{e_s}{c}A \right) \psi + f(|\psi|)\psi = 0,$$

with superconducting currents:

$$J_s = -\frac{e_s^2}{m_s c^2} |\psi|^2 A - \frac{e_s \hbar}{2m_s c} i(\psi^* \nabla \psi - \psi \nabla \psi^*).$$

In view of (3.5), J_s becomes

$$(3.13) \quad J_s = -\frac{e_s^2}{m_s c^2} \zeta^2 A + \frac{e_s \hbar}{m_s c} \zeta^2 \nabla \varphi.$$

Also by (3.5), (3.12) is rewritten as

$$(3.14) \quad \operatorname{div} \left(\hbar \zeta^2 \nabla \varphi - \frac{e_s}{c} \zeta^2 A \right) = 0,$$

$$(3.15) \quad -\frac{\hbar^2}{2m_s} \left[\Delta \zeta - \zeta |\nabla \varphi|^2 \right] + \frac{e_s^2}{2m_s c^2} A^2 \zeta + f(\zeta) \zeta = 0.$$

In (3.14), we have used the Coulomb gauge

$$\operatorname{div} A = 0.$$

By (3.13) and (3.14), we obtain that $\operatorname{div} J_s = 0$.

The above two systems of equations: (3.10)–(3.11), and (3.14)–(3.15) are the pattern formation equations in the physical space for superfluidity, superconductivity and BEC. These are important equations. On the one hand, they dictate two important physical structures of condensate particles:

- (3.16)
 - *physical picture of the distribution function $\rho = \zeta^2$, and*
 - *topological structure of the flow determined by the velocity field $\frac{\hbar}{m} \nabla \varphi$.*

On the other hand, the pattern formation equations are fundamental equations for studying quantum phase transitions. In fact, quantum phase transition deals with the phenomena of structure changes in the physical space of pattern formation equations (3.16) of condensate systems at critical control parameters.

4. THEORY FOR HIGH-TEMPERATURE SUPERCONDUCTIVITY

4.1. Physical mechanism of superconductivity. In the previous section, we introduced Quantum Condensation Mechanism 3.1. For superconducting systems, this mechanism can be explicitly stated as follows:

- (4.1)
 - *there is a critical temperature $T_c > 0$ such that for $T < T_c$, a large number of free electrons in the system form bosonic electron-pairs; and*

- *the interactions between the bosonic electron-pairs and other particles in the system are negligible, so that the electron-pairs share a common interaction potential Φ .*
- (4.2) *This common potential Φ enables the electron-pairs to occupy a single wave function field ψ , forming the condensate, and Φ is the potential given by the interaction between the lattice of the system and the external field.*

This is the common physical mechanism of superconductivity. The classical BCS theory [8] can be used to explain superconductivity in low temperatures. The Cooper pair theory provides the mechanism of electron-phonon interaction in low temperature superconductors, and a physical mechanism for the Bose electron-pair as required in (4.1). However, the BCS theory is applicable only for $T < 35\text{K}$.

However, in 1986, Bednorz and Müller [10] discovered superconductivity in a lanthanum-based cuprate perovskite material, which had a transition temperature of 35 K. Bednorz and Müller's work opened the door for high T_c superconductors. The current highest temperature of superconductors reaches $T_c = 130$, and $T_c = 160\text{K}$ under high pressure.

The discovery of high T_c superconductors shows the limitation of the BCS theory. In fact, in the BCS theory, the Cooper electron-pair is explained as

(4.3) the electron-phonon interaction,

whose physical meaning (picture) is hard to understand. This interaction is reminiscent to the Yukawa interaction mechanism in quantum field theory. The electron-phonon interaction (4.3) means that by exchanging phonons, electrons induce an attractive interaction; the physical picture of such an interaction is not clear. Phonon is a hypothetical pseudo-particle. In fact, in quantum field theory, to compute the binding energy Δ of the Cooper pair, a quantized energy needs to be introduced as the binding energy, and is called phonon. In other words, phonon is essentially a hypothesis, written in the form (4.3), and is introduced for computing Δ . This hypothesis can be expressed in the following easier understandable fashion:

(4.4) *In addition to the repulsive Coulomb potential between electrons, the deformation of lattice induces an attractive interaction potential.*

In summary, both (4.3) and (4.4) are different expressions of the same hypothesis. For the BCS theory, regardless of the validity of this hypothesis, the key issue is if there is a good agreement between

experiments and the computational results derived under the existence assumption of the binding energy Δ between electrons. Indeed, such agreement is only observed in low temperature cases, but not in high temperature cases. Hence it is worth pointing out that

the validity of a result does not ensure the correctness of the hypothesis introduced to derive the result, since reasons for the same result are usually not unique.

