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# A TWO-DIMENSIONAL POLYNOMIAL MAPPING WITH A WANDERING FATOU COMPONENT

MATTHIEU ASTORG, XAVIER BUFF, ROMAIN DUJARDIN, HAN PETERS,  
AND JASMIN RAISSY

ABSTRACT. We show that there exist polynomial endomorphisms of  $\mathbb{C}^2$ , possessing a wandering Fatou component. These mappings are polynomial skew-products, and can be chosen to extend holomorphically of  $\mathbb{P}^2(\mathbb{C})$ . We also find real examples with wandering domains in  $\mathbb{R}^2$ . The proof relies on parabolic implosion techniques, and is based on an original idea of M. Lyubich.

## INTRODUCTION

Let  $P : \mathbb{C}^k \rightarrow \mathbb{C}^k$  be a polynomial mapping. In this article we consider  $P$  as a dynamical system, that is, we study the behavior of the sequence of iterates  $(P^{on})_{n \in \mathbb{N}}$ . A case of particular interest is when  $P$  extends as a holomorphic endomorphism of  $\mathbb{P}^k(\mathbb{C})$ . As a matter of expositional simplicity, let us assume for the moment that this property holds. The *Fatou set*  $\mathcal{F}_P$  is classically defined as the largest open subset of  $\mathbb{P}^k(\mathbb{C})$  in which the sequence of iterates is locally equicontinuous (or *normal*, according to the usual terminology). Its complement, the *Julia set*, is where chaotic dynamics takes place. A *Fatou component* is a connected component of  $\mathcal{F}_P$ .

When the dimension  $k$  equals 1, the Non Wandering Domain Theorem due to Sullivan [Su] asserts that every Fatou component is eventually periodic. This work is fundamental for at least two reasons.

- First, the result opens the way to a complete description of the dynamics in the Fatou set: the orbit of any point in the Fatou set eventually lands in a (super-)attracting basin, a parabolic basin or a rotation domain.
- Secondly, the proof introduced quasi-conformal mappings as a new tool in this research area, leading to many subsequent developments.

There are many variants and generalizations of Sullivan's Theorem in several areas of one-dimensional dynamics. For instance it was shown by Eremenko and Lyubich [EL2] and Goldberg and Keen [GK] that entire mappings with finitely many singular values have no wandering domains. On the other hand, Baker [Ba], prior to Sullivan's result, gave the first example of an entire mapping with a wandering domain. Simple explicit entire mappings with Fatou components wandering to infinity were given in [Su, §9] and [He, §II.11], while more elaborate examples with varied dynamical behaviors were presented in [EL1]. More recently, Bishop [Bi]

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constructed an example with a bounded singular set. In all cases, the orbit of the wandering domain is unbounded.

In the real setting, the question of (non-)existence of wandering intervals has a long and rich history. It started with Denjoy's theory of linearization of circle diffeomorphisms [De]: a  $C^2$ -diffeomorphism of the circle with irrational rotation number has no wandering intervals (hence it is linearizable), whereas this result breaks down for  $C^1$  diffeomorphisms. More recent results include homeomorphisms with various degrees of regularity and flatness of critical points.

For interval maps, the non-existence of wandering intervals for unimodal maps with negative Schwarzian was established by Guckenheimer [G], and later extended to several classes of interval and circle maps in [Ly, BL, MMS]. In particular, the result of Martens, de Melo and van Strien implies the non-existence of wandering intervals for polynomials on the real line.

One difficulty is to define a notion of Fatou set in this context. Let us just note here that for a real polynomial, the Fatou set as defined in [MMS] contains the intersection of the complex Fatou set with the real line but could *a priori* be larger.

The problem was also studied in the non-Archimedean setting, in particular in the work of Benedetto [Be] and Trucco [T].

◇

The question of the existence of wandering Fatou components makes sense in higher dimension, and was put forward by many authors since the 1990's (see e.g. [FS2]). Higher dimensional transcendental mappings with wandering domains can be constructed from one-dimensional examples by simply taking products. An example of a transcendental biholomorphic map in  $\mathbb{C}^2$  with a wandering Fatou component oscillating to infinity was constructed by Fornæss and Sibony in [FS1].

For higher dimensional polynomials and rational mappings, it is widely acknowledged that quasi-conformal methods break down, so a direct approach to generalize Sullivan's Theorem fails. Besides this observation, little was known about the problem so far.

Recently, Misha Lyubich suggested that polynomial skew products with wandering domains might be constructed by using parabolic implosion techniques. The idea was to combine slow convergence to an invariant fiber and parabolic transition in the fiber direction, to produce orbits shadowing those of a Lavaurs map (see below for a more precise description).

In this paper, we bring this idea to completion, thereby providing the first examples of higher dimensional polynomial mappings with wandering domains.

**Main Theorem.** *There exists an endomorphism  $P : \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$ , induced by a polynomial skew-product mapping  $P : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ , possessing a wandering Fatou component.*

Let us point out that the orbits in these wandering domains are bounded. The approach is in fact essentially local. A more detailed statement is the following (see below for the definition of Lavaurs maps).

**Main Theorem (precise form).** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  and  $g : \mathbb{C} \rightarrow \mathbb{C}$  be polynomials of the form*

$$(1) \quad f(z) = z + z^2 + O(z^3) \quad \text{and} \quad g(w) = w - w^2 + O(w^3).$$

If the Lavaurs map  $\mathcal{L}_f : \mathcal{B}_f \rightarrow \mathbb{C}$  has an attracting fixed point, then the skew-product map  $P : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by

$$(2) \quad P(z, w) := \left( f(z) + \frac{\pi^2}{4}w, g(w) \right)$$

admits a wandering Fatou component.

Notice that if  $f$  and  $g$  have the same degree,  $P$  extends to a holomorphic self-map on  $\mathbb{P}^2(\mathbb{C})$ . Observe also that examples in arbitrary dimension  $k \geq 2$  may be obtained from this result by simply considering products mappings of the form  $(P, Q)$ , where  $Q$  admits a fixed Fatou component.

Before explaining what the Lavaurs map is, let us give some explicit examples satisfying the assumption of the Main Theorem.

**Example 1.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be the cubic polynomial  $f(z) = z + z^2 + az^3$ , and  $g$  be as in (1). If  $r > 0$  is sufficiently small and  $a \in D(1 - r, r)$ , then the polynomial skew-product  $P$  defined in (2) admits a wandering Fatou component.

Numerical experiments suggest that the value  $a = 0.95$  works (see Figure 5 on page 31).

In view of the results of [MMS] cited above, it is also of interest to look for real polynomial mappings with wandering Fatou domains intersecting  $\mathbb{R}^2$ . Our method also provides such examples.

**Example 2.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be the degree 4 polynomial defined by

$$f(z) := z + z^2 + bz^4 \quad \text{with } b \in \mathbb{R}.$$

There exist parameters  $b \in (-8/27, 0)$  such that if  $g$  is as in (1), the polynomial skew-product  $P$  defined in (2) admits a wandering Fatou component intersecting  $\mathbb{R}^2$ .

Numerical experiments suggest that the parameter  $b = -0.2136$  satisfies this property. We illustrate this phenomenon graphically in Figure 1 for the mapping  $P$  defined by

$$(3) \quad (z, w) \mapsto \left( z + z^2 - 0.2136z^4 + \frac{\pi^2}{4}w, w - w^2 \right)$$

The set of points  $(z, w) \in \mathbb{R}^2$  with bounded orbit is contained in the rectangle  $(-3, 3) \times (0, 1)$ . The topmost image in Figure 1 displays this set of points. The bottom left image displays the window  $-3 < z < 3$  and  $1/1003^2 < w < 1/999^2$  (this is a thin horizontal slice of the upper image, greatly magnified in the  $w$ -direction); the orbit of a point  $(z_0, w_0)$  contained in the wandering domain is indicated. The coordinate  $w_0$  is close to  $1/1000^2$  and we plotted the first 2001 and the next 2003 iterates. The points are indicated in black except  $(z_0, w_0)$ ,  $P^{\circ 2001}(z_0, w_0)$  and  $P^{\circ 2001+2003}(z_0, w_0)$  which are colored in red. The small square around  $(z_0, w_0)$  is blown up to make the zoom on the bottom right image. The wandering domain is contained in the grey open neighborhood of  $(z_0, w_0)$  of points with bounded orbits. These specific values are explained by the proof of the Main Theorem (see Proposition A below).

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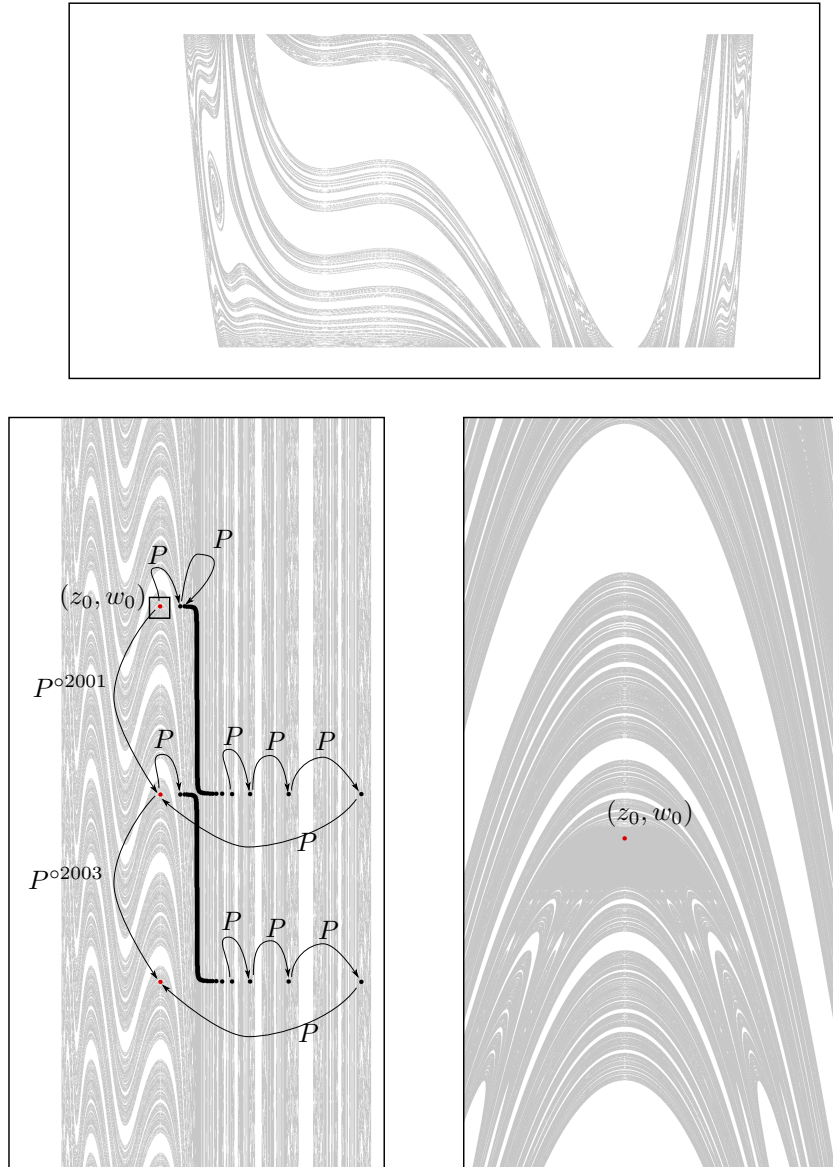


FIGURE 1. A real wandering domain for the map  $P$  defined in (3).

Using skew-products to construct new examples is natural as it allows to build on one-dimensional dynamics. This idea was already used several times in holomorphic dynamics (see e.g. [J, Du]) and beyond.

Fatou components of polynomial skew-products were studied in several earlier works. Lilov showed in his thesis [Li] that skew-products do not have wandering components near a super-attracting invariant fiber. In [PV] it was shown that the argument used in [Li] breaks down near an attracting fiber. The construction in [PV] uses a repelling fixed point in the invariant fiber with multiplier equal to the

inverse of the multiplier of the attracting fiber. This resonance between multipliers induces a dynamical behavior that cannot occur in one-dimensional dynamics.

In the same vein, an invariant fiber at the center of a Siegel disk was used in [BFP] to construct a non-recurrent Fatou component with limit set isomorphic to a punctured disk. In that construction the invariant fiber also contains a Siegel disk, but with the opposite rotation number.

We note that the construction presented in this paper uses an invariant parabolic fiber which again contains a parabolic point.

◇

To explain the notion of Lavaurs map and the strategy of the proof, we need to recall some facts on parabolic dynamics (see Appendix A for more details). Let  $f$  be a polynomial of the form

$$f(z) = z + z^2 + az^3 + O(z^4) \quad \text{for some } a \in \mathbb{C}.$$

and denote by

$$\mathcal{B}_f := \{z \in \mathbb{C} ; f^{\circ n}(z) \xrightarrow[n \rightarrow +\infty]{\neq} 0\}$$

the parabolic basin of 0. It is known that there is an *attracting Fatou coordinate*  $\phi_f : \mathcal{B}_f \rightarrow \mathbb{C}$  which conjugates  $f$  to the translation  $T_1$  by 1:

$$\phi_f \circ f = T_1 \circ \phi_f.$$

This Fatou coordinate may be normalized by<sup>1</sup>

$$\phi_f(z) = -\frac{1}{z} - (1-a) \log\left(-\frac{1}{z}\right) + o(1) \quad \text{as } \Re(-1/z) \rightarrow +\infty.$$

Likewise, there is a *repelling Fatou parameterization*  $\psi_f : \mathbb{C} \rightarrow \mathbb{C}$  satisfying

$$\psi_f \circ T_1 = f \circ \psi_f,$$

which may be normalized by

$$-\frac{1}{\psi_f(Z)} = Z + (1-a) \log(-Z) + o(1) \quad \text{as } \Re(Z) \rightarrow -\infty.$$

The (phase 0) Lavaurs map  $\mathcal{L}_f$  is defined by<sup>2</sup>

$$\mathcal{L}_f := \psi_f \circ \phi_f : \mathcal{B}_f \rightarrow \mathbb{C}.$$

Mappings of this kind appear when considering high iterates of small perturbations of  $f$ : this phenomenon is known as *parabolic implosion*, and will play a key role in this paper. The reader is referred to the text of Douady [Do] for a gentle introduction to this topic, and to [Sh] for a more detailed presentation by Shishikura. (Semi-)parabolic implosion was recently studied in the context of dissipative polynomial automorphisms of  $\mathbb{C}^2$  by Bedford, Smillie and Ueda [BSU] (see also [DL]).

Let us already point out that since our results do not fit into the classical framework, our treatment of parabolic implosion will be essentially self-contained. As it turns out, our computations bear some similarity with those of [BSU].

<sup>1</sup>The branch of log used in this normalization as well as in the next one is the branch defined in  $\mathbb{C} \setminus \mathbb{R}^-$  which vanishes at 1

<sup>2</sup>The reader who is familiar with Lavaurs maps should notice that the choice of phase was determined by the normalization of Fatou coordinates.

We can now explain the strategy of the proof of the Main Theorem. Let  $\mathcal{B}_g$  be the parabolic basin of 0 under iteration of  $g$ . If  $w \in \mathcal{B}_g$ , then  $g^{\circ m}(w)$  converges to 0 like  $1/m$ . We want to choose  $(z_0, w_0) \in \mathcal{B}_f \times \mathcal{B}_g$  so that the first coordinate of  $P^{\circ m}(z_0, w_0)$  returns close to the attracting fixed point of  $\mathcal{L}_f$  infinitely many times. The proof is designed so that the return times are the integers  $n^2$  for  $n \geq n_0$ . So, we have to analyze the orbit segment between  $n^2$  and  $(n+1)^2$ , which is of length  $2n+1$ .

For large  $n$ , the first coordinate of  $P$  along this orbit segment is approximately

$$f(z) + \varepsilon^2 \quad \text{with} \quad \frac{\pi}{\varepsilon} = 2n.$$

The Lavaurs Theorem from parabolic implosion asserts that if  $\frac{\pi}{\varepsilon} = 2n$ , then for large  $n$ , the  $(2n)^{\text{th}}$  iterate of  $f(z) + \varepsilon^2$  is approximately equal to  $\mathcal{L}_f(z)$  on  $\mathcal{B}_f$ .

Our setting is slightly different since  $\varepsilon$  keeps decreasing along the orbit. Indeed on the first coordinate we are taking the composition of  $2n+1$  transformations

$$f(z) + \varepsilon_k^2 \quad \text{with} \quad \frac{\pi}{\varepsilon_k} \simeq 2n + \frac{k}{n} \quad \text{and} \quad 1 \leq k \leq 2n+1.$$

The main step of the proof of the Main Theorem consists in a detailed analysis of this perturbed situation. We show that the decay of  $\varepsilon_k$  is counterbalanced by taking *exactly* one additional iterate of  $P$ . The precise statement is the following.

