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# Proper base change over henselian pairs 

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#### Abstract

We discuss a question which appears in [Séminaire de Géométrie Algébrique du Bous Marie- Théorie des topos et cohomologie étale des schémas, Exposé XII, Remarks 6.13] concerning proper base change. In particular, we propose a solution in a particular non-affine case.


## 1 Introduction

The question we would like to answer is the following one ${ }^{2}$ :
Question 1. Let $\left(X, X_{0}\right)$ be an henselian couple (in the sense of Definition 1). Is it true that (ii) and (iii) in [2, Exposé XII, Proposition 6.5] hold with $\mathbb{L}=\mathbb{P}$ and for every $n$ ?

This question appears in [2, Exposé XII, Remarks 6.13].
We can restate it as follows:
Question 2. Let $\left(X, X_{0}\right)$ be an henselian couple. Is it true that
a. the base change functor induces an equivalence between the category of étale coverings of $X$ and the category of étale coverings of $X_{0}$ ?
b. for any torsion étale sheaf $\mathscr{F}$ and for any integer $n$, the morphism

$$
H^{n}(X, \mathscr{F}) \longrightarrow H^{n}\left(X_{0}, \mathscr{F}_{\mid X_{0}}\right)
$$

is an isomorphism?

[^0]Remark 1. 1. When $X$ is proper and finitely presented over an henselian $\operatorname{ring}(A, m)$ and $X_{0}=X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / m)$, we know that the answer to Question 1 is affirmative. This is the proper base change theorem in étale cohomology.
2. The case $\left(X, X_{0}\right)=(\operatorname{Spec}(A), \operatorname{Spec}(A / I))^{3}$, for $(A, I)$ an henselian pair was studied an solved by R. Elkik in [4] and by O. Gabber in [5].

We propose a solution in the following situation:
$(\dagger) \quad$ Let $X$ be proper over a noetherian affine scheme $\operatorname{Spec}(A)$ and $X_{0}=$ $X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / I)$ for some ideal $I \subseteq A$.

We will see that, under these assumptions, $\left(X, X_{0}\right)$ is an henselian couple for which Question 1 has a positive answer. To achieve this, we will first generalize [1, Theorem 3.1] to the following form:

Theorem 1. Let $(A, I)$ be an henselian pair. Let $S=\operatorname{Spec}(A)$ and let $f$ : $X \longrightarrow S$ be a proper finitely presented morphism. Let $X_{0}=X \times_{S} S_{0}$, where $S_{0}=\operatorname{Spec}(A / I)$. Then

$$
\begin{gathered}
E^{\prime} t_{f}(X) \longrightarrow \overline{E ́ t}_{f}\left(X_{0}\right) \\
Z \mapsto Z \times_{S} S_{0}
\end{gathered}
$$

is an equivalence of categories.
Here $E t_{f}(W)$ denotes the category of finite étale schemes over $W$. The key tools for the proof are Artin's approximation theory and [12, Tag 0AH5], which combined with [1, Corollary 1.8] yields the following theorem
Theorem 2. Let $(A, I)$ be an henselian pair with $A$ noetherian. Let $\hat{A}$ be the I-adic completion of $A$ and assume that one of the following hypothesis is satisfied:

1. $A \longrightarrow \hat{A}$ is a regular ring map;
2. $A$ is a G-ring;
3. $(A, I)$ is the henselization ${ }^{4}$ of a pair $(B, J)$, where $B$ is a noetherian $G$ ring.
Let $\mathscr{F}$ be a functor which is locally of finite presentation ${ }^{5}$

$$
\text { A-algebras } \longrightarrow \text { Sets }
$$

Given any $\hat{\xi} \in \mathscr{F}(\hat{A})$ and any $N \in \mathbb{N}$, there exists an element $\xi \in \mathscr{F}(A)$ such that

$$
\xi \equiv \hat{\xi} \bmod I^{N}
$$

i.e. $\xi$ and $\hat{\xi}$ have the same image in $\mathscr{F}\left(A / I^{N}\right) \cong \mathscr{F}\left(\hat{A} / \hat{I}^{N}\right)$

[^1]Remark 2. In Theorem 2 we have that $3 . \Rightarrow$ 2. $\Rightarrow$ 1. See [12, Tag 0AH5].
Remark 3. Theorem 1, joined with [5, Corollary 1], gives us a positive answer to Question 1 when $\left(X, X_{0}\right)$ is proper and finitely presented over an henselian pair. Moreover, we will see how we can always reduce to this case from situation ( $\dagger$ ).