4.2. PID interaction potential of electrons. The classical theory of electromagnetism shows that there is a repulsive interaction force between two electrons:

$$F = \frac{e^2}{r^2} \quad (e \text{ is the electric charge}),$$

preventing the paring of electrons, called screening of Coulomb repulsion. The formation of Cooper electron-pairs in the BCS theory makes the hypothesis that the lattice deformation induces the electron-phonon attractive interaction: when free electrons are near the Fermi surface,

$$\text{attractive force } F_1 > \text{Coulomb force } F.$$

Notice also that there exists Coulomb repulsive force between two Cooper pairs:

$$(4.5) \quad F = \frac{4e^2}{r^2},$$

which will cause collision and scattering between Cooper-pairs, preventing condensation and superconducting behaviors of particles. In other words, the repulsive force (4.5) makes the mechanism (4.2) impossible.

In [88], using the principle of interaction dynamics (PID) and the $SU(N)$ gauge theory for multi-particle systems, the authors obtained a layered electromagnetic interaction potential formula, which can be used here to derive conditions to ensure the validity of the mechanism (4.1) and (4.2) for superconductivity. For the sake of completion, we recapitulate this formula below.

Consider a particle A with electric charge $Q = ne_0$ with either $e_0 < 0$ or $e_0 > 0$. Let there be a test electron e at distance r from particle A . By the field equations [88, (6.5.20) and (6.5.21)] for N particles, the $SU(N)$ ($N = n + 1$) electromagnetic potentials generated by particle A acting on the electron:

$$(4.6) \quad \{A_\mu^a \mid 0 \leq \mu \leq 3, \quad 1 \leq a \leq N^2 - 1\}$$

satisfy the following field equations, using the standard $SU(N)$ generators to ensure that $Tr(\tau_a \tau_b^\dagger) = 2\delta_{ab}$:

$$(4.7) \quad \partial^\nu A_{\nu\mu}^a - \frac{e_0}{\hbar c} \lambda_{bc}^a g^{\alpha\beta} A_{\alpha\mu}^b A_\beta^c - e_0 \bar{\Psi} \gamma_\mu \tau^a \Psi = [\partial_\mu - \frac{1}{2} k^2 x_\mu] \phi^a,$$

$$(4.8) \quad i\gamma^\mu \left(\partial_\mu + \frac{ie_0}{\hbar c} A_\mu^b \tau_b \right) \Psi - \frac{c}{\hbar} M \Psi = 0,$$

where M is the mass matrix of the particles, $\Psi = (\Psi_1, \dots, \Psi_N)^T$, Ψ_i ($1 \leq i \leq N$) represent i wave function (Dirac spinors), $(g^{\alpha\beta})$ is the Minkowski metric, λ_{bc}^a are the structure constants of $SU(N)$, ϕ^a are the dual potentials of $\{A_\mu^a\}$, γ^μ are the Dirac matrices, $\gamma_\mu = g_{\mu\alpha} \gamma^\alpha$, $\tau^a = \tau_a$, and

$$A_{\nu\mu}^a = \partial_\nu A_\mu^a - \partial_\mu A_\nu^a + \frac{e_0}{\hbar c} \lambda_{bc}^a A_\mu^b A_\nu^c.$$

We note here that the field equations [88, (6.5.20) and (6.5.21)] for N particles are derived from first principles: principle of Lorentz invariance, principle of gauge invariance, Einstein's principle of general relativity, principle of representation invariance (PRI), principle of Lagrangian dynamics and principle of interaction dynamics (PID).

We now derive from (4.7) and (4.8) the electron interaction potential. Thanks to the PRI applied to $SU(N)$, there is an $SU(N)$ representation tensor $\{\theta_a \mid 1 \leq a \leq N^2 - 1\}$ such that

$$(4.9) \quad A_\mu = \theta_a A_\mu^a \quad (\mu = 0, 1, 2, 3)$$

representing the total electromagnetic potential induced by the N electron charges. Namely,

$$\begin{aligned} A_0 & \quad \text{electric potential,} \\ A & = (A_1, A_2, A_3) \quad \text{magnetic potential.} \end{aligned}$$

