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RANK THEOREM IN INFINITE DIMENSION AND LAGRANGE MULTIPLIERS

JOEL BLOT

ABSTRACT. We use an extension to the infinite dimension of the rank theorem of the differential calculus to establish a Karush-Huhn-Tucker theorem for optimization problems in Banach spaces. We provide an application to variational problems on bounded processes under equality constraints.

Keywords : Rank theorem, Lagrange multipliers, Banach spaces.

Classification MSC 2010: 49K27, 49K30.

1. INTRODUCTION

In a first time, we recall the extension of the rank theorem to the infinite dimension as it is established in [4] (Section 3).

In a second time we established a Karush-Kuhn-Tucker theorem in Banach spaces for maximization problems under equality constraints (Section 4) like the following one

$$(M) \begin{cases} \text{Maximize} & J(x) \\ \text{when} & f(x) = 0. \end{cases}$$

where the functional J and the mapping f are defined on an open subset of a Banach space, and the mapping f takes its values in a Banach space. Among the classical multiplier rules, when the differential of f at the solution is surjective we obtain a Karush-Kuhn-Tucker theorem, i.e. there is no multiplier in front of the criterion, when the image of the differential of f at the solution is closed, we obtain a multiplier rule with a multiplier in front of the criterion. Our result treats an intermediate case; our assumption which generalises the constancy of the rank of the differentials of f around the solution is stronger than the closure of the image of the differential, and it is weaker than the surjectivity of the differential, moreover under our condition we avoid the presence of a multiplier in front of the criterion.

In a third time, in Section 5, we establish results on the space of the bounded sequences with values in a Banach space to prepare the following section.

Lastly, in Section 6, we apply our Karush-Kuhn-Tucker theorem to a variational problem in infinite horizon and in discrete time on a space of bounded sequences under equality constraints.

2. NOTATION

When E and F are sets, when $f : E \rightarrow F$ is a function, when $E_0 \subset E$ and $F_0 \subset F$ are such that $f(E_0) \subset F_0$, we define the *abridgement* of f (relatively to E_0 and F_0) as $abf; E_0 \rightarrow F_0$ by setting $abf(x) := f(x)$; this notation comes from [9] (p. 12).

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When X is a topological space, $\mathcal{O}(X)$ denotes the topology of X , and when $x \in X$, $\mathcal{O}_x(X)$ denotes the set of the open neighborhoods of x in X .

The topological interior is denoted by Int .

When E is a Banach space and E_1, E_2 are closed vector subspaces of E , the writing $E = E_1 \oplus^a E_2$ (respectively $E = E_1 \oplus E_2$) means the algebraic (respectively topological) direct sum, i.e. the mapping $(x_1, x_2) \mapsto x_1 + x_2$, from $E_1 \times E_2$ into E is an isomorphism (respectively a topological isomorphism). When E_1 is a closed vector subspace of E , to say that E_1 is *topologically complemented* in E means that there exists E_2 , a closed vector subspace of E such that $E = E_1 \oplus E_2$.

The letter D denotes the Fréchet differentiation, and when $i \in \{1, 2\}$, D_i denotes the partial Fréchet differentiation of a mapping defined on a product space $E_1 \times E_2$ with respect to the i^{th} variable.

When X and Y are real Banach spaces, and when $A \in \mathcal{O}(X)$, $C^1(A, Y)$ denotes the space of the continuously Fréchet differentiable mappings from A into Y .

When E is a set, $E^{\mathbb{N}}$ denotes the space of the sequences (defined on \mathbb{N}) with values in E , and $E^{\mathbb{N}_*}$ denotes the space of the sequences (defined on $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$) with values in E .

An element of $E^{\mathbb{N}}$ will be denoted by $\underline{x} = (x_t)_{t \in \mathbb{N}}$.

When E is a normed space, when $\underline{x} \in \ell^\infty(E) := \ell^\infty(\mathbb{N}, E)$, we set $\|\underline{x}\|_\infty := \sup_{t \in \mathbb{N}} \|x_t\|$.

We also write $B_\infty(\underline{x}, r) := \{\underline{u} \in \ell^\infty(E) : \|\underline{u} - \underline{x}\|_\infty < r\}$.

3. THE RANK THEOREM IN INFINITE DIMENSION

In this section we recall the Rank Theorem in infinite dimension and we establish several consequences of this theorem which are useful for the sequel. Under the assumptions of the Rank Theorem we describe the tangent space of a level set.

X and Y are real Banach spaces. The following result is established in Theorem 1 and in Theorem 5 of [4].

Theorem 3.1. *Let $A \in \mathcal{O}(X)$, $f \in C^1(A, Y)$, $\hat{x} \in A$, and we set $\hat{y} := f(\hat{x})$.*

We assume that $E_2 := \text{Ker} Df(\hat{x})$ is topologically complemented in X and $F_1 := \text{Im} Df(\hat{x})$ is topologically complemented in Y ; and so $X = E_1 \oplus E_2$ where E_1 is a closed vector subspace of X , and $Y = F_1 \oplus F_2$ where F_2 is a closed vector subspace of Y . We also assume that the following condition is fulfilled.

$$\exists A_0 \in \mathcal{O}(A) \text{ s.t. } \forall x \in A_0, \text{Im} Df(x) \cap F_2 = \{0\}. \quad (3.1)$$

Then the following assertions hold.

- (i) $\exists V_1 \in \mathcal{O}_{D_1 f_1(\hat{x})^{-1}(\hat{y}_1)}(E_1)$, $\exists V_2 \in \mathcal{O}_{\hat{x}_2}(E_2)$, $\exists B \in \mathcal{O}_{\hat{x}}(X)$, $\exists \psi : V_1 \times V_2 \rightarrow B$ a C^1 diffeomorphism,
 $\exists W \in \mathcal{O}_{(\hat{y}_1, 0)}(Y)$, $\exists \Omega_1 \in \mathcal{O}_{\hat{y}_1}(F_1)$, $\exists \Omega_2 \in \mathcal{O}_{\hat{y}_2}(F_2)$, $\exists \phi : W \rightarrow \Omega_1 \times \Omega_2$ a C^1 diffeomorphism such that $\phi^{-1} \circ f \circ \psi = Df(\hat{x})$ on $V_1 \times V_2$.
- (ii) $\exists G_1 \in \mathcal{O}_{\hat{x}_1}(E_1)$, $\exists G_2 \in \mathcal{O}_{\hat{x}_2}(E_2)$, $\exists \xi \in C^1(G_2, G_1)$ such that $[f = \hat{y}] \cap (G_1 \times G_2) = \{(\xi(x_2), x_2) : x_2 \in G_2\}$.