## 2 Proof of Theorem 1

This proof is an adaption of the one given in the local case by Artin (see [1, Theorem 3.1]). This generalization is possible thanks to Popescu's characterization of regular morphisms between noetherian rings, which provides us Theorem 2 as a corollary.
First we reduce to the case where $A$ is the henselization of a finitely presented $\mathbb{Z}$-algebra. in order to do this, we need the following two preliminary lemmas.

Lemma 1. Let $S=\operatorname{Spec}(A)$ and let $g: X \longrightarrow S$ be a proper morphism of finite presentation. Then the functor

$$
\mathscr{F}: \text { A-Algebras } \longrightarrow \text { Sets }
$$

$$
B \mapsto\left\{\text { finite étale coverings of } \operatorname{Spec}(B) \times_{S} X\right\} / \text { isomorphism }
$$

is locally of finite presentation.
Proof. See the beginning of the proof of [1, Theorem 3.1].
Lemma 2. Let $S=\operatorname{Spec}(A)$ and let $g: X \longrightarrow S$ be a proper morphism of finite presentation. Let $Z_{1} \rightarrow X$ and $Z_{2} \rightarrow X$ be two finite étale covers of $X$. Then the functor

$$
\begin{gathered}
\mathscr{G}: A \text {-algebras } \longrightarrow \text { Sets } \\
B \mapsto \operatorname{Hom}_{X \times_{S} \operatorname{Spec}(B)}\left(Z_{1} \times_{S} \operatorname{Spec}(B), Z_{2} \times_{S} \operatorname{Spec}(B)\right)
\end{gathered}
$$

is locally of finite presentation.
Proof. The lemma is a straightforward consequence of [7, Theorem 8.8.2.(i)].
Let $(A, I)$ be an henselian pair and write $A$ as a direct $\operatorname{limit} \underset{\rightarrow}{\lim } A_{i}$, where each $A_{i}$ is a subalgebra of $A$ that is finitely generated over $\mathbb{Z}$. Let $\left(\overrightarrow{A_{i}^{h}},\left(I \cap A_{i}\right)^{h}\right)$ be the henselization of $\left(A_{i},\left(I \cap A_{i}\right)\right)$ for each $i$. Then by [11, Chapter XI, Proposition 2] $\underset{\longrightarrow}{\lim }\left(A_{i}^{h},\left(I \cap A_{i}\right)^{h}\right)$ is an henselian pair. It is easy to see that

$$
(A, I)=\underset{\longrightarrow}{\lim }\left(A_{i}^{h},\left(I \cap A_{i}\right)^{h}\right)
$$

Write $S_{i}=\operatorname{Spec}\left(A_{i}^{h}\right)$ for every index $i$. Then

$$
S=\underset{\swarrow}{\lim } S_{i}
$$

By [7, Thereom 8.8.2. (ii)] we know that $X$ comes from a finitely presented scheme $X_{i_{0}}$ for some index $i_{0}$, i.e. $X \cong X_{i_{0}} \times{ }_{S_{i_{0}}} S$. Moreover, by [7, Theorem 8.10.5], we can assume that $X_{i_{0}}$ is also proper over $S_{i_{0}}$. As the functor

$$
\mathscr{F}: A_{i_{0}}^{h}-\text { Algebras } \longrightarrow \text { Sets }
$$

$B \mapsto\left\{\right.$ finite étale coverings of $\left.\operatorname{Spec}(B) \times{ }_{S_{i_{0}}} X_{i_{0}}\right\} /$ isomorphism
is locally of finite presentation, we have that

$$
\mathscr{F}(A)=\underset{\longrightarrow}{\lim } \mathscr{F}\left(A_{i}^{h}\right)
$$

Therefore, every finite étale cover of $X$ comes from a finite étale cover of $X_{i}=$ $S_{i} \times{ }_{S_{i}} X_{i_{0}}$ for a suitable index $i$.

Remark 4. All schemes $X_{i_{0}} \times{ }_{S_{i_{0}}} S_{i}$ and $X \cong X_{i_{0}} \times{ }_{S_{i_{0}}} S$ are quasi-compact and quasi-separated, as they are proper over affine schemes.