Let the radius of particle be r_A , $\Phi(r) = A_0(r)$ be the electric potential acting on the test electron e . The additivity of electric potential shows that the $\mu = 0$ component of (4.7) is linear, taking the following form; see [88] for details:

$$(4.10) \quad \begin{aligned} -\Delta\Phi & = e_0 Q_0 - \frac{1}{4} k^2 c\tau\phi, \\ -\Delta\phi + k^2\phi & = -e_0 \text{div}\vec{Q}, \end{aligned}$$

where $c\tau$ represents the wave-length of the electron, k is the electric charge interaction parameter,

$$\phi = \theta_a \phi^a, \quad (Q_0, \vec{Q}) = Q_\mu = -\theta_a \bar{\Psi} \gamma_\mu \tau^a \Psi,$$

and θ_a is as in (4.9). Here

$$(4.11) \quad Q_0 = nB_0 \left(\frac{r_e}{r_A} \right)^3 \delta(r), \quad \operatorname{div} \vec{Q} = \frac{B_1}{r_A} \delta(r),$$

where r_A , r_e are radii the particle A and the electron respectively, $\delta(r)$ is the Dirac function, B_0 , B_1 are parameters.

Then with the same procedure and derivation in [88, the layered strong interaction formula (4.5.41)], we obtain from (4.10)–(4.11) the following

$$(4.12) \quad \Phi(r) = nB_0 \left(\frac{r_e}{r_A} \right)^3 e_0 \left[\frac{1}{r} - A_0(1 + kr)e^{-kr} \right],$$

where B_0 is as (4.11), k is as (4.10), and A_0 can be expressed explicitly as

$$A_0 = \frac{k^2 c \tau r_A^2 B_1}{4nr_e^3 B_0}.$$

Here A_0 depends on the types of particle A and the test electron, and is determined by experiments.

Formula (4.12) represents the electric potential induced by particle A , where

$$(4.13) \quad q_A = nB_0 \left(\frac{r_e}{r_A} \right)^3 e_0$$

represents the effective electric charge of A , and B_0 depends on the particle type of A . The physical meaning of the effective charge is apparent in (4.14) and (4.15) below. For two effective charges q_1 and q_2 , the electric potential between them is

$$(4.14) \quad V_{12}(r) = q_1 q_2 \left[\frac{1}{r} - A_{12}(1 + kr)e^{-kr} \right],$$

where A_{12} is a parameter depending on the types of the two particles. The force between the two particles with effective charges q_1 and q_2 is then given by

$$(4.15) \quad F_{12} = -q_1 q_2 \frac{d}{dr} \left[\frac{1}{r} - A_{12}(1 + kr)e^{-kr} \right].$$

4.3. Condition for formation of electron-pairs. The field equations (4.7) and (4.8), together with the electron interaction potential and force formulas (4.14) and (4.15), form the theoretical basis of the new theory on electron-pairs in superconductivity in this section.

- 1). *Electric charge interaction between two electrons*

When both particles are electrons, $n = 1$, $B_0 = 1$, $r_A = r_e$. Then (4.12) becomes

$$(4.16) \quad \Phi_e = \frac{e_0}{r} - e_0 A_0 (1 + kr) e^{-kr},$$

which is the modification of the classical Coulomb potential. The first term in the right hand side is the Coulomb potential, and the second term represents the effect of the dual field ϕ of A_μ , induced from PID.

By (4.16), the interaction force between two electrons is given by

$$(4.17) \quad F_e = -e_0 \nabla \Phi_e = \frac{e_0^2}{r^2} - k^2 A_0 e_0^2 r e^{-kr}.$$

There is a short range attractive force in the right hand side of (4.17) ($A_0 > 0$):

$$f = -k^2 A_0 e_0^2 r e^{-kr} < 0.$$

It is this force f , together with magnetic moment of electrons, that induces a short range attractive force to counteract the repulsive Coulomb force.

2). *Spin magnetic moment interaction of electrons*

Electrons possess spin magnetic moment, with numerical value given by

$$(4.18) \quad \mu_0 = -\frac{e_0 \hbar}{2mc},$$

where $e_0 < 0$ is the electric charge of an electron, and m is the mass of electron. We can define the direction of the spin magnetic moment, as the north pole N , and the opposite direction as the south pole S . Then there is a spin magnetic moment interaction force

$$(4.19) \quad F_m = \pm \frac{\alpha \mu_0^2}{r^2},$$

where α is the magnetic moment interaction coefficient. Physically, we know that

$$(4.20) \quad 0 < \alpha < e_0^2 / \mu_0^2.$$

There are two electron orientation arrangements that can produce the magnetic moment force given by (4.19): one is the parallel arrangement, as shown in Figure 4.1 (a) and (b); the other is collinear arrangement as shown in Figure 4.2 (a) and (b). The plus sign in (4.19) represents repulsive force, corresponding to the cases of Figures 4.1(a) and 4.2(a). Hence only Figures 4.1(b) and 4.2(b) may give rise to superconducting electron-pairs.