Note that the equality of the conclusion (i) can be rewritten as follows (cf. Theorem 1 of [4])

$$\forall (x_1, x_2) \in V_1 \times V_2, \quad \phi^{-1} \circ f \circ \psi(x_1, x_2) = (D_1 f_1(\hat{x})x_1, 0). \quad (3.2)$$

When X and Y are finite-dimensional, the assumptions of Theorem 3.1 are equivalent to the constancy of the rank of $Df(x)$ on a neighborhood of \hat{x} , cf. Proposition 4 in [4].

Now we describe consequences of Theorem 3.1.

Proposition 3.2. *In the setting and under the assumptions of Theorem 3.1, the following assertions hold.*

- (i) $Df_2(\hat{x}) = 0$.
- (ii) $D_2f_1(\hat{x}) = 0$.
- (iii) $\xi(\hat{x}_2) = \hat{x}_1$.
- (iv) $D\xi(\hat{x}_2) = -D_1f_1(\hat{x})^{-1} \circ D_2f_1(\hat{x}) = 0$.
- (v) $\psi(D_1f_1(\hat{x})^{-1}\hat{y}_1, \hat{x}_2) = \hat{x}$.
- (vi) $D\psi(D_1f_1(\hat{x})^{-1}\hat{y}_1, \hat{x}_2) = id_X$.
- (vii) $D\phi^{-1}(\hat{y}) = id_Y$.

Proof. (i) $f_2 = \pi_2 \circ f$ implies $Df_2(\hat{x}) = \pi_2 \circ Df(\hat{x}) = 0$ since $Im Df(\hat{x}) \cap F_2 = \{0\}$.

(ii) $D_2f_1(\hat{x}) = Df_1(\hat{x}) \circ p_2 = 0$ since $E_2 := Ker Df(\hat{x})$.

(iii) Recall that in the proof of Theorem 5 in [4], the function ξ is provided by the application of the Implicit Function Theorem to the equation $f_1(x_1, x_2) = \hat{y}_1$ (with $D_1f_1(\hat{x})$ invertible). The uniqueness which is provided by the Implicit Function Theorem ([7] Theorem 4.7.1 in p. 61) implies that $\xi(\hat{x}_2) = \hat{x}_1$.

(iv) Using (ii) of Theorem 3.1, we have, for all $x_2 \in G_2$, $f_1(\xi(x_2), x_2) = \hat{y}_1$. Differentiating this equality with respect to x_2 at \hat{x}_2 , we obtain $0 = D_1f(\hat{x}) \circ D\xi(\hat{x}_2) + D_2f(\hat{x})$ which implies the announced formulas.

(v) Recall that, in the proof of Theorem 1 in [4] we have $\psi = (ab \sigma)^{-1}$ where $\sigma(x_1, x_2) := (D_1f_1(\hat{x})^{-1}f(x_1, x_2), x_2)$. Hence we have

$$\begin{aligned} \psi(u_1, u_2) = (x_1, x_2) &\iff \sigma(x_1, x_2) = (u_1, u_2) \\ &\iff (D_1f_1(\hat{x})^{-1}f(x_1, x_2) = u_1 \text{ and } x_2 = u_2). \end{aligned}$$

Consequently we have

$$\psi(D_1f_1(\hat{x})^{-1}\hat{y}_1, \hat{x}_2) = \psi(D_1f_1(\hat{x})^{-1}f_1(\hat{x}), \hat{x}_2) = \hat{x}.$$

(vi) From the definition of the function σ , using (i) from above we obtain

$$\begin{aligned} D_1\sigma_1(\hat{x}) &= D_1f_1(\hat{x})^{-1} \circ D_1f_1(\hat{x}) = id_{E_1} \\ D_2\sigma_1(\hat{x}) &= D_1f_1(\hat{x})^{-1} \circ D_2f_1(\hat{x}) = 0 \\ D_1\sigma_2(\hat{x}) &= 0 \\ D_2\sigma_2(\hat{x}) &= id_{E_2}, \end{aligned}$$

and so we obtain $D\sigma(\hat{x}) = id_X$. Hence, using a classical result (Theorem 2.5.2, p. 102 in [1]) we have

$$D\psi(D_1f_1(\hat{x})^{-1}\hat{y}_1, \hat{x}_2) = (D\sigma(\hat{x}))^{-1} = id_X.$$

(vii) Recall that in the proof of Theorem 1 in [4], we have built ϕ as $\phi := (ab \gamma)^{-1}$ where

$$\gamma(y_1, y_2) := (y_1, y_2 + f_2 \circ \psi(D_1f_1(\hat{x})^{-1}\hat{y}_1, \hat{x}_2)).$$

From (iv) we obtain $\gamma(\hat{y}_1, 0) = (\hat{y}_1, f_2(\hat{x})) = \hat{y}$. Using (v) and (i), we also have

$$\begin{aligned} D_1\gamma_1(\hat{y}_1, 0) &= id_{F_1} \\ D_2\gamma_1(\hat{y}_1, 0) &= 0 \\ D_1\gamma_2(\hat{y}_1, 0) &= D_1f_2(\hat{x}) = 0 \\ D_2\gamma_2(\hat{y}_1, 0) &= id_{F_2}, \end{aligned}$$

therefore we have $D\gamma(\hat{y}_1, 0) = id_Y$. Note that we have

$$\begin{aligned} (\phi^{-1}(\hat{y}) = z) &\iff (\gamma(z) = \hat{y}) \\ &\iff (z_1 = \hat{y}_1 \text{ and } z_2 + f_2(\hat{x}) = \hat{y}_2) \\ &\iff (z = (\hat{y}_1, 0)), \end{aligned}$$

and therefore (Theorem 2.5.2, p. 102 in [1]) we obtain $D\phi^{-1}(\hat{y}) = D\gamma(\hat{y}_1, 0) = id_Y$. \square

Proposition 3.3. *In the setting and under the assumptions of Theorem 3.1 we have $T_{\hat{x}}[f = \hat{y}] = KerDf(\hat{x})$.*

Proof. Let $v \in T_{\hat{x}}[f = \hat{y}]$. Hence there exists $c \in C^1((-\epsilon, \epsilon), X)$, where $\epsilon > 0$, such that $c((-\epsilon, \epsilon)) \subset [f = \hat{y}]$, $c(0) = \hat{x}$ and $c'(0) = v$. since $f(c(\theta)) = \hat{y}$ for all $\theta \in (-\epsilon, \epsilon)$, differentiating this equality with respect to θ at 0, we obtain $0 = Df(c(0))c'(0) = Df(\hat{x})v$, and so $v \in KerDf(\hat{x})$. We have proven that $T_{\hat{x}}[f = \hat{y}] \subset KerDf(\hat{x})$.