Let $Z \rightarrow X$ and $W \rightarrow X$ be two finite étale covers of $X$. Then we can assume without loss of generality that they come from two finite étale covers $Z_{i_{0}} \rightarrow X_{i_{0}}, W_{i_{0}} \rightarrow X_{i_{0}}$. Then by Lemma 2 we see that

$$
\underset{\longrightarrow}{\lim } \operatorname{Hom}_{X_{i}}\left(Z_{i}, W_{i}\right)=\operatorname{Hom}_{X}(Z, W)
$$

It is then clear that we can reduce the proof of Theorem 1 to the case where $(A, I)$ is the henselization of a pair $(B, J)$, where $B$ is finitely generated over $\mathbb{Z}$. In particular, $B$ is a G-ring and Theorem 2 holds.

Lemma 3. The functor in Theorem 1 is essentially surjective.
Proof. Consider a finite étale morphism $X_{0}^{\prime} \longrightarrow X_{0}$. Label $\hat{A}$ the completion of $A$ with respect to the ideal $I$ and let $\hat{S}=\operatorname{Spec}(\hat{A}), \hat{X}=X \times_{S} \hat{S}$. Notice that $\hat{A}$ is a complete separated ring by Krull's theorem (see [3, Theorem 10.17]). By [8, Theorem 18.3.4], we have that the functor

$$
\begin{gathered}
E t_{f}(\hat{X}) \longrightarrow \dot{E} t_{f}\left(X_{0}\right) \\
Z \mapsto Z \times_{S} S_{0}
\end{gathered}
$$

is an equivalence of categories. Then there exists some $\left[\hat{X}^{\prime} \longrightarrow \hat{X}\right] \in \mathscr{F}(\hat{A})$ such that

$$
\hat{X}^{\prime} \times_{\hat{S}} S_{0} \cong X_{0}^{\prime}
$$

By Theorem 2 we get that there exists some finite étale morphism $X^{\prime} \longrightarrow X$ which is congruent modulo $I$ to $\hat{X}^{\prime} \longrightarrow \hat{X}$, i.e.

$$
X^{\prime} \times_{S} S_{0} \cong X_{0}^{\prime}
$$

It remains only to show that the functor in Theorem 1 is fully faithful.

Lemma 4. The functor in Theorem 1 is fully faithful.
Proof. Let $X^{\prime}$ and $X^{\prime \prime}$ be two finite étale schemes over $X$ and let $\phi \in \operatorname{Hom}_{X}\left(X^{\prime}, X^{\prime \prime}\right)$. The morphism $\phi$ corresponds uniquely to its graph $\Gamma_{\phi}: X^{\prime} \longrightarrow X^{\prime} \times_{X} X^{\prime \prime}$, which is an open immersion as both $X^{\prime}$ and $X^{\prime \prime}$ are of finite type over $X$ and as $X^{\prime \prime}$ is étale over $X$ (see [9, Corollaire 3.4]). Also notice that $\Gamma_{\phi}$ is a closed immersion (see [10, Exercise 3.3.10]). If we assume that $X^{\prime}$ is connected and nonempty, $\phi$ corresponds uniquely to a connected component of $X^{\prime} \times_{X} X^{\prime \prime}$ of degree one over $X^{\prime}$. The degree of such a component can be measured at any point of $X^{\prime}$. We conclude therefore by applying the next lemma to a component of $X^{\prime} \times{ }_{X} X^{\prime \prime}$.
Lemma 5. $X$ is nonempty and connected if and only if the same is true for $X_{0}$.

Proof. We are given the following cartesian square


If $X$ is connected and nonempty, then $f(X) \subseteq S$ is a nonempty closed subset of $S$ (as $f$ is proper). Let $J$ be an ideal of $A$ that identifies $f(X)$. Let $f(x)=$ $p \in V(J)$ be a closed point of $S$. As $I$ is contained in the Jacobson radical of $A$, the prime ideal $p$ lies in $S_{0}$. Then


In particuar, $X_{0}$ is nonempty. Furthermore, as this argument can be used for any connected component of $X$, if $X$ is disconnected then also $X_{0}$ is disconnected. Conversely, assume that $X_{0}$ is disconnected. Label $C_{0}$ a nonempty connected component of $X_{0}$. As the scheme $X_{0}$ is quasi-compact, $C_{0}$ is open and closed in $X_{0}$. Therefore, $C_{0} \longrightarrow X_{0}$ is a finite étale morphism. By Lemma 3, there exists a finite étale morphism $C \longrightarrow X$ which induces $C_{0} \longrightarrow X_{0}$. As $C_{0}$ is connected and nonempty, the same is true for $C$. The morphism $C \longrightarrow X$ is therefore of degree 1 at every point of $C$. As it is also finite and étale, it is both an open and a closed immersion, i.e. $C$ is a connected component of $X$. If $C=X$, we would get $C_{0}=X_{0}$, a contradiction. Then $X$ is disconnected. Finally, it is clear that if $X_{0}$ is nonempty, $X$ is nonempty too.

