



**HAL**  
open science

## Positive isotropic curvature and self-duality in dimension 4

Thomas Richard, Harish Seshadri

► **To cite this version:**

Thomas Richard, Harish Seshadri. Positive isotropic curvature and self-duality in dimension 4. *Manuscripta mathematica*, 2015, 149 (3-4), pp.443 - 457. 10.1007/s00229-015-0790-2 . hal-01611591

**HAL Id: hal-01611591**

**<https://hal.science/hal-01611591v1>**

Submitted on 11 Oct 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Positive isotropic curvature and self-duality in dimension 4

Thomas Richard and Harish Seshadri

We study a positivity condition for the curvature of oriented Riemannian 4-manifolds: The half-*PIC* condition. It is a slight weakening of the positive isotropic curvature (*PIC*) condition introduced by M. Micallef and J. Moore.

We observe that the half-*PIC* condition is preserved by the Ricci flow and satisfies a maximality property among all Ricci flow invariant positivity conditions on the curvature of oriented 4-manifolds.

We also study some geometric and topological aspects of half-*PIC* manifolds.

## 1. Introduction

Let  $(M, g)$  be an oriented Riemannian 4-manifold. We say that  $M$  has *positive isotropic curvature on self-dual 2-forms* (which we abbreviate as *PIC*<sub>+</sub>) if the complex linear extension of the curvature operator  $R : \Lambda^2 T_p M \otimes \mathbb{C} \rightarrow \Lambda^2 T_p M \otimes \mathbb{C}$  satisfies

$$\langle R(v \wedge w), v \wedge w \rangle > 0$$

for every  $p \in M$  and  $v \wedge w \in \Lambda^2_+ T_p M \otimes \mathbb{C} \cap S_0$ , where  $\langle \cdot, \cdot \rangle$  denotes the Hermitian inner product on  $\Lambda^2 T_p M \otimes \mathbb{C}$ ,  $\Lambda^2_+ T_p M$  denotes the self-dual 2-forms at  $p$  and  $S_0 \subset \Lambda^2 T_p M \otimes \mathbb{C}$  denotes the set of complex 2-forms that can be written as  $v \wedge w$  where  $v, w \in T_p M \otimes \mathbb{C}$  span a complex plane which is isotropic for the complex bilinear extension of the Riemannian metric on  $T_p M \otimes \mathbb{C}$ .

This condition is a variation of the *PIC* condition introduced by Micallef and Moore in [19], which requires positivity for all  $v \wedge w$  such  $\{v, w\}$  spans an isotropic complex plane, without requiring  $v \wedge w$  to be self dual.

One has the following alternative definitions (see Proposition 2.10):

1. If “Scal” denotes scalar curvature,  $I : \Lambda^2 TM \rightarrow \Lambda^2 TM$  is the identity operator and  $R_{\mathcal{W}_+} : \Lambda^2_+ TM \rightarrow \Lambda^2_+ TM$  denotes the self-dual part of the Weyl tensor of  $R$  then

$$\frac{\text{Scal}}{6} I - R_{\mathcal{W}_+} > 0$$

on  $\Lambda^2_+ TM$ .

2.  $M$  is  $PIC_+$  if for any  $p \in M$  and any *oriented* orthonormal basis  $(e_1, e_2, e_3, e_4)$  of  $T_pM$ , we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$$

where  $R_{ijkl} = \langle R(e_i \wedge e_j), e_k \wedge e_l \rangle$ .

One has similar definitions in the nonnegative case, which we denote by  $NNIC_+$  and the case of  $\wedge^2 T_pM$ , which we denote by  $PIC_-$  and  $NNIC_-$ . If a metric is either  $PIC_+$  or  $PIC_-$  we say that it is *half-PIC*. As examples we note that:

1. Any anti-self-dual 4-manifold i.e., a manifold with  $R_{\mathcal{W}_+} = 0$ , with positive scalar curvature is  $PIC_+$ .
2. All 4-manifolds with positive isotropic curvature are  $PIC_+$ . This includes  $\mathbb{S}^4$ ,  $\mathbb{S}^3 \times \mathbb{S}^1$ . It is also known that connected sums of manifolds with positive isotropic curvature admit metric with positive isotropic curvature.
3. Any Kähler 4-manifold whose scalar curvature is nonnegative is  $NNIC_+$  but not  $PIC_+$  (with its standard orientation, see Proposition 2.13). Thus  $\mathbb{C}\mathbb{P}^2$  is  $NNIC_+$ . Moreover, since  $\mathbb{C}\mathbb{P}^2$  is half conformally flat (it satisfies  $R_{\mathcal{W}_-} = 0$ ), it is also  $PIC_-$ .

These examples show that the  $PIC_+$  condition is strictly weaker than the usual  $PIC$  condition.

By applying Wilking's criterion [24] it is easy to see (see Proposition 2.11) that the  $NNIC_{\pm}$  conditions are preserved by Ricci flow. We were informed by S. Brendle that R. Hamilton knew this fact and the proof is similar to Hamilton's proof that  $PIC$  is preserved by the Ricci flow in dimension 4 (The fact that  $PIC$  is preserved by Ricci flow in all dimensions is a relatively recent result due to S. Brendle-R. Schoen [6] and H.T. Nguyen [21]). The main result of this note is that the  $NNIC_{\pm}$  conditions are the minimal Ricci flow invariant nonnegativity conditions in dimension 4. More precisely, if  $S_B^2 \Lambda^2 \mathbb{R}^4$  denotes the space of algebraic curvature operators in dimension 4, we have the following:

**Theorem 1.1.** *Let  $\mathcal{C} \subsetneq \mathcal{C}_{\text{Scal}} \subset S_B^2 \Lambda^2 \mathbb{R}^4$  be an oriented Ricci flow invariant curvature cone.  $\mathcal{C}$  is then contained in one of the cones  $NNIC_+$  or  $NNIC_-$ . Moreover, if  $\mathcal{C}$  is an unoriented curvature cone, it is contained in the  $NNIC$  cone.*