To prove the inverse inclusion, we consider $v = (v_1, v_2) \in KerDf(\hat{x})$, therefore we have $v_1 = 0$ after the definition of E_1, E_2 , and so $v = (0, v_2)$. Using Theorem 3.1, ii, since G_2 is open, there exists $\epsilon > 0$ such that, for all $\theta \in (-\epsilon, \epsilon)$, we have $\hat{x}_2 + \theta v_2 \in G_2$. We define $c : (-\epsilon, \epsilon) \rightarrow X$ by setting $c(\theta) := (\xi(\hat{x}_2 + \theta v_2), \hat{x}_2 + \theta v_2) \in [f = \hat{y}] \cap (G_1 \times G_2)$. Note that we have $c(0) = (\xi(\hat{x}_2), \hat{x}_2) = \hat{x}$ after Proposition 3.2, iii. Note that we have $c'(0) = (D\xi(\hat{x}_2)v_2, v_2) = (0, v_2) = v$ after Proposition 3.2, iv, and so we have $v \in T_{\hat{x}}[f = \hat{y}]$. We have proven that $KerDf(\hat{x}) \subset T_{\hat{x}}[f = \hat{y}]$, and so we have proven the announced equality. \square

4. LAGRANGE MULTIPLIERS IN BANACH SPACES

In this section we establish a Karush-Khun-Tucker theorem for problem (\mathcal{M}) (written in Introduction) by using the Rank Theorem in infinite dimension. The interest of this result is to avoid a surjectivity on the differential of the equality constraint and nevertheless to avoid the presence of a multiplier in front of the criterion.

Let X and Y be real Banach spaces, $A \in \mathcal{O}(A)$, $J : A \rightarrow \mathbb{R}$ be a functional, and $f : A \rightarrow Y$ be a mapping. We consider the problem (\mathcal{M}) .

Theorem 4.1. *Let \hat{x} be a local solution of the problem (\mathcal{M}) . We assume that the following conditions are fulfilled.*

- (a) *J is Fréchet differentiable at \hat{x} and f is of class C^1 on a neighborhood of \hat{x} .*
- (b) *$E_2 := KerDf(\hat{x})$ is topologically complemented in X ; $X = E_1 \oplus E_2$ where E_1 is a closed vector subspace of X . $F_1 := ImDf(\hat{x})$ is topologically complemented in Y ; $Y = F_1 \oplus F_2$ where F_2 is a closed vector subspace of Y .*
- (c) *There exists $A_0 \in \mathcal{O}_{\hat{x}}(A)$ such that, for all $x \in A_0$, $ImDf(x) \cap F_2 = \{0\}$.*

Then there exists $\Lambda \in Y^$ such that $Df(\hat{x}) = \Lambda \circ Df(\hat{x})$.*

Proof. Using Theorem 3.1, ii, we introduce the functional $\Gamma : G_2 \rightarrow \mathbb{R}$ by setting $\Gamma(x_2) := J(\xi(x_2), x_2)$, and we easily verify that \hat{x}_2 is a local solution of the following maximization problem

$$(M1) \begin{cases} \text{Maximize} & \Gamma(x_2) \\ \text{when} & x_2 \in G_2. \end{cases}$$

This problem has no constraint, and so the first order necessary optimality condition provides the equality $D\Gamma(\hat{x}_2) = 0$. Using the Chain Rule and Proposition 3.2, we obtain

$$D\Gamma(\hat{x}_2) = D_1J(\hat{x}) \circ D\xi(\hat{x}_2) + D_2J(\hat{x}) = D_2J(\hat{x}).$$

Hence we have proven

$$D_2J(\hat{x}) = 0. \quad (4.1)$$

We set

$$\Lambda_1 := D_1J(\hat{x}) \circ D_1f_1(\hat{x})^{-1}. \quad (4.2)$$

From Proposition 3.2, we know that $D_2f_1(\hat{x}) = 0$, and so, using (4.2) we obtain

$$D_2J(\hat{x}) = \Lambda_1 \circ D_2f_1(\hat{x}). \quad (4.3)$$

From (4.2) it is easy to verify the following equality.

$$D_1J(\hat{x}) = \Lambda_1 \circ D_1f_1(\hat{x}). \quad (4.4)$$

Using (4.3) and (4.4), for all $(v_1, v_2) \in E_1 \times E_2$, we obtain

$$\begin{aligned} DJ(\hat{x})(v_1, v_2) &= D_1J(\hat{x})v_1 + D_2J(\hat{x})v_2 \\ &= \Lambda_1 \circ D_1f_1(\hat{x})v_1 + \Lambda_1 \circ D_2f_1(\hat{x})v_2 \\ &= \Lambda_1(Df_1(\hat{x})(v_1, v_2)), \end{aligned}$$

therefore we have established

$$DJ(\hat{x}) = \Lambda_1 \circ Df_1(\hat{x}). \quad (4.5)$$

From Proposition 3.2 we know that $Df_2(\hat{x}) = 0$, and so from (4.5) we obtain

$$DJ(\hat{x}) = \Lambda_1 \circ Df_1(\hat{x}) + 0 \circ Df_2(\hat{x}) = \Lambda \circ Df(\hat{x})$$

where $\Lambda \in Y^*$ is defined by $\Lambda(y_1, y_2) := \Lambda_1y_1 + 0y_2$. \square

Notice that in Theorem 4.1 we have not a multiplier in front of the differential of the criterion. Such a result in finite-dimensional spaces is proven in [8] (Proposition 2.3).

5. ON SPACES OF BOUNDED SEQUENCES

In this section we establish several results on the space of the bounded sequences to prepare the using of Karush-Khun-Tucker theorem of the previous section in such sequence spaces.