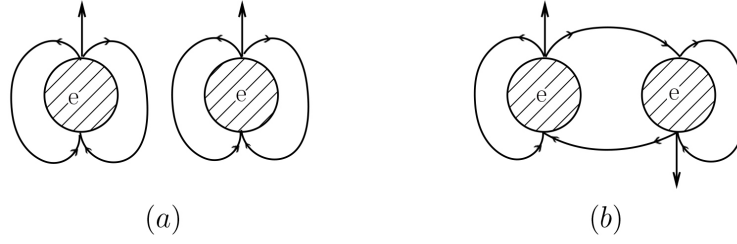


FIGURE 4.1. (a) Parallel orientation leads to repulsive magnetic moment force; and (b) anti-parallel orientation leads to attractive magnetic moment force.

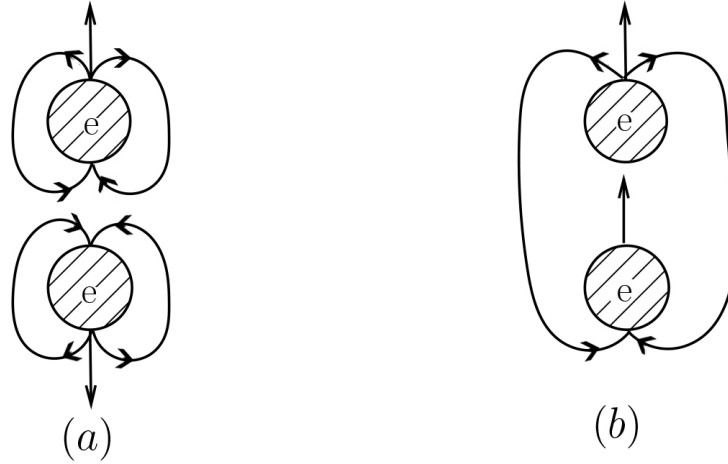


FIGURE 4.2. (a) Anti-collinear orientation leads to repulsive magnetic moment force; and (b) collinear orientation leads to attractive magnetic moment force.

3). *Stability of orientation arrangements of electrons*

For the two electron orientation arrangements that produce attractive spin magnetic moment force, the antiparallel electron-pair shown in Figure 4.1(b) is stable, and the one in Figure 4.2 (b) is unstable. There are two main reasons behind the stability and instability of these two cases. First, since the collinear arrangement of an electron-pair possesses electric charge $2e_0$ and spin $J = 1$, it induces a magnetic moment with intensity: $\mu = -2e_0\hbar/mc$. This causes magnetic moment interaction between electron-pairs. Such long arrangement is also the source of instability, and two reversed parallel electron-pairs as shown in Figure 4.2(b) can break form two electron-pairs with the structure

as shown in Figure 4.1 (b). Second, by the Heisenberg uncertainty relation, for an electron-pair, the distance between the two electrons and the kinetic energy $p = mv$ of the electrons satisfy the following inequality:

$$v \geq \frac{\hbar}{2mr}.$$

This implies that the two electrons in an electron-pair must rotate around a mass center. Hence the rotating arrangement of electrons in Figure 4.1 (b) is stable, and the one in Figure 4.2(b) is unstable. We arrive then at the following conclusion:

(4.21) only electron-pairs with total spin $J = 0$ is stable.

4). *Attracting shell region of electron-pairs*

By (4.17) and (4.19), the interaction force formula of two anti-parallel electrons is given by

$$(4.22) \quad F = \frac{e_0^2}{r^2} \left(1 - \frac{\alpha\mu_0^2}{e_0^2} \right) - k^2 A_0 r e^{-kr} e_0^2.$$

The interval where F takes negative values defines the attraction shell region of two electrons:

$$(4.23) \quad \bar{r}_0 < r < \bar{r}_1 \quad (\bar{r}_0 > 0),$$

where \bar{r}_0, \bar{r}_1 are two positive roots of $F(r) = 0$, satisfying

$$(4.24) \quad \frac{r^3}{e^{kr}} = \frac{1}{k^2 A_0} \left(1 - \frac{\alpha\mu_0^2}{e_0^2} \right).$$