This is proved in Section 3. We note that the above result applies to *all* Ricci flow invariant cones. In [13] a similar result is proved in dimensions  $\geq 5$ : *All the Ricci flow invariant cones constructed by Wilking [24] are contained in the  $NNIC$  cone.*

In Section 5, we show that compact Einstein 4-manifolds which are  $PIC_+$  are rigid:

**Proposition 1.2.** *A compact oriented Einstein  $PIC_-$  4-manifold is isometric, up to scaling, to  $\mathbb{S}^4$  or  $\mathbb{C}\mathbb{P}^2$  with their standard metrics. Moreover, an Einstein  $NNIC_-$  manifold is either  $PIC_-$  or flat or a negatively oriented Kähler-Einstein surface with nonnegative scalar curvature.*

The method of proof we use is based on a Bochner type formula for Einstein curvature operators arising from Ricci flow, which was introduced by S. Brendle in [4]. Our method of proof is based on a subsequent work of Brendle [5]). Another proof can be given using the equality case of a theorem of M. Gursky and C. LeBrun in [12].

In Section 4 we consider the topological implications of the  $PIC_+$  condition. These results are obtained using standard geometric techniques, and their proofs are thus only sketched.

The first observation is a modification of a result of Micallef-Wang [20]

**Proposition 1.3.** *Let  $M_i$ ,  $i = 1, 2$ , be compact oriented 4-manifolds admitting metrics with  $PIC_+$ . Then the connected sum  $M_1 \# M_2$  admits a metric with  $PIC_+$ .*

Next, a standard Bochner formula argument shows:

**Proposition 1.4.** *If an oriented 4-manifold  $(M^4, g)$  is compact and  $PIC_+$ , then  $b_2^+(M) = 0$ . If  $(M, g)$  is  $NNIC_+$  then  $b_+(M) \leq 3$ , moreover, if  $b_+(M) \geq 2$  then  $(M, g)$  is either flat or isometric to a K3 surface.*

We conclude this introduction with some speculative remarks. It would be natural to try to extend Proposition 1.2 to Ricci solitons. Note that such a result is known for  $PIC$  solitons. More generally, it might be interesting to study if a Ricci flow with surgery procedure holds for  $PIC_+$  manifolds. Note that 4-manifolds with positive isotropic curvature have been classified using Ricci flow with surgery (see [14] and [8]). A similar classification might yield a diffeomorphism classification of 4-manifolds with  $PIC_+$ .

## 2. Elementary observations about the half-PIC cone

### 2.1. Curvature operators and curvature cones

**Definition 2.1.** *The space of algebraic curvature operators  $S_B^2 \Lambda^2 \mathbb{R}^n$  is the space of symmetric endomorphisms  $R$  of  $\Lambda^2 \mathbb{R}^n$  which satisfy the first Bianchi identity:*

$$\langle R(x \wedge y), z \wedge t \rangle + \langle R(z \wedge x), y \wedge t \rangle + \langle R(y \wedge z), x \wedge t \rangle = 0$$

for all  $x, y, z, t \in \mathbb{R}^n$ .

*Remark 2.2.* Here, as in the rest of the paper,  $\mathbb{R}^n$  is endowed with its standard Euclidean structure, and the scalar product on  $\Lambda^2 \mathbb{R}^n$  is given by:

$$\langle x \wedge y, z \wedge t \rangle = \langle x, z \rangle \langle y, t \rangle - \langle x, t \rangle \langle y, z \rangle.$$

Similarly if  $(M, g)$  is a Riemannian manifold,  $\Lambda^2 TM$  will be equipped with the inner product coming from the Riemannian metric on  $TM$ .

The Ricci morphism:  $\rho : S_B^2 \Lambda^2 \mathbb{R}^n \rightarrow S^2 \mathbb{R}^n$  is defined by

$$\langle \rho(R)x, y \rangle = \sum_{i=1}^n \langle R(x \wedge e_i), y \wedge e_i \rangle$$

where  $(e_i)_{1 \leq i \leq n}$  is an orthonormal basis of  $\mathbb{R}^n$ .  $R$  is said to be Einstein if  $\rho(R)$  is a multiple of the identity operator  $\text{id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Similarly, the scalar curvature of an algebraic curvature operator is just twice its trace.

The action of  $O(n, \mathbb{R})$  on  $\mathbb{R}^n$  induces the following action of  $O(n, \mathbb{R})$  on  $S_B^2 \Lambda^2 \mathbb{R}^n$ :

$$\langle g.R(x \wedge y), z \wedge t \rangle = \langle R(gx \wedge gy), gz \wedge gt \rangle. \quad (1)$$

Recall that the representation of  $O(n, \mathbb{R})$  given by its action on  $S_B^2 \Lambda^2 \mathbb{R}^n$  is decomposed into irreducible representations in the following way:

$$S_B^2 \Lambda^2 \mathbb{R}^n = \mathbb{R}I \oplus (S_0^2 \mathbb{R}^n \wedge \text{id}) \oplus \mathcal{W} \quad (2)$$

where the space of Weyl curvature operators  $\mathcal{W}$  is the kernel of the Ricci endomorphism  $\rho$  and  $S_0^2 \mathbb{R}^n \wedge \text{id}$  is the image of the space of traceless endomorphisms of  $\mathbb{R}^n$  under the application  $A_0 \mapsto A_0 \wedge \text{id}$ . The wedge product of two symmetric operators  $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by

$$(A \wedge B)(x \wedge y) = \frac{1}{2} (Ax \wedge By + Bx \wedge Ay).$$

This corresponds to the half of the Kulkarni-Nomizu product of  $A$  and  $B$  viewed as quadratic forms.

In dimension 2, only the first summand of (2) exists. In dimension 3 the  $\mathcal{W}$  factor is 0.