Let X and Y be real Banach spaces. We know that $\ell^\infty(X)$ is a Banach space when it is endowed with the norm $\|\underline{x}\|_\infty := \sup_{t \in \mathbb{N}} \|x_t\|$. When we consider, for all $t \in \mathbb{N}$, a subset $E_t \subset X$, we set

$$\ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) := \{\underline{x} \in \ell^\infty(X) : \forall t \in \mathbb{N}, x_t \in E_t\}. \quad (5.1)$$

Lemma 5.1. *For all $t \in \mathbb{N}$ let E_t and F_t be two closed vector subspaces of X such that $E_t \oplus F_t = X$. Then we have*

$$\ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \oplus \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right) = \ell^\infty(X).$$

Proof. Since $\ell^\infty(X)$ is a vector space, we have

$$\ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) + \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right) \subset \ell^\infty(X). \quad (5.2)$$

Since the E_t and the F_t are vector spaces, it is clear that $\{\mathbf{0}\} \subset \ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \cap \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right)$. If $\underline{x} \in \ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \cap \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right)$, then, for all $t \in \mathbb{N}$, we have $x_t \in E_t \cap F_t = \{0\}$, which implies that $x_t = 0$ for all $t \in \mathbb{N}$. And so we have proven

$$\ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \cap \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right) = \{\mathbf{0}\}. \quad (5.3)$$

If $\underline{u} \in \ell^\infty(X)$ then, for all $t \in \mathbb{N}$, there exists a unique $(x_t, y_t) \in E_t \times F_t$ such that $u_t = x_t + y_t$. Since, for all $t \in \mathbb{N}$, we have $\|x_t\| \leq \|\underline{u}\|_\infty$ and $\|y_t\| \leq \|\underline{u}\|_\infty$, we obtain

$$\ell^\infty(X) \subset \ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) + \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right). \quad (5.4)$$

From (5.2), (5.3) and (5.4), we obtain

$$\ell^\infty(X) = \ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \oplus^a \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right). \quad (5.5)$$

Let $(\underline{x}^k)_{k \in \mathbb{N}} \in \ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right)^\mathbb{N}$ be a sequence such that $\lim_{k \rightarrow +\infty} \|\underline{x}^k - \underline{z}\|_\infty = 0$, where $\underline{z} \in \ell^\infty(X)$. Hence, for all $t \in \mathbb{N}$, for all $k \in \mathbb{N}$, we have $0 \leq \|x_t^k - z_t\| \leq \|\underline{x}^k - \underline{z}\|_\infty$ which implies that $\lim_{k \rightarrow +\infty} x_t^k = z_t$. Since E_t is closed in X , we can say that $z_t \in E_t$, and consequently we have proven

$$\ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \text{ is closed in } \ell^\infty(X). \quad (5.6)$$

Using a similar reasoning we also have

$$\ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right) \text{ is closed in } \ell^\infty(X). \quad (5.7)$$

From (5.5), we know that the operator

$$\Sigma : \ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \times \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right) \rightarrow \ell^\infty(X),$$

defined by $\Sigma(\underline{x}, \underline{y}) := \underline{x} + \underline{y}$, is a linear bijection. From (5.6) and (5.7), we know that $\ell^\infty\left(\prod_{t \in \mathbb{N}} E_t\right) \times \ell^\infty\left(\prod_{t \in \mathbb{N}} F_t\right)$ is a Banach space. It is easy to see that Σ is continuous, and then, using the Banach Isomorphism Theorem, we obtain that Σ is a topological isomorphism, and the conclusion follows. \square

Proposition 5.2. *For all $t \in \mathbb{N}$, let $T_t \in \mathcal{L}(X, Y)$. We assume that the following condition is fulfilled: $\forall \underline{x} \in \ell^\infty(X)$, $(T_t(x_t))_{t \in \mathbb{N}} \in \ell^\infty(Y)$.*

This assumption permits us to define the linear operator $\mathcal{T} : \ell^\infty(X) \rightarrow \ell^\infty(Y)$ by setting $\mathcal{T}(\underline{x}) := (T_t(x_t))_{t \in \mathbb{N}}$. Then the following assertions hold.

- (i) $Im\mathcal{T} = \ell^\infty\left(\prod_{t \in \mathbb{N}} ImT_t\right)$.
- (ii) If, $\forall t \in \mathbb{N}$, ImT_t is closed in Y , then $Im\mathcal{T}$ is closed in $\ell^\infty(Y)$.
- (iii) If, $\forall t \in \mathbb{N}$, ImT_t is topologically complemented in Y , then $Im\mathcal{T}$ is topologically complemented in $\ell^\infty(Y)$.
- (iv) If, $\forall t \in \mathbb{N}$, $KerT_t$ is topologically complemented in X , then $Ker\mathcal{T}$ is topologically complemented in $\ell^\infty(X)$.
- (v) If $\sup_{t \in \mathbb{N}} \|T_t\|_{\mathcal{L}} < +\infty$ then \mathcal{T} is continuous from $\ell^\infty(X)$ into $\ell^\infty(Y)$.

Proof. (i) Note that we have

$$\begin{aligned} \underline{z} \in \text{Im}\mathcal{T} &\iff (\exists \underline{x} \in \ell^\infty(X) \text{ s.t. } \mathcal{T}(\underline{x}) = \underline{z}) \\ &\iff (\exists \underline{x} \in \ell^\infty(X) \text{ s.t. } \forall t \in \mathbb{N}, z_t = T_t(x_t)) \\ &\iff \underline{z} \in \ell^\infty(\prod_{t \in \mathbb{N}} \text{Im}T_t). \end{aligned}$$

(ii) Let $(\underline{x}^k)_{k \in \mathbb{N}} \in (\text{Im}\mathcal{T})^\mathbb{N}$ such that $\lim_{k \rightarrow +\infty} \|\underline{x}^k - \underline{w}\|_\infty = 0$, where $\underline{w} \in \ell^\infty(Y)$. Hence, for all $t \in \mathbb{N}$, we have $0 \leq \|x_t^k - w_t\| \leq \|\underline{x}^k - \underline{w}\|_\infty$ which implies that $w_t \in \text{Im}T_t$ since $\text{Im}T_t$ is closed in Y . Consequently we have $\underline{w} \in \ell^\infty(\prod_{t \in \mathbb{N}} \text{Im}T_t)$.

