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RANK THEOREM IN INFINITE DIMENSION AND LAGRANGE MULTIPLIERS

JOËL BLOT

ABSTRACT. We use an extension to the infinite dimension of the Rank Theorem of the differential calculus to establish a Lagrange theorem for optimization problems in Banach spaces. We provide an application to variational problems on a space of bounded sequences under equality constraints.

Keywords : Rank theorem, Lagrange multipliers, Banach spaces.

Classification MSC 2010: 49K27, 49K30.

1. INTRODUCTION

Firstly we recall the extension of the rank theorem to the infinite dimension as it is established in [5] (Section 3).

Secondly we establish a theorem of existence of Lagrange multipliers in Banach spaces for maximization problems under equality constraints (Section 4) like the following one

$$(M) \begin{cases} \text{Maximize} & J(x) \\ \text{subject to} & f(x) = 0 \end{cases}$$

where the functional J and the mapping f are defined on an open subset of a Banach space, and the mapping f takes its values in a Banach space. Among the classical multiplier rules, when the differential of f at the solution is surjective we obtain the existence of Lagrange multipliers without a multiplier before the criterion, i.e. \hat{x} being a solution of (M) , we have $DJ(\hat{x}) = \lambda \circ Df(\hat{x})$ where λ belongs to the topological dual space of the co-domain of f , and when we have only the closedness of the image of the differential of f at the solution, there exists a multiplier rule with a multiplier before the criterion, i.e. $\lambda_0 DJ(\hat{x}) = \lambda \circ Df(\hat{x})$ with $(\lambda_0, \lambda) \neq (0, 0)$ and $\lambda_0 \in \mathbb{R}$. Our aim is to establish a necessary optimality condition in the form $DJ(\hat{x}) = \lambda \circ Df(\hat{x})$ without assuming the surjectivity of $Df(\hat{x})$. To reach this aim we establish a result by using assumptions which are issued from a generalization to the infinite-dimensional Banach spaces of the classical Rank Theorem.

Thirdly, in Section 5, we establish results on the space of the bounded sequences with values in a Banach space to prepare the following section.

Lastly, in Section 6, we apply our theorem on Lagrange multipliers to a variational problem in infinite horizon and in discrete time on a space of bounded sequences under equality constraints.

2. NOTATION

When E and F are sets, when $f : E \rightarrow F$ is a mapping, when $E_0 \subset E$ and $F_0 \subset F$ are such that $f(E_0) \subset F_0$, we define the *abridgement* of f (relatively to E_0

and F_0) as abf ; $E_0 \rightarrow F_0$ by setting $abf(x) := f(x)$; this notation comes from [13] (p. 12).

When X is a topological space, $\mathcal{O}(X)$ denotes the topology of X , and when $x \in X$, $\mathcal{O}_x(X)$ denotes the set of the open neighborhoods of x in X .

The topological interior is denoted by Int .

When E is a Banach space and E_1, E_2 are closed vector subspaces of E , the writting $E = E_1 \oplus^a E_2$ (respectively $E = E_1 \oplus E_2$) means the algebraic (respectively topological) direct sum, i.e. the mapping $(x_1, x_2) \mapsto x_1 + x_2$, from $E_1 \times E_2$ into E is an isomorphism of vector spaces (respectively an isomorphism of topological vector spaces). When E_1 is a closed vector subspace of E , to say that E_1 is *topologically complemented* in E means that there exists a closed vector subspace of E , say E_2 , such that $E = E_1 \oplus E_2$.

The letter D denotes the Fréchet differentiation, and when $i \in \{1, 2\}$, D_i denotes the partial Fréchet differentiation of a mapping defined on a product space $E_1 \times E_2$ with respect to the i^{th} variable.

When X and Y are real Banach spaces, $\mathcal{L}(X, Y)$ denotes the space of the linear continuous mappings from X into Y , and if $L \in \mathcal{L}(X, Y)$, its norm is $\|L\|_{\mathcal{L}} := \sup\{\|L(x)\| : x \in X, \|x\| \leq 1\}$. The topological dual space of X , $\mathcal{L}(X, \mathbb{R})$, is denoted by X^* . When $L \in \mathcal{L}(X, Y)$, the adjoint of L is $L^* \in \mathcal{L}(Y^*, X^*)$ defined by $L^*\varphi := \varphi \circ L$ for all $\varphi \in Y^*$. When $M \subset X$, the orthogonal of M is $M^\perp := \{\varphi \in X^* : \forall x \in M, \varphi(x) = 0\}$.

When $A \in \mathcal{O}(X)$, $C^1(A, Y)$ denotes the space of the continuously Fréchet differentiable mappings from A into Y . When $f : A \rightarrow Y$ is a mapping and $y \in Y$, we set $[f = y] := \{x \in A : f(x) = y\}$.

When $M \subset X$ and $a \in M$, the tangent space of M at a is $T_a M := \{\alpha'(0) : \alpha \in C^1((-\epsilon, \epsilon), M), \alpha(0) = a\}$.

We consider Ω as \mathbb{N} or $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$. When E is a set, E^Ω denotes the space of the sequences defined on Ω with values in E .

An element of E^Ω will be denoted by $\underline{x} = (x_t)_{t \in \Omega}$.

When E is a normed space, $\underline{x} \in \ell^\infty(\Omega, E)$ denotes the set of the bounded sequences from Ω into E ; we set $\|\underline{x}\|_\infty := \sup_{t \in \Omega} \|x_t\|$.