Starting in dimension 4, all three components exist. Moreover in dimension 4, if we restrict the  $O(4, \mathbb{R})$  action to an  $SO(4, \mathbb{R})$  action, the decomposition (2) is no longer irreducible: the Weyl part splits into self-dual and anti-self-dual parts:

$$S_B^2 \Lambda^2 \mathbb{R}^4 = \mathbb{R}I \oplus (S_0^2 \mathbb{R}^4 \wedge \text{id}) \oplus \mathcal{W}_+ \oplus \mathcal{W}_-. \quad (3)$$

This comes from the fact that the adjoint representation of  $SO(4, \mathbb{R})$  on  $\mathfrak{so}(4, \mathbb{R}) \simeq \Lambda^2 \mathbb{R}^4$  split into two three dimensional irreducible components, which correspond to self-dual and anti-self-dual 2-forms. See the Appendix for more details.

We will write  $R = R_I + R_0 + R_{\mathcal{W}}$  to denote the decomposition of a curvature operator along the three components of (2). Similarly we will write  $R = R_I + R_0 + R_{\mathcal{W}_+} + R_{\mathcal{W}_-}$  for the decomposition of  $R \in S_B^2 \Lambda^2 \mathbb{R}^4$  along (3).

**Definition 2.3.** *An oriented (resp. unoriented) curvature cone is a closed convex cone  $\mathcal{C} \subset S_B^2 \Lambda^2 \mathbb{R}^n$  such that:*

- $\mathcal{C}$  is invariant under the action of  $SO(n, \mathbb{R})$  (resp.  $O(n, \mathbb{R})$ ) given by (1).
- The identity operator  $I : \Lambda^2 \mathbb{R}^n \rightarrow \Lambda^2 \mathbb{R}^n$  is in the interior of  $\mathcal{C}$ .

*Remark 2.4.* The condition that  $I$  is in the interior of  $\mathcal{C}$  implies that  $\mathcal{C}$  has full dimension.

Each of these cones defines a nonnegativity condition for the curvature of Riemannian manifold in the following way: the curvature operator  $R$  of a Riemannian manifold  $(M, g)$  is a section of the bundle  $S_B^2 \Lambda^2 TM$  which is built from  $TM$  the same way  $S_B^2 \Lambda^2 \mathbb{R}^n$  is

built from  $\mathbb{R}^n$ . For each  $x \in M$ , one can choose an orthonormal basis of  $T_x M$  to build an isomorphism between  $S_B^2 \Lambda^2 T_x M$  and  $S_B^2 \Lambda^2 \mathbb{R}^n$ . Thanks to the  $O(n, \mathbb{R})$ -invariance of  $\mathcal{C}$ , this allows us to embed  $\mathcal{C}$  in  $S_B^2 \Lambda^2 T_x M$  in a way which is independent of the basis of  $T_x M$  we started with.

We then say that  $(M, g)$  has  $\mathcal{C}$ -nonnegative curvature if for any  $x \in M$  the curvature operator of  $(M, g)$  at  $x$  belongs to the previously discussed embedding of  $\mathcal{C}$  in  $S_B^2 \Lambda^2 T_x M$ . Similarly,  $(M, g)$  is said to have positive  $\mathcal{C}$ -curvature if its curvature operator at each point is in the interior of  $\mathcal{C}$ . By definition, the sphere  $\mathbb{S}^n$  has positive  $\mathcal{C}$ -curvature for all curvature cones  $\mathcal{C}$ .

## 2.2. Ricci flow invariant curvature cones

Let  $Q$  be the quadratic vector field on  $S_B^2 \Lambda^2 \mathbb{R}^n$  defined by

$$Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^\#.$$

Here,  $\mathbf{R}^2$  is just the square of  $\mathbf{R}$  seen as an endomorphism of  $\Lambda^2 \mathbb{R}^n$ .  $\mathbf{R}^\#$  is defined in the following way:

$$\langle \mathbf{R}^\# \eta, \eta \rangle = -\frac{1}{2} \sum_{i=1}^{n(n-1)/2} \langle [\eta, \mathbf{R}([\eta, \mathbf{R}(\omega_i)])], \omega_i \rangle$$

where  $(\omega_i)_{i=1 \dots n(n-1)/2}$  is an orthonormal basis of  $\Lambda^2 \mathbb{R}^n$  and the Lie bracket  $[\cdot, \cdot]$  on  $\Lambda^2 \mathbb{R}^n$  comes from its identification with  $\mathfrak{so}(n, \mathbb{R})$  given by:

$$\phi : x \wedge y \mapsto (u \mapsto \langle x, u \rangle y - \langle y, u \rangle x).$$

This expression for  $\mathbf{R}^\#$  can be found in [2].

**Definition 2.5.** A curvature cone  $\mathcal{C}$  is said to be Ricci flow invariant if for any  $\mathbf{R} \in \partial \mathcal{C}$  of  $\mathcal{C}$ ,  $Q(\mathbf{R}) \in T_{\mathbf{R}} \mathcal{C}$ , the tangent cone at  $\mathbf{R}$  to  $\mathcal{C}$ .

*Remark 2.6.* This condition is equivalent to the fact that the solutions to the ODE  $\frac{d}{dt} \mathbf{R} = Q(\mathbf{R})$  which start inside  $\mathcal{C}$  stay in  $\mathcal{C}$  for positive times.

We now discuss the relevance of this notion to the actual Ricci flow equation. This is the content of Hamilton's maximum principle:

**Theorem 2.7.** (R. Hamilton [14]) Let  $n \geq 2$  and  $\mathcal{C} \subset S_B^2 \Lambda^2 \mathbb{R}^n$  be a Ricci flow invariant curvature cone. If  $(M^n, g(t))_{t \in [0, T]}$  is a Ricci flow on a compact manifold such that  $(M, g(0))$  has  $\mathcal{C}$ -nonnegative curvature, then for  $t \in [0, T)$ ,  $(M, g(t))$  has  $\mathcal{C}$ -nonnegative curvature.