**Proposition A.** *As  $n \rightarrow +\infty$ , the sequence of maps*

$$\mathbb{C}^2 \ni (z, w) \mapsto P^{\circ 2n+1}(z, g^{\circ n^2}(w)) \in \mathbb{C}^2$$

*converges locally uniformly in  $\mathcal{B}_f \times \mathcal{B}_g$  to the map*

$$\mathcal{B}_f \times \mathcal{B}_g \ni (z, w) \mapsto (\mathcal{L}_f(z), 0) \in \mathbb{C} \times \{0\}.$$

See Figure 2 for a graphical illustration of this Proposition.

From this point, the proof of the Main Theorem is easily completed: if  $\xi$  is an attracting fixed point of  $\mathcal{L}_f$  and if  $(z_0, w_0) \in \mathcal{B}_f \times \mathcal{B}_g$  is chosen so that  $P^{\circ n_0^2}(z_0, w_0)$  is close to  $(\xi, 0)$  for some large  $n_0$ , then  $P^{\circ (n_0+1)^2}(z_0, w_0)$  gets closer to  $(\xi, 0)$  and we can repeat the process to get that the sequence  $(P^{\circ n^2}(z_0, w_0))_{n \geq 0}$  converges to  $(\xi, 0)$ . Since this reasoning is valid on an open set of initial conditions, these points belong to some Fatou component. Simple considerations show that it cannot be preperiodic, and the result follows.

Let us observe that by construction, the  $\omega$ -limit set of any point in the wandering Fatou component consists of a single two-sided orbit of  $(\xi, 0)$  under  $P$ , plus the origin.

We give two different approaches for constructing examples satisfying the assumptions of the Main Theorem. The next proposition corresponds to Example 1.

**Proposition B.** *Consider the cubic polynomial  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined by*

$$f(z) := z + z^2 + az^3 \quad \text{with} \quad a \in \mathbb{C}.$$

*If  $r > 0$  is sufficiently close to 0 and  $a \in D(1-r, r)$ , then the Lavaurs map  $\mathcal{L}_f : \mathcal{B}_f \rightarrow \mathbb{C}$  admits an attracting fixed point.*

To prove this proposition we construct a fixed point for the Lavaurs map by perturbation from the degenerate situation  $a = 1$  and estimate its multiplier by a residue computation.

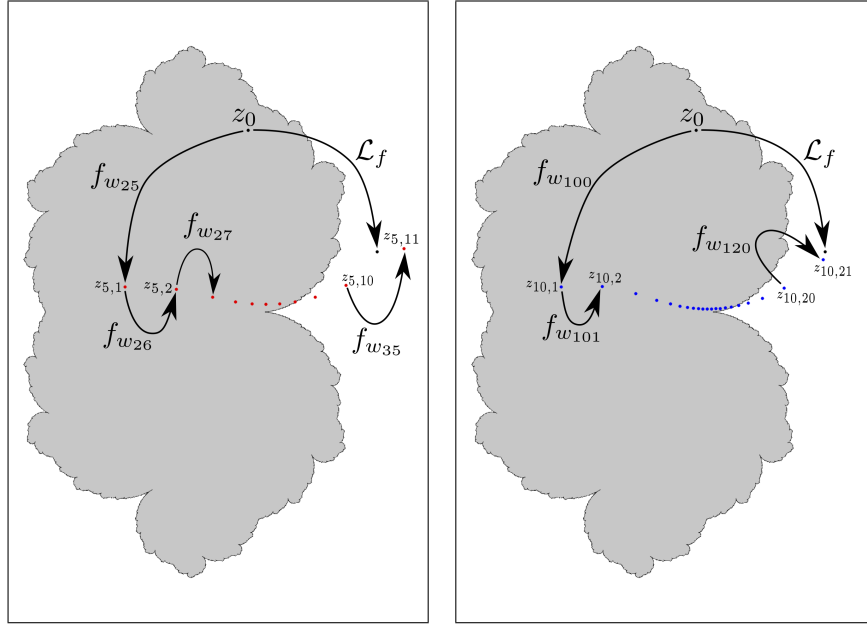


FIGURE 2. Illustration of Proposition A for  $f(z) = z + z^2 + 0.95z^3$  and  $g(w) = w - w^2$ . The parabolic basin  $\mathcal{B}_f$  is colored in grey. It is invariant under  $f$ , but not under  $f_w := f + \frac{\pi^2}{4}w$  for  $w \neq 0$ . The Lavaurs map  $\mathcal{L}_f$  is defined on  $\mathcal{B}_f$ . The point  $z_0 = -0.05 + 0.9i$  and its image  $\mathcal{L}_f(z_0)$  are indicated. The other points are the points  $z_{n,k}$  which are defined by  $P^{\circ k}(z_0, w_{n^2}) = (z_{n,k}, w_{n^2+k})$  for  $1 \leq k \leq 2n + 1$  and  $w_m = g^{\circ m}(1/2)$ . If  $n$  is large enough, the point  $z_{n,2n+1}$  is close to  $\mathcal{L}_f(z_0)$ . Left:  $n = 5$ , there are 11 red points. Right:  $n = 10$ , there are 21 blue points.

To construct the real examples of Example 2, we use the Intermediate Value Theorem to find a real Lavaurs map with a real periodic critical point.

**Proposition C.** *Consider the degree 4 polynomial  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined by*

$$f(z) := z + z^2 + bz^4 \quad \text{with } b \in \mathbb{R}.$$

*Then there exists a parameter  $b \in (-8/27, 0)$  such that the Lavaurs map  $\mathcal{L}_f$  has a fixed critical point in  $\mathbb{R} \cap \mathcal{B}_f$ .*

In particular this fixed point is super-attracting so we are in the situation of the Main Theorem.

◇

The plan of the paper is the following. In Section 1 we show how to deduce the Main Theorem from Proposition A. In Section 2 we develop a formalism of *approximate Fatou coordinates* in our context, which is necessary for proving Proposition A, which is itself established in Section 3. The proofs in Sections 2 and 3 are rather technical, and are divided into a number of steps. A detailed outline will be given there. Propositions B and C are respectively established in Sections 4 and 5. Appendix A summarizes the main properties of Fatou coordinates and Lavaurs maps.



## ACKNOWLEDGEMENTS

This project grew up from discussions between Misha Lyubich and Han Peters in relation with the work [LP]. It evolved to the current list of authors at the occasion of a meeting of the ANR project LAMBDA in Université Paris-Est Marne-la-Vallée in April 2014. We are grateful to Misha Lyubich for his crucial input which ultimately led to the present work.

## 1. EXISTENCE OF WANDERING DOMAINS

In this section, we prove the Main Theorem assuming Proposition A.

Let  $\xi \in \mathcal{B}_f$  be an attracting fixed point of the Lavaurs map  $\mathcal{L}_f$ . Let  $V$  be a disk centered at  $\xi$ , chosen that  $\mathcal{L}_f(V)$  is compactly contained in  $V$ . It follows that  $\mathcal{L}_f^{\circ k}(V)$  converges to  $\xi$  as  $k \rightarrow \infty$ . Let also  $W \Subset \mathcal{B}_g$  be an arbitrary disk.

Denote by  $\pi_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$  the first coordinate projection, that is  $\pi_1(z, w) := z$ . According to Proposition A, there exists  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$ ,

$$\pi_1 \circ P^{\circ(2n+1)}(V \times g^{\circ n^2}(W)) \Subset V.$$

Let  $U$  be a connected component of the open set  $P^{-n_0^2}(V \times g^{\circ n_0^2}(W))$ .

**Lemma 1.1.** *The sequence  $(P^{\circ n^2})_{n \geq 0}$  converges locally uniformly to  $(\xi, 0)$  on  $U$ .*

*Proof.* An easy induction shows that for every integer  $n \geq n_0$ ,

$$(4) \quad P^{\circ n^2}(U) \subseteq V \times g^{\circ n^2}(W).$$

Indeed this holds by assumption for  $n = n_0$ . Now if the inclusion is true for some  $n \geq n_0$ , then

$$\begin{aligned} \pi_1 \circ P^{\circ(n+1)^2}(U) &= \pi_1 \circ P^{\circ(2n+1)}(P^{\circ n^2}(U)) \\ &\subset \pi_1 \circ P^{\circ(2n+1)}(V \times g^{\circ n^2}(W)) \subset V, \end{aligned}$$

from which (4) follows.

From this we get that the sequence  $(P^{\circ n^2})_{n \geq 0}$  is uniformly bounded, hence normal, on  $U$ . Also, any cluster value of this sequence of maps is constant and of the form  $(z, 0)$  for some  $z \in V$ . In addition,  $(z, 0)$  is a limit value (associated to a subsequence  $(n_k)$ ) if and only if  $(\mathcal{L}_f(z), 0)$  is a limit value (associated to the subsequence  $(1 + n_k)$ ). We infer that the set of cluster limits is totally invariant under  $\mathcal{L}_f : V \rightarrow V$ , therefore it must be reduced to the attracting fixed point  $\xi$  of  $\mathcal{L}_f$ , and we are done.  $\square$

**Corollary 1.2.** *The domain  $U$  is contained in the Fatou set of  $P$ .*

*Proof.* It is well-known in our context that the sequence  $(P^{\circ m})_{m \geq 0}$  is locally bounded on  $U$  if and only if there exists a subsequence  $(m_k)$  such that  $(P^{\circ m_k}|_U)_{k \geq 0}$  has the same property. Indeed since  $\overline{W}$  is compact, there exists  $R > 0$  such that if  $|z| > R$ , then for every  $w \in W$ ,  $(z, w)$  escapes locally uniformly to infinity under iteration. The result follows.  $\square$

*Proof of the Main Theorem.* Let  $\Omega$  be the component of the Fatou set  $\mathcal{F}_P$  containing  $U$ . According to Lemma 1.1, for any integer  $i \geq 0$ , the sequence  $(P^{\circ(n^2+i)})_{n \geq 0}$  converges locally uniformly to  $P^{\circ i}(\xi, 0) = (f^{\circ i}(\xi), 0)$  on  $U$ , hence on  $\Omega$ . Therefore, the sequence  $(P^{\circ n^2})_{n \geq 0}$  converges locally uniformly to  $(f^{\circ i}(\xi), 0)$  on  $P^{\circ i}(\Omega)$ .

As a consequence, if  $i, j$  are nonnegative integers such that  $P^{\circ i}(\Omega) = P^{\circ j}(\Omega)$ , then  $f^{\circ i}(\xi) = f^{\circ j}(\xi)$ , from which we deduce that  $i = j$ . Indeed,  $\xi$  belongs to the parabolic basin  $\mathcal{B}_f$ , and so, it is not (pre)periodic under iteration of  $f$ . This shows that  $\Omega$  is not (pre)periodic under iteration of  $P$ : it is a wandering Fatou component for  $P$ .  $\square$

## 2. APPROXIMATE FATOU COORDINATES

In this section we study the phenomenon of *persistence of Fatou coordinates* in our non-autonomous setting. As in the Main Theorem, we consider polynomial mappings  $f, g : \mathbb{C} \rightarrow \mathbb{C}$  of the form

$$f(z) = z + z^2 + az^3 + O(z^4) \quad \text{and} \quad g(w) = w - w^2 + O(w^3).$$

We put

$$f_w(z) = f(z) + \frac{\pi^2}{4}w.$$

We want to show that there exist changes of variables  $\varphi_w$  and  $\varphi_{g(w)}$  which are in a sense approximations to the Fatou coordinates of  $f$  in appropriate domains, and such that  $\varphi_{g(w)} \circ f_w \circ \varphi_w^{-1}$  is close to a translation. These change of variables are normalized so that  $\varphi_{g(w)} \circ f_w \circ \varphi_w^{-1}$  is roughly defined in a vertical strip of width 1 and the translation vector is  $\sqrt{w}/2$ . They will be given by explicit formulas: in this respect our approach is similar to that of [BSU]. Precise error estimates are required in order to ultimately prove Proposition A in the next section.

**2.1. Notation.** The following notation will be used throughout this section and the next one (see also Figure 3).

First, choose  $R > 0$  large enough so that  $F : Z \mapsto -1/f(-1/Z)$  is univalent on  $\mathbb{C} \setminus \overline{D}(0, R)$ ,

$$\sup_{|Z| > R} |F(Z) - Z - 1| < \frac{1}{10} \quad \text{and} \quad \sup_{|Z| > R} |F'(Z) - 1| < \frac{1}{10}.$$

Denote by  $\mathbb{H}_R$  the right half-plane  $\mathbb{H}_R := \{Z \in \mathbb{C} ; \Re(Z) > R\}$  and by  $-\mathbb{H}_R$  the left half-plane  $-\mathbb{H}_R := \{Z \in \mathbb{C} ; \Re(Z) < -R\}$ . Define the *attracting petal*  $P_f^{\text{att}}$  by

$$P_f^{\text{att}} = \left\{ z \in \mathbb{C} ; \Re\left(-\frac{1}{z}\right) > R \right\}$$

Then,

- the restriction of the attracting Fatou coordinate  $\phi_f$  to the attracting petal  $P_f^{\text{att}}$  is univalent, and
- the restriction of the repelling Fatou parameterization  $\psi_f$  to the left half-plane  $-\mathbb{H}_R$  is univalent.

We use the notation  $\psi_f^{-1}$  only for the inverse branch  $\psi_f^{-1} : P_f^{\text{rep}} \rightarrow -\mathbb{H}_R$  of  $\psi_f$  on the repelling petal  $P_f^{\text{rep}} := \psi_f(-\mathbb{H}_R)$ . Recall that

$$\phi_f \circ f = T_1 \circ \phi_f, \quad f \circ \psi_f = \psi_f \circ T_1,$$

$$\phi_f(z) \sim -\frac{1}{z} \quad \text{as} \quad \Re\left(-\frac{1}{z}\right) \rightarrow +\infty \quad \text{and} \quad \psi_f(Z) \sim -\frac{1}{Z} \quad \text{as} \quad \Re(Z) \rightarrow -\infty.$$

Next, for  $r > 0$  we set  $B_r := D(r, r)$  and fix  $r > 0$  small enough that

$$B_r \subset \mathcal{B}_g \quad \text{and} \quad g(B_r) \subset B_r.$$

For the remainder of the article, we assume that  $w \in B_r$ , whence  $g^{\circ m}(w) \rightarrow 0$  as  $m \rightarrow +\infty$ . The notation  $\sqrt{w}$  stands for the branch of the square root on  $B_r$  that has positive real part.

Finally, we fix a real number

$$(5) \quad \frac{1}{2} < \alpha < \frac{2}{3}.$$

The relevance of this range of values for  $\alpha$  will be made clearer during the proof. For  $w \in B_r$ , we let

$$r_w := |w|^{(1-\alpha)/2} \xrightarrow{w \rightarrow 0} 0 \quad \text{and} \quad R_w := |w|^{-\alpha/2} \xrightarrow{w \rightarrow 0} +\infty.$$

Define  $\mathcal{R}_w$  to be the rectangle

$$(6) \quad \mathcal{R}_w := \left\{ Z \in \mathbb{C} ; \frac{r_w}{10} < \Re(Z) < 1 - \frac{r_w}{10} \text{ and } -\frac{1}{2} < \Im(Z) < \frac{1}{2} \right\},$$

and let  $D_w^{\text{att}}$  and  $D_w^{\text{rep}}$  be the disks

$$D_w^{\text{att}} := D\left(R_w, \frac{R_w}{10}\right) \quad \text{and} \quad D_w^{\text{rep}} := D\left(-R_w, \frac{R_w}{10}\right).$$

In this section, the notation  $O(h)$  means a quantity that is defined for  $w \in B_r$  close enough to zero and is bounded by  $C \cdot h$  for a constant  $C$  which does not depend on  $w$ . As usual,  $o(h)$  means a quantity such that  $o(h)/h$  converges to zero as  $w \rightarrow 0$ .

**2.2. Properties of approximate Fatou coordinates.** Our purpose in this paragraph is to state precisely the properties of the approximate Fatou coordinates, in an axiomatic fashion. The actual definitions as well as the proofs will be detailed afterwards.

The claim is that there exists a family of domains  $(V_w)$  and univalent maps  $(\varphi_w : V_w \rightarrow \mathbb{C})$  parameterized by  $w \in B_r$ , satisfying the following three properties.

**Property 1** (Comparison with the attracting Fatou coordinate). As  $w \rightarrow 0$  in  $B_r$ ,

$$D_w^{\text{att}} \subset \phi_f(V_w \cap P_f^{\text{att}}) \quad \text{and} \quad \sup_{Z \in D_w^{\text{att}}} \left| \frac{2}{\sqrt{w}} \cdot \varphi_w \circ \phi_f^{-1}(Z) - Z \right| \rightarrow 0.$$

This means that  $\frac{2}{\sqrt{w}}\varphi_w$  approximates the Fatou coordinate on the attracting side. A similar result holds on the repelling side.

