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# RANK THEOREM IN INFINITE DIMENSION AND LAGRANGE MULTIPLIERS 

JOËL BLOT


#### Abstract

We use an extension to the infinite dimension of the Rank Theorem of the differential calculus to establish a Lagrange theorem for optimization problems in Banach spaces. We provide an application to variational problems on a space of bounded sequences under equality constraints.


Keywords : Rank theorem, Lagrange multipliers, Banach spaces.
Classification MSC 2010: 49K27, 49K30.

## 1. Introduction

Firstly we recall the extension of the rank theorem to the infinite dimension as it is established in [5] (Section 3).
Secondly we establish a theorem of existence of Lagrange multipliers in Banach spaces for maximization problems under equality constraints (Section 4) like the following one

$$
(M) \begin{cases}\text { Maximize } & J(x) \\ \text { subject to } & f(x)=0\end{cases}
$$

where the functional $J$ and the mapping $f$ are defined on an open subset of a Banach space, and the mapping $f$ takes its values in a Banach space. Among the classical multiplier rules, when the differential of $f$ at the solution is surjective we obtain the existence of Lagrange multipliers without a multiplier before the criterion, i.e. $\hat{x}$ being a solution of $(M)$, we have $D J(\hat{x})=\lambda \circ D f(\hat{x})$ where $\lambda$ belongs to the topological dual space of the co-domain of $f$, and when we have only the closedness of the image of the differential of $f$ at the solution, there exists a multiplier rule with a multiplier before the criterion, i.e. $\lambda_{0} D J(\hat{x})=\lambda \circ D f(\hat{x})$ with $\left(\lambda_{0}, \lambda\right) \neq(0,0)$ and $\lambda_{0} \in \mathbb{R}$. Our aim is to establish a necessary optimality condition in the form $D J(\hat{x})=\lambda \circ D f(\hat{x})$ wihtout assuming the surjectivity of $D f(\hat{x})$. To reach this aim we establish a result by using assumptions which are issued from a generalization to the infinite-dimensional Banach spaces of the classical Rank Theorem.
Thirdly, in Section 5, we establish results on the space of the bounded sequences with values in a Banach space to prepare the following section.
Lastly, in Section 6, we apply our theorem on Lagrange multipliers to a variational problem in infinite horizon and in discrete time on a space of bounded sequences under equality constraints.

## 2. Notation

When $E$ and $F$ are sets, when $f: E \rightarrow F$ is a mapping, when $E_{0} \subset E$ and $F_{0} \subset F$ are such that $f\left(E_{0}\right) \subset F_{0}$, we define the abridgement of $f$ (relatively to $E_{0}$
and $F_{0}$ ) as abf; $E_{0} \rightarrow F_{0}$ by setting $a b f(x):=f(x)$; this notation comes from [13] (p. 12).

When $X$ is a topological space, $\mathcal{O}(X)$ denotes the topology of $X$, and when $x \in X$, $\mathcal{O}_{x}(X)$ denotes the set of the open neighborhoods of $x$ in $X$.
The topological interior is denoted by Int.
When $E$ is a Banach space and $E_{1}, E_{2}$ are closed vector subspaces of $E$, the writting $E=E_{1} \oplus^{a} E_{2}$ (respectively $E=E_{1} \oplus E_{2}$ ) means the algebraic (respectively topological) direct sum, i.e. the mapping $\left(x_{1}, x_{2}\right) \mapsto x_{1}+x_{2}$, from $E_{1} \times E_{2}$ into $E$ is an isomorhism of vector spaces (respectively an isomorphism of topological vector spaces). When $E_{1}$ is a closed vector subspace of $E$, to say that $E_{1}$ is topologically complemented in $E$ means that there exists a closed vector subspace of $E$, say $E_{2}$, such that $E=E_{1} \oplus E_{2}$.
The letter $D$ denotes the Fréchet differentiation, and when $i \in\{1,2\}, D_{i}$ denotes the partial Fréchet differentiation of a mapping defined on a product space $E_{1} \times E_{2}$ with respect to the $i^{\text {th }}$ variable.
When $X$ and $Y$ are real Banach spaces, $\mathcal{L}(X, Y)$ denotes the space of the linear continuous mappings from $X$ into $Y$, and if $L \in \mathcal{L}(X, Y)$, its norm is $\|L\|_{\mathcal{L}}:=$ $\sup \{\|L(x)\|: x \in X,\|x\| \leq 1\}$. The topological dual space of $X, \mathcal{L}(X, \mathbb{R})$, is denoted by $X^{*}$. When $L \in \mathcal{L}(X, Y)$, the adjoint of $L$ is $L^{*} \in \mathcal{L}\left(Y^{*}, X^{*}\right)$ defined by $L^{*} \varphi:=\varphi \circ L$ for all $\varphi \in Y^{*}$. When $M \subset X$, the orthogonal of $M$ is $M^{\perp}:=\{\varphi \in$ $\left.X^{*}: \forall x \in M, \varphi(x)=0\right\}$.
When $A \in \mathcal{O}(X), C^{1}(A, Y)$ denotes the space of the continuously Fréchet differentiable mappings from $A$ into $Y$. When $f: A \rightarrow Y$ is a mapping and $y \in Y$, we set $[f=y]:=\{x \in A: f(x)=y\}$.
When $M \subset X$ and $a \in M$, the tangent space of $M$ at $a$ is $T_{a} M:=\left\{\alpha^{\prime}(0): \alpha \in\right.$ $\left.C^{1}((-\epsilon, \epsilon), M), \alpha(0)=a\right\}$.
We consider $\Omega$ as $\mathbb{N}$ or $\mathbb{N}_{*}:=\mathbb{N} \backslash\{0\}$. When $E$ is a set, $E^{\Omega}$ denotes the space of the sequences defined on $\Omega$ with values in $E$.
An element of $E^{\Omega}$ will be denoted by $\underline{x}=\left(x_{t}\right)_{t \in \Omega}$.
When $E$ is a normed space, $\underline{x} \in \ell^{\infty}(\Omega, E)$ denotes the set of the bounded sequences from $\Omega$ into $E$; we set $\|\underline{x}\|_{\infty}:=\sup _{t \in \Omega}\left\|x_{t}\right\|$.
When $A \subset E, \ell_{\infty}(\Omega, A):=\left\{\underline{x} \in \ell_{\infty}(\Omega, E): \forall t \in \Omega, x_{t} \in A\right\}$.
We also write $B_{\infty}(\underline{x}, r):=\left\{\underline{u} \in \ell_{\infty}(\Omega, E):\|\underline{u}-\underline{x}\|_{\infty}<r\right\}$.
$\ell_{1}(\Omega, E)$ denotes the space of the sequences $\underline{x}=\left(x_{t}\right)_{t \in \Omega} \in E^{\Omega}$ such that $\sum_{t \in \Omega}\left\|p_{t}\right\|<+\infty$.
$c_{0}(\Omega, E)$ denotes the space of the sequences $\underline{x}=\left(x_{t}\right)_{t \in \Omega} \in E^{\Omega}$ such that
$\lim _{t \rightarrow+\infty} x_{t}=0$.