Next we recall Wilking's method for generating Ricci flow invariant cones. We begin with the isomorphism  $\phi : \Lambda^2 \mathbb{R}^n \rightarrow \mathfrak{so}(n, \mathbb{R})$  given by

$$\phi(u \wedge v)(x) = \langle u, x \rangle v - \langle v, x \rangle u \quad x \in \mathbb{R}^n.$$

Under the above identification of  $\Lambda^2 \mathbb{R}^n$  with  $\mathfrak{so}(n, \mathbb{R})$  the inner product on  $\mathfrak{so}(n, \mathbb{R})$  is given by  $\langle A, B \rangle = -\frac{1}{2} \text{tr}(AB)$ . Extend this inner product to a Hermitian form

on  $\mathfrak{so}(n, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{so}(n, \mathbb{C})$ . We also extend any algebraic curvature operator  $R \in S^2(\mathfrak{so}(n, \mathbb{R}))$  to a complex linear map  $\mathfrak{so}(n, \mathbb{C}) \rightarrow \mathfrak{so}(n, \mathbb{C})$ . Denoting the extensions by the same symbols, one has :

**Theorem 2.8** ([24]). *Let  $S$  be a subset of the complex Lie algebra  $\mathfrak{so}(n, \mathbb{C})$ . If  $S$  is invariant under the adjoint action of the corresponding complex Lie group  $SO(n, \mathbb{C})$ , then the curvature cone*

$$\mathcal{C}(S) = \{R \in S^2(\mathfrak{so}(n, \mathbb{R})) \mid \langle R(X), X \rangle \geq 0 \text{ for all } X \in S\}$$

is Ricci flow invariant.

We also have Wilking's generalization of the Brendle-Schoen strong maximum principle [7] (which is based on a maximum principle of J. M. Bony [3]):

**Theorem 2.9** ([24]). *Let  $S$  be an  $Ad_{SO(n, \mathbb{C})}$  invariant subset of  $\mathfrak{so}(n, \mathbb{C})$  and  $(M, g)$  be a compact  $n$ -manifold with nonnegative  $\mathcal{C}(S)$ -curvature. Let  $g(t)$  be the solution to Ricci flow starting at  $g$ . For  $p \in M$  and  $t > 0$ , let  $S_t(p) \subset \Lambda^2 T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$  be the subset corresponding to  $S$  at time  $t$  i.e.,  $S_t(p) = \rho_{g(t)}^{-1}(S)$  and let*

$$T_t(p) := \{x \in S_t(p) \mid \langle R(t)(x), x \rangle_t = 0\}.$$

Then the set  $\bigcup_{p \in M} T_t(p)$  is invariant under parallel transport.

### 2.3. The half-PIC cone

**Proposition 2.10.**  $R \in S_B^2 \Lambda^2 \mathbb{R}^n$  is  $PIC_+$  if and only if the symmetric operator on  $\Lambda_+^2 \mathbb{R}^4$  defined by the quadratic form  $\langle R\eta, \eta \rangle$  is 2-positive. This is also equivalent to the condition  $\frac{\text{Scal}}{6} - R_{\mathcal{W}_+} > 0$ .

As a corollary, we get that a  $PIC_+$  curvature operator has positive scalar curvature and a  $NNIC_+$  curvature operator with zero scalar curvature has vanishing  $R_{\mathcal{W}_+}$ . Similar statements hold for  $PIC_-$  and  $NNIC_-$ .

*Proof.* This comes from Micallef and Moore [19]. Let  $\omega = (e_1 + ie_2) \wedge (e_3 + ie_4) \in \Lambda^2 \mathbb{C}^4$  where  $(e_i)_{i=1..4}$  is a oriented orthonormal basis of  $\mathbb{R}^4$ . Then :

$$\begin{aligned} \langle R\omega, \omega \rangle &= \langle R(e_1 \wedge e_3 - e_2 \wedge e_4), e_1 \wedge e_3 - e_2 \wedge e_4 \rangle \\ &\quad + \langle R(e_1 \wedge e_4 + e_2 \wedge e_3), e_1 \wedge e_4 + e_2 \wedge e_3 \rangle. \end{aligned}$$

Since  $e_1 \wedge e_3 - e_2 \wedge e_4$  and  $e_1 \wedge e_4 + e_2 \wedge e_3$  are in  $\Lambda_+^2 \mathbb{R}^4$  and any pair of orthonormal 2-forms in  $\Lambda_+^2 \mathbb{R}^4$  can be written in this way for some oriented orthonormal basis of  $\mathbb{R}^4$ , this shows that  $R$  is  $PIC_+$  if and only if the quadratic form  $\eta \mapsto \langle R\eta, \eta \rangle$  on  $\Lambda_+^2 \mathbb{R}^4$  is 2-positive.

To see the second equivalence, first notice that  $\langle R\eta, \eta \rangle = \langle (R_I + R_{\mathcal{W}_+})\eta, \eta \rangle$  for any  $\eta \in \Lambda_+^2 \mathbb{R}^4$ . So  $R$  is  $PIC_+$  if and only if  $\frac{\text{Scal}}{12} + R_{\mathcal{W}_+}$  is 2-positive. Let  $\lambda_+ \leq \mu_+ \leq \nu_+$  be the eigenvalues of  $R_{\mathcal{W}_+}$ .  $\frac{\text{Scal}}{12} + R_{\mathcal{W}_+}$  is 2-positive if and only if  $(\frac{\text{Scal}}{12} + \mu_+) + (\frac{\text{Scal}}{12} + \lambda_+) > 0$ . The tracelessness of  $R_{\mathcal{W}_+}$  implies that  $R$  is  $PIC_+$  if and only if :  $\frac{\text{Scal}}{6} - \nu_+ > 0$ , which is exactly the condition  $\frac{\text{Scal}}{6} - R_{\mathcal{W}_+} > 0$ .  $\square$

**Proposition 2.11.** *The cone  $\mathcal{C}_{IC_+} = \{R \in S_B^2 \Lambda^2 \mathbb{R}^4 \mid R \text{ is } NNIC_+\}$  is a Ricci flow invariant curvature cone.*