**Property 2** (Comparison with the repelling Fatou coordinate). As  $w \rightarrow 0$  in  $B_r$ , we have that

$$1 + \frac{\sqrt{w}}{2} \cdot D_w^{\text{rep}} \subset \varphi_w(V_w \cap P_f^{\text{rep}})$$

and

$$\sup_{Z \in D_w^{\text{rep}}} \left| \psi_f^{-1} \circ \varphi_w^{-1} \left( 1 + \frac{\sqrt{w}}{2} Z \right) - Z \right| \rightarrow 0.$$

Finally, the last property asserts that  $\frac{2}{\sqrt{w}}\varphi_w$  is almost a Fatou coordinate.

**Property 3** (Approximate conjugacy to a translation). As  $w \rightarrow 0$  in  $B_r$ , we have that

$$\mathcal{R}_w \subset \varphi_w(V_w), \quad f_w \circ \varphi_w^{-1}(\mathcal{R}_w) \subset V_{g(w)}$$

and

$$\sup_{Z \in \mathcal{R}_w} \left| \varphi_{g(w)} \circ f_w \circ \varphi_w^{-1}(Z) - Z - \frac{\sqrt{w}}{2} \right| = o(w).$$

To improve the readability of the proof, which involves several changes of coordinates, we adopt the following typographical convention:

- block letters (like  $z, V_w, \dots$ ) are used for objects which are thought of as living in the initial coordinates;
- script like letters (like  $Z, \mathcal{R}_w, \dots$ ) are used for objects living in approximate Fatou coordinates;
- the coordinate  $Z$  is used for the actual Fatou coordinate.

This gives rise to expressions like  $\phi_f(z) = Z$  or  $\varphi_w(z) = Z$ .

**2.3. Definition of the approximate Fatou coordinates.** The skew-product  $P$  fixes the origin and leaves the line  $\{w = 0\}$  invariant. We may wonder whether there are other *parabolic invariant curves* near the origin, in the sense of Écalle [E] and Hakim [Ha].

**Question.** Do there exist holomorphic maps  $\xi^\pm : B_r \rightarrow \mathbb{C}$  such that  $\xi^\pm(w) \rightarrow 0$  as  $w \rightarrow 0$  and such that  $f_w \circ \xi^\pm(w) = \xi^\pm \circ g(w)$  for  $w \in B_r$ ?

We shall content ourselves with the following weaker result.

**Lemma 2.1.** *Let  $\zeta^\pm : B_r \rightarrow \mathbb{C}$  be defined by*

$$\zeta^\pm(w) = \pm c_1 \sqrt{w} + c_2 w \quad \text{with} \quad c_1 = \frac{\pi i}{2} \quad \text{and} \quad c_2 = \frac{a\pi^2}{8} - \frac{1}{4}.$$

Then,

$$f_w \circ \zeta^\pm(w) = \zeta^\pm \circ g(w) + O(w^2).$$

*Proof.* An elementary computation shows that

$$f_w \circ \zeta^\pm(w) = \pm c_1 \sqrt{w} + \left( c_2 + c_1^2 + \frac{\pi^2}{4} \right) w \pm (ac_1^3 + 2c_1 c_2) w \sqrt{w} + O(w^2).$$

On the other hand,

$$\sqrt{g(w)} = \sqrt{w} - \frac{1}{2} w \sqrt{w} + O(w^2),$$

so

$$\zeta \circ g(w) = \pm c_1 \sqrt{w} + c_2 w \mp \frac{c_1}{2} w \sqrt{w} + O(w^2).$$

Thus the result follows from our choice of  $c_1$  and  $c_2$  since

$$c_2 + c_1^2 + \frac{\pi^2}{4} = c_2 \quad \text{and} \quad ac_1^3 + 2c_1 c_2 = -\frac{c_1}{2}. \quad \square$$

Let  $\psi_w : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C}) \setminus \{\zeta^+(w), \zeta^-(w)\}$  be the universal cover defined by

$$(7) \quad \psi_w(Z) := \frac{\zeta^-(w) \cdot e^{2\pi i Z} - \zeta^+(w)}{e^{2\pi i Z} - 1} = ic_1 \sqrt{w} \cot(\pi Z) + c_2 w.$$

This universal cover restricts to a univalent map on the vertical strip

$$\mathcal{S}_0 := \{Z \in \mathbb{C} ; 0 < \Re(Z) < 1\},$$

with inverse given by

$$\psi_w^{-1}(z) = \frac{1}{2\pi i} \log \left( \frac{z - \zeta^+(w)}{z - \zeta^-(w)} \right).$$

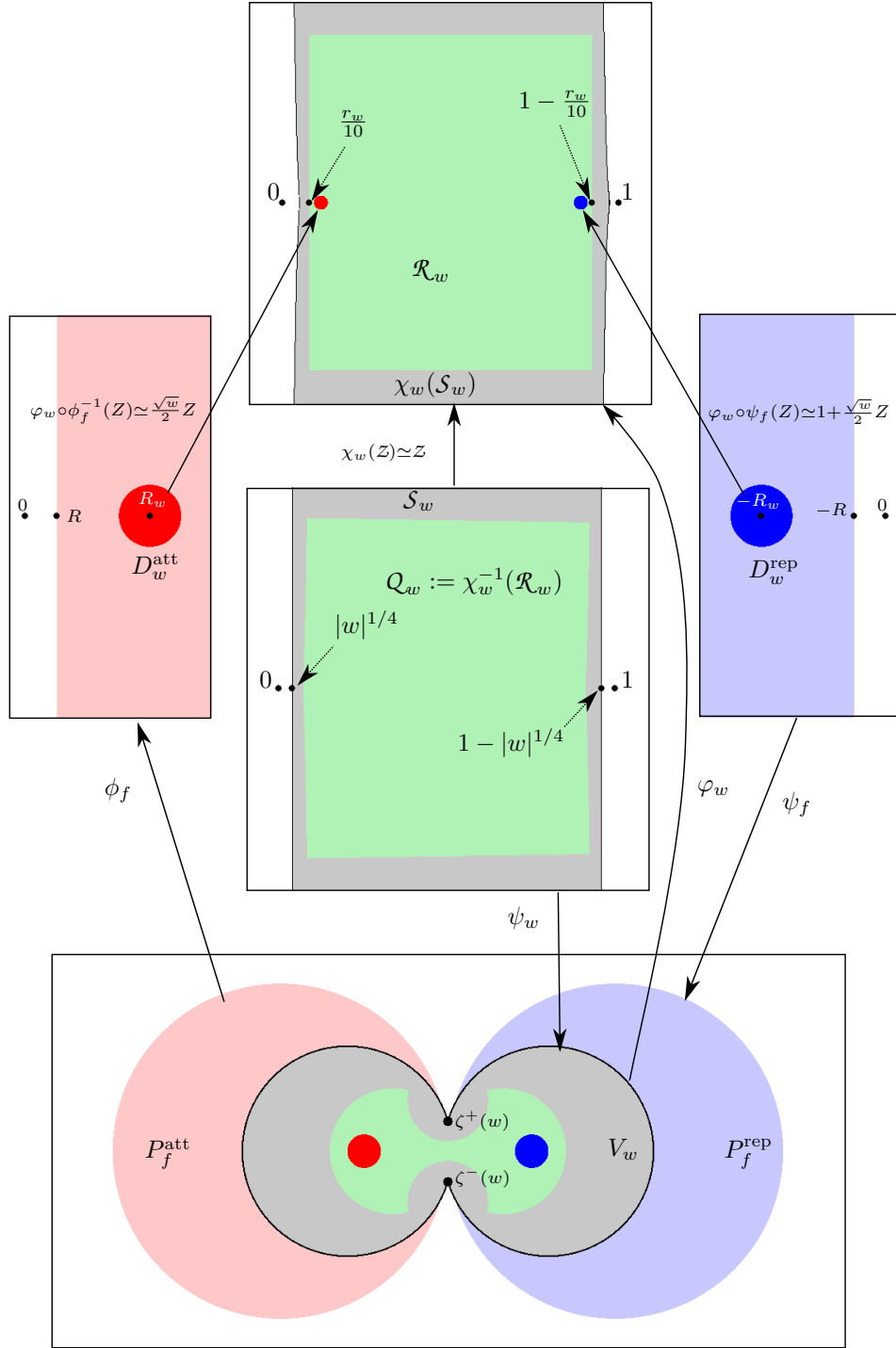


FIGURE 3. Changes of coordinates used in the proof.

In this formula,  $\log(\cdot)$  stands for the branch of the logarithm defined on  $\mathbb{C} \setminus \mathbb{R}^+$  and such that  $\log(-1) = \pi i$ .

For  $w \in B_r$ , let  $\chi_w : \mathcal{S}_0 \rightarrow \mathbb{C}$  be the map defined by

$$(8) \quad \chi_w(\mathcal{Z}) := \mathcal{Z} - \frac{\sqrt{w}(1-a)}{2} \log\left(\frac{2 \sin(\pi \mathcal{Z})}{\pi \sqrt{w}}\right),$$

where in this formula the branch of  $\log$  is defined on  $\frac{1}{\sqrt{w}}(\mathbb{C} \setminus \mathbb{R}^-)$  and vanishes at 1.

Now introduce

$$\mathcal{S}_w := \{\mathcal{Z} \in \mathbb{C} ; |w|^{1/4} < \Re(\mathcal{Z}) < 1 - |w|^{1/4}\}$$

and its image under  $\psi_w$ :

$$V_w := \psi_w(\mathcal{S}_w) \subset \mathbb{C}.$$

**Lemma 2.2.** *If  $w \in B_r$  is close enough to 0, then  $\chi_w : \mathcal{S}_w \rightarrow \mathbb{C}$  is univalent. In addition,  $\chi_w$  is close to the identity in the following sense:*

$$\text{if } \mathcal{Z} \in \mathcal{S}_w \cap \{\Im(\mathcal{Z}) < 1\} \text{ then } \chi_w(\mathcal{Z}) = \mathcal{Z} + O(|w|^{1/2} \log |w|) = \mathcal{Z} + o(r_w).$$

*Proof.* Observe that

$$\chi_w'(\mathcal{Z}) = 1 - \frac{\sqrt{w}(1-a)\pi}{2} \cot(\pi \mathcal{Z}) \quad \text{and} \quad \sup_{\mathcal{Z} \in \mathcal{S}_w} |\cot(\pi \mathcal{Z})| \in O(|w|^{-1/4}).$$

As a consequence,

$$\sup_{\mathcal{Z} \in \mathcal{S}_w} |\chi_w'(\mathcal{Z}) - 1| \in O(|w|^{1/4}).$$

If  $\chi_w(\mathcal{Z}_1) = \chi_w(\mathcal{Z}_2)$ , then

$$|\mathcal{Z}_2 - \mathcal{Z}_1| = |(\chi_w(\mathcal{Z}_1) - \mathcal{Z}_1) - (\chi_w(\mathcal{Z}_2) - \mathcal{Z}_2)| \leq \sup_{[\mathcal{Z}_1, \mathcal{Z}_2]} |\chi_w'(\mathcal{Z}) - 1| \cdot |\mathcal{Z}_2 - \mathcal{Z}_1|.$$

When  $w$  is sufficiently close to 0, the supremum is smaller than 1 and we necessarily have  $\mathcal{Z}_1 = \mathcal{Z}_2$ .

The second assertion of the lemma follows directly from the definition of  $\chi_w$  and the fact that on  $\mathcal{S}_w \cap \{\Im(\mathcal{Z}) < 1\}$ ,  $|\sin(\pi \mathcal{Z})| \geq c |w|^{1/4}$  for some positive constant  $c$ .  $\square$

From now on, we assume that  $w$  is sufficiently close to 0 so that  $\chi_w : \mathcal{S}_w \rightarrow \mathbb{C}$  is univalent.

**Definition 2.3.** *The approximate Fatou coordinates  $\varphi_w$  are the maps*

$$\varphi_w := \chi_w \circ \psi_w^{-1} : V_w \rightarrow \mathbb{C} \quad \text{with } w \in B_r.$$

We need to prove that these approximate Fatou coordinates satisfy Properties 1, 2 and 3.

Observe that there is a slight abuse in terminology here since the actual approximation to the Fatou coordinate is given by  $\frac{2}{\sqrt{w}}\varphi_w$ , as stated in Property 1. This normalization is chosen so that in the new coordinates, the parabolic transition (which requires approximately  $2/\sqrt{w}$  steps) occurs in a vertical strip of width 1 (see Figure 3).

**2.4. Comparison with the attracting Fatou coordinate.** In this paragraph, we prove that the approximate Fatou coordinate  $\varphi_w$  satisfies Property 1, namely when  $w \rightarrow 0$  in  $B_r$ ,  $D_w^{\text{att}} \subset \phi_f(V_w)$  and

$$(9) \quad \sup_{Z \in D_w^{\text{att}}} \left| \frac{2}{\sqrt{w}} \cdot \varphi_w \circ \phi_f^{-1}(Z) - Z \right| \rightarrow 0.$$

Recall that  $R_w = |w|^{-\alpha/2}$ ,  $r_w = |w|^{1/2} R_w = |w|^{(1-\alpha)/2}$ , and  $D_w^{\text{att}} = D(R_w, R_w/10)$ .

*Proof of Property 1.*

**Step 1.** Let us first prove that  $D_w^{\text{att}} \subset \phi_f(V_w)$ . Note that  $R_w \rightarrow +\infty$  as  $w \rightarrow 0$ , hence  $D_w^{\text{att}} \subset \phi_f(P_f^{\text{att}})$  for  $w$  close to 0. If  $z \in \phi_f^{-1}(D_w^{\text{att}})$ , then

$$\phi_f(z) = -\frac{1}{z} + o\left(\frac{1}{z}\right) = O(R_w).$$

In addition,

$$\zeta^\pm(w) = \pm \frac{\pi i}{2} \sqrt{w} (1 + o(1)) \quad \text{and} \quad \frac{\zeta^\pm(w)}{z} = O(r_w).$$

In particular,

$$\begin{aligned} \log\left(\frac{z - \zeta^+(w)}{z - \zeta^-(w)}\right) &= \log\left(1 - \frac{\zeta^+(w)}{z}\right) - \log\left(1 - \frac{\zeta^-(w)}{z}\right) \\ &= -\frac{\zeta^+(w)}{z} + \frac{\zeta^-(w)}{z} + O(r_w^2) = -\pi i \frac{\sqrt{w}}{z} + O(r_w^2). \end{aligned}$$

Since  $\alpha > 1/2$ , we have that  $r_w^2 = |w|^{1-\alpha} = o(|w|^{1/2})$ , and it follows that

$$(10) \quad \mathcal{Z} := \frac{1}{2\pi i} \log\left(\frac{z - \zeta^+(w)}{z - \zeta^-(w)}\right) = -\frac{\sqrt{w}}{2z} + O(r_w^2) = -\frac{\sqrt{w}}{2z} + o(|w|^{1/2}).$$

So, the real part of  $\mathcal{Z}$  is comparable to  $r_w$  and since  $|w|^{1/4} = o(r_w)$ , we get that  $\mathcal{Z} \in \mathcal{S}_w$ , whence  $z = \psi_w(\mathcal{Z}) \in V_w$  for  $w \in B_r$  close enough to 0.

**Step 2.** We now establish (9). Note that

$$\sup_{Z \in D_w^{\text{att}}} \left| \frac{2}{\sqrt{w}} \cdot \varphi_w \circ \phi_f^{-1}(Z) - Z \right| = \sup_{z \in \phi_f^{-1}(D_w^{\text{att}})} \left| \frac{2}{\sqrt{w}} \cdot \varphi_w(z) - \phi_f(z) \right|.$$

Observe first that when  $w$  tends to 0, the domain  $\phi_f^{-1}(D_w^{\text{att}})$  also tends to 0. So, if  $z \in \phi_f^{-1}(D_w^{\text{att}})$ , then

$$\phi_f(z) = -\frac{1}{z} - (1-a) \log\left(-\frac{1}{z}\right) + o(1).$$

On the other hand,

$$\frac{2 \sin(\pi \mathcal{Z})}{\pi \sqrt{w}} = \frac{2}{\sqrt{w}} (\mathcal{Z} + o(\mathcal{Z})) = -\frac{1}{z} + o\left(\frac{1}{z}\right).$$

Thus, together with the estimate (10) we infer that

$$\begin{aligned}
\varphi_w(z) = \chi_w(\mathcal{Z}) &= \mathcal{Z} - \frac{\sqrt{w} \cdot (1-a)}{2} \log \left( \frac{2 \sin(\pi \mathcal{Z})}{\pi \sqrt{w}} \right) \\
&= -\frac{\sqrt{w}}{2z} + o(|w|^{1/2}) - \frac{\sqrt{w} \cdot (1-a)}{2} \log \left( -\frac{1}{z} + o\left(\frac{1}{z}\right) \right) \\
(11) \quad &= \frac{\sqrt{w}}{2} \left( -\frac{1}{z} - (1-a) \log \left( -\frac{1}{z} \right) + o(1) \right) \\
&= \frac{\sqrt{w}}{2} (\phi_f(z) + o(1)),
\end{aligned}$$

which completes the proof.  $\square$

**2.5. Comparison with the repelling Fatou coordinate.** In this paragraph, we deal with Property 2, that is, we wish to prove that as  $w \rightarrow 0$  in  $B_r$ ,

$$\mathcal{D}'_w := 1 + \frac{\sqrt{w}}{2} \cdot D_w^{\text{rep}} \subset \varphi_w \left( V_w \cap P_f^{\text{rep}} \right)$$

and

$$\sup_{Z \in D_w^{\text{rep}}} \left| \psi_f^{-1} \circ \varphi_w^{-1} \left( 1 + \frac{\sqrt{w}}{2} Z \right) - Z \right| \rightarrow 0.$$

The proof is rather similar to that of Property 1.