Here the condition for (4.24) having two positive roots is that the maximum of r^3/e^{kr} is greater than the right-hand side of the above equation:

$$\left(\frac{3}{e} \right)^3 > \frac{k}{A_0} \left(1 - \frac{\alpha\mu_0^2}{e_0^2} \right),$$

where $e \simeq 2.718$. By (4.18), the above condition can be rewritten as

$$(4.25) \quad A_0 > \left(\frac{e}{3} \right)^3 k \left(1 - \frac{\alpha\hbar^2}{4m^2 c^2} \right),$$

which is the condition to ensure the existence of the attracting shell region (4.23).

5). *Attracting potential of electron-pairs*

The attracting interaction potential of magnetic moment for an electron-pair is

$$V_m = -\frac{\alpha\mu_0^2}{r} = -\frac{\alpha\hbar^2}{4m^2 c^2 r} e_0^2.$$

By (4.16), the total interaction potential of an anti-parallel electron-pair is

$$(4.26) \quad V = \frac{e_0^2}{r} \left(1 - \frac{\alpha \hbar^2}{4m^2 c^2} \right) - e_0^2 A_0 (1 + kr) e^{-kr}.$$

By the relation (4.22) between the potential V and the force F :

$$F = -\frac{d}{dr} V,$$

The first zero \bar{r}_0 of F in (4.23) is the minimum point V :

$$(4.27) \quad V(\bar{r}_0) = \min V(r).$$

The condition for an anti-parallel electron-pair to possess a shell region of attracting potential is $V(\bar{r}_0) < 0$, i.e.

$$(4.28) \quad A_0 > \frac{e^{k\bar{r}_0}}{\bar{r}_0(1 + k\bar{r}_0)} \left(1 - \frac{\alpha \hbar^2}{4m^2 c^2} \right).$$

This is the condition for an electron-pair to possess attracting interaction potential.

It is clear that we can view \bar{r}_0 as the theoretical radius of electron-pairs. Let

$$(4.29) \quad A_0 = A/\bar{r}_0,$$

then (4.28) becomes

$$(4.30) \quad A > \frac{e^{k\bar{r}_0}}{1 + k\bar{r}_0} \left(1 - \frac{\alpha \hbar^2}{4m^2 c^2} \right).$$

Obviously if this condition holds true, then (4.25) holds true as well for $k\bar{r}_0 > 0$.

4.4. Binding energy of superconducting electron-pairs. The key ingredient of the BCS theory of superconductivity is to postulate the existence of the binding energy of electron-pairs for $T < T_c$:

$$(4.31) \quad \begin{aligned} \text{binding energy of electron-pair} &= 2\Delta(T) \quad \text{for } T < T_c, \\ \Delta(T_c) &= 0. \end{aligned}$$

The relation between $\Delta(T)$ and the energy spectrum $E(p)$ of the electron-pair is

$$(4.32) \quad E(p) = \sqrt{(E_p - E_F)^2 + \Delta(T)}.$$

where E_F is the energy of Fermi level, and $E_p = \frac{1}{2m}p^2$ is the kinetic energy of the particle. Also, Δ is determined by the following equation, called energy-gap equation:

$$(4.33) \quad N(0)V \int_0^\infty \frac{1}{(2\pi\hbar)^3 E(p)} \left[1 - \frac{2}{1 + \exp(E(p)/kT)} \right] dp = 1,$$

where $E(p)$ is as in (4.32), $N(0)$ is the Fermi energy density with temperature at absolute zero, and V is the intensity of attracting portion of potential of electrons.

The above three formulas (4.31)–(4.33) are the three main ingredients of the BCS theory, and the remaining components of the BCS theory can be derived from these three formulas. By $\Delta(T_c) = 0$, we derive from the energy-gap equation (4.33) the following formula for the critical temperature T_c :

$$(4.34) \quad kT_c = 1.14\hbar\omega_D \exp[-1/N(0)V],$$

where ω_D is the Debye frequency, $N(0)$ and V are as in (4.33). The highest critical temperature formula (4.34). Hence for high T_c superconductivity, the BCS theory is invalid, and in particular for high T_c superconductivity, the electron-phonon interaction mechanism for electron-pairs is not convincing.

We now establish the new theory of binding energy of electrons, suitable for high T_c superconductivity, based on the new interaction theory of electrons established earlier in this article.