*Proof.* We remark that :

$$\mathcal{C}_{IC_+} = \{R \mid \forall \omega \in S, \langle R\omega, \omega \rangle \geq 0\}$$

with  $S = \{\omega \in \mathfrak{so}(4, \mathbb{C}) \mid \text{trace}(\omega^2) = 0, \omega \in \Lambda_+^2 \mathbb{C}^4\}$ . This comes from the fact that  $\omega = \omega_1 + i\omega_2 \in S$  if and only if  $|\omega_1|^2 = |\omega_2|^2 = 1$  and  $\langle \omega_1, \omega_2 \rangle = 1$ , and thus  $R$  is nonnegative on  $S$  if and only if it is 2-nonnegative on  $\Lambda_+^2 \mathbb{R}^4$ . Thus  $\mathcal{C}$  is a Wilking cone and is therefore Ricci flow invariant (see [24]).  $\square$

*Remark 2.12.* This result contradicts Theorem 1.1 in [13], which says that any Wilking cone is contained in the cone of curvature operators with nonnegative isotropic curvature. The proof is however valid in dimension  $n \geq 5$ . The point is that it uses (p. 4 of [13]) the simplicity of the Lie algebra  $\mathfrak{so}(n, \mathbb{C})$ , which holds only if  $n \neq 4$ . However, one should notice that the cone  $\mathcal{C}_{IC_+}$  is only  $SO(4, \mathbb{R})$ -invariant. We will prove in Section 3, that, regardless of the Wilking condition, the cone of curvature operators with nonnegative isotropic curvature is maximal among  $O(4, \mathbb{R})$ -invariant Ricci flow invariant cones strictly smaller than the cone of operators with nonnegative scalar curvature.

The cone  $\mathcal{C}_{IC_-}$  is defined similarly and is also Ricci flow invariant. Moreover, we have that  $\mathcal{C}_{IC_+} \cap \mathcal{C}_{IC_-}$  is the cone of curvature operators with nonnegative isotropic curvature.

An oriented 4-manifold  $(M, g)$  is said to be positive Kähler if  $g$  is Kähler for some almost complex structure  $J$  and the orientation of  $M$  induced by  $J$  coincide with the given orientation of  $M$ . This is equivalent to the Kähler form being a section of the bundle  $\Lambda_+^2 T^*M$ . Similarly  $(M, g)$  is negative Kähler if  $g$  is Kähler for some almost complex structure  $J$  and the orientation of  $M$  induced by  $J$  is the opposite of the given orientation of  $M$ . With these condition,  $\mathbb{C}\mathbb{P}^2$  is positive Kähler.

**Proposition 2.13.** *Any positive Kähler manifold with nonnegative scalar curvature is  $NNIC_+$  but not  $PIC_+$ . A  $NNIC_-$  positive Kähler manifold is biholomorphic to  $\mathbb{C}\mathbb{P}^2$ .*

*Proof.* A theorem of Derdziński (see [9]) shows that if  $(M^4, g)$  is positive Kähler and of real dimension 4, then the eigenvalues of  $R_{\mathcal{W}_+}$  are  $\frac{\text{Scal}}{6}$ ,  $-\frac{\text{Scal}}{12}$  and  $-\frac{\text{Scal}}{12}$ , so  $\frac{\text{Scal}}{6} - R_{\mathcal{W}_+}$  is nonnegative (with a zero eigenvalue). Hence  $(M^4, g)$  is  $NNIC_+$  and not  $PIC_+$ .

If in addition  $(M^4, g)$  is  $NNIC_-$ , then  $(M^4, g)$  is Kähler and has nonnegative isotropic curvature, earlier results of one of the authors ([23]) imply that  $M^4$  is actually biholomorphic to  $\mathbb{C}\mathbb{P}^2$ .  $\square$

### 3. Half-PIC as a maximal Ricci flow invariant curvature condition

In this section we prove:



**Theorem 3.1.** *Let  $\mathcal{C} \subsetneq \mathcal{C}_{\text{Scal}} \subset S_B^2 \Lambda^2 \mathbb{R}^4$  be an oriented Ricci flow invariant curvature cone.  $\mathcal{C}$  is then contained in one of the cones  $\text{PIC}_+$  or  $\text{PIC}_-$ . Moreover, if  $\mathcal{C}$  is an unoriented curvature cone, it is contained in the  $\text{PIC}$  cone.*

We will need a couple of results from [22].

**Proposition 3.2** (Proposition A.5 from [22]). *If an oriented curvature cone  $\mathcal{C}$  contains a non zero  $R \in \mathcal{W}_+$  (resp.  $R \in \mathcal{W}_-$ ), then  $\mathcal{C}$  contains  $\mathcal{W}_+$  (resp.  $\mathcal{W}_-$ ).*

**Proposition 3.3** (Proposition 3.6 from [22]). *If an oriented curvature cone  $\mathcal{C}$  contains  $\mathcal{W}$  and is Ricci flow invariant, then  $\mathcal{C}$  is the cone  $\mathcal{C}_{\text{Scal}}$  of curvature operators whose scalar curvature is nonnegative.*

We begin with a lemma:

**Lemma 3.4.** *Let  $\mathcal{C}$  be an oriented curvature cone and  $R \in \mathcal{C}$ , then  $R_I + R_{\mathcal{W}_+} \in \mathcal{C}$ .*

*Proof.* Consider the subgroup  $G = \mathbb{S}^3 \subset SO(4, \mathbb{R})$  defined in the appendix. It acts trivially on  $\Lambda_+^2 \mathbb{R}^4$  and irreducibly on  $\Lambda_-^2 \mathbb{R}^4$ . Set  $\tilde{R} = \int_G g.R dg$  where  $dg$  is the Haar measure on  $G$ . Since  $\mathcal{C}$  is convex and invariant under  $SO(4, \mathbb{R}) \supset G$ ,  $\tilde{R} \in \mathcal{C}$ . We claim that  $\tilde{R} = R_I + R_{\mathcal{W}_+}$ .

Since  $G$  acts trivially on  $\mathbb{R}I$  and  $\mathcal{W}_+$ , we have  $\tilde{R}_I = R_I$  and  $\tilde{R}_{\mathcal{W}_+} = R_{\mathcal{W}_+}$ .