*Proof of Property 2.*

**Step 1.** Let us first prove that for  $w \in B_r$  close enough to 0, the disk  $\mathcal{D}'_w$  is contained in  $\varphi_w(V_w)$ . Note that with  $r_w = |w|^{1/2} R_w = |w|^{(1-\alpha)/2}$  as before, we have

$$\mathcal{D}'_w = D \left( 1 - \frac{\sqrt{w} R_w}{2}, \frac{r_w}{20} \right).$$

Since  $\alpha > 1/2$ , we have that  $|w|^{1/4} = o(r_w)$ . Furthermore,  $\Re(\sqrt{w}) > \frac{\sqrt{2}}{2} |w|^{1/2}$  for  $w \in B_r$ , hence

$$\mathcal{D}''_w := D \left( 1 - \frac{\sqrt{w} R_w}{2}, \frac{r_w}{10} \right) \subset S_w.$$

In addition, by Lemma 2.2,  $\chi_w(\mathcal{Z}) = \mathcal{Z} + o(r_w)$  for  $\mathcal{Z} \in \mathcal{D}''_w$ , so

$$\mathcal{D}'_w \subset \chi_w(\mathcal{D}''_w) \subset \chi_w(S_w) = \varphi_w(V_w).$$

**Step 2.** Given  $Z \in D_w^{\text{rep}}$ , we set

$$\mathcal{X} := \chi_w^{-1} \left( 1 + \frac{\sqrt{w}}{2} Z \right).$$

Note that  $\sqrt{w}Z$  has modulus equal to  $r_w$ . Also, put

$$(12) \quad z := \varphi_w^{-1} \left( 1 + \frac{\sqrt{w}}{2} Z \right) = ic_1 \sqrt{w} \cot(\pi \mathcal{X}) + c_2 w.$$

By Lemma 2.2 we have that

$$\mathcal{X} - 1 = \frac{\sqrt{w}}{2} Z \cdot (1 + o(1)) = O(r_w),$$

hence

$$\cot(\pi \mathcal{X}) = \cot(\pi(\mathcal{X} - 1)) = \frac{2}{\pi \sqrt{w} Z} (1 + o(1)).$$



Remembering that  $c_1 = \pi i/2$ , from (12) we deduce that

$$z = \frac{2ic_1}{\pi Z} \cdot (1 + o(1)) \quad \text{with} \quad \frac{2ic_1}{\pi Z} = -\frac{1}{Z} \in D\left(|w|^{\alpha/2}, \frac{|w|^{\alpha/2}}{2}\right).$$

So when  $w \in B_r$  is close enough to 0, we find that  $z \in P_f^{\text{rep}}$  and

$$\psi_f^{-1}(z) = -\frac{1}{z} - (1-a) \log\left(\frac{1}{z}\right) + o(1).$$

Moreover,

$$\frac{2 \sin(\pi X)}{\pi \sqrt{w}} = -\frac{2 \sin(\pi(X-1))}{\pi \sqrt{w}} = -\frac{2}{\sqrt{w}} \left( \frac{\sqrt{w}}{2} Z \cdot (1 + o(1)) \right) = \frac{1}{z} + o\left(\frac{1}{z}\right).$$

Finally, as in (11), we compute

$$\begin{aligned} Z &= \frac{2}{\sqrt{w}} (\chi_w(X) - 1) = -\frac{1}{z} + o(1) - (1-a) \log\left(\frac{1}{z} + o\left(\frac{1}{z}\right)\right) \\ &= \psi_f^{-1}(z) + o(1) \\ &= \psi_f^{-1} \circ \varphi_w^{-1} \left( 1 + \frac{\sqrt{w}}{2} Z \right) + o(1). \end{aligned}$$

This completes the proof of Property 2.  $\square$

**2.6. Approximate translation property.** In this paragraph, we prove that the approximate Fatou coordinate  $\varphi_w$  satisfies Property 3, that is: as  $w \rightarrow 0$  in  $B_r$ , the inclusions  $\mathcal{R}_w \subset \varphi_w(V_w)$  and  $f_w \circ \varphi_w^{-1}(\mathcal{R}_w) \subset V_{g(w)}$  hold (recall that the rectangle  $\mathcal{R}_w$  was defined in (6)), and

$$\sup_{Z \in \mathcal{R}_w} \left| \varphi_{g(w)} \circ f_w \circ \varphi_w^{-1}(Z) - Z - \frac{\sqrt{w}}{2} \right| = o(w).$$

*Outline of the proof.* Let

$$\psi^0 := \psi_w, \quad \psi^2 := \psi_{g(w)}, \quad \chi^0 := \chi_w, \quad \text{and} \quad \chi^2 := \chi_{g(w)}.$$

To handle the fact that  $f_w \circ \zeta^\pm$  is not exactly equal to  $\zeta^\pm \circ g$  (see Lemma 2.1), rather than dealing directly with  $\psi_2 \circ f_w \circ \psi_0^{-1}$ , we introduce an intermediate change of coordinates

$$\psi^1 : \mathbb{C} \ni Z \mapsto \frac{f_w(\zeta^-(w)) \cdot e^{2\pi i Z} - f_w(\zeta^+(w))}{e^{2\pi i Z} - 1} \in \mathbb{P}^1(\mathbb{C}) \setminus \{f_w(\zeta^+(w)), f_w(\zeta^-(w))\}.$$

Let  $\mathcal{H}$  be the horizontal strip

$$\mathcal{H} := \{Z \in \mathbb{C} ; -1 < \Im(Z) < 1\}.$$

We will see that there are lifts  $\mathcal{F}^0 : \mathcal{S}_w \rightarrow \mathbb{C}$ ,  $\mathcal{F}^1 : \mathcal{H} \rightarrow \mathbb{C}$  and a map  $\mathcal{F} : \mathcal{R}_w \rightarrow \mathbb{C}$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathcal{R}_w & \xrightarrow{\mathcal{F}} & \mathbb{C} \\
 & \nearrow \varphi_w & \uparrow \chi^0 & & \uparrow \chi^2 \\
 & & \mathcal{Q}_w & \xrightarrow{\mathcal{F}^0} & \mathcal{F}^0(\mathcal{Q}_w) & \xrightarrow{\mathcal{F}^1} & \mathcal{S}_{g(w)} & \xrightarrow{\varphi_{g(w)}} & \mathbb{C} \\
 & & \downarrow \psi^0 & & \downarrow \psi^1 & & \downarrow \psi^2 & & \downarrow \psi^2 \\
 & & \psi^0(\mathcal{Q}_w) & \xrightarrow{f_w} & V_{g(w)} & \xrightarrow{\text{id}} & V_{g(w)} & & \mathbb{C}
 \end{array}$$

**Step 1.** We prove that  $\mathcal{R}_w \subset \varphi_w(V_w) = \chi^0(\mathcal{S}_w)$  and that

$$\sup_{Z \in \mathcal{R}_w} |\varphi_w^{-1}(Z)| = O(|w|^{\alpha/2}).$$

Define  $\mathcal{Q}_w := (\chi^0)^{-1}(\mathcal{R}_w) \subset \mathcal{S}_w$ .

**Step 2.** We define  $\mathcal{F}^0$  on  $\mathcal{S}_w$  and prove that for  $Z \in \mathcal{Q}_w$ ,

$$(13) \quad \mathcal{F}^0(Z) = Z + \frac{\sqrt{w}}{2} + \frac{\pi(1-a)w}{4} \cot(\pi Z) + o(w).$$

In particular, for  $w$  close enough to 0,  $\mathcal{F}^0(Z) = Z + O(|w|^{1/2})$ , hence  $\mathcal{F}^0(\mathcal{Q}_w) \subset \mathcal{H}$ .

**Step 3.** We define  $\mathcal{F}^1$  on  $\mathcal{H}$  and prove that for  $Z \in \mathcal{H}$ ,

$$\mathcal{F}^1(Z) = Z + o(w).$$

In particular, for  $w$  close enough to 0,  $\mathcal{F}^1 \circ \mathcal{F}^0(Z) = \mathcal{F}^0(Z) + o(w)$ , from which we deduce that

$$\mathcal{F}^1 \circ \mathcal{F}^0(\mathcal{Q}_w) \subset \mathcal{S}_{g(w)} \quad \text{whence} \quad f_w \circ \varphi_w^{-1}(\mathcal{R}_w) \subset V_{g(w)}.$$

**Step 4.** We use  $\chi^0$  and  $\chi^2$  to eliminate the third term on the right hand side of (13). Specifically, we define  $\mathcal{F}$  on  $\mathcal{R}_w$  and prove that for  $Z \in \mathcal{R}_w$ ,

$$\mathcal{F}(Z) = Z + \frac{\sqrt{w}}{2} + o(w).$$

Thus Property 3 is established.  $\square$

2.6.1. *Proof of Step 1.* We prove that  $\mathcal{R}_w \subset \varphi_w(V_w)$  and that

$$\sup_{Z \in \mathcal{R}_w} |\varphi_w^{-1}(Z)| = O(|w|^{\alpha/2}).$$

Let  $\mathcal{R}'_w \subset \mathcal{S}_w$  be the rectangle

$$\mathcal{R}'_w := \left\{ Z \in \mathcal{S}_w ; \frac{r_w}{20} < \Re(Z) < 1 - \frac{r_w}{20} \text{ and } -1 < \Im(Z) < 1 \right\},$$

with, as before,  $r_w := |w|^{(1-\alpha)/2}$ . We see that  $\mathcal{R}_w \subset \mathcal{R}'_w$  and the distance between the boundaries is  $r_w/20$ . On the other hand, by Lemma 2.2, for  $Z \in \mathcal{R}'_w$ ,  $\chi_w(Z) = Z + o(r_w)$ . From this it follows that  $\chi_w(\partial \mathcal{R}'_w)$  surrounds  $\mathcal{R}_w$ , whence

$$\mathcal{R}_w \subset \chi_w(\mathcal{R}'_w) \subset \chi_w(\mathcal{S}_w) = \varphi_w(V_w),$$

as desired.

To prove the estimate on  $\varphi_w^{-1}(\mathcal{Z})$ , define  $\mathcal{Q}_w := \chi_w^{-1}(\mathcal{R}_w)$ . Since  $\varphi_w = \chi_w \circ \psi_w^{-1}$ , we see that  $\varphi_w^{-1}(\mathcal{R}_w) = \psi_w(\mathcal{Q}_w)$ . The above sequence of inclusions shows that  $\mathcal{Q}_w \subset \mathcal{R}'_w$ . Thus from

$$\psi_w(\mathcal{Z}) = ic_1\sqrt{w} \cot(\pi\mathcal{Z}) + c_2w,$$

we infer that

$$\begin{aligned} \sup_{\mathcal{Z} \in \mathcal{R}_w} |\varphi_w^{-1}(\mathcal{Z})| &= \sup_{\mathcal{Z} \in \mathcal{Q}_w} |\psi_w(\mathcal{Z})| \leq \sup_{\mathcal{Z} \in \mathcal{R}'_w} |\psi_w(\mathcal{Z})| \\ &= O\left(\frac{|w|^{1/2}}{r_w}\right) + O(w) = O(|w|^{\alpha/2}). \end{aligned}$$

This completes the proof of Step 1.

2.6.2. *Proof of Step 2.* We define  $\mathcal{F}^0$  on  $\mathcal{S}_w$  and prove that for  $\mathcal{Z} \in \mathcal{Q}_w$ ,

$$\mathcal{F}^0(\mathcal{Z}) = \mathcal{Z} + \frac{\sqrt{w}}{2} + \frac{\pi(1-a)w}{4} \cot(\pi\mathcal{Z}) + o(w) = \mathcal{Z} + O(|w|^{1/2}).$$

**Step 2.1.** We first define  $\mathcal{F}^0$ . It will be convenient to set  $w = \varepsilon^2$  so that expansions with respect to  $\sqrt{w}$  become expansions with respect to  $\varepsilon$ . Set

$$\zeta_0^\pm(\varepsilon) := \zeta^\pm(\varepsilon^2) = \pm c_1\varepsilon + c_2\varepsilon^2, \quad \zeta_1^\pm(\varepsilon) := f_{\varepsilon^2} \circ \zeta_0^\pm(\varepsilon) \quad \text{and} \quad \zeta_2^\pm(\varepsilon) := \zeta^\pm \circ g(\varepsilon^2).$$

Choose  $r_1 > 0$  small enough so that the only preimage of 0 by  $f$  within  $D(0, 2r_1)$  is 0. Choose  $r_2 > 0$  so that for  $\varepsilon \in D(0, r_2)$ , the only preimage of  $\zeta_1^\pm(\varepsilon) = f_{\varepsilon^2}(\zeta_0^\pm(\varepsilon))$  under  $f_{\varepsilon^2}$  within  $D(0, r_1)$  is  $\zeta_0^\pm(\varepsilon)$ . The function

$$(z, \varepsilon) \mapsto \frac{f_{\varepsilon^2}(z) - \zeta_1^+(\varepsilon)}{z - \zeta_0^+(\varepsilon)} \cdot \frac{z - \zeta_0^-(\varepsilon)}{f_{\varepsilon^2}(z) - \zeta_1^-(\varepsilon)}$$

extends holomorphically to  $\Delta := D(0, r_1) \times D(0, r_2)$  and does not vanish there. In addition, it identically takes the value 1 for  $\varepsilon = 0$ . We set

$$u : (z, \varepsilon) \mapsto \frac{1}{2\pi i} \log \left( \frac{f_{\varepsilon^2}(z) - \zeta_1^+(\varepsilon)}{z - \zeta_0^+(\varepsilon)} \cdot \frac{z - \zeta_0^-(\varepsilon)}{f_{\varepsilon^2}(z) - \zeta_1^-(\varepsilon)} \right)$$

where the branch of logarithm is chosen so that  $u(z, 0) \equiv 0$ . Consider the map  $\mathcal{F}^0$  defined on  $\mathcal{S}_w$  by

$$\mathcal{F}^0(\mathcal{Z}) := \mathcal{Z} + u(\psi^0(\mathcal{Z}), \varepsilon).$$

Then, for  $\mathcal{Z} \in \mathcal{S}_w$ , set

$$z := \psi^0(\mathcal{Z}) \in V_w \quad \text{so that} \quad \mathcal{Z} = \frac{1}{2\pi i} \log \left( \frac{z - \zeta_0^+(\varepsilon)}{z - \zeta_0^-(\varepsilon)} \right).$$

As  $\mathcal{Z}$  ranges in  $\mathcal{S}_w$ ,  $z$  avoids the points  $\zeta_0^\pm(\varepsilon)$  and remains in a small disk around 0, thus  $f_{\varepsilon^2}(z)$  avoids the points  $\zeta_1^\pm(\varepsilon) = f_{\varepsilon^2}(\zeta_0^\pm(\varepsilon))$ . So we can define

$$\mathcal{Z}_1 := \frac{1}{2\pi i} \log \left( \frac{f_{\varepsilon^2}(z) - \zeta_1^+(\varepsilon)}{f_{\varepsilon^2}(z) - \zeta_1^-(\varepsilon)} \right)$$

where the branch is chosen so that

$$\mathcal{Z}_1 - \mathcal{Z} = u(z, \varepsilon) = u(\psi^0(\mathcal{Z}), \varepsilon) = \mathcal{F}^0(\mathcal{Z}) - \mathcal{Z}.$$

We therefore have

$$\psi^1 \circ \mathcal{F}^0(\mathcal{Z}) = \psi^1(\mathcal{Z}_1) = f_{\varepsilon^2}(z) = f_{\varepsilon^2} \circ \psi^0(\mathcal{Z}).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_w & \xrightarrow{\mathcal{F}^0} & \mathbb{C} \\ \psi^0 \downarrow & & \downarrow \psi^1 \\ V_w & \xrightarrow{f_w} & \mathbb{P}^1(\mathbb{C}). \end{array}$$

**Step 2.2.** We now prove that for  $(z, \varepsilon) \in \Delta$ , the following estimate is true:

$$2\pi i u(z, \varepsilon) = 2c_1 \varepsilon - 2c_1(1-a)\varepsilon z + O(\varepsilon z^2) + O(\varepsilon^3).$$