When $A \subset E$, $\ell_\infty(\Omega, A) := \{\underline{x} \in \ell_\infty(\Omega, E) : \forall t \in \Omega, x_t \in A\}$.

We also write $B_\infty(\underline{x}, r) := \{\underline{u} \in \ell_\infty(\Omega, E) : \|\underline{u} - \underline{x}\|_\infty < r\}$.

$\ell_1(\Omega, E)$ denotes the space of the sequences $\underline{x} = (x_t)_{t \in \Omega} \in E^\Omega$ such that $\sum_{t \in \Omega} \|p_t\| < +\infty$.

$c_0(\Omega, E)$ denotes the space of the sequences $\underline{x} = (x_t)_{t \in \Omega} \in E^\Omega$ such that $\lim_{t \rightarrow +\infty} x_t = 0$.

3. THE RANK THEOREM IN INFINITE DIMENSION

In this section we recall the Rank Theorem in infinite dimension and we establish several consequences of this theorem which are useful for the sequel. Under the assumptions of the Rank Theorem we describe the tangent space of a level set. As we indicate in [5], there exist other generalizations of the classical Rank Theorem to the infinite-dimensional Banach spaces.

X and Y are real Banach spaces. The following result is given in [4] and it is established in Theorem 1 and in Theorem 5 of [5].

Theorem 3.1. *Let $A \in \mathcal{O}(X)$, $f \in C^1(A, Y)$, $\hat{x} \in A$, and we set $\hat{y} := f(\hat{x})$. We assume that $E_2 := \text{Ker}Df(\hat{x})$ is topologically complemented in X and $F_1 := \text{Im}Df(\hat{x})$ is closed and topologically complemented in Y ; and so $X = E_1 \oplus E_2$ where E_1 is a closed vector subspace of X , and $Y = F_1 \oplus F_2$ where F_2 is a closed vector subspace of Y . We also assume that the following condition is fulfilled:*

$$(*) \quad \exists A_0 \in \mathcal{O}_{\hat{x}}(A) \text{ s.t. } \forall x \in A_0, \text{Im}Df(x) \cap F_2 = \{0\}.$$

Then the following assertions hold.

- (i) $\exists V_1 \in \mathcal{O}_{D_1f_1(\hat{x})^{-1}(\hat{y}_1)}(E_1)$, $\exists V_2 \in \mathcal{O}_{\hat{x}_2}(E_2)$, $\exists B \in \mathcal{O}_{\hat{x}}(X)$, $\exists \psi : V_1 \times V_2 \rightarrow B$ a C^1 diffeomorphism, $\exists W \in \mathcal{O}_{(\hat{y}_1, 0)}(X)$, $\exists \Omega_1 \in \mathcal{O}_{\hat{y}_1}(F_1)$, $\exists \Omega_2 \in \mathcal{O}_{\hat{y}_2}(F_2)$, $\exists \phi : W \rightarrow \Omega_1 \times \Omega_2$ a C^1 diffeomorphism such that $\phi^{-1} \circ f \circ \psi = Df(\hat{x})$ on $V_1 \times V_2$.
- (ii) $\exists G_1 \in \mathcal{O}_{\hat{x}_1}(E_1)$, $\exists G_2 \in \mathcal{O}_{\hat{x}_2}(E_2)$, $\exists \xi \in C^1(G_2, G_1)$ such that $[f = \hat{y}] \cap (G_1 \times G_2) = \{(\xi(x_2), x_2) : x_2 \in G_2\}$.

Note that the equality of the conclusion (i) can be rewritten as follows (cf. Theorem 1 of [5])

$$\forall (x_1, x_2) \in V_1 \times V_2, \quad \phi^{-1} \circ f \circ \psi(x_1, x_2) = (D_1f_1(\hat{x})x_1, 0) \quad (3.1)$$

which a kind of local linearization of f .

When X and Y are finite-dimensional, the assumption $(*)$ of Theorem 3.1 is equivalent to the constancy of the rank of $Df(x)$ on a neighborhood of \hat{x} , cf. Proposition 4 in [5]. We can provide a heuristic meaning of the condition $(*)$: when x is closed to \hat{x} , due to the continuity of Df , $Df(x)$ is closed to $Df(\hat{x})$ and $\text{Im}Df(x)$ cannot be "smaller" than $\text{Im}Df(\hat{x})$; this is a consequence of the openness of the set of the invertible linear continuous operators in the normed space of the linear continuous operators. And so to have $\text{Im}Df(x)$ isomorphic to $\text{Im}Df(\hat{x})$, it suffices to forbid $\text{Im}Df(x)$ to be too "big". The condition $\text{Im}Df(x) \cap F_2 = \{0\}$ is a way to forbid $\text{Im}Df(x)$ to be too "big".

Now we describe consequences of Theorem 3.1.

Proposition 3.2. *In the setting and under the assumptions of Theorem 3.1, the following assertions hold.*

- (i) $Df_2(\hat{x}) = 0$.
- (ii) $D_2f_1(\hat{x}) = 0$.
- (iii) $\xi(\hat{x}_2) = \hat{x}_1$.
- (iv) $D\xi(\hat{x}_2) = -D_1f_1(\hat{x})^{-1} \circ D_2f_1(\hat{x}) = 0$.

Proof. $\pi_2 : Y \rightarrow F_2$ and $p_2 : X \rightarrow E_2$ denote the projections.

(i) $f_2 = \pi_2 \circ f$ implies $Df_2(\hat{x}) = \pi_2 \circ Df(\hat{x}) = 0$ since $\text{Im}Df(\hat{x}) \cap F_2 = \{0\}$.