## 3. The rank theorem in infinite dimension

In this section we recall the Rank Theorem in infinite dimension and we establish several consequences of this theorem which are useful for the sequel. Under the assumptions of the Rank Theorem we describe the tangent space of a level set. As we indicate in [5], there exist other generalizations of the classical Rank Theorem to the infinite-dimensional Banach spaces.
$X$ and $Y$ are real Banach spaces. The following result is given in 4 and it is established in Theorem 1 and in Theorem 5 of [5.

Theorem 3.1. Let $A \in \mathcal{O}(X), f \in C^{1}(A, Y), \hat{x} \in A$, and we set $\hat{y}:=f(\hat{x})$.
We assume that $E_{2}:=\operatorname{Ker} D f(\hat{x})$ is topologically complemented in $X$ and $F_{1}:=$ $\operatorname{ImD} D(\hat{x})$ is closed and topologically complemented in $Y$; and so $X=E_{1} \oplus E_{2}$ where $E_{1}$ is a closed vector subspace of $X$, and $Y=F_{1} \oplus F_{2}$ where $F_{2}$ is a closed vector subspace of $Y$. We also assume that the following condition is fulfilled:

$$
\begin{equation*}
\exists A_{0} \in \mathcal{O}_{\hat{x}}(A) \text { s.t. } \forall x \in A_{0}, \operatorname{Im} D f(x) \cap F_{2}=\{0\} . \tag{*}
\end{equation*}
$$

Then the following assertions hold.
(i) $\exists V_{1} \in \mathcal{O}_{D_{1} f_{1}(\hat{x})^{-1}\left(\hat{y}_{1}\right)}\left(E_{1}\right), \exists V_{2} \in \mathcal{O}_{\hat{x}_{2}}\left(E_{2}\right), \exists B \in \mathcal{O}_{\hat{x}}(X), \exists \psi: V_{1} \times V_{2} \rightarrow B$ $a C^{1}$ diffeormorphism, $\exists W \in \mathcal{O}_{\left(\hat{y}_{1}, 0\right)}(X), \exists \Omega_{1} \in \mathcal{O}_{\hat{y}_{1}}\left(F_{1}\right), \exists \Omega_{2} \in \mathcal{O}_{\hat{y}_{2}}\left(F_{2}\right)$, $\exists \phi: W \rightarrow \Omega_{1} \times \Omega_{2}$ a $C^{1}$ diffeomorphism such that $\phi^{-1} \circ f \circ \psi=D f(\hat{x})$ on $V_{1} \times V_{2}$.
(ii) $\exists G_{1} \in \mathcal{O}_{\hat{x}_{1}}\left(E_{1}\right), \exists G_{2} \in \mathcal{O}_{\hat{x}_{2}}\left(E_{2}\right), \exists \xi \in C^{1}\left(G_{2}, G_{1}\right)$ such that $[f=\hat{y}] \cap\left(G_{1} \times G_{2}\right)=\left\{\left(\xi\left(x_{2}\right), x_{2}\right): x_{2} \in G_{2}\right\}$.

Note that the equality of the conclusion (i) can be rewritten as follows (cf. Theorem 1 of [5])

$$
\begin{equation*}
\forall\left(x_{1}, x_{2}\right) \in V_{1} \times V_{2}, \quad \phi^{-1} \circ f \circ \psi\left(x_{1}, x_{2}\right)=\left(D_{1} f_{1}(\hat{x}) x_{1}, 0\right) \tag{3.1}
\end{equation*}
$$

which a kind of local linearization of $f$.
When $X$ and $Y$ are finite-dimensional, the assumption $(*)$ of Theorem 3.1 is equivalent to the constancy of the rank of $D f(x)$ on a neighborhood of $\hat{x}$, cf. Proposition 4 in [5]. We can provide a heuristic meaning of the condition $(*)$ : when $x$ is closed to $\hat{x}$, due to the continuity of $D f, D f(x)$ is closed to $D f(\hat{x})$ and $\operatorname{ImD} f(x)$ cannot be "smaller" than $\operatorname{ImDf}(\hat{x})$; this is a consequence of the openness of the set of the invertible linear continuous operators in the normed space of the linear continuous operators. And so to have $\operatorname{Im} D f(x)$ isomorphic to $\operatorname{Im} D f(\hat{x})$, it suffices to forbid $\operatorname{Im} D f(x)$ to be too "big". The condition $\operatorname{Im} D f(x) \cap F_{2}=\{0\}$ is a way to forbid $\operatorname{ImD}(x)$ to be too "big".

Now we describe consequences of Theorem 3.1.
Proposition 3.2. In the setting and under the assumptions of Theorem 3.1, the following assertions hold.
(i) $D f_{2}(\hat{x})=0$.
(ii) $D_{2} f_{1}(\hat{x})=0$.
(iii) $\xi\left(\hat{x}_{2}\right)=\hat{x}_{1}$.
(iv) $D \xi\left(\hat{x}_{2}\right)=-D_{1} f_{1}(\hat{x})^{-1} \circ D_{2} f_{1}(\hat{x})=0$.

Proof. $\pi_{2}: Y \rightarrow F_{2}$ and $p_{2}: X \rightarrow E_{2}$ denote the projections.
(i) $f_{2}=\pi_{2} \circ f$ implies $D f_{2}(\hat{x})=\pi_{2} \circ D f(\hat{x})=0$ since $\operatorname{Im} D f(\hat{x}) \cap F_{2}=\{0\}$.
(ii) $D_{2} f_{1}(\hat{x})=D f_{1}(\hat{x}) \circ p_{2}=0$ since $E_{2}:=\operatorname{Ker} D f(\hat{x})$.
(iii) From Theorem 3.1(ii), since $f\left(\hat{x}_{1}, \hat{x}_{2}\right)=\hat{y}$, we necessarily obtain $\hat{x}_{1}=\xi\left(\hat{x}_{2}\right)$.
(iv) Using (ii) of Theorem [3.1] we have, for all $x_{2} \in G_{2}, f_{1}\left(\xi\left(x_{2}\right), x_{2}\right)=\hat{y}_{2}$. Differentiating this equality with respect to $x_{2}$ at $\hat{x}_{2}$, we obtain $0=D_{1} f(\hat{x}) \circ$ $D \xi\left(\hat{x}_{2}\right)+D_{2} f(\hat{x})$ which implies the announced formulas.