Indeed, observe that

$$\frac{f_{\varepsilon^2}(z) - \zeta_1^+(\varepsilon)}{z - \zeta_0^+(\varepsilon)} \cdot \frac{z - \zeta_0^-(\varepsilon)}{f_{\varepsilon^2}(z) - \zeta_1^-(\varepsilon)} = \frac{f(z) - f(\zeta_0^+(\varepsilon))}{z - \zeta_0^+(\varepsilon)} \cdot \frac{z - \zeta_0^-(\varepsilon)}{f(z) - f(\zeta_0^-(\varepsilon))},$$

whence

$$2\pi i u(z, \varepsilon) = \log \left( \frac{1 - f(\zeta_0^+(\varepsilon))/f(z)}{1 - \zeta_0^+(\varepsilon)/z} \right) - \log \left( \frac{1 - f(\zeta_0^-(\varepsilon))/f(z)}{1 - \zeta_0^-(\varepsilon)/z} \right).$$

Recall that  $\zeta_0^\pm(\varepsilon) = \pm c_1 \varepsilon + c_2 \varepsilon^2$ , so

$$f(\zeta_0^\pm(\varepsilon)) = \pm c_1 \varepsilon + c_3 \varepsilon^2 + O(\varepsilon^3) \quad \text{with} \quad c_3 := c_2 + c_1^2.$$

Since  $u$  is holomorphic on  $\Delta$  and  $u(z, 0) \equiv 0$ , it admits an expansion of the form  $u(z, \varepsilon) = u_1(z)\varepsilon + u_2(z)\varepsilon^2 + O(\varepsilon^3)$  on  $\Delta$ . To find  $u_1$  and  $u_2$  we write

$$\begin{aligned} \log \left( \frac{1 - f(\zeta_0^\pm(\varepsilon))/f(z)}{1 - \zeta_0^\pm(\varepsilon)/z} \right) &= \pm c_1 \cdot \left( -\frac{1}{f(z)} + \frac{1}{z} \right) \cdot \varepsilon \\ &\quad + \left( -\frac{c_3}{f(z)} - \frac{c_1^2}{2(f(z))^2} + \frac{c_2}{z} + \frac{c_1^2}{2z^2} \right) \cdot \varepsilon^2 + O(\varepsilon^3), \end{aligned}$$

and taking the difference between these two expressions we conclude that

$$\begin{aligned} 2\pi i u(z, \varepsilon) &= 2c_1 \cdot \left( \frac{1}{z} - \frac{1}{f(z)} \right) \cdot \varepsilon + O(\varepsilon^3) \\ &= 2c_1 \varepsilon - 2c_1(1-a)\varepsilon z + O(\varepsilon z^2) + O(\varepsilon^3). \end{aligned}$$

**Step 2.3.** We finally establish (13). If  $Z \in Q_w$  and

$$z := \psi^0(Z) = ic_1 \varepsilon \cot(\pi Z) + c_2 \varepsilon^2 = ic_1 \varepsilon \cot(\pi Z) + o(\varepsilon),$$

it then follows from Step 1 of the proof (see (14)) that  $z = O(|\varepsilon|^\alpha)$ . Since  $\alpha > 1/2$  we see that  $O(\varepsilon z^2) = O(|\varepsilon|^{1+2\alpha}) \subset o(\varepsilon^2)$ .<sup>3</sup> Thus, for  $Z \in Q_w$ ,

$$\begin{aligned} u(\psi^0(Z), \varepsilon) &= \frac{2c_1}{2\pi i} \varepsilon - \frac{c_1^2 \cdot (1-a)}{\pi} \varepsilon^2 \cot(\pi Z) + o(\varepsilon^2) \\ &= \frac{\varepsilon}{2} + \frac{\pi(1-a)\varepsilon^2}{4} \cot(\pi Z) + o(\varepsilon^2). \end{aligned}$$

Since  $\mathcal{F}^0(Z) = Z + u(\psi^0(Z), \varepsilon)$ , we arrive at the desired estimate (13), and the proof of Step 2 is complete.

<sup>3</sup>This computation is responsible for the lower bound in the choice of the value of  $\alpha$ .

2.6.3. *Proof of Step 3.* We define  $\mathcal{F}^1$  on the horizontal strip  $\mathcal{H}$  and prove that

$$\mathcal{F}^1(\mathcal{Z}) = \mathcal{Z} + o(w).$$

The idea is simply that since the distance  $|\zeta_2^\pm - \zeta_1^\pm|$  is much smaller than  $|\zeta_1^+ - \zeta_1^-|$ ,  $(\psi^2)^{-1} \circ \psi^1$  is very close to the identity.

**Step 3.1.** We first define  $\mathcal{F}^1$ . Let  $\mu_1 : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  and  $\mu_2 : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  be the Möbius transformations defined by (recall that  $\varepsilon = \sqrt{w}$ )

$$\mu_1(z) := \frac{z - \zeta_1^+(\varepsilon)}{z - \zeta_1^-(\varepsilon)} \quad \text{and} \quad \mu_2(z) := \frac{z - \zeta_2^+(\varepsilon)}{z - \zeta_2^-(\varepsilon)}.$$

The Möbius transformation  $\mu := \mu_2 \circ \mu_1^{-1} : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  sends  $\mu_1 \circ \zeta_2^+(\varepsilon)$  to 0,  $\mu_1 \circ \zeta_2^-(\varepsilon)$  to  $\infty$  and fixes 1. Set

$$\delta^+ := \mu_1 \circ \zeta_2^+(\varepsilon) = \frac{\zeta_2^+(\varepsilon) - \zeta_1^+(\varepsilon)}{\zeta_2^+(\varepsilon) - \zeta_1^-(\varepsilon)} \quad \text{and} \quad \delta^- := \frac{1}{\mu_1 \circ \zeta_2^-(\varepsilon)} = \frac{\zeta_2^-(\varepsilon) - \zeta_1^-(\varepsilon)}{\zeta_2^-(\varepsilon) - \zeta_1^+(\varepsilon)}.$$

Note that

$$\zeta_2^\pm(\varepsilon) - \zeta_1^\pm(\varepsilon) = O(\varepsilon^4) \quad \text{whereas} \quad \zeta_2^\pm(\varepsilon) - \zeta_1^\mp(\varepsilon) = i\pi\varepsilon \cdot (1 + o(1)),$$

therefore

$$\delta^+ = O(\varepsilon^3) \quad \text{and} \quad \delta^- = O(\varepsilon^3).$$

Thus, the image of the horizontal strip  $\mathcal{H} = \{-1 < \Im(\mathcal{Z}) < 1\}$  under the exponential map

$$\exp : \mathbb{C} \ni \mathcal{Z} \mapsto e^{2\pi i \mathcal{Z}} \in \mathbb{C} \setminus \{0\}$$

avoids  $\delta^+$  and  $1/\delta^-$  and  $\mu : \exp(\mathcal{H}) \rightarrow \mathbb{C} \setminus \{0\}$  lifts to a map  $\mathcal{F}^1 : \mathcal{H} \rightarrow \mathbb{C}$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{F}^1} & \mathbb{C} \\ \downarrow \exp & & \downarrow \exp \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\mu} & \mathbb{P}^1(\mathbb{C}) \\ \uparrow \mu_1 & & \uparrow \mu_2 \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\text{id}} & \mathbb{P}^1(\mathbb{C}) \end{array} \quad \begin{array}{c} \psi^1 \\ \psi^2 \end{array}$$

Since  $\mu(1) = 1$ , the choice of lift is completely determined by requiring  $\mathcal{F}^1(0) = 0$ .

**Step 3.2.** We estimate  $\mathcal{F}^1(\mathcal{Z}) - \mathcal{Z}$ . Since  $\mu(\delta^+) = 0$ ,  $\mu(1/\delta^-) = \infty$  and  $\mu(1) = 1$ , we infer that

$$\mu(z) = z \cdot \frac{1 - \delta^-}{1 - \delta^+} \cdot \frac{1 - \delta^+/z}{1 - \delta^-/z}.$$

As a consequence,

$$\mathcal{F}^1(\mathcal{Z}) - \mathcal{Z} = \log(1 - \delta^-) - \log(1 - \delta^+) + \log(1 - \delta^+/z) + \log(1 - \delta^-/z),$$

where  $\log$  is the principal branch of the logarithm on  $\mathbb{C} \setminus \mathbb{R}^-$  (the arguments of the four logarithms are close to 1). Since  $\delta^+ = O(\varepsilon^3)$  and  $\delta^- = O(\varepsilon^3)$ , we conclude that

$$\sup_{\mathcal{Z} \in \mathcal{H}} |\mathcal{F}^1(\mathcal{Z}) - \mathcal{Z}| = O(\varepsilon^3) \subset o(\varepsilon^2) = o(w),$$

as desired.

**Step 3.3.** Let us show that  $\mathcal{F}^1 \circ \mathcal{F}^0(Q_w) \subset \mathcal{S}_{g(w)}$ . First, we saw in the course of Step 1 that  $Q_w \subset \mathcal{R}'_w$ . Now, when  $w$  is small,

$$|g(w)|^{1/4} = |w|^{1/4} + o(|w|^{1/4}),$$

hence  $\mathcal{R}'_w \subset \mathcal{S}_{g(w)}$  and the distance between the boundaries is comparable to  $|w|^{1/4}$ . Since  $\mathcal{F}^1 \circ \mathcal{F}^0(Z) = Z + O(|w|^{1/2})$  on  $Q_w$  and  $|w|^{1/2} = o(|w|^{1/4})$ , we see that  $\mathcal{F}^1 \circ \mathcal{F}^0(Q_w) \subset \mathcal{S}_{g(w)}$ , as claimed.

From this we deduce that

$$f_w \circ \varphi_w^{-1}(\mathcal{R}_w) = f_w \circ \psi_w(Q_w) = \psi_{g(w)}(\mathcal{F}^1 \circ \mathcal{F}^0(Q_w)) \subset \psi_{g(w)}(\mathcal{S}_{g(w)}) = V_{g(w)},$$

which finishes the proof of Step 3.

2.6.4. *Proof of Step 4.* We define  $\mathcal{F}$  on  $\mathcal{R}_w$  and prove that

$$\mathcal{F}(Z) = Z + \frac{\sqrt{w}}{2} + o(w).$$

Let

$$\mathcal{F} := \chi^2 \circ (\mathcal{F}^1 \circ \mathcal{F}^0) \circ (\chi^0)^{-1} : \mathcal{R}_w \rightarrow \mathbb{C},$$

so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}_w & \xrightarrow{\mathcal{F}} & \mathbb{C} \\ \chi^0 \uparrow & & \uparrow \chi^2 \\ Q_w & \xrightarrow{\mathcal{F}^0} \mathcal{F}^0(Q_w) \xrightarrow{\mathcal{F}^1} & \mathcal{S}_{g(w)}. \end{array}$$

For  $Z \in Q_w$ , define

$$v(Z) := \frac{\sqrt{w}}{2} + \frac{\pi(1-a)w}{4} \cot(\pi Z) = \frac{\sqrt{w}}{2} + o(|w|^{1/2}),$$

where the second equality follows from the fact that  $\cot(\pi Z) = O(r_w^{-1})$  on  $Q_w$ . Now we write

$$\begin{aligned} \mathcal{F}(\chi^0(Z)) &= \chi^2(Z + v(Z) + o(w)) \\ (14) \quad &= Z + v(Z) + o(w) + \frac{\sqrt{g(w)}(1-a)}{2} \cdot \log \left( \frac{2 \sin(\pi Z + \pi v(Z) + o(w))}{\sqrt{g(w)}} \right). \end{aligned}$$

Using

$$\sqrt{g(w)} = \sqrt{w + O(w^2)} = \sqrt{w} + O(|w|^{3/2}),$$

and arguing as in Lemma 2.2, we see that the logarithm in (14) is  $O(\log |w|)$ . Thus we infer that

$$\mathcal{F}(\chi^0(Z)) = Z + v(Z) - \frac{\sqrt{w}(1-a)}{2} \log \left( \frac{2 \sin(\pi Z + \pi v(Z))}{\pi \sqrt{w}} \right) + o(w),$$

as a result:

$$\mathcal{F}(\chi^0(Z)) - \chi^0(Z) = v(Z) - \frac{\sqrt{w}(1-a)}{2} \log \left( \frac{\sin(\pi Z + \pi v(Z))}{\sin(\pi Z)} \right) + o(w).$$

From the estimate  $v(\mathcal{Z}) = \sqrt{w}/2 + o(|w|^{1/2})$ , we deduce

$$\begin{aligned} \frac{\sin(\pi\mathcal{Z} + \pi v(\mathcal{Z}))}{\sin(\pi\mathcal{Z})} &= \frac{\sin(\pi\mathcal{Z}) + \frac{\pi\sqrt{w}}{2} \cos(\pi\mathcal{Z}) + o(|w|^{1/2})}{\sin(\pi\mathcal{Z})} \\ &= 1 + \frac{\pi\sqrt{w}}{2} \cdot \cot(\pi\mathcal{Z}) + o(|w|^{1/2}). \end{aligned}$$

So finally,

$$\begin{aligned} \mathcal{F}(\chi^0(\mathcal{Z})) - \chi^0(\mathcal{Z}) &= \frac{\sqrt{w}}{2} + \frac{\pi(1-a)w}{4} \cot(\pi\mathcal{Z}) \\ &\quad - \frac{\sqrt{w}(1-a)}{2} \cdot \frac{\pi\sqrt{w}}{2} \cot(\pi\mathcal{Z}) + o(w) \\ &= \frac{\sqrt{w}}{2} + o(w). \end{aligned}$$

This completes the proof of Step 4, and accordingly of Property 3 of the approximate Fatou coordinates.  $\square$

### 3. PARABOLIC IMPLOSION

This section is devoted to the proof of Proposition A.

**3.1. Set up and notation.** Let  $C_f$  be a compact subset of  $\mathcal{B}_f$  and  $C_g$  be a compact subset of  $\mathcal{B}_g$ . We need to prove that the sequence of maps

$$\mathbb{C}^2 \ni (z, w) \mapsto P^{\circ 2n+1}(z, g^{\circ n^2}(w)) \in \mathbb{C}^2$$

converges uniformly on  $C_f \times C_g$  to the map

$$C_f \times C_g \ni (z, w) \mapsto (\mathcal{L}_f(z), 0) \in \mathbb{C} \times \{0\}.$$

For  $(z, w) \in C_f \times C_g$  and for  $m \geq 0$ , set

$$w_m := g^{\circ m}(w).$$

This sequence converges uniformly to 0 on  $C_g$  so the difficulty consists in proving that the first coordinate converges uniformly to  $\mathcal{L}_f(z)$ .

To do this, we will have to estimate various quantities which depend on an integer  $k \in [0, 2n+1]$  (corresponding to an iterate  $m = n^2 + k \in [n^2, (n+1)^2]$ ). We adopt the convention that the notation  $o(\cdot)$  or  $O(\cdot)$  stands for an estimate that is uniform on  $C_f \times C_g$ , and depends only on  $n$ , meaning that it is uniform with respect to  $k \in [0, 2n+1]$ .

For  $m_2 \geq m_1 \geq 0$ , we set

$$\mathbf{f}_{m_2, m_1} := f_{w_{m_2-1}} \circ \cdots \circ f_{w_{m_1}} \quad \text{with} \quad f_w(z) := f(z) + \frac{\pi^2}{4}w.$$

By convention, an empty composition is the identity, whence  $\mathbf{f}_{m, m} = \text{id}$ . Then,

$$P^{\circ 2n+1}(z, g^{\circ n^2}(w)) = (\mathbf{f}_{(n+1)^2, n^2}(z), w_{(n+1)^2})$$

so we must prove that

$$\mathbf{f}_{(n+1)^2, n^2}(z) = \mathcal{L}_f(z) + o(1).$$

**3.2. Outline of the proof.** Let us recall that  $R > 0$  was chosen so large that  $F : Z \mapsto -1/f(-1/Z)$  satisfies

$$(15) \quad \sup_{|Z|>R} |F(Z) - Z - 1| < \frac{1}{10} \quad \text{and} \quad \sup_{|Z|>R} |F'(Z) - 1| < \frac{1}{10}.$$

The repelling petal  $P_f^{\text{rep}}$  is the image of  $-\mathbb{H}_R$  under the univalent map  $\psi_f$ , and the notation  $\psi_f^{-1}$  is reserved for the inverse branch  $\psi_f^{-1} : P_f^{\text{rep}} \rightarrow \mathbb{H}_R$ .

Set

$$k_n := \lfloor n^\alpha \rfloor = o(n) \quad \text{where} \quad \frac{1}{2} < \alpha < \frac{2}{3} \text{ is as in (5).}$$

The proof will be divided into four propositions that we state independently, corresponding to three moments of the transition between  $n^2$  and  $(n+1)^2$ . The proofs will be given in §3.4 to 3.7.

We first show that for the first  $k_n$  iterates, the orbit stays close to an orbit of  $f$  (the bound  $\alpha < 2/3$  is important here).