(iii) From the assumption we know that, for all $t \in \mathbb{N}$, there exists a closed vector subspace F_t of Y such that $\text{Im}T_t \oplus F_t = Y$. Using Lemma 5.1, we obtain $\ell^\infty(\prod_{t \in \mathbb{N}} \text{Im}T_t) \oplus \ell^\infty(\prod_{t \in \mathbb{N}} F_t) = \ell^\infty(Y)$. Using (i) we obtain the announced assertion.

(iv) First, when $\underline{v} \in \ell_\infty(X)$, we have

$$\underline{v} \in \text{Ker}\mathcal{T} \iff \mathcal{T}(\underline{v}) = \underline{0} \iff (\forall t \in \mathbb{N}, T_t(v_t) = 0) \iff (\forall t \in \mathbb{N}, v_t \in \text{Ker}T_t),$$

and since \underline{v} is bounded, we obtain

$$\text{Ker}\mathcal{T} = \ell^\infty(\prod_{t \in \mathbb{N}} \text{Ker}T_t). \quad (5.8)$$

For all $t \in \mathbb{N}$, there exists E_t , a closed vector subspace of X , such that $\text{Ker}T_t \oplus E_t = X$. Using 5.8 and Lemma 5.1, we obtain

$$\ell^\infty(X) = \text{Ker}\mathcal{T} \oplus \ell^\infty(E). \quad (5.9)$$

and assertion (iv) is proven.

(v) When $\underline{x} \in \ell^\infty(X)$, we have

$$\|\mathcal{T}(\underline{x})\|_\infty = \sup_{t \in \mathbb{N}} \|T_t(x_t)\| \leq \sup_{t \in \mathbb{N}} (\|T_t\|_{\mathcal{L}} \|x_t\|) \leq (\sup_{t \in \mathbb{N}} \|T_t\|_{\mathcal{L}}) \|\underline{x}\|_\infty.$$

□

Lemma 5.3. *Let $A \in \mathcal{O}(X)$. Then the following assertions hold.*

- (i) $\text{Int}\ell^\infty(A) = \{\underline{x} \in \ell^\infty(A) : \inf_{t \in \mathbb{N}} d(x_t, A^c) > 0\}$, where $d(x_t, A^c) := \inf\{\|x_t - z\| : z \in A^c\}$.
- (ii) let $\underline{x} \in \text{Int}\ell^\infty(A)$. We set $r := \inf_{t \in \mathbb{N}} d(x_t, A^c) > 0$. Then we have $\prod_{t \in \mathbb{N}} B(x_t, \frac{r}{2}) \subset \text{Int}\ell^\infty(A)$.

Proof. Assertion (i) is proven in [5] (Lemma A.1.1). To prove (ii), if $\underline{u} \in \prod_{t \in \mathbb{N}} B(x_t, \frac{r}{2})$, then we have $\|u_t - x_t\| < \frac{r}{2}$, and for all $v \in A^c$, we have

$$r < \|x_t - v\| \leq \|x_t - u_t\| + \|u_t - v\| \leq \frac{r}{2} + \|u_t - v\| \implies r - \frac{r}{2} \leq \|u_t - v\|$$

which implies $d(u_t, A^c) \geq \frac{r}{2} > 0$, and so $\underline{u} \in \text{Int}\ell^\infty(A)$. □

Definition 5.4. *Let $A \in \mathcal{O}(A)$ and $g_t : A \rightarrow Y$ be a mapping for all $t \in \mathbb{N}$. The sequence $(g_t)_{t \in \mathbb{N}}$ is said uniformly equicontinuous on the bounded subsets of A when*

$$\begin{cases} \forall B \in \mathbb{B}(A), \forall \epsilon > 0, \exists \eta_{B, \epsilon} > 0, \forall u, v \in B, \\ \|u - v\| \leq \eta_{B, \epsilon} \implies (\forall t \in \mathbb{N}, \|g_t(u) - g_t(v)\| \leq \epsilon). \end{cases}$$

Proposition 5.5. *Let $A \in \mathcal{O}(A)$ and $g_t : A \rightarrow Y$ be a mapping for all $t \in \mathbb{N}$. We assume that the following condition is fulfilled: $\forall \underline{x} \in \ell^\infty(A), (g_t(x_t))_{t \in \mathbb{N}} \in \ell^\infty(Y)$. This condition permits to define the operator $G : \ell^\infty(A) \rightarrow \ell^\infty(Y)$ by setting $G(\underline{x}) := (g_t(x_t))_{t \in \mathbb{N}}$.*

If, in addition we assume that $(g_t)_{t \in \mathbb{N}}$ is uniformly equicontinuous in the bounded subsets of A , then G is continuous from $\text{Int}\ell^\infty(A)$ into $\ell^\infty(Y)$.

Proof. We arbitrarily fix $\underline{x} \in \text{Int}\ell^\infty(A)$. Hence there exists $r > 0$ such that $B_\infty(\underline{x}, r) \subset \ell^\infty(A)$.

The set $B := \{u \in A : \exists t_u \in \mathbb{N}, \|u - x_{t_u}\| < r\}$ is bounded since, when $u \in B$, $\|u\| \leq \|u - x_{t_u}\| + \|x_{t_u}\| \leq r + \|\underline{x}\|_\infty < +\infty$. We arbitrarily fix $\epsilon > 0$. Let $\underline{z} \in \text{Int}\ell^\infty(A)$ such that $\|\underline{z} - \underline{x}\|_\infty < \min\{r, \eta_{B, \epsilon}\}$ where $\eta_{B, \epsilon}$ is provided by Definition 5.4. Hence we have $z_t \in B$ since $\|z_t - x_t\| < r$ for all $t \in \mathbb{N}$, and we have $\|z_t - x_t\| < \eta_{B, \epsilon}$ which implies $\|g_t(z_t) - g_t(x_t)\| < \epsilon$. Hence we have $\|G(\underline{z}) - G(\underline{x})\|_\infty \leq \epsilon$. We have proven that G is continuous at \underline{x} . \square

Proposition 5.6. *let $A \in \mathcal{O}(A)$ and $(g_t)_{t \in \mathbb{N}} \in C^1(A, Y)^\mathbb{N}$. We assume that the following conditions are fulfilled.*

- (a) $\forall \underline{x} \in \text{Int}\ell^\infty(A)$, $(g_t(x_t))_{t \in \mathbb{N}} \in \ell^\infty(Y)$.
- (b) $\forall \underline{x} \in \text{Int}\ell^\infty(A)$, $(Dg_t(x_t))_{t \in \mathbb{N}} \in \ell^\infty(\mathcal{L}(X, Y))$.
- (c) $(Dg_t)_{t \in \mathbb{N}}$ is uniformly equicontinuous on the bounded subsets of A .