(ii) $D_2f_1(\hat{x}) = Df_1(\hat{x}) \circ p_2 = 0$ since $E_2 := \text{Ker}Df(\hat{x})$.

(iii) From Theorem 3.1(ii), since $f(\hat{x}_1, \hat{x}_2) = \hat{y}$, we necessarily obtain $\hat{x}_1 = \xi(\hat{x}_2)$.

(iv) Using (ii) of Theorem 3.1, we have, for all $x_2 \in G_2$, $f_1(\xi(x_2), x_2) = \hat{y}_2$. Differentiating this equality with respect to x_2 at \hat{x}_2 , we obtain $0 = D_1f(\hat{x}) \circ D\xi(\hat{x}_2) + D_2f(\hat{x})$ which implies the announced formulas. \square

Proposition 3.3. *In the setting and under the assumptions of Theorem 3.1 we have $T_{\hat{x}}[f = \hat{y}] = \text{Ker}Df(\hat{x})$.*

Proof. Let $v \in T_{\hat{x}}[f = \hat{y}]$. Hence there exists $c \in C^1((-\epsilon, \epsilon), X)$, where $\epsilon > 0$, such that $c((-\epsilon, \epsilon)) \subset [f = \hat{y}]$, $c(0) = \hat{x}$ and $c'(0) = v$. Since $f(c(\theta)) = \hat{y}$ for all $\theta \in (-\epsilon, \epsilon)$, differentiating this equality with respect to θ at 0, we obtain $0 = Df(c(0))c'(0) = Df(\hat{x})v$, and so $v \in \text{Ker}Df(\hat{x})$. We have proven that $T_{\hat{x}}[f = \hat{y}] \subset \text{Ker}Df(\hat{x})$.

To prove the inverse inclusion, we consider $v = (v_1, v_2) \in \text{Ker}Df(\hat{x})$. Therefore we have $v_1 = 0$ after the definition of E_1, E_2 , and so $v = (0, v_2)$. Using Theorem 3.1(ii), since G_2 is open, there exists $\epsilon > 0$ such that, for all $\theta \in (-\epsilon, \epsilon)$, we have $\hat{x}_2 + \theta v_2 \in G_2$. We define $c : (-\epsilon, \epsilon) \rightarrow X$ by setting $c(\theta) := (\xi(\hat{x}_2 + \theta v_2), \hat{x}_2 + \theta v_2) \in [f = \hat{y}] \cap (G_1 \times G_2)$. Note that we have $c(0) = (\xi(\hat{x}_2), \hat{x}_2) = \hat{x}$ after Proposition 3.2(iii). Note that we have $c'(0) = (D\xi(\hat{x}_2)v_2, v_2) = (0, v_2) = v$ after Proposition 3.2(iv) and so we have $v \in T_{\hat{x}}[f = \hat{y}]$. We have proven that $\text{Ker}Df(\hat{x}) \subset T_{\hat{x}}[f = \hat{y}]$, and so we have proven the announced equality. \square

4. LAGRANGE MULTIPLIERS IN BANACH SPACES

In this section we establish a theorem of existence of Lagrange multipliers for problem (M) (written in Introduction) by using the Rank Theorem in infinite dimension. The interest of this result is to avoid a surjectivity on the differential of the equality constraint and nevertheless to avoid the presence of a multiplier before the criterion.

Let X and Y be real Banach spaces, $A \in \mathcal{O}(A)$, $J : A \rightarrow \mathbb{R}$ be a functional, and $f : A \rightarrow Y$ be a mapping. We consider the problem (M).

Theorem 4.1. *Let \hat{x} be a local solution of the problem (M). We assume that the following conditions are fulfilled.*

- (a) *J is Fréchet differentiable at \hat{x} and f is of class C^1 on a neighborhood of \hat{x} .*
- (b) *$E_2 := \text{Ker}Df(\hat{x})$ is topologically complemented in X ; i.e. $X = E_1 \oplus E_2$ where E_1 is a closed vector subspace of X . $F_1 := \text{Im}Df(\hat{x})$ is closed and topologically complemented in Y ; i.e. $Y = F_1 \oplus F_2$ where F_2 is a closed vector subspace of Y .*
- (c) *There exists $A_0 \in \mathcal{O}_{\hat{x}}(A)$ such that, for all $x \in A_0$, $\text{Im}Df(x) \cap F_2 = \{0\}$.*

Then there exists $\lambda \in Y^$ such that $DJ(\hat{x}) = \lambda \circ Df(\hat{x})$.*

Proof. Using Theorem 4.1, we know that the set $[f = 0]$ is a C^1 manifold around \hat{x} , and since J is of class C^1 at \hat{x} , the first-order necessary optimality condition is $DJ(\hat{x})h = 0$ when $h \in T_{\hat{x}}[f = 0]$. Using Proposition 3.3, we know that $T_{\hat{x}}[f = 0] = \text{Ker}Df(\hat{x})$, hence we obtain $DJ(\hat{x})h = 0$ when $h \in \text{Ker}Df(\hat{x})$, i.e., $DJ(\hat{x}) \in (\text{Ker}Df(\hat{x}))^\perp$. Since $\text{Im}Df(\hat{x})$ is closed, using Theorem 2.19 in [9] (p. 46), we have $(\text{Ker}Df(\hat{x}))^\perp = \text{Im}Df(\hat{x})^*$, and then we have $DJ(\hat{x}) \in \text{Im}(Df(\hat{x}))^*$, hence there exists $\lambda \in Y^*$ such that $DJ(\hat{x}) = Df(\hat{x})^* \lambda = \lambda \circ Df(\hat{x})$. \square

Notice that in Theorem 4.1 we have not a multiplier before the differential of the criterion. Such a result in finite-dimensional spaces is proven in [10] (Proposition 2.3).