**Proposition 3.1** (Entering the eggbeater). *Assume  $z \in C_f$  and let  $x_n^t$  be defined by  $x_n^t := \mathbf{f}_{n^2+k_n, n^2}(z)$ .<sup>4</sup>*

*Then,  $x_n^t \sim -1/k_n$ , whence  $x_n^t \in \mathcal{B}_f$  for  $n$  large enough. Moreover,*

$$\phi_f(x_n^t) = \phi_f(f^{\circ k_n}(z)) + o(1) \quad \text{as} \quad n \rightarrow +\infty.$$

The next two propositions concern the iterates between  $n^2+k_n$  and  $(n+1)^2-k_n$ .

**Proposition 3.2** (Transition length). *As  $n \rightarrow \infty$ ,*

$$2n \cdot \left( \sum_{m=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w_m}}{2} \right) = 2n - 2k_n + o(1).<sup>5</sup>$$

**Proposition 3.3** (Passing through the eggbeater). *Let  $(x_n^t)_{n \geq 0}$  be a sequence such that  $x_n^t \sim -1/k_n$ , whence  $x_n^t \in \mathcal{B}_f$  for  $n$  large enough. Set*

$$x_n^o := \mathbf{f}_{(n+1)^2-k_n, n^2+k_n}(x_n^t).$$

*Then,  $x_n^o \sim 1/k_n$ , whence  $x_n^o \in P_f^{\text{rep}}$  for  $n$  large enough. Moreover, as  $n \rightarrow \infty$ ,*

$$\psi_f^{-1}(x_n^o) = \phi_f(x_n^t) + 2n \cdot \left( \sum_{m=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w_m}}{2} \right) - 2n + o(1) = \phi_f(x_n^t) - 2k_n + o(1).$$

The last one is similar to Proposition 3.1.

**Proposition 3.4** (Leaving the eggbeater). *Let  $(x_n^o)_{n \geq 0}$  be a sequence contained in  $P_f^{\text{rep}}$  such that  $\psi_f^{-1}(x_n^o) = -k_n + O(1)$  as  $n \rightarrow +\infty$ . Then,*

$$\mathbf{f}_{(n+1)^2, (n+1)^2-k_n}(x_n^o) = f^{\circ k_n}(x_n^o) + o(1) \quad \text{as} \quad n \rightarrow +\infty.$$

<sup>4</sup>The superscript  $\iota$  stands for *incoming*, and in Proposition 3.4 below,  $o$  stands for *outgoing*. This convention was used in [BSU].

<sup>5</sup>Recall that  $\sqrt{w}$  is the square-root with positive real part.



*Proof of Proposition A.* We start with Proposition 3.1: if  $z \in C_f$ , then

$$x_n^t := \mathbf{f}_{n^2+k_n, n^2}(z)$$

satisfies  $x_n^t \sim -1/k_n$  and as  $n \rightarrow +\infty$ ,

$$\phi_f(x_n^t) = \phi_f(f^{\circ k_n}(z)) + o(1) = \phi_f(z) + k_n + o(1).$$

According to Proposition 3.3,

$$x_n^o := \mathbf{f}_{(n+1)^2-k_n, n^2+k_n}(x_n^t) = \mathbf{f}_{(n+1)^2-k_n, n^2}(z)$$

satisfies  $x_n^o \sim 1/k_n$  and as  $n \rightarrow +\infty$ ,

$$\psi_f^{-1}(x_n^o) = \phi_f(z) + k_n - 2k_n + o(1) = \phi_f(z) - k_n + o(1).$$

Finally, since  $\phi_f(z) - k_n + o(1) = -k_n + O(1)$ , Proposition 3.4 implies that as  $n \rightarrow +\infty$ ,

$$\mathbf{f}_{(n+1)^2, n^2}(z) = \mathbf{f}_{(n+1)^2, (n+1)^2-k_n}(x_n^o) = f^{\circ k_n}(x_n^o) + o(1).$$

This in turn finishes the proof of Proposition A because

$$\begin{aligned} f^{\circ k_n}(x_n^o) &= f^{\circ k_n} \circ \psi_f \circ \psi_f^{-1}(x_n^o) = \psi_f\left(\psi_f^{-1}(x_n^o) + k_n\right) \\ &= \psi_f(\phi_f(z) + o(1)) = \mathcal{L}_f(z) + o(1). \quad \square \end{aligned}$$

**3.3. Comparison with classical parabolic implosion.** Propositions 3.1, 3.3 and 3.4 are valid if instead of using the sequence  $(w_m := g^{\circ m}(w))$ , we use the sequence  $(w'_m)$  defined by

$$w'_m := \frac{1}{n^2} \quad \text{if } n^2 \leq m \leq (n+1)^2 - 1.$$

In that case, the only modification is for Proposition 3.2 which has to be replaced by:

$$2n \cdot \left( \sum_{m=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w'_m}}{2} \right) = 2n \cdot \left( \sum_{m=n^2+k_n}^{n^2+2n-k_n} \frac{1}{2n} \right) = 2n - 2k_n + 1 + o(1).$$

Following the proof of Proposition A, we get

$$f_{1/n^2}^{\circ(2n+1)}(z) = \psi_f(\phi_f(z) + 1 + o(1)) = f \circ \mathcal{L}_f(z) + o(1).$$

We thus see that in our non-autonomous context, where the dynamics slowly decelerates as the orbit transits between the eggbeaters, it takes exactly one more iteration to make the transition than in the classical case.

**3.4. Transition length.** In this paragraph, we prove Proposition 3.2, which concerns the dynamics of  $g$  only. We need to show that

$$\sum_{m=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w_m}}{2} = 1 - \frac{k_n}{n} + o\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow +\infty.$$

With  $\phi_g : \mathcal{B}_g \rightarrow \mathbb{C}$  denoting the attracting Fatou coordinate of  $g$ , for all  $k \geq 0$ , we have

$$\phi_g(w_{n^2+k}) = \phi_g(w) + n^2 + k = n^2 + k + O(1).^6$$

<sup>6</sup>Recall that the notations  $o(\cdot)$  and  $O(\cdot)$  mean that the estimates are uniform on  $C_f \times C_g$  and with respect to  $k \in [0, 2n+1]$ .

As a consequence, for  $k \in [k_n, 2n - k_n]$  it holds

$$w_{n^2+k} = \phi_g^{-1}(n^2 + k + O(1)) = \frac{1}{n^2 + k + O(\log n)}$$

and

$$\sqrt{w_{n^2+k}} = \frac{1}{\sqrt{n^2 + k + O(\log n)}} = \frac{1}{n} - \frac{k}{2n^3} + O\left(\frac{\log n}{n^3}\right).$$

It follows that

$$\begin{aligned} \sum_{k=k_n}^{2n-k_n} \sqrt{w_{n^2+k}} &= \frac{2n - 2k_n + 1}{n} - \frac{2n(2n - 2k_n + 1)}{4n^3} + O\left(\frac{\log n}{n^2}\right) \\ &= 2 - \frac{2k_n}{n} + o\left(\frac{1}{n}\right) \end{aligned}$$

and we are done.  $\square$

**3.5. Entering the eggbeater.** In this paragraph, we prove Proposition 3.1, that is: if  $z \in C_f$  and  $x_n^t := \mathbf{f}_{n^2+k_n, n^2}(z)$ , then, as  $n \rightarrow +\infty$ ,

$$x_n^t \sim -\frac{1}{k_n} \quad \text{and} \quad \phi_f(x_n^t) = \phi_f(f^{\circ k_n}(z)) + o(1).$$

**3.5.1. Entering the attracting petal.** Choose  $\kappa_0 \geq 1$  sufficiently large so that

$$f^{\circ \kappa_0}(C_f) \subset P_f^{\text{att}}.$$

For every fixed  $k \geq 0$ , the sequence of polynomials  $(f_{w_{n^2+k}})_{n \geq 0}$  converges locally uniformly to  $f$ . It follows that for every  $k \in [1, \kappa_0]$ , the sequence  $\mathbf{f}_{n^2+k, n^2}$  converges uniformly to  $f^{\circ k}$  on  $C_f$ . In particular, if  $n$  is large enough, then

$$\mathbf{f}_{n^2+k, n^2}(C_f) \subset \mathcal{B}_f \text{ for } k \in [1, \kappa_0], \text{ and } \mathbf{f}_{n^2+\kappa_0, n^2}(C_f) \subset P_f^{\text{att}}.$$

In addition, since  $\mathbf{f}_{n^2+\kappa_0, n^2}(z)$  is close to  $f^{\circ \kappa_0}(z)$ , then for large  $n$  we also have

$$(16) \quad k_n > \frac{10}{|\mathbf{f}_{n^2+\kappa_0, n^2}(z)|} \text{ for } z \in C_f.$$

**3.5.2. The orbit remains in the attracting petal.** We now prove that if  $n$  is large enough and  $k \in [\kappa_0, k_n]$ , then  $\mathbf{f}_{n^2+k, n^2}(C_f) \subset P_f^{\text{att}}$ .

For this purpose, we work in the coordinate  $Z = -1/z$ . For  $m \geq 0$ , consider the rational map  $F_m$  defined by

$$F_m(Z) := -\frac{1}{f_{w_m}(-1/Z)} = F(Z) - \frac{\pi^2 w_m \cdot [F(Z)]^2}{4 + \pi^2 w_m \cdot F(Z)}.$$

This has to be understood as a perturbation of  $F$ . Notice however that the remainder term  $F_m(Z) - F(Z)$  is *not* negligible with respect to  $F(Z)$  as  $Z \rightarrow \infty$ , so we have to control precisely for which values of  $Z$  the remainder is indeed small.

Since  $F(Z) \sim Z$  as  $z \rightarrow \infty$  and since  $w_{n^2+k} \in O(1/n^2)$  for  $k \in [0, k_n]$ , we get

$$\sup_{|Z|=R} |F_{n^2+k}(Z) - F(Z)| = o(1) \text{ and } \sup_{|Z|=2k_n} |F_{n^2+k}(Z) - F(Z)| = O\left(\frac{k_n^2}{n^2}\right) = o(1).$$

In particular, according to the Maximum Principle and the choice of  $R$  - see (15) - for  $n$  large enough, if  $k \in [0, k_n]$  then

$$\sup_{R < |Z| < 2k_n} |F_{n^2+k}(Z) - Z - 1| < \frac{1}{10}.$$

An easy induction on  $k$  shows that for every  $k \in [\kappa_0, k_n]$  and every  $z \in C_f$ ,

$$(17) \quad -\frac{1}{\mathbf{f}_{n^2+k,n^2}(z)} \in \overline{D} \left( -\frac{1}{\mathbf{f}_{n^2+\kappa_0,n^2}(z)} + k - \kappa_0, \frac{k - \kappa_0}{10} \right) \\ \subset \{Z \in \mathbb{C} ; \Re(Z) > R \text{ and } |Z| < 2k_n\}.$$

Indeed, the induction hypothesis clearly holds for  $k = \kappa_0$  and if it holds for some  $k \in [\kappa_0, k_n - 1]$ , then

$$-\frac{1}{\mathbf{f}_{n^2+k+1,n^2}(z)} = F_{n^2+k} \left( -\frac{1}{\mathbf{f}_{n^2+k,n^2}(z)} \right) \in \overline{D} \left( -\frac{1}{\mathbf{f}_{n^2+k,n^2}(z)} + 1, \frac{1}{10} \right) \\ \subset \overline{D} \left( -\frac{1}{\mathbf{f}_{n^2+\kappa_0,n^2}(z)} + k - \kappa_0 + 1, \frac{k - \kappa_0}{10} + \frac{1}{10} \right).$$

If  $Z$  belongs to the latter disk, then

$$\Re(Z) > \Re \left( -\frac{1}{\mathbf{f}_{n^2+\kappa_0,n^2}(z)} \right) + k - \kappa_0 + 1 - \frac{k - \kappa_0 + 1}{10} > R + \frac{9}{10}(k - \kappa_0 + 1) > R$$

and using (16),

$$|Z| < \left| -\frac{1}{\mathbf{f}_{n^2+\kappa_0,n^2}(z)} \right| + k - \kappa_0 + 1 + \frac{k - \kappa_0 + 1}{10} < \frac{1}{10}k_n + \frac{11}{10}k_n < 2k_n.$$

This shows that  $\mathbf{f}_{n^2+k,n^2}(z) \in P_f^{\text{att}}$  for all  $k \in [\kappa_0, k_n]$  and all  $z \in C_f$ .

3.5.3. *Working in attracting Fatou coordinates.* We finally prove that for  $k \in [0, k_n]$

$$\phi_f(\mathbf{f}_{n^2+k,n^2}(z_n)) = \phi_f(f^{\circ k}(z)) + o(1).$$

This is clear for  $k \in [1, \kappa_0]$  since for each fixed  $k$ , the sequence  $(\mathbf{f}_{n^2+k,n^2})$  converges uniformly to  $f^{\circ k}$  on  $C_f$ . So it is enough to prove the estimate for  $k \in [\kappa_0, k_n]$ .

We have that  $\phi'_f(z) \sim 1/z^2$  as  $z \rightarrow 0$  in  $P_f^{\text{att}}$ . Also, we saw in (17) that for  $k \in [\kappa_0, k_n]$ ,

$$\left| \frac{1}{\mathbf{f}_{n^2+k,n^2}(z_n)} \right| \leq 2k_n.$$

It follows that for  $k \in [\kappa_0, k_n - 1]$  and  $z \in C_f$ ,

$$\sup_{I_k} |\phi'_f| \in O(k_n^2) \quad \text{with} \quad I_k := [f \circ \mathbf{f}_{n^2+k,n^2}(z), \mathbf{f}_{n^2+k+1,n^2}(z)],$$

whence

$$\phi_f(\mathbf{f}_{n^2+k+1,n^2}(z)) = \phi_f \left( f(\mathbf{f}_{n^2+k,n^2}(z)) + \frac{\pi^2}{4} w_{n^2+k} \right) \\ = \phi_f \circ f(\mathbf{f}_{n^2+k,n^2}(z)) + w_{n^2+k} \cdot \sup_{I_k} |\phi'_f| \cdot O(1) \\ = \phi_f(\mathbf{f}_{n^2+k,n^2}(z)) + 1 + O\left(\frac{k_n^2}{n^2}\right).$$

As a consequence, for  $k \in [\kappa_0, k_n]$ ,

$$\phi_f(\mathbf{f}_{n^2+k,n^2}(z)) = \phi_f(\mathbf{f}_{n^2+\kappa_0,n^2}(z)) + k - \kappa_0 + O\left(\frac{k_n^3}{n^2}\right) \\ = \phi_f(f^{\circ \kappa_0}(z)) + k - \kappa_0 + o(1) \\ = \phi_f(f^{\circ k}(z)) + o(1),$$

where the second equality follows from the estimate  $\mathbf{f}_{n^2+\kappa_0, n^2}(z) = f^{\circ\kappa_0}(z) + o(1)$  and the fact that  $k_n^3 = O(n^{3\alpha})$  since  $\alpha < 2/3$ .

Taking  $k = k_n$ , we conclude that

$$\phi_f(x_n^t) = \phi_f(\mathbf{f}_{n^2+k_n, n^2}(z)) + o(1) = \phi_f(z) + k_n + o(1) = k_n + O(1)$$

and so,  $x_n^t \sim -1/\phi_f(x_n^t) \sim -1/k_n$  as required. The proof of Proposition 3.1 is completed.  $\square$

**3.6. Passing through the eggbeater.** In this paragraph, we prove Proposition 3.3, that is: if  $(x_n^t)_{n \geq 0}$  is a sequence such that  $x_n^t \sim -1/k_n$  and if

$$x_n^o := \mathbf{f}_{(n+1)^2-k_n, n^2+k_n}(x_n^t),$$

then, as  $n \rightarrow +\infty$ ,

$$x_n^o \sim \frac{1}{k_n} \quad \text{and} \quad \psi_f^{-1}(x_n^o) = \phi_f(x_n^t) + n \cdot \left( \sum_{m=n^2+k_n}^{n^2+2n-k_n} \sqrt{w_m} \right) - 2n + o(1).$$

The proof relies on the formalism of approximate Fatou coordinates introduced in Section 2 and notation thereof (in particular Properties 1, 2 and 3). Figure 4 illustrates the proof.