Theorem 1 follows immediately from Lemma 3 and Lemma 4.

## 3 Henselian couples

Recall that an henselian pair $(A, I)$ is a ring $A$ together with an ideal $I \subseteq$ such that

1. $I$ is contained in the Jacobson ideal of $A$;
2. for every finite $A$ algebra $B$, there is a bijection between the set of idempotent elements of $B$ and the set of idempotent elements of $B \otimes_{A} A / I$.
For more details, see [11].
Let $(A, I)$ be an henselian pair. Then for every finite morphism $\operatorname{Spec}(B)=$ $X \longrightarrow S p e c(A)$, we have a bijection

$$
I d(B)=O f(X)=O f\left(X_{0}\right)=I d(B / I B) \quad \text { where } X_{0}=X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / I)
$$

Here $O f(Z)$ denotes the set of subsets of $Z$ which are both open and closed. This fact suggests the following definition (see [8, Définition 18.5.5]), which is meant to generalize the notion of henselian pair to the non-affine setting.

Definition 1. Let $X$ be a scheme and let $X_{0}$ be a closed subscheme. We say that $\left(X, X_{0}\right)$ is an henselian couple if for every finite morphism $Y \longrightarrow X$ we have a bijection

$$
O f(Y)=O f\left(Y_{0}\right)
$$

where $Y_{0}=Y \times_{X} X_{0}$.
Remark 5. If $X$ is locally noetherian, it is a consequence of [6, Proposition 6.1.4] and [6, Corollaire 6.1.9] that connected sets in $O f(X)$ (resp. $O f\left(X_{0}\right)$ ) are in bijection with $\Pi_{0}(X)$ (resp. $\Pi_{0}\left(X_{0}\right)$ ), the set of connected components of $X$ (resp. $X_{0}$ ).
Remark 6. It is a consequence of [6, Corollary 5.1.8] that ( $X, X_{0}$ ) is an henselian couple if and only if $\left(X_{r e d},\left(X_{0}\right)_{r e d}\right)$ is an henselian couple as well.

Remark 7. It is immediate to observe that if $(A, I)$ is a pair and $(\operatorname{Spec}(A), \operatorname{Spec}(A / I))$ is an henselian couple, then $I$ is contained in the Jacobson radical of A. In fact, if $m \subseteq A$ is a maximal ideal, then we have a bijection

$$
O f(\operatorname{Spec}(A / m))=O f\left(\operatorname{Spec}\left(A / m \otimes_{A} A / I\right)\right)
$$

In particular, $\operatorname{Spec}\left(A / m \otimes_{A} A / I\right)$ can not be the empty scheme. Therefore, as it is a closed subscheme of $\operatorname{Spec}(A / m)$, we must have an equality $\operatorname{Spec}(A / m)=$ $\operatorname{Spec}\left(A / m \otimes_{A} A / I\right)$, whence $I \subseteq m$. Moreover, if $Z \longrightarrow S p e c(A)$ is a finite morphism, then $Z=\operatorname{Spec}(B)$ is affine and the corresponding morphism $A \longrightarrow$ $B$ is finite. Then we have bijections

$$
I d(B)=O f(\operatorname{Spec}(B))=O f(\operatorname{Spec}(B / I B))=I d(B / I B)
$$

We have just showed that an affine henselian couple is an henselian pair. The converse was observed at the beginning of this section.

Lemma 6. Let $(A, I)$ be an henselian pair with $A$ noetherian and let $X$ be a proper $A$-scheme. Set $S=S \operatorname{pec}(A), S_{0}=S p e c(A / I)$ and let $X_{0}=X \times_{S} S_{0}$. Then $\left(X, X_{0}\right)$ is an henselian couple.

Proof. This is a trivial consequence of Theorem 1 and [2, Exposé XII, Proposition 6.5 (i)].