5. ON SPACES OF BOUNDED SEQUENCES

In this section we establish several results on the space of the bounded sequences to prepare the using of the theorem on the Lagrange multipliers of the previous section in such sequence spaces.

First we recall a result on the characterization of the closedness of linear operators.

Proposition 5.1. *Let \mathcal{X} and \mathcal{Y} be two real Banach spaces and $L \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The two following assertions are equivalent.*

- (i) *ImL is closed in \mathcal{Y} .*
- (ii) *$\exists c(L) > 0, \forall y \in ImL, \exists x_y \in \mathcal{X}$ s.t. $L(x_y) = y$ and $\|x_y\| \leq c(L)\|y\|$.*

We can find two different proofs of this result in [3] (Lemma 3.4) and in [6] (Lemma 2.1).

Remark 5.2. *About the constant $c(L)$, under (i), when in addition we assume that $KerL$ is topologically complemented in \mathcal{X} , i.e. $\mathcal{X} = KerL \oplus \mathcal{X}_1$, where \mathcal{X}_1 is a closed vector subspace of \mathcal{X} , we can consider the abridgement $abL : \mathcal{X}_1 \rightarrow ImL$, $abL(x) := L(x)$, which is an isomorphism of Banach spaces, and we have $c(L) = \|(abL)^{-1}\|_{\mathcal{L}}$.*

Let X and Y be real Banach spaces. Let $(T_t)_{t \in \mathbb{N}_*} \in \mathcal{L}(X, Y)^{\mathbb{N}_*}$. We consider the three following conditions on this family of linear operators.

- (C1) $\sup_{t \in \mathbb{N}_*} \|T_t\|_{\mathcal{L}} < +\infty$.
- (C2) For all $t \in \mathbb{N}_*$, $KerT_t$ is topologically complemented in X , i.e. there exists a closed vector subspace of X , say S_t , such that $X = KerT_t \oplus S_t$, and ImT_t is closed and topologocally complemented in Y , i.e. there exists a closed vector subspace of Y , say W_t , such that $Y = ImT_t \oplus W_t$.
- (C3) $\hat{c} := \sup_{t \in \mathbb{N}_*} c(T_t) < +\infty$.

Notice that (C2) is automatically fulfilled when $dimX < +\infty$ and $dimY < +\infty$. We associate to this family the four following sequence spaces.

$$\begin{aligned} \mathcal{K} &:= \{\underline{x} \in \ell^\infty(\mathbb{N}_*, X) : \forall t \in \mathbb{N}_*, x_t \in KerT_t\}. \\ \mathcal{S} &:= \{\underline{x} \in \ell^\infty(\mathbb{N}_*, X) : \forall t \in \mathbb{N}_*, x_t \in S_t\}. \\ \mathcal{I} &:= \{\underline{y} \in \ell^\infty(\mathbb{N}_*, Y) : \forall t \in \mathbb{N}_*, y_t \in ImT_t\}. \\ \mathcal{W} &:= \{\underline{y} \in \ell^\infty(\mathbb{N}_*, Y) : \forall t \in \mathbb{N}_*, y_t \in W_t\}. \end{aligned}$$

The condition (C1), which is equivalent to the boundeness of $(T_t(x_t))_{t \in \mathbb{N}_*}$ when \underline{x} is bounded after the Banach-Steinhaus theorem, permits us to define the operator

$$\mathcal{T} : \ell^\infty(\mathbb{N}_*, X) \rightarrow \ell^\infty(\mathbb{N}_*, Y), \quad \mathcal{T}(\underline{x}) := (T_y(x_t))_{t \in \mathbb{N}_*}. \quad (5.1)$$

Lemma 5.3. *Under (C1, C2, C3) the following assertions hold.*

- (i) $\ell^\infty(\mathbb{N}_*, X) = Ker\mathcal{T} \oplus \mathcal{S}$.
- (ii) $\ell^\infty(\mathbb{N}_*, Y) = Im\mathcal{T} \oplus \mathcal{W}$.

Proof. For each $t \in \mathbb{N}_*$, we consider the mappings $\pi_t : \ell^\infty(\mathbb{N}_*, X) \rightarrow X$ defined by $\pi_t(\underline{x}) := x_t$, and the mapping $\varpi_t : \ell^\infty(\mathbb{N}_*, Y) \rightarrow Y$, defined by $\varpi_t(\underline{y}) := y_t$. Clearly we have $\pi_t \in \mathcal{L}(\ell^\infty(\mathbb{N}_*, X), X)$ and $\varpi_t \in \mathcal{L}(\ell^\infty(\mathbb{N}_*, Y), Y)$. Note that we have

$$\mathcal{K} = \bigcap_{t \in \mathbb{N}_*} \pi_t^{-1}(KerT_t), \mathcal{S} = \bigcap_{t \in \mathbb{N}_*} \pi_t^{-1}(S_t), \mathcal{I} = \bigcap_{t \in \mathbb{N}_*} \varpi_t^{-1}(ImT_t), \mathcal{W} = \bigcap_{t \in \mathbb{N}_*} \varpi_t^{-1}(W_t).$$

Due to the continuity of the π_t and ϖ_t , these four spaces are intersections of closed subsets which implies that \mathcal{K} and \mathcal{S} are closed vector subspaces of $\ell^\infty(\mathbb{N}_*, X)$ and \mathcal{I} and \mathcal{W} are closed vector subspaces of $\ell^\infty(\mathbb{N}_*, Y)$.