1). *Binding energy of electron-pairs*

The previous section has shown that under condition (4.30), two electrons with reversed parallel orientation can form an electron-pair, with radius \bar{r}_0 satisfying (4.27).

In the BCS theory, the binding energy 2Δ of electron-pairs is determined by the energy-gap equation (4.33). The binding energy 2Δ can also be expressed using the potential in (4.26) as follows:

$$(4.35) \quad 2\Delta = -V(\bar{r}_0) - m\bar{v}^2,$$

where $m\bar{v}^2$ stands for the total average kinetic energy of the two electrons. By the the average energy level formula of temperature, we can view the average kinetic energy of the two electrons as a function of temperature:

$$(4.36) \quad \frac{1}{2}m\bar{v}^2 = \varepsilon(T).$$

Hence (4.35)–(4.36) can be written as

$$(4.37) \quad \Delta = \frac{e_0^2}{2\bar{r}_0} \left[A(1 + k\bar{r}_0)e^{-k\bar{r}_0} + \frac{\alpha\hbar^2}{4m^2c^2} - 1 \right] - \varepsilon(T),$$

where A is a parameter as in (4.29)–(4.30).

Formula (4.37) is the formula for the binding energy Δ of electron-pairs of superconductivity, in terms of average kinetic energy of electrons and temperature function $\varepsilon(T)$. At low temperature, Δ as well as $\varepsilon(T)$ are determined by (4.33), showing that the new theory is consistent with the BCS theory at low temperatures. Also, physically, by (4.35), the critical temperature T_c satisfies $\Delta(T_c) = 0$, which amounts to saying that

$$(4.38) \quad \varepsilon(T_c) = \frac{e_0^2}{2\bar{r}_0} \left[A(1 + k\bar{r}_0)e^{-k\bar{r}_0} + \frac{\alpha\hbar^2}{4m^2c^2} - 1 \right],$$

which is the new temperature formula, applicable for both low and high T_c cases.

2). *Verification of mechanism (4.1) of superconductivity*

By (4.36), $\varepsilon(T)$ is an increasing function of T :

$$\frac{d}{dT}\varepsilon(T) > 0.$$

Hence we infer from (4.37) and (4.38) that

$$(4.39) \quad \begin{aligned} \Delta(T) &= 0 && \text{for } T \geq T_c, \\ \frac{d}{dT}\Delta(T) &< 0 && \text{for } T < T_c. \end{aligned}$$

This verifies the mechanism (4.1) of superconductivity. Namely,

$$\begin{aligned} \text{no superconducting electron-pairs emerge} &&& \text{for } T \geq T_c, \\ \text{superconducting electron-pairs emerge} &&& \text{for } T < T_c. \end{aligned}$$

By (4.21), we note that

$$\text{total spin of electron-pairs } J = 0.$$

3). *Verification of mechanism (4.2) of superconductivity*

For $T < T_c$, electron-pairs with radius \bar{r}_0 emerge in a superconducting system. By (4.13), the effective charge of a superconducting electron-pair is

$$q_0 = 2B_0 \left(\frac{r_e}{\bar{r}_0} \right)^3 e_0.$$

The classical radius of an electron is

$$r_e = \frac{e_0^2}{mc^2} = 2.8 \times 10^{-13} \text{cm},$$

and the radius of an electron-pair \bar{r}_0 has the same order of magnitude as the diameter of the particle lattice of the system: $\rho \sim 10^{-8} \text{cm}$. Hence

$$q_0 \sim 10^{-13} B_0 e_0.$$

In other words, the effective electric charge of an electron-pair is negligibly small, almost electric neutral. Hence according to the interaction force formula of electric charge (4.15), the interaction between superconducting electron-pairs and particles in the system are extremely weak. Consequently this verifies the mechanism (4.2) of superconductivity.

4.5. Formulas for the critical temperature T_c . We now compute the critical temperature T_c from both the microscopic and macroscopic approaches. The macroscopic approach is based on the thermodynamic potential (free-energy) in the standard model of thermodynamics, and the microscopic approach is based on (4.37).