Proposition 3.3. In the setting and under the assumptions of Theorem 3.1 we have $T_{\hat{x}}[f=\hat{y}]=\operatorname{KerDf}(\hat{x})$.

Proof. Let $v \in T_{\hat{x}}[f=\hat{y}]$. Hence there exists $c \in C^{1}((-\epsilon, \epsilon), X)$, where $\epsilon>0$, such that $c((-\epsilon, \epsilon)) \subset[f=\hat{y}], c(0)=\hat{x}$ and $c^{\prime}(0)=v$. Since $f(c(\theta))=\hat{y}$ for all $\theta \in(-\epsilon, \epsilon)$, differentiating this equality with respect to $\theta$ at 0 , we obtain $0=$ $D f(c(0)) c^{\prime}(0)=D f(\hat{x}) v$, and so $v \in \operatorname{Ker} D f(\hat{x})$. We have proven that $T_{\hat{x}}[f=\hat{y}] \subset$ $\operatorname{Ker} D f(\hat{x})$.
To prove the inverse inclusion, we consider $v=\left(v_{1}, v_{2}\right) \in \operatorname{Ker} D f(\hat{x})$. Therefore we have $v_{1}=0$ after the definition of $E_{1}, E_{2}$, and so $v=\left(0, v_{2}\right)$. Using Theorem 3.1(ii), since $G_{2}$ is open, there exists $\epsilon>0$ such that, for all $\theta \in(-\epsilon, \epsilon)$, we have $\hat{x}_{2}+\theta v_{2} \in G_{2}$. We define $c:(-\epsilon, \epsilon) \rightarrow X$ by setting $c(\theta):=\left(\xi\left(\hat{x}_{2}+\theta v_{2}\right), \hat{x}_{2}+\theta v_{2}\right) \in$ $[f=\hat{y}] \cap\left(G_{1} \times G_{2}\right)$. Note that we have $c(0)=\left(\xi\left(\hat{x}_{2}\right), \hat{x}_{2}\right)=\hat{x}$ after Proposition 3.2(iii). Note that we have $c^{\prime}(0)=\left(D \xi\left(\hat{x}_{2}\right) v_{2}, v_{2}\right)=\left(0, v_{2}\right)=v$ after Proposition 3.2(iv) and so we have $v \in T_{\hat{x}}[f=\hat{y}]$. We have proven that $\operatorname{Ker} D f(\hat{x}) \subset T_{\hat{x}}[f=\hat{y}]$, ansd so we have proven the announced equality.

## 4. Lagrange multipliers in Banach spaces

In this section we establish a theorem of existence of Lagrange multipliers for problem ( $M$ ) (written in Introduction) by using the Rank Theorem in infinite dimension. The interest of this result is to avoid a surjectivity on the differential of the equality constraint and nevertheless to avoid the presence of a multiplier before the criterion.
Let $X$ and $Y$ be real Banach spaces, $A \in \mathcal{O}(A), J: A \rightarrow \mathbb{R}$ be a functional, and $f: A \rightarrow Y$ be a mapping. We consider the problem $(M)$.

Theorem 4.1. Let $\hat{x}$ be a local solution of the problem (M). We assume that the following conditions are fulfilled.
(a) $J$ is Fréchet differentiable at $\hat{x}$ and $f$ is of class $C^{1}$ on a neighborhood of $\hat{x}$.
(b) $E_{2}:=\operatorname{Ker} D f(\hat{x})$ is topologically complemented in $X$; i.e. $X=E_{1} \oplus E_{2}$ where $E_{1}$ is a closed vector subspace of $X . F_{1}:=\operatorname{ImD}(\hat{x})$ is closed and topologically complemented in $Y$; i.e. $Y=F_{1} \oplus F_{2}$ where $F_{2}$ is a closed vector subspace of $Y$.
(c) There exists $A_{0} \in \mathcal{O}_{\hat{x}}(A)$ such that, for all $x \in A_{0}, \operatorname{Im} D f(x) \cap F_{2}=\{0\}$.

Then there exists $\lambda \in Y^{*}$ such that $D J(\hat{x})=\lambda \circ D f(\hat{x})$.
Proof. Using Theorem 4.1, we know that the set $[f=0]$ is a $C^{1}$ manifold around $\hat{x}$, and since $J$ is of class $C^{1}$ at $\hat{x}$, the first-order necessary optimality condition is $D J(\hat{x}) h=0$ when $h \in T_{\hat{x}}[f=0]$. Using Proposition 3.3, we know that $T_{\hat{x}}[f=$ $0]=\operatorname{Ker} D f(\hat{x})$, hence we obtain $D J(\hat{x}) h=0$ when $h \in \operatorname{Ker} D f(\hat{x})$, i.e., $D J(\hat{x}) \in$ $(\operatorname{Ker} D f(\hat{x}))^{\perp}$. Since $\operatorname{ImD} D(\hat{x})$ is closed, using Theorem 2.19 in [9] (p. 46), we have $(\operatorname{Ker} D f(\hat{x}))^{\perp}=\operatorname{ImD} D(\hat{x})^{*}$, and then we have $D J(\hat{x}) \in \operatorname{Im}(D f(\hat{x}))^{*}$, hence there exists $\lambda \in Y^{*}$ such that $D J(\hat{x})=D f(\hat{x})^{*} \lambda=\lambda \circ D f(\hat{x})$.

Notice that in Theorem4.1 we have not a multiplier before the differential of the criterion. Such a result in finite-dimensional spaces is proven in [10] (Proposition 2.3).

## 5. On spaces of bounded sequences

In this section we establish several results on the space of the bounded sequences to prepare the using of the theorem on the Lagrange multipliers of the previous section in such sequence spaces.
First we recall a result on the characterization of the closedness of linear operators.
Proposition 5.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two real Banach spaces and $L \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The two following assertions are equivalent.
(i) $\operatorname{ImL}$ is closed in $\mathcal{Y}$.
(ii) $\exists c(L)>0, \forall y \in \operatorname{Im} L, \exists x_{y} \in \mathcal{X}$ s.t. $L\left(x_{y}\right)=y$ and $\left\|x_{y}\right\| \leq c(L)\|y\|$.