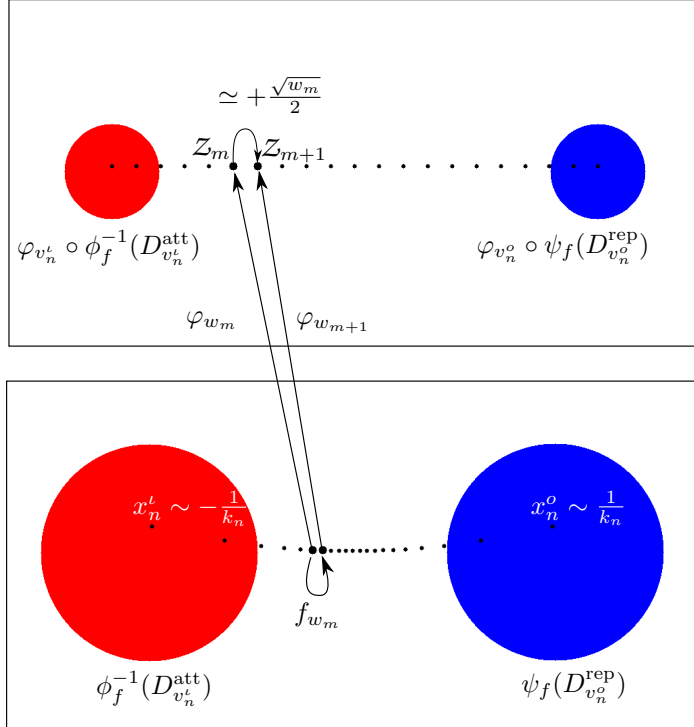


FIGURE 4. The map  $\varphi_{w_{m+1}} \circ f_{w_m} \circ \varphi_{w_m}^{-1}$  is close to a translation by  $\frac{\sqrt{w_m}}{2}$ .

*Proof.* Let  $v_n^t := w_{n^2+k_n}$  which belongs to  $B_r$  for  $n$  large enough.

**Step 1.** If  $x_n^t \sim -1/k_n$ , then for  $n$  large enough,  $x_n^t \in P_f^{\text{att}}$ . Set  $Y_n := \phi_f(x_n^t)$  and note that

$$Y_n \sim -\frac{1}{x_n^t} \sim k_n \sim n^\alpha \sim |v_n^t|^{-\alpha/2} \quad \text{whence} \quad Y_n \in D_{v_n^t}^{\text{att}} \quad \text{for } n \text{ large enough.}$$

According to Property 1, for  $n$  large enough,  $x_n^t = \phi_f^{-1}(Y_n) \in V_{v_n^t}$  and

$$(18) \quad \frac{2}{\sqrt{v_n^t}} \cdot \varphi_{v_n^t}(x_n^t) = \frac{2}{\sqrt{v_n^t}} \cdot \varphi_{v_n^t} \circ \phi_f^{-1}(Y_n) = Y_n + o(1) = \phi_f(x_n^t) + o(1).$$

**Step 2.** We will now prove by induction on  $m$  that for all  $m \in [n^2+k_n, (n+1)^2-k_n]$ ,

$$\mathbf{f}_{m, n^2+k_n}(x_n^t) \in V_{w_m}$$

and

$$Z_m := \varphi_{w_m} \circ \mathbf{f}_{m, n^2+k_n}(x_n^t) = \varphi_{w_m}(x_n^t) + \sum_{j=n^2+k_n}^{m-1} \left( \frac{\sqrt{w_j}}{2} + o\left(\frac{1}{n^2}\right) \right).$$

Indeed, for  $m = n^2 + k_n$ , we have that  $w_m = v_n^t$  and according to Step 1,

$$\mathbf{f}_{m, n^2+k_n}(x_n^t) = x_n^t \in V_{v_n^t} = V_{w_m},$$

so the induction hypothesis holds in this case.

Now, assume the induction hypothesis holds for some  $m \in [n^2 + k_n, (n+1)^2 - k_n - 1]$ . According to Step 1,

$$(19) \quad \varphi_{v_n^t}(x_n^t) = \frac{\sqrt{v_n^t}}{2} \cdot (\phi_f(x_n^t) + o(1)) = \frac{k_n}{2n} + o\left(\frac{k_n}{n}\right).$$

In addition,

$$\sqrt{w_m} = \frac{1}{\sqrt{n^2 + O(n)}} = \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

It follows that

$$Z_m = \frac{k_n}{2n} + o\left(\frac{k_n}{n}\right) + (m - n^2 - k_n) \cdot \left( \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \right) = \frac{m - n^2}{2n} + o\left(\frac{k_n}{n}\right),$$

and therefore

$$\frac{k_n}{2n} + o\left(\frac{k_n}{n}\right) \leq \Re(Z_m) \leq 1 - \frac{k_n}{2n} + o\left(\frac{k_n}{n}\right) \quad \text{and} \quad \Im(Z_m) = o(1).$$

Since  $r_{w_m} = |w_m|^{(1-\alpha)/2} \sim k_n/n$ , we see that for large  $n$ ,  $Z_m \in \mathcal{R}_{w_m}$ . According to Property 3,

$$\mathbf{f}_{m+1, n^2+k_n}(x_n^t) = f_{w_m} \circ \mathbf{f}_{m, n^2+k_n}(x_n^t) \in V_{w_{m+1}}$$

and

$$\begin{aligned} Z_{m+1} &= \phi_{w_{m+1}} \circ f_{w_m} \circ \phi_{w_m}^{-1}(Z_m) = Z_m + \frac{\sqrt{w_m}}{2} + o(w_m) \\ &= \varphi_{v_n^t}(x_n^t) + \sum_{j=n^2+k_n}^{m-1} \left( \frac{\sqrt{w_j}}{2} + o\left(\frac{1}{n^2}\right) \right) + \frac{\sqrt{w_m}}{2} + o\left(\frac{1}{n^2}\right) \\ &= \varphi_{v_n^t}(x_n^t) + \sum_{j=n^2+k_n}^m \left( \frac{\sqrt{w_j}}{2} + o\left(\frac{1}{n^2}\right) \right). \end{aligned}$$

**Step 3.** We now specialize to the case  $m := (n+1)^2 - k_n$  and set

$$v_n^o := w_{(n+1)^2 - k_n} \quad \text{and} \quad x_n^o := \mathbf{f}_{(n+1)^2 - k_n, n^2 + k_n}(x_n^t).$$

According to Step 2 of the proof,  $x_n^o \in V_{v_n^o}$  and

$$(20) \quad \varphi_{v_n^o}(x_n^o) = \varphi_{v_n^t}(x_n^t) + \sum_{j=n^2+k_n}^{n^2+2n-k_n} \left( \frac{\sqrt{w_j}}{2} + o\left(\frac{1}{n^2}\right) \right).$$

In particular, by using (19) and Proposition 3.2 we get

$$(21) \quad \varphi_{v_n^o}(x_n^o) = \frac{k_n}{2n} + o\left(\frac{k_n}{n}\right) + 1 - \frac{k_n}{n} + o\left(\frac{1}{n}\right) = 1 - \frac{k_n}{2n} + o\left(\frac{k_n}{n}\right)$$

Set

$$X_n := \frac{2}{\sqrt{v_n^o}} \cdot (\varphi_{v_n^o}(x_n^o) - 1) \quad \text{so that} \quad \varphi_{v_n^o}(x_n^o) = 1 + \frac{\sqrt{v_n^o}}{2} \cdot X_n.$$

Since  $2/\sqrt{v_n^o} \sim 2n$ , from (21) we deduce that  $X_n = -k_n \cdot (1 + o(1))$ . Since in addition  $k_n \sim (v_n^o)^{(1-\alpha)/2}$  it follows that for  $n$  large enough,  $X_n \in D_{v_n^o}^{\text{rep}}$ . Thus we compute

$$\begin{aligned} \psi_f^{-1}(x_n^o) &= \psi_f^{-1} \circ \varphi_{v_n^o}^{-1} \left( 1 + \frac{\sqrt{v_n^o}}{2} \cdot X_n \right) = X_n + o(1) \\ &= \frac{2}{\sqrt{v_n^o}} \cdot \left( \varphi_{v_n^t}(x_n^t) + \left( \sum_{j=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w_j}}{2} \right) + o\left(\frac{1}{n}\right) - 1 \right) \\ &= \frac{2}{\sqrt{v_n^o}} \cdot \left( \frac{\sqrt{v_n^t}}{2} (\phi_f(x_n^t) + o(1)) + \left( \sum_{j=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w_j}}{2} \right) + o\left(\frac{1}{n}\right) - 1 \right) \\ &= \phi_f(x_n^t) + 2n \cdot \left( \sum_{j=n^2+k_n}^{n^2+2n-k_n} \frac{\sqrt{w_j}}{2} \right) - 2n + o(1) \end{aligned}$$

where we deduce the first line by Property 2, the second line follows by (20), the third line holds thanks to Property 1, and the final line holds since  $v_n^t \sim v_n^o \sim \frac{1}{n^2}$ . This completes the proof.  $\square$

**3.7. Leaving the eggbeater.** In this paragraph, we prove Proposition 3.4, that is: if  $(x_n^o)_{n \geq 0}$  is a sequence contained in the repelling petal  $P_f^{\text{rep}}$  and if

$$\psi_f^{-1}(x_n^o) = -k_n + O(1),$$

then, as  $n \rightarrow +\infty$ ,

$$\mathbf{f}_{(n+1)^2, (n+1)^2 - k_n}(x_n^o) = f^{o k_n}(x_n^o) + o(1).$$

Set  $x_{n,0}^o := x_n^o$  and for  $k \in [1, k_n]$ , set

$$x_{n,k}^o := \mathbf{f}_{(n+1)^2 - k_n + k, (n+1)^2 - k_n}(x_n^o)$$

3.7.1. *Within the repelling petal.* Let  $\kappa_1$  be an integer such that for all  $n \geq 0$ ,

$$\Re(\psi_f^{-1}(x_n^o)) + k_n + R < \kappa_1.$$

We prove by induction on  $k$  that for  $n$  large enough, if  $k \in [0, k_n - \kappa_1]$ , then

$$x_{n,k}^o \in P_f^{\text{rep}} \quad \text{and} \quad \psi_f^{-1}(x_{n,k}^o) = \psi_f^{-1}(x_n^o) + k + k \cdot \mathcal{O}\left(\frac{k_n^2}{n^2}\right).$$

First, the induction hypothesis clearly holds for  $k = 0$ . So, let us assume it is true for some  $k \in [0, k_n - \kappa_1 - 1]$ . As in Proposition 3.1, since  $\alpha < 2/3$  and  $k \leq k_n$ ,  $k\mathcal{O}(k_n^2/n^3) \subset \mathcal{o}(1)$ . It follows that for large  $n$ ,

$$\Re(\psi_f^{-1}(x_{n,k}^o)) = \Re(\psi_f^{-1}(x_n^o)) + k + \mathcal{o}(1) < k + \kappa_1 - k_n - R \leq -R - 1.$$

Since  $x_{n,k+1}^o = f_{w_{(n+1)^2 - k_n + k}}(x_{n,k}^o) = f(x_{n,k}^o) + \mathcal{o}(1)$  and  $f \circ \psi_f = \psi_f \circ T_1$ , taking  $n$  larger if necessary,  $x_{n,k+1}^o$  belongs to the repelling petal  $P_f^{\text{rep}}$ .

Next, since

$$(\psi_f^{-1})'(z) \sim \frac{1}{z^2} \quad \text{as } z \rightarrow 0 \text{ in } P_f^{\text{rep}},$$

as in Proposition 3.1 we see that

$$\begin{aligned} \psi_f^{-1}(x_{n,k+1}^o) &= \psi_f^{-1}\left(f(x_{n,k}^o) + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &= \psi_f^{-1} \circ f(x_{n,k}^o) + \mathcal{O}\left(\frac{1}{n^2|x_{n,k}^o|^2}\right) \\ &= \psi_f^{-1}(x_{n,k}^o) + 1 + \mathcal{O}\left(\frac{k_n^2}{n^2}\right) \\ &= \psi_f^{-1}(x_n^o) + k + 1 + (k + 1) \cdot \mathcal{O}\left(\frac{k_n^2}{n^2}\right), \end{aligned}$$

and the proof of the induction is completed.

3.7.2. *Leaving the repelling petal.* By the previous step we have that

$$\psi_f^{-1}(x_{n,k_n - \kappa_1}^o) = \psi_f^{-1}(x_n^o) + k_n - \kappa_1 + \mathcal{o}(1) = \psi_f^{-1} \circ f^{\circ k_n - \kappa_1}(x_n^o) + \mathcal{o}(1).$$

Applying  $\psi_f$  on both sides yields

$$x_{n,k_n - \kappa_1}^o = f^{\circ k_n - \kappa_1}(x_n^o) + \mathcal{o}(1).$$

Since the sequence of polynomials  $\mathbf{f}_{(n+1)^2, (n+1)^2 - \kappa_1}$  converges locally uniformly to  $f^{\circ \kappa_1}$ , we deduce that

$$x_{n,k_n}^o = \mathbf{f}_{(n+1)^2, (n+1)^2 - \kappa_1}(x_n^o) = f^{\circ k_n}(x_n^o) + \mathcal{o}(1),$$

thereby concluding the proof of Proposition 3.4.  $\square$

#### 4. A LAVAURS MAP WITH AN ATTRACTING FIXED POINT

This section is devoted to the proof of Proposition B. Given  $a \in \mathbb{C}$ , let  $f_a : \mathbb{C} \rightarrow \mathbb{C}$  be the cubic polynomial defined by

$$f_a(z) = z + z^2 + az^3.$$

We must show that if  $r > 0$  is sufficiently close to 0 and  $a \in D(1 - r, r)$ , then the Lavaurs map  $\mathcal{L}_{f_a} : \mathcal{B}_f \rightarrow \mathbb{C}$  admits an attracting fixed point. Notice that since  $\mathcal{L}_{f_a}$  commutes with  $f_a$ , it therefore has infinitely many of them.

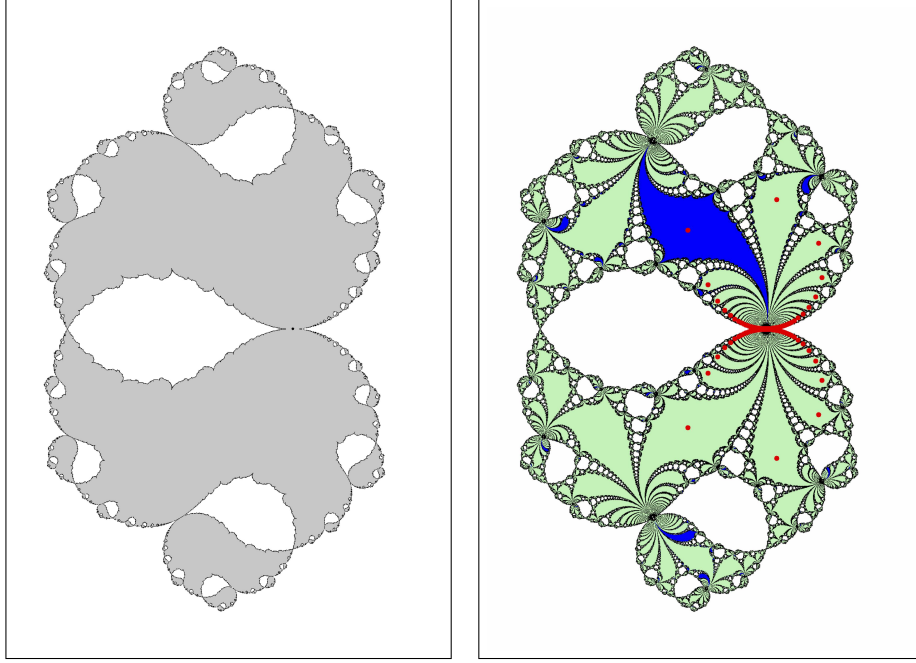


FIGURE 5. Behavior of  $\mathcal{L}_f$  for  $f(z) = z + z^2 + 0.95z^3$ . Left: the set of points  $z \in \mathcal{B}_f$  whose image by  $\mathcal{L}_f$  remains in  $\mathcal{B}_f$ . The restriction of  $\mathcal{L}_f$  to the bounded white domains is a covering above  $\mathbb{C} \setminus \overline{\mathcal{B}_f}$ . Right: the Lavaurs map  $\mathcal{L}_f$  has two complex conjugate sets of attracting fixed points. The fixed points of  $\mathcal{L}_f$  are indicated as red points and their basins of attraction are colored (blue for one of the fixed points, and green for the others).

Set

$$\mathcal{U}_a := \psi_{f_a}^{-1}(\mathcal{B}_{f_a}) \quad \text{and} \quad \mathcal{E}_a := \phi_{f_a} \circ \psi_{f_a} : \mathcal{U}_a \rightarrow \mathbb{C}.$$

This is an open set containing an upper half-plane and a lower half-plane. Note that  $\psi_{f_a} : \mathcal{U}_a \rightarrow \mathcal{B}_{f_a}$  semi-conjugates  $\mathcal{E}_a$  to  $\mathcal{L}_{f_a}$ . Since  $\psi_{f_a}$  is univalent in a left half-plane, it is enough to show that  $\mathcal{E}_a$  has an attracting fixed point with real part arbitrarily close to  $-\infty$ . Since  $\mathcal{E}_a$  commutes with the translation by 1, it is therefore enough to show that  $\mathcal{E}_a : \mathcal{U}_a \rightarrow \mathbb{C}$  has an attracting fixed point.