Lemma 7. Let $X$ be a scheme and let $X_{0}$ be a closed subscheme. Let $A$ be a noetherian ring and assume that $X$ is proper over $\operatorname{Spec}(A)$. Also assume that $X_{0}=X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / I)$ for some ideal $I \subseteq A$. Put $J=\operatorname{ker}(B=$ $\left.\mathscr{O}_{X}(X) \longrightarrow \mathscr{O}_{X_{0}}\left(X_{0}\right)\right)$. If $(B, J)$ is an henselian pair, then $\left(X, X_{0}\right)$ is an henselian couple.

Proof. Let $\left(A^{h}, I^{h}\right)$ be the henselization of the couple $(A, I)$ given by [12, Tag 0A02]. Then we have the following diagram

which induces the following diagram of pairs:


The morphism $\psi$ is the one induced by the universal property of $\left(A^{h}, I^{h}\right)$. As

$$
\operatorname{Hom}_{\text {Rings }}\left(A^{h}, B\right)=\operatorname{Hom}_{\text {Schemes }}\left(X, \operatorname{Spec}\left(A^{h}\right)\right)
$$

the homomorphism $\psi$ identifies a unique morphism of schemes $\phi: X \longrightarrow$ $\operatorname{Spec}\left(A^{h}\right)$. Thus we get the following commutative diagram


Moreover, by [12, Tag 0AGU], we get that

$$
\gamma^{-1}(\operatorname{Spec}(A / I))=\operatorname{Spec}\left(A^{h} \otimes_{A} A / I\right)=\operatorname{Spec}\left(A^{h} / I^{h}\right)
$$

whence

$$
X \times_{\operatorname{Spec}\left(A^{h}\right)} \operatorname{Spec}\left(A^{h} / I^{h}\right)=X_{0}
$$

Therefore, the couple $\left(X, X_{0}\right)$ lies over the henselian couple $\left(\operatorname{Spec}\left(A^{h}\right), \operatorname{Spec}\left(A^{h} / I^{h}\right)\right)$. Furthermore, $A^{h}$ is a noetherian ring (see [12, Tag 0AGV]). Finally, as $f$ is a proper morphism and $\gamma$ is separated, we get that $\phi$ is proper as well by [10, Proposition 3.3.16]. Then we can conclude that $\left(X, X_{0}\right)$ is an henselian couple by the previous lemma.

The previous lemma tells us that, under some appropriate hypothesis, if the pair

$$
\left(\mathscr{O}_{X}(X), \operatorname{ker}\left(\mathscr{O}_{X}(X) \longrightarrow \mathscr{O}_{X_{0}}\left(X_{0}\right)\right)\right)
$$

is henselian, then $\left(X, X_{0}\right)$ is an henselian couple. It is natural to ask if the converse is true, i.e. if given an henselian couple $\left(X, X_{0}\right)$ the associated pair is henselian. An answer is provided by the next lemma.

Lemma 8. Let $X$ be a quasi-compact and quasi-separated scheme and let $i$ : $X_{0} \longrightarrow X$ be a closed immersion such that $\left(X, X_{0}\right)$ is an henselian couple. Then $(B, J)=\left(\mathscr{O}_{X}(X), \operatorname{ker}\left(\mathscr{O}_{X}(X) \longrightarrow \mathscr{O}_{X_{0}}\left(X_{0}\right)\right)\right)$ is an henselian pair.

Proof. By [12, Tag 09XI], it is sufficient to show that for every étale ring map $B \longrightarrow C$ together with a $B$-morphism $\sigma: C \longrightarrow B / J$, there exists a $B$ morphism $C \longrightarrow B$ which lifts $\sigma$.
Consider the cartesian diagram


As $\operatorname{Spec}(C) \longrightarrow \operatorname{Spec}(B)$ is étale and separated, the morphism $X_{C} \longrightarrow X$ is étale and separated as well. Then, by [8, Proposition 18.5.4], we have a bijection

$$
\Gamma\left(X_{C} / X\right) \longrightarrow \Gamma\left(X_{C} \times_{X} X_{0} / X_{0}\right)
$$

between the sections of $X_{C} \longrightarrow X$ and those of $X_{C} \times_{X} X_{0} \longrightarrow X_{0}$.
Observation 1. The universal property of $X_{C} \times_{X} X_{0}$ tells us that

$$
\Gamma\left(X_{C} \times_{X} X_{0} / X_{0}\right) \cong \operatorname{Hom}_{X}\left(X_{0}, X_{C}\right)
$$