1). *Formula for T_c based on the standard model*

First we recall that the thermodynamic potential for superconductivity. Under no external magnetic field and homogeneous conditions, the potential functional for superconductivity is in the following form; see [83] for its detailed derivation:

$$(4.40) \quad F = -g_0|\psi|^2 + \frac{1}{2}g_1|\psi|^4 - \frac{\alpha_0}{2}TS^2 - \alpha_1TS|\psi|^2 - ST,$$

where $g_0 > 0$ is the binding potential of condensates, g_1 is the coupling constant of particles, α_0 and α_1 are entropy coupling constants. The entropy is determined by

$$(4.41) \quad \frac{\partial F}{\partial S} = 0 \Rightarrow S = -\frac{1}{\alpha_0} - \frac{\alpha_1}{\alpha_0}|\psi|^2.$$

On the other hand, we have

$$(4.42) \quad -\frac{\partial}{\partial \psi^*} F(\psi) = g_0\psi - g_1|\psi|^2\psi + \alpha_1TS\psi.$$

In view of (4.41) for entropy S , we have

$$(4.43) \quad -\frac{\partial}{\partial \psi^*} F(\psi) = \beta_1\psi - \left(g_1 + \frac{\alpha_1^2}{\alpha_0}T\right)|\psi|^2\psi,$$

where β_1 represents the eigenvalue of the linearized operator of (4.43), given by

$$\beta_1(T) = g_0 - \frac{\alpha_1}{\alpha_0}T.$$

By the theory of phase transition dynamics [87], $\beta_1(T) = 0$ provides the critical temperature:

$$(4.44) \quad T_c = \alpha_0 g_0 / \alpha_1.$$

Consider T_c in (4.44). If $T < T_c$, by (4.43) and

$$\frac{\partial}{\partial \psi^*} F(\psi) = 0,$$

we deduce the solution for superconducting phase

$$(4.45) \quad |\psi|^2 = \frac{\beta_1(T)}{g_1 + \alpha_1^2 T / \alpha_0} = \frac{\alpha_1}{\alpha_0} \frac{(T_c - T)}{g_1 + \alpha_1^2 T / \alpha_0}.$$

Plugging (4.45) into (4.41), we obtain that

$$S = \begin{cases} -\frac{1}{\alpha_0} - \left(\frac{\alpha_1}{\alpha_0}\right)^2 \frac{(T_c - T)}{g_1 + \alpha_1^2 T / \alpha_0} + S_0(T) & \text{for } T < T_c, \\ S_0(T) & \text{for } T_c < T. \end{cases}$$

Therefore, we derive that

$$(4.46) \quad \Delta C_v = T_c \left[\frac{\partial S}{\partial T} \Big|_{T_c^-} - \frac{\partial S_0}{\partial T} \right] = \left(\frac{\alpha_1}{\alpha_0}\right)^2 \frac{T_c}{g_1 + \alpha_1^2 T_c / \alpha_0},$$

where ΔC_V is the increment of heat capacity at T_c . By (4.46), we have

$$(4.47) \quad \frac{\alpha_0^2}{\alpha_1^2} = \frac{T_c}{g_1 \Delta C_V} (1 - \alpha_0 \Delta C_V),$$

where it is required that

$$(4.48) \quad \alpha_0 \Delta C_V < 1.$$

This is the condition for superconductivity to occur. By (4.44) and (4.47), we deduce that

$$(4.49) \quad T_c = \frac{g_0^2}{g_1 \Delta C_V} (1 - \alpha_0 \Delta C_V).$$

This is the formula for critical temperature T_c based on the standard model.

Remark 4.1. The formula (4.49) of the critical temperature T_c depends on the parameters g_0 , g_1 , α_0 and ΔC_V . Here ΔC_V is the increment of heat capacity at the critical temperature T_c , which is a theoretical value without taking into consideration of fluctuations and does not reflect the real measurement. \square

2). *Formula for T_c based on microscopic theory*

Formula (4.38) provides the theoretical basis for computing T_c . To obtain a formula for T_c , we need consider the expression for the function $\varepsilon(T)$. First, $\varepsilon(T)$ is defined by (4.36), and represents the average energy level of electron gas in the superconducting system. By the statistical theory of heat [90], we can derive an expression for $\varepsilon(T)$ as follows:

$$(4.50) \quad \varepsilon(T) = \gamma k_B T, \quad k_B \text{ is the Boltzmann constant,}$$

where γ is called the energy ratio of the system, and can be expressed as

$$(4.51) \quad \gamma = \frac{N^2}{N_1 N_2 (1 + \theta)}.$$

Here N is the total number of particles in the system, N_1 , N_2 are respectively the numbers of free electrons and lattice particles such that $N_1 + N_2 = N$, and θ is

$$(4.52) \quad \theta = \frac{\text{average energy level of lattice particles}}{\text{average energy level of free electrons}}.$$

We shall prove (4.50)–(4.52) later.