We can find two different proofs of this result in [3] (Lemma 3.4) and in [6] (Lemma 2.1).

Remark 5.2. About the constant $c(L)$, under (i), when in addition we assume that $\operatorname{Ker} L$ is topologically complemented in $\mathcal{X}$, i.e. $\mathcal{X}=\operatorname{Ker} L \oplus \mathcal{X}_{1}$, where $\mathcal{X}_{1}$ is a closed vector subspace of $\mathcal{X}$, we can consider the abridgement abL: $\mathcal{X}_{1} \rightarrow \operatorname{ImL}$, $a b L(x):=L(x)$, which is an isomorphism of Banach spaces, and we have $c(L)=$ $\left\|(a b L)^{-1}\right\|_{\mathcal{L}}$.

Let $X$ and $Y$ be real Banach spaces. Let $\left(T_{t}\right)_{t \in \mathbb{N}_{*}} \in \mathcal{L}(X, Y)^{\mathbb{N}_{*}}$. We consider the three following conditions on this family of linear operators.
(C1) $\sup _{t \in \mathbb{N}_{*}}\left\|T_{t}\right\|_{\mathcal{L}}<+\infty$.
(C2) For all $t \in \mathbb{N}_{*}, \operatorname{Ker} T_{t}$ is topologically complemented in $X$, i.e. there exists a closed vector subspace of $X$, say $S_{t}$, such that $X=\operatorname{Ker} T_{t} \oplus S_{t}$, and $\operatorname{Im} T_{t}$ is closed and topologocally complemented in $Y$, i.e. there exists a closed vector subspace of $Y$, say $W_{t}$, such that $Y=I m T_{t} \oplus W_{t}$.
(C3) $\hat{c}:=\sup _{t \in \mathbb{N}_{*}} c\left(T_{t}\right)<+\infty$.
Notice that (C2) is automatically fulfilled when $\operatorname{dim} X<+\infty$ and $\operatorname{dim} Y<+\infty$.
We associate to this family the four following sequence spaces.

$$
\begin{gathered}
\mathcal{K}:=\left\{\underline{x} \in \ell^{\infty}\left(\mathbb{N}_{*}, X\right): \forall t \in \mathbb{N}_{*}, x_{t} \in \operatorname{Ker} T_{t}\right\} . \\
\mathcal{S}:=\left\{\underline{x} \in \ell^{\infty}\left(\mathbb{N}_{*}, X\right): \forall t \in \mathbb{N}_{*}, x_{t} \in S_{t}\right\} \\
\mathcal{I}:=\left\{\underline{y} \in \ell^{\infty}\left(\mathbb{N}_{*}, Y\right): \forall t \in \mathbb{N}_{*}, y_{t} \in \operatorname{Im} T_{t}\right\} . \\
\mathcal{W}:=\left\{\underline{y} \in \ell^{\infty}\left(\mathbb{N}_{*}, Y\right): \forall t \in \mathbb{N}_{*}, y_{t} \in W_{t}\right\}
\end{gathered}
$$

The condition (C1), which is equivalent to the boundeness of $\left(T_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}}$ when $\underline{x}$ is bounded after the Banach-Steinhaus theorem, permits us to define the operator

$$
\begin{equation*}
\mathcal{T}: \ell^{\infty}\left(\mathbb{N}_{*}, X\right) \rightarrow \ell^{\infty}\left(\mathbb{N}_{*}, Y\right), \quad \mathcal{T}(\underline{x}):=\left(T_{y}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}} \tag{5.1}
\end{equation*}
$$

Lemma 5.3. Under (C1, C2, C3) the following assertions hold.
(i) $\ell^{\infty}\left(\mathbb{N}_{*}, X\right)=\operatorname{Ker} \mathcal{T} \oplus \mathcal{S}$.
(ii) $\ell^{\infty}\left(\mathbb{N}_{*}, Y\right)=\operatorname{Im} \mathcal{T} \oplus \mathcal{W}$.

Proof. For each $t \in \mathbb{N}_{*}$, we consider the mappings $\pi_{t}: \ell^{\infty}\left(\mathbb{N}_{*}, X\right) \rightarrow X$ defined by $\pi_{t}(\underline{x}):=x_{t}$, and the mapping $\varpi_{t}: \ell^{\infty}\left(\mathbb{N}_{*}, Y\right) \rightarrow Y$, defined by $\varpi_{t}(\underline{y}):=y_{t}$. Clearly we have $\pi_{t} \in \mathcal{L}\left(\ell^{\infty}\left(\mathbb{N}_{*}, X\right), X\right)$ and $\varpi_{t} \in \mathcal{L}\left(\ell^{\infty}\left(\mathbb{N}_{*}, Y\right), Y\right)$. Note that we have

$$
\mathcal{K}=\bigcap_{t \in \mathbb{N}_{*}} \pi_{t}^{-1}\left(\operatorname{Ker} T_{t}\right), \mathcal{S}=\bigcap_{t \in \mathbb{N}_{*}} \pi_{t}^{-1}\left(S_{t}\right), \mathcal{I}=\bigcap_{t \in \mathbb{N}_{*}} \varpi_{t}^{-1}\left(\operatorname{Im} T_{t}\right), \mathcal{W}=\bigcap_{t \in \mathbb{N}_{*}} \varpi_{t}^{-1}\left(W_{t}\right)
$$

Due to the continuity of the $\pi_{t}$ and $\varpi_{t}$, these four spaces are intersections of closed subsets which implies that $\mathcal{K}$ and $\mathcal{S}$ are closed vector subspaces of $\ell^{\infty}\left(\mathbb{N}_{*}, X\right)$ and $\mathcal{I}$ and $\mathcal{W}$ are closed vector subspaces of $\ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$.