Observation 2. Let $\mathscr{J} \subseteq \mathscr{O}_{X}$ be the sheaf of ideals associated to $X_{0}$. Then we have a short exact sequence of $\mathscr{O}_{X}$-modules

$$
0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{O}_{X} \longrightarrow i_{*} \mathscr{O}_{X_{0}} \longrightarrow 0
$$

Applying the global sections functor, we get an exact sequence

$$
0 \longrightarrow J=\mathscr{J}(X) \longrightarrow \mathscr{O}_{X}(X)=B \longrightarrow \mathscr{O}_{X_{0}}\left(X_{0}\right)
$$

Hence, we have an homomorphism

$$
B / J \longrightarrow \mathscr{O}_{X_{0}}\left(X_{0}\right)
$$

Therefore, we get a morphism of schemes

$$
X_{0} \longrightarrow \operatorname{Spec}\left(\mathscr{O}_{X_{0}}\left(X_{0}\right)\right) \longrightarrow \operatorname{Spec}(B / J)
$$

Also notice that the diagram

is commutative.
Now consider the diagram


Label $\tilde{\alpha}: X_{0} \longrightarrow X_{C}$ the $X$-morphism provided by the universal property of $X_{C}$ and let $\alpha: X \longrightarrow X_{C}$ be the corresponding $X$-morphism in $\Gamma\left(X_{C} / X\right)$. Consider the following commutative diagram

and the corresponding commutative diagram in Rings:


It is then clear that $\psi$ is the $B$-morphism we were looking for. This concludes the proof of the lemma.

Corollary 1. Let $\left(X, X_{0}\right)$ be an henselian couple. Assume that $X$ is proper over a noetherian ring $A$ and that $X_{0}=X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / I)$ for some ideal $I \subseteq A$. Then $\left(X, X_{0}\right)$ is proper over an henselian pair.

Proof. As $X$ is proper over $\operatorname{Spec}(A)$, it is a quasi-compact and quasi-separated scheme. Hence, by Lemma $8,\left(\mathscr{O}_{X}(X), \operatorname{ker}\left(\mathscr{O}_{X}(X) \longrightarrow \mathscr{O}_{X_{0}}\left(X_{0}\right)\right)\right)$ is an henselian pair. Therefore, by the same construction described in Lemma 7, we get that $\left(X, X_{0}\right)$ is proper over $\left(A^{h}, I^{h}\right)$.

Corollary 2. Let $\left(X, X_{0}\right)$ be a couple and assume that $X$ is proper over a noetherian ring $A$ and that $X_{0}=X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / I)$ for some ideal $I \subseteq A$. Then $\left(X, X_{0}\right)$ is an henselian couple if and only if $\left(\mathscr{O}_{X}(X), \operatorname{ker}\left(\mathscr{O}_{X}(X) \longrightarrow\right.\right.$ $\left.\mathscr{O}_{X_{0}}\left(X_{0}\right)\right)$ ) is an henselian pair.

By Remark 3 every henselian couple ( $X, X_{0}$ ) which arises as in Lemma 6 satisfies conditions 2. and 3. in [2, Exposé XII, Proposition 6.5] with $\mathbb{L}=\mathbb{P}$ and for every $n$. Then, applying Corollary 1 , we get the following result:

Theorem 3. Let $\left(X, X_{0}\right)$ be an henselian couple. Assume that $X$ is proper over a noetherian ring $A$ and that $X_{0}=X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A / I)$ for some ideal $I \subseteq A$. Then conditions 2. and 3. in [2, Exp. XII, Remarks 6.13] are satisfied with $\mathbb{L}=\mathbb{P}$ and for every $n$.

This gives a positive answer to Question 1 if we assume that hypothesis ( $\dagger$ ) hold.

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[^0]:    ${ }^{1}$ the content of this note was obtained before the author started to receive the above mentioned scholarship, but it was organized in the present manuscript afterwards.
    ${ }^{2}$ for the notation and the exact definition of the objects that are involved we refer to the original source.

[^1]:    ${ }^{3}$ an affine henselian couple is the same thing as an henselian pair. See Remark 7.
    ${ }^{4}$ here the henselization is the left adjoint to the inclusion functor Henselian Pairs $\longrightarrow$ Pairs
    ${ }^{5}$ see [1, Definition 1.5]