We consider the operator $G : \text{Int}\ell^\infty(A) \rightarrow \ell^\infty(Y)$ defined by $G(\underline{x}) := (g_t(x_t))_{t \in \mathbb{N}}$. Then the following assertions hold for all $\underline{x} \in \text{Int}\ell^\infty(A)$.

- (i) G is Fréchet differentiable at \underline{x} , and, for all $\underline{v} \in \ell^\infty(X)$, we have $DG(\underline{x})\underline{v} = (Dg_t(x_t)v_t)_{t \in \mathbb{N}}$.
- (ii) $G \in C^1(\text{Int}\ell^\infty(A), \ell^\infty(Y))$.
- (iii) $\text{Im}Dg(\underline{x}) = \ell^\infty(\prod_{t \in \mathbb{N}} \text{Im}Dg_t(x_t))$.
- (iv) If, for all $t \in \mathbb{N}$, $\text{Im}Dg_t(x_t)$ is closed in Y , then $\text{Im}DG(\underline{x})$ is closed in $\ell^\infty(Y)$.
- (v) If, for all $t \in \mathbb{N}$, $\text{Im}Dg_t(x_t)$ is topologically complemented in Y , then $\text{Im}DG(\underline{x})$ is topologically complemented in $\ell^\infty(Y)$.

Proof. (i) We arbitrarily fix $\underline{x} \in \text{Int}\ell^\infty(A)$. Using Proposition 5.2, the linear operator $\mathcal{T} : \ell^\infty(X) \rightarrow \ell^\infty(Y)$, defined by $\mathcal{T}\underline{h} := (Dg_t(x_t)h_t)_{t \in \mathbb{N}}$, is well-defined and continuous.

We can use Lemma 5.3, ii, and assert that $\prod_{t \in \mathbb{N}} B(x_t, \frac{r}{2}) \subset \text{Int}\ell^\infty(A)$. Using the set $B := \{u \in A : \exists t_u \in \mathbb{N}, \|u - x_{t_u}\| < r\}$, we have yet seen that B is bounded in A and we have $B(x_t, \frac{r}{2}) \subset B$.

We arbitrarily fix $\epsilon > 0$, and, using assumption (c), we consider $\eta_{B, \epsilon} > 0$ provided by Definition 5.4 for the sequence $(Dg_t)_{t \in \mathbb{N}}$.

We arbitrarily fix $\underline{h} \in \ell^\infty(X)$ such that $\|\underline{h}\|_\infty < \min\{\frac{r}{2}, \eta_{B, \epsilon}\}$. Hence, $\forall t \in \mathbb{N}$, $\forall z_t \in [x_t, x_t + h_t] \subset B(x_t, \frac{r}{2})$, we have $\|z_t - x_t\| < \eta_{B, \epsilon}$ which implies $\|Dg_t(z_t) - Dg_t(x_t)\|_{\mathcal{L}} < \epsilon$.

Now using the Mean Value Inequality as established in [2] (Corollary 1, p. 144), we obtain, for all $t \in \mathbb{N}$,

$$\|g_t(x_t + h_t) - g_t(x_t) - Dg_t(x_t)h_t\| \leq \sup_{z_t \in [x_t, x_t + h_t]} \|Dg_t(z_t) - Dg_t(x_t)\|_{\mathcal{L}} \|h_t\| \leq \epsilon \|h_t\|, \text{ and taking the sup on the } t \in \mathbb{N}, \text{ we obtain}$$

$$\|G(\underline{x} + \underline{h}) - G(\underline{x}) - \mathcal{T}\underline{h}\|_\infty \leq \epsilon \|\underline{h}\|_\infty.$$

Hence we have proven that G is Fréchet differentiable at \underline{x} and that $DG(\underline{x})\underline{h} = (Dg_t(x_t)h_t)_{t \in \mathbb{N}}$.

Using the operator $\mathcal{T}\underline{h} := (Dg_t(x_t)h_t)_{t \in \mathbb{N}}$, assertions (ii)-(v) are straightforward consequences of Proposition 5.2. \square

6. A VARIATIONAL PROBLEM

In this section we consider a maximization problem in infinite horizon and in discrete time under holonomic constraints. The unknown variable is a bounded sequence with values in \mathbb{R}^n . Using the results of the previous sections we obtain a first-order necessary optimality condition in the form of a nonhomogeneous Euler-Lagrange equation.

Let A be a nonempty subset of \mathbb{R}^n . For all $t \in \mathbb{N}$, let $u_t : A \times A \rightarrow \mathbb{R}$ and for all $t \in \mathbb{N}_*$ let $g_t : A \rightarrow \mathbb{R}^k$ be functions. We fix a vector $\sigma \in A$, a real number $\beta \in (0, 1)$ and we consider the following variational problem under holonomic constraints.

$$(\mathcal{V}) \begin{cases} \text{Maximize} & J(\underline{x}) := \sum_{t=0}^{+\infty} \beta^t u_t(x_t, x_{t+1}) \\ \text{when} & \underline{x} \in \ell^\infty(A), x_0 = \sigma \\ \text{and when} & \forall t \in \mathbb{N}_*, g_t(x_t) = 0. \end{cases}$$

Theorem 6.1. *Let $\hat{\underline{x}}$ be a solution of (\mathcal{V}) . We assume that the following conditions are fulfilled.*