It is easy to verify that

$$
\ell^{\infty}\left(\mathbb{N}_{*} X\right)=\mathcal{K} \oplus^{a} \mathcal{S}, \quad \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)=\mathcal{I} \oplus^{a} \mathcal{W}
$$

Since $\ell^{\infty}\left(\mathbb{N}_{*}, X\right)$ and $\ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$ are Banach spaces and since $\mathcal{K}, \mathcal{S}, \mathcal{I}, \mathcal{W}$ are closed vector subspaces, from the Inverse Mapping Theorem of Banach-Schauder, 11 (Corollary 1.5, p. 388), we obtain

$$
\begin{equation*}
\ell^{\infty}\left(\mathbb{N}_{*}, X\right)=\mathcal{K} \oplus \mathcal{S}, \quad \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)=\mathcal{I} \oplus \mathcal{W} \tag{5.2}
\end{equation*}
$$

We easily verify that $\mathcal{K}=\operatorname{Ker} \mathcal{T}$, and from (5.2) we obtain (i).
About the image, we see that $\operatorname{Im} \mathcal{T} \subset \mathcal{I}$. if $\underline{y} \in \mathcal{I}$ then, for all $t \in \mathbb{N}_{*}$, we have $y_{t} \in \operatorname{Im} T_{t}$. From (C2), using Proposition5.1, we can say that there exists $x_{t, y_{t}} \in X$ such that $T_{t}\left(x_{t, y_{t}}\right)=y_{t}$ and $\left\|x_{t, y_{t}}\right\| \leq c\left(T_{t}\right)\left\|y_{t}\right\|$. Hence, setting $\underline{x}_{\underline{y}}:=\left(x_{t, y_{t}}\right)_{t \in \mathbb{N}_{*}}$, we have $\mathcal{T}\left(\underline{x}_{\underline{y}}\right)=\underline{y}$ and $\left\|\underline{x}_{\underline{y}}\right\|_{\infty} \leq \hat{c}\|\underline{y}\|$, therefore $\underline{x}_{\underline{y}} \in \ell\left(\mathbb{N}_{*}, X\right)$ and $\mathcal{T}\left(\underline{x}_{\underline{y}}\right)=\underline{y}$, i.e. $\underline{y} \in \operatorname{Im} \mathcal{T}$. We have proven that $\mathcal{I} \subset \operatorname{Im} \mathcal{T}$, and consequently $\mathcal{I}=\operatorname{Im} \mathcal{T} \overline{\text { From (5.2) }}$ we obtain (ii).

Lemma 5.4. Let $A \in \mathcal{O}(X)$. Then the following assertions hold.
(i) $\operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right)=\left\{\underline{x} \in \ell^{\infty}\left(\mathbb{N}_{*}, A\right): \inf _{t \in \mathbb{N}_{*}} d\left(x_{t}, A^{c}\right)>0\right\}$, where $d\left(x_{t}, A^{c}\right):=\inf \left\{\left\|x_{t}-z\right\|: z \in A^{c}\right\}$.
(ii) Let $\underline{x} \in \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right)$. We set $r:=\inf _{t \in \mathbb{N}_{*}} d\left(x_{t}, A^{c}\right)>0$. Then we have $\prod_{t \in \mathbb{N}_{*}} B\left(x_{t}, \frac{r}{2}\right) \subset \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right)$.
Proof. Assertion (i) is proven in [7] (Lemma A.1.1). To prove (ii),
if $\underline{u} \in \prod_{t \in \mathbb{N}} B\left(x_{t}, \frac{r}{2}\right)$, then we have $\left\|u_{t}-x_{t}\right\|<\frac{r}{2}$, and for all $v \in A^{c}$, we have

$$
r<\left\|x_{t}-v\right\| \leq\left\|x_{t}-u_{t}\right\|+\left\|u_{t}-v\right\| \leq \frac{r}{2}+\left\|u_{t}-v\right\| \Longrightarrow r-\frac{r}{2} \leq\left\|u_{t}-v\right\|
$$

which implies $d\left(u_{t}, A^{c}\right) \geq \frac{r}{2}>0$, and so $\underline{u} \in \operatorname{Int} \ell^{\infty}\left(\mathbb{N}_{*}, A\right)$.
Definition 5.5. Let $A \in \mathcal{O}(X)$ and $g_{t}: A \rightarrow Y$ be a mapping for all $t \in \mathbb{N}_{*}$. The sequence $\left(g_{t}\right)_{t \in \mathbb{N}_{*}}$ is said uniformly equicontinuous on the bounded subsets of A when

$$
\left\{\begin{array}{l}
\forall B \in \mathbb{B}(A), \forall \epsilon>0, \exists \eta_{B, \epsilon}>0, \forall u, v \in B \\
\|u-v\| \leq \eta_{B, \epsilon} \Longrightarrow\left(\forall t \in \mathbb{N}_{*},\left\|g_{t}(u)-g_{t}(v)\right\| \leq \epsilon\right)
\end{array}\right.
$$

Proposition 5.6. Let $A \in \mathcal{O}(X)$ and $g_{t}: A \rightarrow Y$ be a mapping for all $t \in \mathbb{N}_{*}$. We assume that the following condition is fulfilled: $\forall \underline{x} \in \ell^{\infty}(A),\left(g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}} \in$ $\ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$.
This condition permits to define the operator $G: \ell^{\infty}\left(\mathbb{N}_{*}, A\right) \rightarrow \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$ by setting $G(\underline{x}):=\left(g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}}$.
If, in addition we assume that $\left(g_{t}\right)_{t \in \mathbb{N}}$ is uniformly equicontinuous in the bounded subsets of $A$, then $G$ is continuous from $\operatorname{Int} \ell^{\infty}\left(\mathbb{N}_{*}, A\right)$ into $\ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$.

Proof. We arbitrarily fix $\underline{x} \in \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right)$. Hence there exists $r>0$ such that $B_{\infty}(\underline{x}, r) \subset \ell^{\infty}\left(\mathbb{N}_{*}, A\right)$.
The set $B:=\left\{u \in A: \exists t_{u} \in \mathbb{N}_{*},\left\|u-x_{t_{u}}\right\|<r\right\}$ is bounded since, when $u \in B$, $\|u\| \leq\left\|u-x_{t_{u}}\right\|+\left\|x_{t_{u}}\right\| \leq r+\|\underline{x}\|_{\infty}<+\infty$. We arbitrarily fix $\epsilon>0$. Let $\underline{z} \in \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right)$ such that $\|\underline{z}-\underline{x}\|_{\infty}<\min \left\{r, \eta_{B, \epsilon}\right\}$ where $\eta_{B, \epsilon}$ is provided by

Definition 5.5. Hence we have $z_{t} \in B$ since $\left\|z_{t}-x_{t}\right\|<r$ for all $t \in \mathbb{N}_{*}$, and we have $\left\|z_{t}-x_{t}\right\|<\eta_{B, \epsilon}$ which implies $\left\|g_{t}\left(z_{t}\right)-g_{t}\left(x_{t}\right)\right\|<\epsilon$. Hence we have $\|G(\underline{z})-G(\underline{x})\|_{\infty} \leq \epsilon$. We have proven that $G$ is continuous at $\underline{x}$.