It is easy to verify that

$$\ell^\infty(\mathbb{N}_* X) = \mathcal{K} \oplus^a \mathcal{S}, \quad \ell^\infty(\mathbb{N}_*, Y) = \mathcal{I} \oplus^a \mathcal{W}.$$

Since $\ell^\infty(\mathbb{N}_*, X)$ and $\ell^\infty(\mathbb{N}_*, Y)$ are Banach spaces and since \mathcal{K} , \mathcal{S} , \mathcal{I} , \mathcal{W} are closed vector subspaces, from the Inverse Mapping Theorem of Banach-Schauder, [11] (Corollary 1.5, p. 388), we obtain

$$\ell^\infty(\mathbb{N}_*, X) = \mathcal{K} \oplus \mathcal{S}, \quad \ell^\infty(\mathbb{N}_*, Y) = \mathcal{I} \oplus \mathcal{W}. \quad (5.2)$$

We easily verify that $\mathcal{K} = \text{Ker}\mathcal{T}$, and from (5.2) we obtain (i).

About the image, we see that $\text{Im}\mathcal{T} \subset \mathcal{I}$. if $\underline{y} \in \mathcal{I}$ then, for all $t \in \mathbb{N}_*$, we have $y_t \in \text{Im}T_t$. From (C2), using Proposition 5.1, we can say that there exists $x_{t,y_t} \in X$ such that $T_t(x_{t,y_t}) = y_t$ and $\|x_{t,y_t}\| \leq c(T_t)\|y_t\|$. Hence, setting $\underline{x}_{\underline{y}} := (x_{t,y_t})_{t \in \mathbb{N}_*}$, we have $\mathcal{T}(\underline{x}_{\underline{y}}) = \underline{y}$ and $\|\underline{x}_{\underline{y}}\|_\infty \leq \hat{c}\|\underline{y}\|$, therefore $\underline{x}_{\underline{y}} \in \ell(\mathbb{N}_*, X)$ and $\mathcal{T}(\underline{x}_{\underline{y}}) = \underline{y}$, i.e. $\underline{y} \in \text{Im}\mathcal{T}$. We have proven that $\mathcal{I} \subset \text{Im}\mathcal{T}$, and consequently $\mathcal{I} = \text{Im}\mathcal{T}$. From (5.2) we obtain (ii). \square

Lemma 5.4. *Let $A \in \mathcal{O}(X)$. Then the following assertions hold.*

- (i) $\text{Int}\ell^\infty(\mathbb{N}_*, A) = \{\underline{x} \in \ell^\infty(\mathbb{N}_*, A) : \inf_{t \in \mathbb{N}_*} d(x_t, A^c) > 0\}$, where $d(x_t, A^c) := \inf\{\|x_t - z\| : z \in A^c\}$.
- (ii) Let $\underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$. We set $r := \inf_{t \in \mathbb{N}_*} d(x_t, A^c) > 0$. Then we have $\prod_{t \in \mathbb{N}_*} B(x_t, \frac{r}{2}) \subset \text{Int}\ell^\infty(\mathbb{N}_*, A)$.

Proof. Assertion (i) is proven in [7] (Lemma A.1.1). To prove (ii), if $\underline{u} \in \prod_{t \in \mathbb{N}} B(x_t, \frac{r}{2})$, then we have $\|u_t - x_t\| < \frac{r}{2}$, and for all $v \in A^c$, we have

$$r < \|x_t - v\| \leq \|x_t - u_t\| + \|u_t - v\| \leq \frac{r}{2} + \|u_t - v\| \implies r - \frac{r}{2} \leq \|u_t - v\|$$

which implies $d(u_t, A^c) \geq \frac{r}{2} > 0$, and so $\underline{u} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$. \square

Definition 5.5. *Let $A \in \mathcal{O}(X)$ and $g_t : A \rightarrow Y$ be a mapping for all $t \in \mathbb{N}_*$. The sequence $(g_t)_{t \in \mathbb{N}_*}$ is said uniformly equicontinuous on the bounded subsets of A when*

$$\left\{ \begin{array}{l} \forall B \in \mathbb{B}(A), \forall \epsilon > 0, \exists \eta_{B,\epsilon} > 0, \forall u, v \in B, \\ \|u - v\| \leq \eta_{B,\epsilon} \implies (\forall t \in \mathbb{N}_*, \|g_t(u) - g_t(v)\| \leq \epsilon). \end{array} \right.$$

Proposition 5.6. *Let $A \in \mathcal{O}(X)$ and $g_t : A \rightarrow Y$ be a mapping for all $t \in \mathbb{N}_*$. We assume that the following condition is fulfilled: $\forall \underline{x} \in \ell^\infty(A), (g_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, Y)$.*

This condition permits to define the operator $G : \ell^\infty(\mathbb{N}_, A) \rightarrow \ell^\infty(\mathbb{N}_*, Y)$ by setting $G(\underline{x}) := (g_t(x_t))_{t \in \mathbb{N}_*}$.*

If, in addition we assume that $(g_t)_{t \in \mathbb{N}}$ is uniformly equicontinuous in the bounded subsets of A , then G is continuous from $\text{Int}\ell^\infty(\mathbb{N}_, A)$ into $\ell^\infty(\mathbb{N}_*, Y)$.*

Proof. We arbitrarily fix $\underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$. Hence there exists $r > 0$ such that $B_\infty(\underline{x}, r) \subset \ell^\infty(\mathbb{N}_*, A)$.