We now prove that  $\tilde{R}_0 = 0$ . We have that  $R_0 = A_0 \wedge \text{id}$  for some  $A_0 \in S_0^2 \mathbb{R}^4$ . Then  $g.R_0 = g^{-1}A_0g \wedge \text{id}$ . This implies that  $\tilde{R}_0 = \tilde{A}_0 \wedge \text{id}$  where  $\tilde{A}_0 = \int_G g^{-1}A_0g dg$ . As described in the appendix,  $G$ , viewed as the group of unit quaternions, acts on  $\mathbb{R}^4$ , identified with the quaternions  $\mathbb{H}$ , by left multiplication. This implies that, for any  $u \in \mathbb{R}^4$ :

$$\langle \tilde{A}_0 u, u \rangle = \int_G \langle A_0(gu), gu \rangle dg = \int_{\|x\|=\|u\|} \langle A_0 x, x \rangle dx$$

where  $dx$  is the usual measure on the sphere of radius  $\|u\|$  scaled to have total mass 1. Since the last integral is just a multiple of the trace of  $A_0$ , we have shown that  $\tilde{A}_0 = 0$ .

It remains to show that  $\tilde{R}_{\mathcal{W}_-} = 0$ . But the action of  $G$  on the three dimensional space  $\Lambda_-^2 \mathbb{R}^4$  is isomorphic to the standard action of  $\mathbb{S}^3$  on  $\mathbb{R}^3$  by rotations. Thus the action of  $G$  on  $\mathcal{W}_-$  is isomorphic to the irreducible action of  $\mathbb{S}^3$  on traceless symmetric  $3 \times 3$  matrices. This implies that  $\mathcal{W}_-$  is irreducible as a representation of  $G$ , thus  $\tilde{R}_{\mathcal{W}_-} = \int_G g.R_{\mathcal{W}_-} dg = 0$ .  $\square$

*Proof of Theorem 3.1.* We assume that  $\mathcal{C}$  is contained in neither  $\mathcal{C}_{\text{IC}_+}$  nor  $\mathcal{C}_{\text{IC}_-}$  and we will show that  $\mathcal{C} = \mathcal{C}_{\text{Scal}}$ . By Proposition 3.3. It is enough to show that  $\mathcal{C}$  contains  $\mathcal{W}$ .

Since  $\mathcal{C}$  is not contained in  $\text{NNIC}_+$ , we can find some  $R \in \mathcal{C}$  such that  $R_{\mathcal{E}_+} = R_I + R_{\mathcal{W}_+}$  is not 2-nonnegative. Thanks to Lemma 3.4, we can assume that  $R = R_{\mathcal{E}_+}$ . Let  $v_1, v_2, v_3 \in \Lambda_+^2 \mathbb{R}^4$  be the eigenvectors of  $R$  and  $\mu_1 \leq \mu_2 \leq \mu_3$  be the associated eigenvalues. We have that  $\mu_1 + \mu_2 + \mu_3 > 0$  and  $\mu_1 + \mu_2 < 0$ .

The action on  $SO(4)$  on  $\Lambda_+^2 \mathbb{R}^4$  is transitive on oriented orthonormal basis, thus we can find some  $g \in SO(4, \mathbb{R})$  such that  $ge_1 = e_2$ ,  $ge_2 = -e_1$  and  $ge_3 = e_3$ . We consider the curvature operator  $\tilde{R} = \frac{1}{2}(R + g.R)$ . It has eigenvectors  $e_1, e_2$  and  $e_3$  with associated

eigenvalues  $\tilde{\mu}_1 = \tilde{\mu}_2 < 0 < \tilde{\mu}_3$ . From this we deduce that up to scaling  $\tilde{R} + |\tilde{\mu}_1|I$  is the curvature operator of  $\overline{\mathbb{C}\mathbb{P}^2}$ . Thus the curvature operator of  $\overline{\mathbb{C}\mathbb{P}^2}$  is in the interior of  $\mathcal{C}$ . Arguing as in the proof of corollary 0.3 in [22], we get that  $\mathcal{W}_+ \subset \mathcal{C}$ .

Similarly, using that  $\mathcal{C}$  contains some  $R$  which is not in  $NNIC_-$ , one can show that the curvature operator of  $\mathbb{C}\mathbb{P}^2$  is in the interior of  $\mathcal{C}$  and get that  $\mathcal{W}_- \subset \mathcal{C}$ . We have thus proved that  $\mathcal{W} \subset \mathcal{C}$ . This proves the first part of the theorem.

The unoriented case follows: any orientation reversing element of  $O(4, \mathbb{R})$  will exchange  $\Lambda_+^2 \mathbb{R}^4$  and  $\Lambda_-^2 \mathbb{R}^4$ , consequently it will exchange the cones  $PIC_+$  and  $PIC_-$ . Thus a cone which is invariant under the full  $O(4, \mathbb{R})$  will have to be contained in the intersection of the  $PIC_+$  and  $PIC_-$  cones, which is exactly the  $PIC$  cone.  $\square$

## 4. The topology of half-PIC manifolds

**Theorem 4.1.** *The connected sum of two  $PIC_+$  manifolds admits a  $PIC_+$  metric.*

*Proof.* This follows from [13] since the curvature operator of  $\mathbb{R} \times \mathbb{S}^3$  is in the interior of  $\mathcal{C}_{IC_+}$ . See also the recent work of Hoelzel [17].  $\square$

The following Bochner-Weitzenböck formula for self-dual 2-forms  $\omega_+$  can be found in [10, appendix C] :

$$\Delta_d \omega_+ = \Delta \omega_+ + 2 \left( \frac{\text{Scal}}{6} - R_{\mathcal{W}_+} \right) \omega_+.$$

From this we easily deduce:

**Theorem 4.2.** *Let  $(M, g)$  be a compact oriented 4-manifold.*

- *If  $(M, g)$  is  $PIC_+$  then  $b_+(M) = 0$ .*
- *If  $(M, g)$  is  $NNIC_+$  then  $b_+(M) \leq 3$ , moreover, if  $b_+(M) \geq 2$  then  $(M, g)$  is either flat or isometric to a K3 surface.*

## 5. Rigidity of Einstein half-PIC manifolds

We recall the classical rigidity theorem for half-conformally flat Einstein 4-manifolds:

**Theorem 5.1.** *(N. Hitchin [16], T. Friedrich - H. Kurke [11]) Let  $(M, g)$  be a compact oriented half-conformally flat Einstein 4-manifold.*

1. *If  $M$  has positive scalar curvature then it is isometric, up to scaling, to  $S^4$  or  $\mathbb{C}P^2$  with their canonical metrics.*
2. *If  $M$  is scalar flat then it is either flat or its universal cover is isometric to a K3 metric with its Calabi-Yau metric.*

The following result can be regarded as a generalization of the above theorem. This result also follows from the work of C. LeBrun - M. Gursky in [12].