Proposition 5.7. Let $A \in \mathcal{O}(X)$ and $\left(g_{t}\right)_{t \in \mathbb{N}_{*}} \in C^{1}(A, Y)^{\mathbb{N}_{*}}$. We assume that the following conditions are fulfilled.
(a) $\forall \underline{x} \in$ Int $^{\infty}\left(\mathbb{N}_{*}, A\right),\left(g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}} \in \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$.
(b) $\forall \underline{x} \in \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right),\left(D g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}} \in \ell^{\infty}\left(\mathbb{N}_{*}, \mathcal{L}(X, Y)\right)$.
(c) $\left(D g_{t}\right)_{t \in \mathbb{N}_{*}}$ is uniformly equicontinuous on the bounded subsets of $A$.

We consider the operator $G: \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right) \rightarrow \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$ defined by $G(\underline{x}):=$ $\left(g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}}$.
Then the following assertions hold for all $\underline{x} \in \operatorname{Int} \ell^{\infty}\left(\mathbb{N}_{*}, A\right)$.
(i) $G$ is Fréchet differentiable at $\underline{x}$, and, for all $\underline{v} \in \ell^{\infty}\left(\mathbb{N}_{*}, X\right)$, we have $D G(\underline{x}) \underline{v}=\left(D g_{t}\left(x_{t}\right) v_{t}\right)_{t \in \mathbb{N}_{*}}$.
(ii) $G \in C^{1}\left(\operatorname{Int} \ell^{\infty}\left(\mathbb{N}_{*}, A\right), \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)\right)$.

Proof. We arbitrarily fix $\underline{x} \in \operatorname{Int} \ell^{\infty}\left(\mathbb{N}_{*}, A\right)$. Using (b), the linear operator $\mathcal{T}$ : $\ell^{\infty}\left(\mathbb{N}_{*}, X\right) \rightarrow \ell^{\infty}\left(\mathbb{N}_{*} Y\right)$, defined by $\mathcal{T} \underline{h}:=\left(D g_{t}\left(x_{t}\right) h_{t}\right)_{t \in_{*} \mathbb{N}}$, is well-defined and continuous.
We can use Lemma 5.4(ii) and assert that $\prod_{t \in \mathbb{N}_{*}} B\left(x_{t}, \frac{r}{2}\right) \subset$ Int $^{\infty}\left(\mathbb{N}_{*}, A\right)$. Using the set $B:=\left\{u \in A: \exists t_{u} \in \mathbb{N}_{*},\left\|u-x_{t}\right\|<r\right\}$, we have yet seen that $B$ is bounded in $A$ and we have $B\left(x_{t}, \frac{r}{2}\right) \subset B$.
We arbitrarily fix $\epsilon>0$, and, using assumption (c), we consider $\eta_{B, \epsilon}>0$ provided by Definition 5.5 for the sequence $\left(D g_{t}\right)_{t \in \mathbb{N}_{*}}$.
We arbitrarily fix $\underline{h} \in \ell^{\infty}\left(\mathbb{N}_{*}, X\right)$ such that $\|\underline{h}\|_{\infty}<\min \left\{\frac{r}{2}, \eta_{B, \epsilon}\right\}$. Hence, $\forall t \in \mathbb{N}_{*}$, $\forall z_{t} \in\left[x_{t}, x_{t}+h_{t}\right] \subset B\left(x_{t}, \frac{r}{2}\right)$, we have $\left\|z_{t}-x_{t}\right\|<\eta_{B, \epsilon}$ which implies $\| D g_{t}\left(z_{t}\right)-$ $D g_{t}\left(x_{t}\right) \|_{\mathcal{L}}<\epsilon$.
Now using the Mean Value Inequality as established in (Corollary 1, p. 144), we obtain, for all $t \in \mathbb{N}_{*}$,
$\left\|g_{t}\left(x_{t}+h_{t}\right)-g_{t}\left(x_{t}\right)-D g_{t}\left(x_{t}\right) h_{t}\right\| \leq \sup _{z_{t} \in\left[x_{t}, x_{t}+h_{t}\right]}\left\|D g_{t}\left(z_{t}\right)-D g_{t}\left(x_{t}\right)\right\|_{\mathcal{L}}\left\|h_{t}\right\| \leq$ $\epsilon\left\|h_{t}\right\|$, and taking the sup on the $t \in \mathbb{N}_{*}$, we obtain

$$
\| G\left(\underline{x}+\underline{h}-G(\underline{x})-\mathcal{T} \underline{h}\left\|_{\infty} \leq \epsilon\right\| \underline{h} \|_{\infty}\right.
$$

Hence we have proven that $G$ is Fréchet differentiable at $\underline{x}$ and that $D G(\underline{x}) \underline{h}=$ $\left(D g_{t}\left(x_{t}\right) h_{t}\right)_{t \in \mathbb{N}_{*}}$.
Applying Proposition 5.6 to $D G$, we obtain the continuity of $D G$.
Proposition 5.8. Let $A \in \mathcal{O}(X)$ and $\left(g_{t}\right)_{t \in \mathbb{N}_{*}} \in C^{1}(A, Y)^{\mathbb{N}_{*}}$. We assume that the conditions (a), (b), (c) of Proposition 5.7 are fulfilled. Let $\underline{\hat{x}}=\left(\hat{x}_{t}\right)_{t \in \mathbb{N}_{*}} \in$ Int $\ell^{\infty}\left(\mathbb{N}_{*}, A\right)$. We assume that the following conditions are fulfilled.
(d) $\sup _{t \in \mathbb{N}_{*}}\left\|D g\left(\hat{x}_{t}\right)\right\|_{\mathcal{L}}<+\infty$.
(e) For all $t \in \mathbb{N}_{*}$ there exist a closed vector subspace of $X$, say $S_{t}$, and a closed vector subspace of $Y$, say $W_{t}$, such that $X=\operatorname{Ker} D g\left(\hat{x}_{t}\right) \oplus S_{t}$ and $Y=\operatorname{ImD} D\left(\hat{x}_{t}\right) \oplus W_{t}$.
(f) $\hat{c}:=\sup _{t \in \mathbb{N}_{*}} c\left(D g\left(\hat{x}_{t}\right)\right)<+\infty$.

Then we have $\ell^{\infty}\left(\mathbb{N}_{*}, X\right)=\operatorname{Ker} D G(\underline{\hat{x}}) \oplus \mathcal{S}$ and $\ell^{\infty}\left(\mathbb{N}_{*}, Y\right)=\operatorname{ImD} G(\underline{\hat{x}}) \oplus \mathcal{W}$, with $\mathcal{S}:=\left\{\underline{v} \in \ell^{\infty}\left(\mathbb{N}_{*}, X\right): \forall t \in \mathbb{N}_{*}, v_{t} \in \bar{S}_{t}\right\}$ and $\mathcal{W}:=\left\{\underline{w} \in \ell^{\infty}\left(\mathbb{N}_{*}, Y\right): \forall t \in \mathbb{N}_{*}, w_{t} \in\right.$ $\left.W_{t}\right\}$.