The set $B := \{u \in A : \exists t_u \in \mathbb{N}_*, \|u - x_{t_u}\| < r\}$ is bounded since, when $u \in B$, $\|u\| \leq \|u - x_{t_u}\| + \|x_{t_u}\| \leq r + \|\underline{x}\|_\infty < +\infty$. We arbitrarily fix $\epsilon > 0$. Let $\underline{z} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$ such that $\|\underline{z} - \underline{x}\|_\infty < \min\{r, \eta_{B,\epsilon}\}$ where $\eta_{B,\epsilon}$ is provided by

Definition 5.5. Hence we have $z_t \in B$ since $\|z_t - x_t\| < r$ for all $t \in \mathbb{N}_*$, and we have $\|z_t - x_t\| < \eta_{B,\epsilon}$ which implies $\|g_t(z_t) - g_t(x_t)\| < \epsilon$. Hence we have $\|G(\underline{z}) - G(\underline{x})\|_\infty \leq \epsilon$. We have proven that G is continuous at \underline{x} . \square

Proposition 5.7. *Let $A \in \mathcal{O}(X)$ and $(g_t)_{t \in \mathbb{N}_*} \in C^1(A, Y)^{\mathbb{N}_*}$. We assume that the following conditions are fulfilled.*

- (a) $\forall \underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$, $(g_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, Y)$.
- (b) $\forall \underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$, $(Dg_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, \mathcal{L}(X, Y))$.
- (c) $(Dg_t)_{t \in \mathbb{N}_*}$ is uniformly equicontinuous on the bounded subsets of A .

We consider the operator $G : \text{Int}\ell^\infty(\mathbb{N}_*, A) \rightarrow \ell^\infty(\mathbb{N}_*, Y)$ defined by $G(\underline{x}) := (g_t(x_t))_{t \in \mathbb{N}_*}$.

Then the following assertions hold for all $\underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$.

- (i) G is Fréchet differentiable at \underline{x} , and, for all $\underline{v} \in \ell^\infty(\mathbb{N}_*, X)$, we have $DG(\underline{x})\underline{v} = (Dg_t(x_t)v_t)_{t \in \mathbb{N}_*}$.
- (ii) $G \in C^1(\text{Int}\ell^\infty(\mathbb{N}_*, A), \ell^\infty(\mathbb{N}_*, Y))$.

Proof. We arbitrarily fix $\underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$. Using (b), the linear operator $\mathcal{T} : \ell^\infty(\mathbb{N}_*, X) \rightarrow \ell^\infty(\mathbb{N}_*, Y)$, defined by $\mathcal{T}\underline{h} := (Dg_t(x_t)h_t)_{t \in \mathbb{N}_*}$, is well-defined and continuous.

We can use Lemma 5.4(ii) and assert that $\prod_{t \in \mathbb{N}_*} B(x_t, \frac{r}{2}) \subset \text{Int}\ell^\infty(\mathbb{N}_*, A)$. Using the set $B := \{u \in A : \exists t_u \in \mathbb{N}_*, \|u - x_t\| < r\}$, we have yet seen that B is bounded in A and we have $B(x_t, \frac{r}{2}) \subset B$.

We arbitrarily fix $\epsilon > 0$, and, using assumption (c), we consider $\eta_{B,\epsilon} > 0$ provided by Definition 5.5 for the sequence $(Dg_t)_{t \in \mathbb{N}_*}$.

We arbitrarily fix $\underline{h} \in \ell^\infty(\mathbb{N}_*, X)$ such that $\|\underline{h}\|_\infty < \min\{\frac{r}{2}, \eta_{B,\epsilon}\}$. Hence, $\forall t \in \mathbb{N}_*$, $\forall z_t \in [x_t, x_t + h_t] \subset B(x_t, \frac{r}{2})$, we have $\|z_t - x_t\| < \eta_{B,\epsilon}$ which implies $\|Dg_t(z_t) - Dg_t(x_t)\|_{\mathcal{L}} < \epsilon$.

Now using the Mean Value Inequality as established in [1] (Corollary 1, p. 144), we obtain, for all $t \in \mathbb{N}_*$,

$\|g_t(x_t + h_t) - g_t(x_t) - Dg_t(x_t)h_t\| \leq \sup_{z_t \in [x_t, x_t + h_t]} \|Dg_t(z_t) - Dg_t(x_t)\|_{\mathcal{L}} \|h_t\| \leq \epsilon \|h_t\|$, and taking the sup on the $t \in \mathbb{N}_*$, we obtain

$$\|G(\underline{x} + \underline{h}) - G(\underline{x}) - \mathcal{T}\underline{h}\|_\infty \leq \epsilon \|\underline{h}\|_\infty.$$

Hence we have proven that G is Fréchet differentiable at \underline{x} and that $DG(\underline{x})\underline{h} = (Dg_t(x_t)h_t)_{t \in \mathbb{N}_*}$.