**Theorem 5.2.** *An Einstein  $PIC_-$  4-manifold is isometric, up to scaling, to  $\mathbb{S}^4$  or  $\mathbb{C}\mathbb{P}^2$  with their standard metrics. Moreover, an Einstein  $NNIC_-$  manifold is either  $PIC_-$  or flat or a negatively oriented Kähler-Einstein surface with nonnegative scalar curvature.*

*Proof.* We first deal with the case where  $(M^4, g)$  is  $PIC_-$ . By Theorem 5.1 it is enough to show that  $R_{\mathcal{W}_-} = 0$ . The proof of this fact closely follows the proof by Brendle in [5] that Einstein manifolds which have certain Ricci flow invariant positive curvature properties are spherical space-forms.

We can assume that  $\text{Ric} = 3g$ . Let  $\kappa > 0$  be the largest real number such that  $\tilde{R} = R - \kappa I$  is  $NNIC_-$ . Note that the scalar curvature of  $\tilde{R}$  is a constant equal to  $12(1 - \kappa)$ , in particular,  $\kappa \in (0, 1]$ . Since  $(M, g)$  is Einstein, Proposition 3 in [4] shows that:

$$\Delta R + 2Q(R) = 6R.$$

We compute :

$$\begin{aligned} \Delta \tilde{R} + 2Q(\tilde{R}) &= \Delta R + 2Q(R) - 4\kappa B(R, I) + 2\kappa^2 Q(I) \\ &= 6R - 12\kappa I + 6\kappa^2 I \\ &= 6\tilde{R} + 6\kappa(\kappa - 1)I. \end{aligned}$$

Here we used that since  $R$  is Einstein and has scalar curvature 12,  $B(R, I) = B(R_I, I) = B(I, I) = Q(I) = 3I$  (these identities come from [2]).

Note that at each point we have that  $\Delta \tilde{R}$ ,  $Q(\tilde{R})$  and  $-\tilde{R}$  are in  $T_{\tilde{R}}\mathcal{C}$ , where  $\mathcal{C}$  is the cone of  $NNIC_-$  curvature operators. This follows from the fact that  $\mathcal{C}$  is a convex Ricci flow invariant cone. This implies that  $6\kappa(\kappa - 1)I \in T_{\tilde{R}}\mathcal{C}$ .

We claim that the scalar curvature of  $\tilde{R}$  is zero, this will imply that  $\tilde{R}_{\mathcal{W}_-} = R_{\mathcal{W}_-}$  vanishes.

Assume that the scalar curvature of  $\tilde{R}$  is positive. This implies that  $\kappa < 1$ . Consequently  $-I \in T_{\tilde{R}}\mathcal{C}$ , and we can find  $\varepsilon > 0$  such that  $\tilde{R} - \varepsilon I \in \mathcal{C}$ . Thus  $\tilde{R} = (\tilde{R} - \varepsilon I) + \varepsilon I$  is in the interior of  $\mathcal{C}$  since  $I$  is in the interior. This contradicts the definition of  $\kappa$ .

We now deal with the borderline cases. First, since the only locally reducible Einstein 4-manifold with nonnegative scalar curvature are quotients of  $\mathbb{R}^4$  and  $\mathbb{S}^2 \times \mathbb{S}^2$ , and both of them are negative Kähler, we can assume that  $(M, g)$  is irreducible. So the only thing to show is that if  $(M, g)$  is irreducible  $NNIC_-$  and not negative Kähler, then it is  $PIC_-$ .

Using Berger's classification of holonomy groups, we see that  $(M, g)$  is irreducible not negative Kähler if and only if the holonomy group of  $(M, g)$   $G \subset SO(4, \mathbb{R})$  contains the group  $\mathbb{S}_-^3 \subset SO(4, \mathbb{R})$  defined in the appendix. This implies in particular that  $G$  acts irreducibly on  $\Lambda_-^2 \mathbb{R}^4$ .

It follows from the strong maximum principle that the set :

$$K = \{\omega \in \Lambda_-^2 T^*M \mid \frac{\text{Scal}}{6}|\omega|^2 - \langle R_{\mathcal{W}_-} \omega, \omega \rangle = 0\}$$

is invariant under parallel transport. The irreducibility of the action of  $G$  implies that  $K$  is either 0, in which case  $(M, g)$  is  $PIC_-$ , or the whole  $\Lambda_-^2 T^*M$ . This implies that  $\text{Scal} = 0$  and  $R_{\mathcal{W}_-} = 0$  and hence, by Theorem 5.1,  $(M, g)$  is a negative K3 surface, hence negative Kähler, a contradiction.  $\square$

## A. $SO(4, \mathbb{R})$ , $\Lambda^2 \mathbb{R}^4$ and the curvature of 4 manifolds

In this section we recall a small number of classical facts related to the splitting of  $\Lambda^2 TM$  in dimension 4. For practical purposes, we interpret them using quaternions.