Proof. This result is a consequence of Proposition 5.7 and of Lemma 5.3 with $T_{t}=D g\left(\hat{x}_{t}\right)$.

Notice that (e) is automatically fulfilled when $\operatorname{dim} X<+\infty$ and $\operatorname{dim} Y<+\infty$.

## 6. A variational problem

In this section we consider a maximization problem in infinite horizon and in discrete time under holonomic constraints. The unknown variable is a bounded sequence with values in a real Banach space $X$. Using the results of the previous sections we obtain a first-order necessary optimality condition in the form of a nonhomogeneous Euler-Lagrange equation.
Let $A$ be a nonempty subset of $X$. For all $t \in \mathbb{N}$, let $u_{t}: A \times A \rightarrow \mathbb{R}$ and for all $t \in \mathbb{N}_{*}$ let $g_{t}: A \rightarrow Y$ be functions, where $Y$ is a real Banach space. We fix a vector $\sigma \in A$, a real number $\beta \in(0,1)$ and we consider the following variational problem under holonomic constraints.

$$
(\mathcal{V})\left\{\begin{aligned}
\text { Maximize } & J(\underline{x}):=\sum_{t=0}^{+\infty} \beta^{t} u_{t}\left(x_{t}, x_{t+1}\right) \\
\text { subject to } & \underline{x} \in \ell^{\infty}(\mathbb{N}, A), x_{0}=\sigma \\
\text { and } & \forall t \in \mathbb{N}_{*}, g_{t}\left(x_{t}\right)=0 .
\end{aligned}\right.
$$

The problems of Calculus of Variations or of Optimal Control in discrete time and in infinite horizon are very usual in Economics and in Management; see for instance [8, 3, 7] and references therein.

Theorem 6.1. Let $\underline{\hat{x}}$ be a solution of $(\mathcal{V})$. We assume that the following conditions are fulfilled
(A1) $\left(\hat{x}_{t}, \hat{x}_{t+1}\right)_{t \in \mathbb{N}} \in \operatorname{Int}^{\infty}(\mathbb{N}, A \times A)$.
(A2) $\forall t \in \mathbb{N}, u_{t} \in C^{1}(A \times A, \mathbb{R})$.
(A3) $\forall(\underline{x}, \underline{y}) \in \operatorname{Int}^{\infty}(\mathbb{N}, A \times A),\left(u_{t}\left(x_{t}, y_{t}\right)\right)_{t \in \mathbb{N}} \in \ell_{\infty}(\mathbb{N}, \mathbb{R})$.
(A4) $\forall(\underline{x}, \underline{y}) \in \operatorname{Int}^{\infty}(\mathbb{N}, A \times A),\left(D u_{t}\left(x_{t}, y_{t}\right)\right)_{t \in \mathbb{N}} \in \ell^{\infty}\left(\mathbb{N},(X \times X)^{*}\right)$.
(A5) $\left(D u_{t}\right)_{t \in \mathbb{N}}$ is uniformly equicontinuous on the bounded subsets of $A \times A$.
(A6) $\underline{\hat{x}} \in \operatorname{Int}^{\infty}(\mathbb{N}, A)$.
(A7) $\forall t \in \mathbb{N}_{*}, g_{t} \in C^{1}(A, \mathbb{R})$.
(A8) $\forall \underline{x} \in \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right),\left(g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}} \in \ell^{\infty}\left(\mathbb{N}_{*}, Y\right)$.
(A9) $\forall \underline{x} \in \operatorname{Int}^{\infty}\left(\mathbb{N}_{*}, A\right),\left(D g_{t}\left(x_{t}\right)\right)_{t \in \mathbb{N}_{*}} \in \ell^{\infty}\left(\mathbb{N}_{*}, \mathcal{L}(X, Y)\right)$.
(A10) $\left(D g_{t}\right)_{t \in \mathbb{N}_{*}}$ is uniformly equicontinuous on the bounded subsets of $A$.
(A11) For all $t \in \mathbb{N}_{*}$, there exists a closed vector subspace of $X$, say $S_{t}$, such that $X=\operatorname{Ker} D g_{t}\left(\hat{x}_{t}\right) \oplus S_{t}$, and there exists a closed vector subspace of $Y$, say $W_{t}$, such that $Y=\operatorname{ImD} g_{t}\left(\hat{x}_{t}\right) \oplus W_{t}$.
(A12) $\sup _{t \in \mathbb{N}_{*}} c\left(D g_{t}\left(\hat{x}_{t}\right)\right)<+\infty$.
(A13) $\exists r>0, \forall \underline{x} \in B_{\infty}(\underline{\hat{x}}, r), \forall t \in \mathbb{N}_{*}, \operatorname{ImD} g_{t}\left(x_{t}\right) \cap W_{t}=\{0\}$.
Then there exists a sequence $\left(p_{t}\right)_{t \in \mathbb{N}_{*}} \in \ell^{1}\left(\mathbb{N}_{*}, Y^{*}\right)$ which satisfies, for all $t \in \mathbb{N}_{*}$, the following equality

$$
\beta^{t} D_{1} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right)+\beta^{t-1} D_{2} u_{t-1}\left(\hat{x}_{t-1}, \hat{x}_{t}\right)=p_{t} \circ D g_{t}\left(\hat{x}_{t}\right) .
$$

Proof. For each $t \in \mathbb{N}_{*}$ we consider the following static problem.

$$
\left(\mathcal{R}_{t}\right) \begin{cases}\text { Maximize } & J_{t}\left(x_{t}\right):=\beta^{t-1} u_{t-1}\left(\hat{x}_{t-1}, x_{t}\right)+\beta^{t} u_{t}\left(x_{t}, \hat{x}_{t+1}\right) \\ \text { subject to } & x_{t} \in A, \quad g_{t}\left(x_{t}\right)=0\end{cases}
$$

Since $\underline{\hat{x}}$ is a solution of $(\mathcal{V})$, proceeding by contradiction, it is easy to see that $\hat{x}_{t}$ is a solution of $\left(\mathcal{R}_{t}\right)$. Under our assumptions we can use Theorem4.1 on $\left(\mathcal{R}_{t}\right)$ and assert that there exists $q_{t} \in Y^{*}$ such the following equality holds.