Applying Proposition 5.6 to DG , we obtain the continuity of DG . \square

Proposition 5.8. *Let $A \in \mathcal{O}(X)$ and $(g_t)_{t \in \mathbb{N}_*} \in C^1(A, Y)^{\mathbb{N}_*}$. We assume that the conditions (a), (b), (c) of Proposition 5.7 are fulfilled. Let $\hat{\underline{x}} = (\hat{x}_t)_{t \in \mathbb{N}_*} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$. We assume that the following conditions are fulfilled.*

- (d) $\sup_{t \in \mathbb{N}_*} \|Dg(\hat{x}_t)\|_{\mathcal{L}} < +\infty$.
- (e) For all $t \in \mathbb{N}_*$ there exist a closed vector subspace of X , say S_t , and a closed vector subspace of Y , say W_t , such that $X = \text{Ker}Dg(\hat{x}_t) \oplus S_t$ and $Y = \text{Im}Dg(\hat{x}_t) \oplus W_t$.
- (f) $\hat{c} := \sup_{t \in \mathbb{N}_*} c(Dg(\hat{x}_t)) < +\infty$.

Then we have $\ell^\infty(\mathbb{N}_*, X) = \text{Ker}DG(\hat{\underline{x}}) \oplus \mathcal{S}$ and $\ell^\infty(\mathbb{N}_*, Y) = \text{Im}DG(\hat{\underline{x}}) \oplus \mathcal{W}$, with $\mathcal{S} := \{\underline{v} \in \ell^\infty(\mathbb{N}_*, X) : \forall t \in \mathbb{N}_*, v_t \in S_t\}$ and $\mathcal{W} := \{\underline{w} \in \ell^\infty(\mathbb{N}_*, Y) : \forall t \in \mathbb{N}_*, w_t \in W_t\}$.

Proof. This result is a consequence of Proposition 5.7 and of Lemma 5.3 with $T_t = Dg(\hat{x}_t)$. \square

Notice that (e) is automatically fulfilled when $\dim X < +\infty$ and $\dim Y < +\infty$.

6. A VARIATIONAL PROBLEM

In this section we consider a maximization problem in infinite horizon and in discrete time under holonomic constraints. The unknown variable is a bounded sequence with values in a real Banach space X . Using the results of the previous sections we obtain a first-order necessary optimality condition in the form of a nonhomogeneous Euler-Lagrange equation.

Let A be a nonempty subset of X . For all $t \in \mathbb{N}$, let $u_t : A \times A \rightarrow \mathbb{R}$ and for all $t \in \mathbb{N}_*$ let $g_t : A \rightarrow Y$ be functions, where Y is a real Banach space. We fix a vector $\sigma \in A$, a real number $\beta \in (0, 1)$ and we consider the following variational problem under holonomic constraints.

$$(\mathcal{V}) \begin{cases} \text{Maximize} & J(\underline{x}) := \sum_{t=0}^{+\infty} \beta^t u_t(x_t, x_{t+1}) \\ \text{subject to} & \underline{x} \in \ell^\infty(\mathbb{N}, A), x_0 = \sigma \\ \text{and} & \forall t \in \mathbb{N}_*, g_t(x_t) = 0. \end{cases}$$

The problems of Calculus of Variations or of Optimal Control in discrete time and in infinite horizon are very usual in Economics and in Management; see for instance [8], [3], [7] and references therein.

Theorem 6.1. *Let \hat{x} be a solution of (\mathcal{V}) . We assume that the following conditions are fulfilled*

- (A1) $(\hat{x}_t, \hat{x}_{t+1})_{t \in \mathbb{N}} \in \text{Int} \ell^\infty(\mathbb{N}, A \times A)$.
- (A2) $\forall t \in \mathbb{N}, u_t \in C^1(A \times A, \mathbb{R})$.
- (A3) $\forall (\underline{x}, \underline{y}) \in \text{Int} \ell^\infty(\mathbb{N}, A \times A)$, $(u_t(x_t, y_t))_{t \in \mathbb{N}} \in \ell_\infty(\mathbb{N}, \mathbb{R})$.
- (A4) $\forall (\underline{x}, \underline{y}) \in \text{Int} \ell^\infty(\mathbb{N}, A \times A)$, $(Du_t(x_t, y_t))_{t \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, (X \times X)^*)$.
- (A5) $(Du_t)_{t \in \mathbb{N}}$ is uniformly equicontinuous on the bounded subsets of $A \times A$.
- (A6) $\hat{x} \in \text{Int} \ell^\infty(\mathbb{N}, A)$.
- (A7) $\forall t \in \mathbb{N}_*, g_t \in C^1(A, \mathbb{R})$.
- (A8) $\forall \underline{x} \in \text{Int} \ell^\infty(\mathbb{N}_*, A)$, $(g_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, Y)$.
- (A9) $\forall \underline{x} \in \text{Int} \ell^\infty(\mathbb{N}_*, A)$, $(Dg_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, \mathcal{L}(X, Y))$.
- (A10) $(Dg_t)_{t \in \mathbb{N}_*}$ is uniformly equicontinuous on the bounded subsets of A .
- (A11) For all $t \in \mathbb{N}_*$, there exists a closed vector subspace of X , say S_t , such that $X = \text{Ker} Dg_t(\hat{x}_t) \oplus S_t$, and there exists a closed vector subspace of Y , say W_t , such that $Y = \text{Im} Dg_t(\hat{x}_t) \oplus W_t$.
- (A12) $\sup_{t \in \mathbb{N}_*} c(Dg_t(\hat{x}_t)) < +\infty$.
- (A13) $\exists r > 0$, $\forall \underline{x} \in B_\infty(\hat{x}, r)$, $\forall t \in \mathbb{N}_*$, $\text{Im} Dg_t(x_t) \cap W_t = \{0\}$.