We identify  $\mathbb{R}^4$  with the quaternions  $\mathbb{H}$  by  $(x, y, z, t) \mapsto x + iy + jz + kt$ . We denote by  $\mathbb{S}^3$  the group (isomorphic to  $SU(2)$ ) of unit quaternions. The classical double cover of  $SO(4, \mathbb{R})$  by  $\mathbb{S}^3 \times \mathbb{S}^3$  is given by :

$$\begin{aligned} \pi : \mathbb{S}^3 \times \mathbb{S}^3 &\rightarrow SO(4, \mathbb{R}) \\ (q_1, q_2) &\mapsto (x \in \mathbb{H} \mapsto q_1 x q_2^{-1}). \end{aligned}$$

This shows that the Lie algebra of  $SO(4, \mathbb{R})$  (which is isomorphic to  $\Lambda^2 \mathbb{R}^4$ ) splits as a direct sum :

$$\Lambda^2 \mathbb{R}^4 = \mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{so}(3, \mathbb{R})$$

This decomposition is exactly the same as the decomposition  $\Lambda^2 \mathbb{R}^4 = \Lambda_-^2 \mathbb{R}^4 \oplus \Lambda_+^2 \mathbb{R}^4$  of two-forms into anti-self-dual and self-dual parts.

We denote by  $\mathbb{S}_+^3 \subset SO(4, \mathbb{R})$  (resp.  $\mathbb{S}_-^3 \subset SO(4, \mathbb{R})$ ) the image of  $\{1\} \times \mathbb{S}^3$  (resp.  $\mathbb{S}^3 \times \{1\}$ ) under  $\pi$ .  $\mathbb{S}_+^3$  acts irreducibly on  $\Lambda_+^2 \mathbb{R}^4$  and trivially on  $\Lambda_-^2 \mathbb{R}^4$ .

From this decomposition, the following decomposition of the representation of  $SO(4, \mathbb{R})$  on the space  $S_B^2 \Lambda^2 \mathbb{R}^4$  of algebraic curvature operators holds :

$$S_B^2 \Lambda^2 \mathbb{R}^4 = \mathbb{R} I \oplus S_0^2 \mathbb{R}^n \wedge \text{id} \oplus \mathcal{W}_+ \oplus \mathcal{W}_-,$$

where  $I : \Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^2 \mathbb{R}^4$  is the identity operator,  $S_0^2 \mathbb{R}^n \wedge \text{id}$  is the space of pure traceless Ricci tensor, and  $\mathcal{W}_+$  (resp.  $\mathcal{W}_-$ ) is the space of self-dual (resp. anti-self-dual) Weyl curvature tensors. Note that  $\mathcal{W}_+$  (resp.  $\mathcal{W}_-$ ) is in fact the space of traceless symmetric operators on  $\Lambda_+^2 \mathbb{R}^4$  (resp.  $\Lambda_-^2 \mathbb{R}^4$ ). If  $R$  is a curvature operator, we will write its decomposition as  $R = R_I + R_0 + R_{\mathcal{W}_+} + R_{\mathcal{W}_-}$ .

## References

- [1] A. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume **10**, Springer-Verlag.
- [2] C. Böhm, B. Wilking, *Manifolds with positive curvature operators are space forms*, Annals of Math. **167** (2008), 1079-1097.
- [3] J. M. Bony, *Principe du maximum, in égalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble) **19** (1969), 277 – 304.
- [4] S. Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. J. **151** (2010), 1-21.

- [5] S. Brendle, *Einstein metrics and preserved curvature conditions for the Ricci flow*, Complex and Differential Geometry Conference at Hannover 2009, Springer Proceedings in Mathematics Volume 8 (2011), 81-85.
- [6] S. Brendle, R. Schoen, *Classification of manifolds with weakly  $\frac{1}{4}$ -pinched curvatures*, Acta Mathematica, 200 (2008), 287-307.
- [7] S. Brendle, R. Schoen, *Manifolds with  $\frac{1}{4}$ -pinched curvatures are space forms*, Journal of AMS, 22 (2009), 287-307.
- [8] B.-L. Chen, X.-P. Zhu, *Ricci flow with surgery on four-manifolds with positive isotropic curvature*, J. Diff. Geom. 74 (2006), 177-264.
- [9] A. Derdzinski, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Comp. Math. 49 (1983), 405-433.
- [10] D. Freed, K. Uhlenbeck, *Instantons and four-manifolds*, Mathematical Sciences Research Institute Publications, vol. 1 (1991).
- [11] T. Friedrich, H. Kurke, *Compact four-dimensional self dual Einstein manifolds with positive scalar curvature*, Math. Nachr. 106 (1982) 271-299.
- [12] M. Gursky, C. LeBrun, *On Einstein Manifolds of Positive Sectional Curvature*, Annals of Global Analysis and Geometry 17 (1999) 315-328.
- [13] H. S. Gururaja, S. Maity, H. Seshadri, *On Wilking's criterion for the Ricci flow*, Math. Z. 274 (2013) 471-481.
- [14] R. Hamilton, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. 5 (1997) 1-92.
- [15] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Diff. Geom, 22 (1986), 153-179.
- [16] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Diff. Geom. 9 (1974) 435-441.
- [17] S. Hoelzel, *Surgery stable curvature conditions*, arXiv:1303.6531.
- [18] Ni, L. and N. Wallach (2008) *Four-Dimensional Gradient Shrinking Solitons*, International Mathematics Research Notices, Vol. 2008, Article ID rnm152.
- [19] M. Micallef, J. Moore *Minimal Two-Spheres and the Topology of Manifolds with Positive Curvature on Totally Isotropic Two-Planes*, Annals of Math. (1988), 127, 199-227.
- [20] M. Micallef, M . Wang, *Metrics with nonnegative isotropic curvature*, Duke Math. J. 72, Number 3 (1993), 649-672.

- [21] H. T. Nguyen, *Isotropic curvature and the Ricci flow*, Int. Math. Res. Not., **3** (2010), 536-558.
- [22] T. Richard, H. Seshadri, *Noncoercive Ricci flow invariant cones*, arXiv:1308.1190.
- [23] H. Seshadri, *Manifolds with nonnegative isotropic curvature*, Comm. Analysis and Geometry (2009), **4**, 621-635.
- [24] B. Wilking, *A Lie algebraic approach to Ricci flow invariant curvature conditions and Harnack inequalities*, J. Reine Angew. Mathematik (2013), **679**, 223-247.