$$
\begin{equation*}
\left.\beta^{t-1} D_{2} u_{t-1} \hat{x}_{t-1}, \hat{x}_{t}\right)+\beta^{t} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right)=q_{t} \circ D g_{t}\left(\hat{x}_{t}\right) \tag{6.1}
\end{equation*}
$$

Under (A11), we know that, for every $y \in Y$, there exists a unique $(z, w) \in$ $\operatorname{ImD} g_{t}\left(\hat{x}_{t}\right) \times W_{t}$ such that $y=z+w$. We define $p_{t} \in Y^{*}$ by setting $p_{t}(y):=q_{t}(z)$. Hence we deduce from (6.1) the following equality:

$$
\begin{equation*}
\beta^{t-1} D_{2} u_{t-1}\left(\hat{x}_{t-1}, \hat{x}_{t}\right)+\beta^{t} D_{1} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right)=p_{t} \circ D g_{t}\left(\hat{x}_{t}\right) \tag{6.2}
\end{equation*}
$$

Now we ought to prove that $\left(p_{t}\right)_{t \in \mathbb{N}_{*}} \in \ell_{1}\left(\mathbb{N}_{*}, Y^{*}\right)$.
Let $\underline{y} \in \ell_{\infty}\left(\mathbb{N}_{*}, Y\right)$. Using Proposition 5.8, there exists $\underline{x} \in \ell_{\infty}\left(\mathbb{N}_{*}, X\right)$ and $\underline{w} \in \mathcal{W}$ such that $\underline{y}=D G(\underline{\hat{x}}) \underline{x}+\underline{w}$, i.e. $y_{t}=D g_{t}\left(\hat{x}_{t}\right) x_{t}+w_{t}$. Hence we have

$$
\begin{aligned}
p_{t}\left(y_{t}\right) & =p_{t} \circ D g_{t}\left(\hat{x}_{t}\right) x_{t}+p_{t}\left(w_{t}\right) \\
& =p_{t} \circ D g_{t}\left(\hat{x}_{t}\right) x_{t}+0 \\
& =\beta^{t-1} D_{2} u_{t-1}\left(\hat{x}_{t-1}, \hat{x}_{t}\right) x_{t}+\beta^{t} D_{1} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right) x_{t}
\end{aligned}
$$

The sequences $\left(D_{2} u_{t-1}\left(\hat{x}_{t-1}, \hat{x}_{t}\right)\right)_{t \in \mathbb{N}_{*}}$ and $\left(D_{1} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right)\right)_{t \in \mathbb{N}_{*}}$ are bounded after (A4), and since $\left(x_{t}\right)_{t \in \mathbb{N}_{*}}$ is bounded, the sequences $\left(D_{2} u_{t-1}\left(\hat{x}_{t-1}, \hat{x}_{t}\right) x_{t}\right)_{t \in \mathbb{N}_{*}}$ and $\left(D_{1} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right) x_{t}\right)_{t \in \mathbb{N}_{*}}$ are bounded.
Since $\beta \in(0,1)$, the series $\sum_{t \geq 1} \beta^{t-1} D_{2} u_{t}\left(\hat{x}_{t-1}, \hat{x}_{t}\right) x_{t}$ and $\sum_{t \geq 0} \beta^{t} D_{1} u_{t}\left(\hat{x}_{t}, \hat{x}_{t+1}\right) x_{t}$ are convergent in $\mathbb{R}$, and consequently the series $\sum_{t \geq 1} p_{t}\left(y_{t}\right)$ is convergent in $\mathbb{R}$. Since $c_{0}\left(\mathbb{N}_{*}, Y\right) \subset \ell_{\infty}\left(\mathbb{N}_{*}, Y\right)$, we obtain that the series $\sum_{t \geq 1} p_{t}\left(y_{t}\right)$ is convergentin $\mathbb{R}$ when $\underline{y} \in c_{0}\left(\mathbb{N}_{*}, Y\right)$. Using $[12](\operatorname{assertion}(\alpha)$ in page 247$)$, we can assert that $\left(p_{t}\right)_{t \in \mathbb{N}_{*}} \bar{\in} \ell_{1}\left(\mathbb{N}_{*}, Y^{*}\right)$.

Notice that (A3), (A4), (A8), (A9) and (A11) are automatically fulfilled when $\operatorname{dim} X<+\infty$ and $\operatorname{dim} Y<+\infty$. When $\operatorname{dim} Y<+\infty$, using a represention of the topological dual space of $\ell_{\infty}\left(\mathbb{N}_{*}, Y^{*}\right)$ (as a direct of $\ell_{1}\left(\mathbb{N}_{*}, Y^{*}\right)$ and of another subspace) which is given in [2] Chapter 15 , Section 15.8) we can process as in [8] (Chapter 3) and to obtain the sequence $\left(p_{t}\right)_{t b \in \mathbb{N}_{*}}$ as the component in $\ell_{1}\left(\mathbb{N}_{*}, Y^{*}\right)$ of an element of $\ell_{\infty}\left(\mathbb{N}_{*}, Y^{*}\right)^{*}$.

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## References

[1] Alexeev VM, Tihomirov VM, Fomin SV (1982) Commande optimale (French) MIR, Moscow
[2] Aliprantis CD, Border KC (1999) Infinite dimensional analysis, 2nd edn. Springer, Berlin
[3] Bachir M, Blot J (2017) Infinite dimensional multipliers and Pontryagin principles for discretetime problems, Pure Appl. Funct. Anal. 2, 411-420.
[4] Blot J (1985) Le théorème du rang en dimension infinie (French), C.R. Acad. Sci. Paris Sér. I Math. 301, 755-757
[5] Blot J (1986) The rank theorem in infinite dimension, Nonlinear Anal 10.1009-1020
[6] Blot J, Cieutat P (2016) Completeness of sums of subspaces of bounded functions and applications, Commun. Math. Anal. 19, 43-61
[7] Blot J, Crettez B (2004) On the smoothness of optimal paths, Decis Econ Finance 27.1-34
[8] Blot J, Hayek N (2014) Infinite horizon optimal control in the discrete-time framework. Springer, New York
[9] Brezis H (2011) Functional analysis, Sobolev spaces and partial differential equations, Springer, New York
[10] Janin R (1984) Directional derivative of the marginal function in nonlinear programming, Mathematical Programming Study 21.110-126
[11] Lang S (1993) Real and functional analysis, third edn, Springer-Verlag, New York
[12] Leonard IE (1976) Banach sequence spaces, J. Math. Anal. Appl. 84, 245-265
[13] Rohlin VA, Fuchs DB (1981) Premier cours de topologie (French), MIR, Moscow

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