- (A1) $(\hat{x}_t, \hat{x}_{t+1})_{t \in \mathbb{N}} \in \text{Int}\ell^\infty(A \times A)$.
- (A2) $\forall t \in \mathbb{N}, u_t \in C^1(A \times A, \mathbb{R})$.
- (A3) $\forall (\underline{x}, \underline{y}) \in \text{Int}\ell^\infty(A \times A), (u_t(x_t, y_t))_{t \in \mathbb{N}} \in \ell^\infty(\mathbb{R})$.
- [A4] $\forall (\underline{x}, \underline{y}) \in \text{Int}\ell^\infty(A \times A), (Du_t(x_t, y_t))_{t \in \mathbb{N}} \in \ell^\infty((\mathbb{R}^n \times \mathbb{R}^n)^*)$.
- (A5) $(Du_t)_{t \in \mathbb{N}}$ is uniformly equicontinuous on the bounded subsets of $A \times A$.
- (A6) $\hat{\underline{x}} \in \text{Int}\ell^\infty(A)$.
- (A7) $\forall t \in \mathbb{N}, g_t \in C^1(A, \mathbb{R}^k)$.
- (A8) $\forall \underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A), (g_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{R}^k)$.
- (A9) $\forall \underline{x} \in \text{Int}\ell^\infty(\mathbb{N}_*, A), (Dg_t(x_t))_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k))$.
- (A10) $(Dg_t)_{t \in \mathbb{N}_*}$ is uniformly equicontinuous on the bounded subsets of A .
- (A11) $\exists r > 0, \forall t \in \mathbb{N}_*, \exists F_t$ vector subspace of $\mathbb{R}^n, \forall \underline{x} \in B_\infty(\hat{\underline{x}}, r), \forall t \in \mathbb{N}_*, \text{Im} Dg_t(x_t) \cap F_t = \{0\}$.

Then there exists a sequence $(p_t)_{t \in \mathbb{N}_*} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{k*})$ which satisfies, for all $t \in \mathbb{N}_*$, the following equality

$$\beta^t D_1 u_t(\hat{x}_t, \hat{x}_{t+1}) + \beta^{t-1} D_2 u_t(\hat{x}_{t-1}, \hat{x}_t) = p_t \circ Dg_t(\hat{x}_t).$$

Proof. We consider the operator $\Delta : \ell^\infty(\mathbb{R}^n) \rightarrow \ell^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ defined by, for all $\underline{x} \in \ell^\infty(\mathbb{R}^n)$,

$$\Delta(\underline{x}) := ((x_t, x_{t+1}))_{t \in \mathbb{N}}. \quad (6.1)$$

Δ is linear continuous and consequently, it is of class C^1 , and we have, for all $\underline{x}, \underline{v} \in \ell^\infty(\mathbb{R}^n)$,

$$D\Delta(\underline{x})\underline{v} = ((v_t, v_{t+1}))_{t \in \mathbb{N}}. \quad (6.2)$$

Under (A2, A3, A4, A5), using Proposition 5.6, we can assert that the operator $U : \text{Int}\ell^\infty(A \times A) \rightarrow \ell^\infty(\mathbb{R})$, defined by

$$U(\underline{x}, \underline{y}) := (u_t(x_t, y_t))_{t \in \mathbb{N}}, \quad (6.3)$$

is of class C^1 on $\text{Int}\ell^\infty(A \times A)$, and for all $(\underline{x}, \underline{y}) \in \text{Int}\ell^\infty(A \times A)$, for all $(\underline{v}, \underline{w}) \in \ell^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we have

$$DU(\underline{x}, \underline{y})(\underline{v}, \underline{w}) = (Du_t(x_t, y_t)(v_t, w_t))_{t \in \mathbb{N}}. \quad (6.4)$$

The operator $S : \ell^\infty(\mathbb{R}) \rightarrow \mathbb{R}$, defined by

$$S(\underline{r}) := \sum_{t=0}^{+\infty} \beta^t r_t, \quad (6.5)$$

is linear continuous since, setting $\underline{\beta} := (\beta^t)_{t \in \mathbb{N}} \in \ell_1(\mathbb{R})$, we have $|S(\underline{r})| \leq \|\beta\|_1 \cdot \|\underline{r}\|_\infty$. Hence S is of class C^1 , and we have, for all $\underline{r}, \underline{s} \in \ell_\infty(\mathbb{R})$,

$$DS(\underline{r})\underline{s} = S(\underline{s}) = \sum_{t=0}^{+\infty} \beta^t s_t. \quad (6.6)$$

Note that $J = S \circ U \circ \Delta$ is of class C^1 as a composition of three mappings of class C^1 . Using (6.2), (6.4), (6.6) and the chain rule, for all $\underline{x} \in \text{Int}\ell^\infty(A)$ and for all $\underline{h} \in \ell^\infty(\mathbb{R}^n)$, we obtain

$$\begin{aligned} DJ(\underline{x})\underline{h} &= DS(U \circ \Delta(\underline{x}) \circ DU(\Delta(\underline{x}) \circ D\Delta(\underline{x})\underline{h})) \\ &= S(DU(\Delta(\underline{x})\Delta(\underline{h}))) \\ &= \sum_{t=0}^{+\infty} \beta^t (Du_t(x_t, x_{t+1})(h_t, h_{t+1})) \implies \\ DJ(\underline{x})\underline{h} &= \sum_{t=0}^{+\infty} \beta^t (D_1u_t(x_t, x_{t+1})h_t + D_2u_t(x_t, x_{t+1})h_{t+1}). \end{aligned} \quad (6.7)$$

Doing a change of variable on the second term of the previous expression, we can also write this differential as follows.

$$DJ(\underline{x})\underline{h} = \left. \begin{aligned} &D_1u_0(x_0, x_1)h_0 + \\ &\sum_{t=1}^{+\infty} (\beta^t D_1u_t(x_t, x_{t+1}) + \beta^{t-1} D_2u_{t-1}(x_{t-1}, x_t))h_t. \end{aligned} \right\} \quad (6.8)$$

We introduce the operator $G_1 : \text{Int}\ell^\infty(\mathbb{N}_*, A) \rightarrow \ell^\infty(\mathbb{N}_*, \mathbb{R}^k)$ by setting, for all $(x_t)_{t \in \mathbb{N}_*} \in \text{Int}\ell^\infty(\mathbb{N}_*, A)$,

$$G_1((x_t)_{t \in \mathbb{N}_*}) := (g_t(x_t))_{t \in \mathbb{N}_*}. \quad (6.9)$$

We introduce the mapping $g_0 : A \rightarrow \mathbb{R}^n$ defined, for all $x \in \mathbb{R}^n$, by

$$g_0(x) := x - \sigma. \quad (6.10)$$

Note that g_0 is affine on a finite dimensional space which implies that g_0 is of class C^1 and

$$Dg_0 = id_{\mathbb{R}^n}. \quad (6.11)$$

We introduce the operator $G : \text{Int}\ell^\infty(A) \rightarrow \mathbb{R}^n \times \ell^\infty(\mathbb{N}_*, \mathbb{R}^k)$ by setting, for all $\underline{x} \in \text{Int}\ell^\infty(A)$,

$$G(\underline{x}) := (g_0(x_0), G_1((x_t)_{t \in \mathbb{N}_*})). \quad (6.12)$$