Then there exists a sequence $(p_t)_{t \in \mathbb{N}_*} \in \ell^1(\mathbb{N}_*, Y^*)$ which satisfies, for all $t \in \mathbb{N}_*$, the following equality

$$\beta^t D_1 u_t(\hat{x}_t, \hat{x}_{t+1}) + \beta^{t-1} D_2 u_{t-1}(\hat{x}_{t-1}, \hat{x}_t) = p_t \circ Dg_t(\hat{x}_t).$$

Proof. For each $t \in \mathbb{N}_*$ we consider the following static problem.

$$(\mathcal{R}_t) \begin{cases} \text{Maximize} & J_t(x_t) := \beta^{t-1} u_{t-1}(\hat{x}_{t-1}, x_t) + \beta^t u_t(x_t, \hat{x}_{t+1}) \\ \text{subject to} & x_t \in A, g_t(x_t) = 0. \end{cases}$$

Since \hat{x} is a solution of (\mathcal{V}) , proceeding by contradiction, it is easy to see that \hat{x}_t is a solution of (\mathcal{R}_t) . Under our assumptions we can use Theorem 4.1 on (\mathcal{R}_t) and assert that there exists $q_t \in Y^*$ such the following equality holds.

$$\beta^{t-1}D_2u_{t-1}(\hat{x}_{t-1}, \hat{x}_t) + \beta^t u_t(\hat{x}_t, \hat{x}_{t+1}) = q_t \circ Dg_t(\hat{x}_t). \quad (6.1)$$

Under (A11), we know that, for every $y \in Y$, there exists a unique $(z, w) \in \text{Im}Dg_t(\hat{x}_t) \times W_t$ such that $y = z + w$. We define $p_t \in Y^*$ by setting $p_t(y) := q_t(z)$. Hence we deduce from (6.1) the following equality:

$$\beta^{t-1}D_2u_{t-1}(\hat{x}_{t-1}, \hat{x}_t) + \beta^t D_1u_t(\hat{x}_t, \hat{x}_{t+1}) = p_t \circ Dg_t(\hat{x}_t). \quad (6.2)$$

Now we ought to prove that $(p_t)_{t \in \mathbb{N}_*} \in \ell_1(\mathbb{N}_*, Y^*)$.

Let $\underline{y} \in \ell_\infty(\mathbb{N}_*, Y)$. Using Proposition 5.8, there exists $\underline{x} \in \ell_\infty(\mathbb{N}_*, X)$ and $\underline{w} \in \mathcal{W}$ such that $\underline{y} = DG(\hat{x})\underline{x} + \underline{w}$, i.e. $y_t = Dg_t(\hat{x}_t)x_t + w_t$. Hence we have

$$\begin{aligned} p_t(y_t) &= p_t \circ Dg_t(\hat{x}_t)x_t + p_t(w_t) \\ &= p_t \circ Dg_t(\hat{x}_t)x_t + 0 \\ &= \beta^{t-1}D_2u_{t-1}(\hat{x}_{t-1}, \hat{x}_t)x_t + \beta^t D_1u_t(\hat{x}_t, \hat{x}_{t+1})x_t \end{aligned}$$

The sequences $(D_2u_{t-1}(\hat{x}_{t-1}, \hat{x}_t))_{t \in \mathbb{N}_*}$ and $(D_1u_t(\hat{x}_t, \hat{x}_{t+1}))_{t \in \mathbb{N}_*}$ are bounded after (A4), and since $(x_t)_{t \in \mathbb{N}_*}$ is bounded, the sequences $(D_2u_{t-1}(\hat{x}_{t-1}, \hat{x}_t)x_t)_{t \in \mathbb{N}_*}$ and $(D_1u_t(\hat{x}_t, \hat{x}_{t+1})x_t)_{t \in \mathbb{N}_*}$ are bounded.

Since $\beta \in (0, 1)$, the series $\sum_{t \geq 1} \beta^{t-1}D_2u_{t-1}(\hat{x}_{t-1}, \hat{x}_t)x_t$ and $\sum_{t \geq 0} \beta^t D_1u_t(\hat{x}_t, \hat{x}_{t+1})x_t$ are convergent in \mathbb{R} , and consequently the series $\sum_{t \geq 1} p_t(y_t)$ is convergent in \mathbb{R} . Since $c_0(\mathbb{N}_*, Y) \subset \ell_\infty(\mathbb{N}_*, Y)$, we obtain that the series $\sum_{t \geq 1} p_t(y_t)$ is convergent in \mathbb{R} when $\underline{y} \in c_0(\mathbb{N}_*, Y)$. Using [12](assertion (α) in page 247), we can assert that $(p_t)_{t \in \mathbb{N}_*} \in \ell_1(\mathbb{N}_*, Y^*)$. \square

Notice that (A3), (A4), (A8), (A9) and (A11) are automatically fulfilled when $\dim X < +\infty$ and $\dim Y < +\infty$. When $\dim Y < +\infty$, using a representation of the topological dual space of $\ell_\infty(\mathbb{N}_*, Y^*)$ (as a direct of $\ell_1(\mathbb{N}_*, Y^*)$ and of another subspace) which is given in [2] Chapter 15, Section 15.8) we can proceed as in [8] (Chapter 3) and to obtain the sequence $(p_t)_{t \in \mathbb{N}_*}$ as the component in $\ell_1(\mathbb{N}_*, Y^*)$ of an element of $\ell_\infty(\mathbb{N}_*, Y^*)^*$.

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