It is clear that θ defined by (4.52) depends on the temperature, and consequently γ depends on the temperature as well. Let γ_0 be the temperature ratio of the electron gas at the critical temperature T_c , then by (4.50)–(4.51), the formula for T_c in (4.38) can be rewritten as

$$(4.53) \quad k_B T_c = \frac{E_0}{\gamma_0} = \frac{N_1 N_2 (1 + \theta_0)}{N^2} E_0.$$

Here E_0 is the intrinsic energy of electron-pairs, expressed as

$$(4.54) \quad E_0 = \frac{e_0^2}{2\bar{r}_0} \left[A(1 + k\bar{r}_0)e^{-k\bar{r}_0} + \frac{\alpha\hbar^2}{4m^2c^2} - 1 \right].$$

Formulas (4.53) and (4.54) provide the needed formula for T_c based on the microscopic theory.

3). *Derivation of the expression for $\varepsilon(T)$*

First we recall the average energy level temperature formula in [90]:

$$(4.55) \quad k_B T = \frac{1}{N} \sum_n \frac{a_n \varepsilon_n}{1 + \beta_n \ln \varepsilon_n} \left(1 - \frac{a_n}{N} \right),$$

where ε_n is the n -th energy level, a_n is the number of particles on the energy level ε_n , $N = \sum_n a_n$ is the total number of particles. A superconducting system can be considered as a binary system of free electron gases and lattice-particles. Therefore, (4.55) can be written as

$$(4.56) \quad \frac{N_1 N_2}{N^2} \frac{\varepsilon_{\text{electron}}}{1 + \beta_1 \ln \varepsilon_{\text{electron}}} + \frac{N_1 N_2}{N^2} \frac{\varepsilon_{\text{lattice}}}{1 + \beta_2 \ln \varepsilon_{\text{lattice}}},$$

where N_1 , N_2 are respectively the numbers of free electrons and lattice-particles in the system, $\varepsilon_{\text{electron}}$ and $\varepsilon_{\text{lattice}}$ are the average energy levels of electrons and lattice-particles. We approximately take $\beta_1, \beta_2 = 0$, then the above expression reduces to

$$(4.57) \quad k_B T = \frac{N_1 N_2}{N^2} \varepsilon_{\text{electron}} + \frac{N_1 N_2}{N^2} \varepsilon_{\text{lattice}}.$$

Let $\theta = \varepsilon_{\text{lattice}}/\varepsilon_{\text{electron}}$, then we infer from (4.57) that

$$(4.58) \quad k_B T = \frac{N_1 N_2}{N^2} (1 + \theta) \varepsilon_{\text{electron}},$$

where $\varepsilon_{\text{electron}} = \frac{1}{2} m \bar{v}^2$ is the average kinetic energy. Therefore, (4.58) implies (4.50).

4). *Theoretical analysis of T_c*

From (4.53), we can deduce the following property of high T_c superconducting materials:

the energy level for the lattice-particle is much larger than that of free electrons, and consequently $\theta \gg 1$.

We have derived basic analysis of the new microscopic theory of superconductivity, and further analysis needs to consider the effect and physical meaning of $\beta_1, \beta_2 \neq 0$ in (4.56).

Also, the formula (4.49) based on the standard model indicates that the critical temperature T_c is a decreasing function of the increment of heat capacity. We now demonstrate that the microscopic theory of T_c can reveal such property as well. Recall the heat capacity formula

$$C_V = \frac{\partial E}{\partial T}, \quad E \text{ is the internal energy.}$$

By (4.50), we have

$$N_1 k_B \gamma = N_1 \frac{\partial \varepsilon}{\partial T} = \text{heat capacity of electron gas.}$$

The phase transition is due to condensation of the electron gas, and is independent of the lattice. Hence the heat capacity for the lattice at T_c is continuous. Hence at T_c , γ_0 is given by

$$\gamma_0 \sim \Delta C_V + \delta \quad (\delta = \gamma(T_c^-)).$$

We infer then from (4.53) that

$$k_B T_c \sim \frac{E_0}{\Delta C_V + \delta}.$$

This provides the same property as the macroscopic formula (4.49) based on the standard model.

In summary, (4.49) is derived using the standard model of thermodynamics, and the microscopic formula (4.53) is derived using PID interaction theory of electrons and the statistical theory of heat. With different origins, these two formulas possess the same property.

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