Using (A7, A8, A9, A10) and Proposition 5.6 we know that G_1 is of class C^1 on $\text{Int}\ell^\infty(\mathbb{N}_*, \mathbb{R}^k)$, and so since g_0 is also of class C^1 , we can assert that G is of class C^1 , and we have, for all $\underline{x} \in \text{Int}\ell^\infty(A)$ and for all $\underline{v} \in \ell^\infty(\mathbb{R}^n)$,

$$DG(\underline{x})\underline{v} = (v_0, (Dg_t(x_t)v_t)_{t \in \mathbb{N}_*}). \quad (6.13)$$

Using Proposition 5.2, (iv), since $\text{Ker}Dg_0(\hat{x}_0) = \{0\}$ and since, for all $t \in \mathbb{N}_*$, there exists E_t , a vector subspace of \mathbb{R}^n , such that $\text{Ker}Dg_t(\hat{x}_t) \oplus E_t = \mathbb{R}^n$, by setting $E := \{0\} \times \ell^\infty(\prod_{t \in \mathbb{N}_*} E_t)$, we obtain

$$\text{Ker}DG(\hat{\underline{x}}) \oplus E = \ell^\infty(\mathbb{R}^n). \quad (6.14)$$

Using Proposition 5.2, (iii), and (A11), and, setting $F := \{0\} \times (\prod_{t \in \mathbb{N}_*} F_t)$, we obtain

$$\text{Im}DG(\hat{\underline{x}}) \oplus \ell^\infty(F) = \mathbb{R}^n \times \ell^\infty(\mathbb{N}_*, \mathbb{R}^k). \quad (6.15)$$

From (A11), when $\underline{x} \in B_\infty(\hat{\underline{x}}, r)$, if $\underline{v} \in \text{Im}DG(\underline{x}) \cap \ell^\infty(F)$, we have $v_0 = 0$ and, for all $t \in \mathbb{N}_*$, $v_t \in Dg_t(x_t) \cap F_t = \{0\}$ which implies $\underline{v} = \underline{0}$. Hence we have proven

$$\forall \underline{x} \in B_\infty(\hat{\underline{x}}, r), \quad \text{Im}DG(\underline{x}) \cap \ell^\infty(F) = \{\underline{0}\}. \quad (6.16)$$

Note that $\hat{\underline{x}}$ is a solution of the following problem

$$\begin{cases} \text{Maximize} & J(\underline{x}) \\ \text{when} & G(\underline{x}) = \underline{0}. \end{cases}$$

Using the Fréchet differentiability of J and the continuous Fréchet differentiability of G and (6.15), (6.16), we see that the assumptions of Theorem 4.1 are fulfilled, and so we can use it and assert that there exists $M \in (\mathbb{R}^n \times \ell^\infty(\mathbb{N}_*, \mathbb{R}^k))^*$ such that

$$DJ(\hat{\underline{x}}) = M \circ DG(\hat{\underline{x}}). \quad (6.17)$$

Note that $M = \mu_0 \otimes \Lambda$ with $\mu_0 \in \mathbb{R}^{n*}$ and $\Lambda \in \ell^\infty(\mathbb{N}_*, \mathbb{R}^k)^*$. Using [3] (Chapter 16, sections 16.7 and 16.8) and [6] (Lemma 3.5), we know that $\Lambda = \underline{p} \oplus \Theta$ where $\underline{p} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{k*})$ and there exists $\xi \in \mathbb{R}^{k*}$ such that, for all $(x_t)_{t \in \mathbb{N}_*} \in c(\mathbb{N}_*, \mathbb{R}^k)$, $\Theta((x_t)_{t \in \mathbb{N}_*}) = \xi(\lim_{t \rightarrow +\infty} x_t)$.

Using (6.8) and (6.13) we can write (6.17) in the following form, for all $\underline{v} \in \ell_\infty(\mathbb{R}^n)$,

$$\left. \begin{aligned} & D_1 u_0(\hat{x}_0, \hat{x}_1) v_0 + \\ & \sum_{t=1}^{+\infty} (\beta^t D_1 u_t(\hat{x}_t, \hat{x}_{t+1}) + \beta^{t-1} D_2 u_{t-1}(\hat{x}_{t-1}, \hat{x}_t)) v_t \\ & = \mu_0(v_0) + \sum_{t=1}^{+\infty} p_t \circ Dg_t(\hat{x}_t) v_t + \Theta((v_t)_{t \in \mathbb{N}_*}). \end{aligned} \right\}$$

When $v_t = 0$ for all $t \geq 1$, we obtain $D_1 u_0(\hat{x}_0, \hat{x}_1) = \mu_0$, and consequently, we obtain, for all $(v_t)_{t \in \mathbb{N}_*} \in \ell^\infty(\mathbb{N}_*, \mathbb{R}^n)$,

$$\left. \begin{aligned} & \sum_{t=1}^{+\infty} (\beta^t D_1 u_t(\hat{x}_t, \hat{x}_{t+1}) + \beta^{t-1} D_2 u_{t-1}(\hat{x}_{t-1}, \hat{x}_t) - p_t \circ Dg_t(\hat{x}_t)) v_t \\ & = \Theta((v_t)_{t \in \mathbb{N}_*}). \end{aligned} \right\} \quad (6.18)$$

From $z \in \mathbb{R}^n$, we built \underline{v}^z by setting $v_s^z := z$ if $s = t$, and $v_s^z := 0$ if $s \neq t$. Hence we obtain $\Theta(\underline{v}^z) = \xi(0) = 0$, and so from (6.18) we obtain

$$(\beta^t D_1 u_t(\hat{x}_t, \hat{x}_{t+1}) + \beta^{t-1} D_2 u_{t-1}(\hat{x}_{t-1}, \hat{x}_t) - p_t \circ Dg_t(\hat{x}_t)) z = 0$$

which implies

$$\beta^t D_1 u_t(\hat{x}_t, \hat{x}_{t+1}) + \beta^{t-1} D_2 u_{t-1}(\hat{x}_{t-1}, \hat{x}_t) = p_t \circ Dg_t(\hat{x}_t)$$

which is the announced formula. \square

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