The open set  $\mathcal{U}_a$  is invariant by  $T_1$  and the map  $\mathcal{E}_a$  commutes with  $T_1$ . The set

$$\mathcal{U} := \{(a, Z) \in \mathbb{C} \times \mathbb{C} ; Z \in \mathcal{U}_a\}$$

is an open subset of  $\mathbb{C}^2$  and the map

$$\mathcal{E} : \mathcal{U} \ni (a, Z) \mapsto \mathcal{E}_a(Z) \in \mathbb{C}$$

is analytic. The universal cover

$$\exp : \mathbb{C} \ni Z \mapsto e^{2\pi i Z} \in \mathbb{C} \setminus \{0\}$$

semi-conjugates  $\mathcal{E}_a$  to a map

$$e_a : U_a \rightarrow \mathbb{C} \setminus \{0\} \quad \text{with} \quad U_a := \exp(\mathcal{U}_a) \subset \mathbb{C} \setminus \{0\}.$$



The open set  $U_a$  is a neighborhood of 0 and  $\infty$  in  $\mathbb{C} \setminus \{0\}$ . The map  $e_a$  has removable singularities at 0 and  $\infty$ , see the proof of Lemma 4.1 below, thus it extends as a map  $e_a : \widehat{U}_a \rightarrow \widehat{\mathbb{C}}$ , where  $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  is the Riemann sphere and  $\widehat{U}_a := U_a \cup \{0, \infty\} \subset \widehat{\mathbb{C}}$ . We set

$$\widehat{U} := \left\{ (a, z) \in \mathbb{C} \times \widehat{\mathbb{C}} ; z \in \widehat{U}_a \right\}.$$

**Lemma 4.1.** *The points 0 and  $\infty$  in  $\widehat{\mathbb{C}}$  are fixed points of  $e_a : \widehat{U}_a \rightarrow \widehat{\mathbb{C}}$ . Both fixed points have multiplier  $e^{2\pi^2(1-a)}$ .*

*Proof.* As  $\Im(Z) \rightarrow +\infty$ , we have that

$$-\frac{1}{\psi_{f_a}(Z)} = Z + (1-a)\log(-Z) + o(1)$$

where  $\log$  is the principal branch of logarithm. Note that  $\log(-Z) = \log(Z) - \pi i$  as  $\Im(Z) \rightarrow +\infty$ . Thus,

$$\begin{aligned} \mathcal{E}_a(Z) &= \phi_{f_a} \circ \psi_{f_a}(Z) = Z + (1-a)\log(-Z) + o(1) \\ &\quad - (1-a)\log(Z + (1-a)\log(-Z) + o(1)) + o(1) \\ &= Z + (1-a)\log(Z) - \pi i(1-a) - (1-a)\log(Z) + o(1) \\ &= Z - \pi i(1-a) + o(1). \end{aligned}$$

As a consequence, as  $z = \exp(Z) \rightarrow 0$ , we have that

$$e_a(z) = e^{2\pi i Z} \cdot e^{2\pi^2(1-a)+o(1)} = e^{2\pi^2(1-a)z} \cdot (1 + o(1)),$$

thus we conclude that 0 is a fixed point of  $e_a$  with multiplier  $e^{2\pi^2(1-a)}$ . A similar argument shows that  $\infty$  is also a fixed point of  $e_a$  with multiplier  $e^{2\pi^2(1-a)}$ .  $\square$

In particular, we see that for  $a = 1$ , the map  $e_1$  has multiple fixed points at 0 and  $\infty$ .

**Lemma 4.2.** *The multiplicity of 0 and  $\infty$  as fixed points of  $e_1$  is 2.*

*Proof.* The mapping  $e_1 : \widehat{U}_1 \rightarrow \widehat{\mathbb{C}}$  is a finite type analytic map in the sense of Epstein (see Appendix A.4). Therefore each attracting petal at 0 or at  $\infty$  must attract the infinite orbit of a singular value of  $e_1$ . Indeed if not, the component  $B$  of the immediate basin containing this petal would avoid the singular values of  $e_1$ . The restriction  $e_1 : B \rightarrow B$  would then be a covering and the corresponding attracting Fatou coordinate would extend to a covering map  $\phi : B \rightarrow \mathbb{C}$ . This would force  $B$  to be isomorphic to  $\mathbb{C}$ , which is not possible since  $B$  is contained in  $\mathbb{C} \setminus \{0\}$ .

According to Proposition A.5, the finite type map  $e_1$  admits exactly two critical values (the images of the critical values of  $f_1$  under the map  $\exp \circ \phi_{f_1}$ ) and two singular values which are respectively fixed at 0 and  $\infty$ . It follows that the number of attracting petals at 0 plus the number of attracting petals at  $\infty$  is at most 2. So this number must be equal to 2 and the result follows.  $\square$

As we perturb  $a$  away from 1, the multiple fixed point at 0 splits into a pair of fixed points of  $e_a$ : one at 0 with multiplier  $e^{2\pi^2(1-a)}$  and another one denoted by

$\xi(a)$ , with multiplier  $\rho(a)$ . We use a classical residue computation to estimate this multiplier. Let  $\gamma$  be a small loop around 0. The Cauchy Residue Formula yields

$$\frac{1}{1 - e^{2\pi^2(1-a)}} + \frac{1}{1 - \rho(a)} = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - e_a(z)} \xrightarrow{a \rightarrow 1} \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - e_1(z)} \in \mathbb{C}.$$

From this it follows that

$$\frac{1}{1 - \rho(a)} = \frac{1}{2\pi^2(1-a)} + O(1) \quad \text{as } a \rightarrow 1.$$

Now observe that

$$\rho(a) \in \mathbb{D} \iff \Re\left(\frac{1}{1 - \rho(a)}\right) > \frac{1}{2},$$

and similarly,

$$a \in D(1-r, r) \iff \Re\left(\frac{1}{2\pi^2(1-a)}\right) > \frac{1}{4\pi^2 r}.$$

As a consequence when  $r > 0$  is sufficiently close to 0 and  $a \in D(1-r, 1)$ , we deduce that  $|\rho(a)| < 1$ , so  $\xi(a)$  is an attracting fixed point.

Let finally  $Z(a)$  be a preimage of  $\xi(a)$  under  $\exp$ , that is  $\exp(Z(a)) = \xi(a)$ . We claim that for  $a$  sufficiently close to 1,  $Z(a)$  is a fixed point of  $\mathcal{E}_a$ . Indeed observe first that  $\mathcal{E}_a(Z(a)) - Z(a)$  is an integer which does not depend on the choice of preimage  $Z(a)$ . Therefore it is sufficient to prove that

$$\lim_{a \rightarrow 1} \mathcal{E}_a(Z(a)) - Z(a) = 0.$$

This may be seen as follows. The function  $\mathcal{E}_a - \text{id}$  is periodic of period 1, hence of the form  $u_a \circ \exp$  for some function  $u_a : \widehat{U}_a \rightarrow \mathbb{C}$ . The function

$$u : \widehat{U} \ni (a, z) \mapsto u_a(z) \in \mathbb{C}$$

is analytic, in particular continuous. So,

$$\lim_{a \rightarrow 1} \mathcal{E}_a(Z(a)) - Z(a) = \lim_{a \rightarrow 1} u(a, \xi(a)) = u(1, 0) = \lim_{\Im(Z) \rightarrow +\infty} \mathcal{E}_1(Z) - Z = 0.$$

The last equality follows from the proof of Lemma 4.1. This shows that for  $a$  sufficiently close to 1,  $Z(a)$  is a fixed point of  $\mathcal{E}_a$  with multiplier  $\rho(a)$ , and the proof of Proposition B is complete.  $\square$

## 5. WANDERING DOMAINS IN $\mathbb{R}^2$

In this section, we prove Proposition C, which shows the existence of real polynomial maps in two complex variables with wandering Fatou components intersecting  $\mathbb{R}^2$ . Let us consider the polynomial  $f(z) := z + z^2 + bz^4$ . We seek a parameter  $b \in (-8/27, 0)$  such that the Lavaurs map  $\mathcal{L}_f$  has a fixed critical point in  $\mathbb{R} \cap \mathcal{B}_f$ .

*Outline of the proof.* Set

$$b := -\frac{1+2c}{4c^3} \quad \text{with } c \in [-3/2, -1/2] \quad \text{and } f_c(z) := z + z^2 - \frac{1+2c}{4c^3} z^4.$$

As  $c$  increases from  $-3/2$  to  $-3/4$ , the corresponding parameter  $b$  decreases from  $-4/27$  to  $-8/27$  and as  $c$  increases from  $-3/4$  to  $-1/2$ , the parameter  $b$  increases from  $-8/27$  to 0. The point  $c$  is a critical point of the polynomial  $f_c$ . As a consequence,

$$\deg_c \phi_{f_c} = \deg_c T_1 \circ \phi_{f_c} = \deg_c \phi_{f_c} \circ f_c = (\deg_{f_c(c)} \phi_{f_c}) \cdot (\deg_c f_c) \geq 2.$$

So,  $c$  is a critical point of  $\phi_{f_c}$ , whence a critical point of  $\mathcal{L}_{f_c}$ .

**Claim 1:** when  $c \in (-3/2, -1/2]$ , the critical point  $c$  belongs to the parabolic basin  $\mathcal{B}_{f_c}$ .

**Claim 2:** the function  $\mathfrak{L} : (-3/2, -1/2] \rightarrow \mathbb{R}$  defined by  $\mathfrak{L}(c) := \mathcal{L}_{f_c}(c)$  is continuous.

**Claim 3:**  $\mathfrak{L}(-1/2) > 0$ .

**Claim 4:** there is a sequence  $c_n$  converging to  $-3/2$  with  $\mathfrak{L}(c_n) < c_n$ .

These four claims are enough to get the desired conclusion. Indeed, the function  $c \mapsto \mathfrak{L}(c) - c$  takes a positive value at  $c = -1/2$  and takes negative values arbitrarily close to  $-3/2$ . Since it is continuous, it follows from the Intermediate Value Theorem that it must vanish somewhere in  $(-3/2, -1/2)$ .  $\square$

Figure 6 shows the graph of the function  $\mathfrak{L} : (-\frac{3}{2}, -\frac{1}{2}) \rightarrow \mathbb{R}$  which intersects the diagonal. As  $c$  tends to  $-\frac{3}{2}$ ,  $\mathfrak{L}(c)$  accumulates the whole interval  $f(\mathbb{R}) = (-\infty, x]$  with  $x := \frac{27}{16} + \frac{9}{8}\sqrt{3} \simeq 3.63$ .

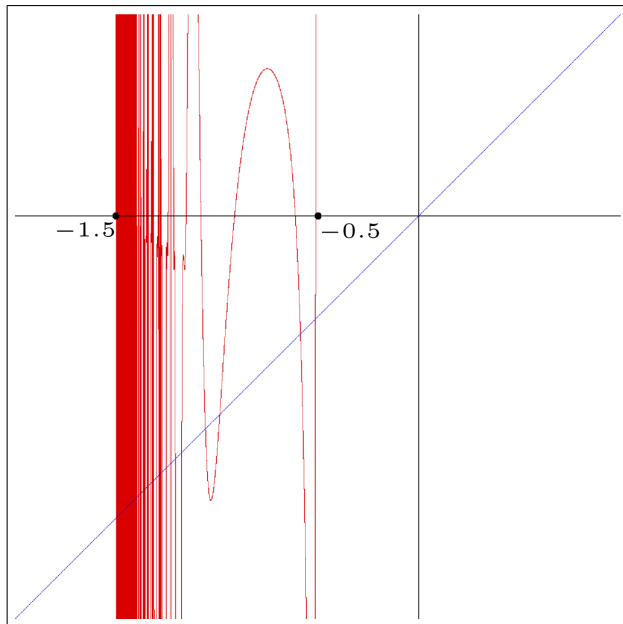


FIGURE 6. The graph of the function  $\mathfrak{L} : c \mapsto \mathcal{L}_{f_c}(c)$ . Each intersection with the diagonal corresponds to a super-attracting fixed point of  $\mathcal{L}_{f_c}$ .

A numerical experiment suggests that the function  $\mathfrak{L}(c) - c$  vanishes for a value of  $c$  close to  $-0.586$ . Accordingly, for

$$f(z) = z + z^2 - 0.2136z^4$$

the Lavaurs map  $\mathcal{L}_f : \mathcal{B}_f \rightarrow \mathbb{C}$  has a real attracting fixed point.

*Proof of Claim 1.* It is enough to show that  $z < f_c(z) \leq 0$  for  $z \in [c, 0)$ . Indeed, if so, then the sequence  $(f_c^n(c))_{n \geq 0}$  stays in  $[c, 0)$  and it is non-decreasing, so it must converge to the unique fixed point in  $[c, 0]$ , namely to the parabolic fixed point 0.

To see that  $f_c - \text{id} > 0$  on  $[c, 0)$ , note that if  $c \in [-3/2, -1/2]$ , then  $b \in [-8/27, 0]$  and if  $z \in [c, 0)$ , then

$$1 + bz^2 \geq 1 + bc^2 = \frac{1}{2} - \frac{1}{4c} \geq \frac{1}{2} + \frac{1}{4} \cdot \frac{2}{3} = \frac{2}{3}.$$

Thus,

$$f_c(z) - z = z^2 + bz^4 = z^2 \cdot (1 + bz^2) \geq \frac{2}{3}z^2 > 0.$$

To see that  $f_c \leq 0$  on  $[c, 0)$ , it is enough to see that  $g(z) := 1 + z + bz^3 \geq 0$  on  $[c, 0)$ . As above, for  $c \in [-3/2, -1/2]$  and  $z \in [c, 0)$ , we have

$$g'(z) = 1 + 3bz^2 \geq 1 + 3bc^2 = -\frac{1}{2} - \frac{3}{4c} \geq -\frac{1}{2} + \frac{3}{4} \cdot \frac{2}{3} = 0.$$

Thus,  $g$  is increasing on  $[c, 0)$  and since

$$g(c) = 1 + c + bc^3 = \frac{3}{4} + \frac{1}{2}c \geq 0.$$

the proof of Claim 1 is completed.  $\square$

*Proof of Claim 2.* For  $c \in \mathbb{C} \setminus 0$  we may consider the attracting Fatou coordinate  $\phi_{f_c}$  and the repelling Fatou parameterization  $\psi_{f_c}$  of  $f_c$  (normalized according to our usual convention, see Appendix A). The formulas (22) and (23) defining  $\phi_{f_c}$  and  $\psi_{f_c}$  as limits show that  $\phi_{f_c}$  and  $\psi_{f_c}$  take real values on the real axis. Define

$$\mathcal{B} := \{(c, z) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C} ; z \in \mathcal{B}_{f_c}\}.$$

Propositions A.1 and A.2 imply that

$$\mathcal{B} \ni (c, z) \mapsto \phi_{f_c}(z) \in \mathbb{C} \quad \text{and} \quad (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \ni (c, Z) \mapsto \psi_{f_c}(Z) \in \mathbb{C}$$

are continuous, as well as their composition

$$\mathcal{L} : \mathcal{B} \ni (c, z) \mapsto \psi_{f_c} \circ \phi_{f_c}(z).$$

Now for  $c \in (-3/2, 1/2]$ , the point  $(c, c)$  belongs to  $\mathcal{B}$  so we conclude that the function  $\mathfrak{L} : c \mapsto \mathcal{L}(c, c)$  is continuous on  $(-3/2, -1/2]$ .  $\square$

*Proof of Claim 3.* Assume  $c = -1/2$  so that  $f := f_{-1/2}$  is the quadratic polynomial  $z \mapsto z + z^2$ . The repelling Fatou parametrization sends points on  $\mathbb{R}$  which are sufficiently close to  $-\infty$  to points on  $\mathbb{R}^+$  which are close to 0. Since  $f(\mathbb{R}^+) = \mathbb{R}^+$  and since  $\psi_f(z) = f^{\circ m} \circ \psi_f(z - m)$  for all  $m \geq 0$ , we see that  $\psi_f(\mathbb{R}) = \mathbb{R}^+$ . As a consequence,  $\mathfrak{L}(-1/2) = \psi_f \circ \phi_f(-1/2) > 0$ .  $\square$

*Proof of Claim 4.* Let us first study the behavior of  $\phi_{f_c}(c)$  when  $c$  is close to  $-3/2$ . Putting  $c = -\frac{3}{2} + t$  we compute

$$f_c(c) = \frac{3}{4}c + \frac{1}{2}c^2 = -\frac{3}{4}t + \mathcal{O}(t^2).$$

Let  $\Phi(c) := \phi_{f_c}(c)$ . Then the asymptotic expansion of  $\phi_{f_c}$  (see §A.1) at 0 yields

$$\Phi(c) = \phi_{f_c} \circ f_c(c) - 1 = \frac{4}{3t} - \log\left(\frac{4}{3t}\right) - 1 + o(1)$$

Thus the sequence of maps  $(\Phi_n)$  defined by :

$$\Phi_n(u) := \Phi\left(-\frac{3}{2} + \frac{4}{3(n+u)}\right) - n + \log n + 1$$

converges uniformly to the identity on compact intervals of  $\mathbb{R}$ .

Now let us consider the map  $f := f_{c_0}$  for  $c_0 := -3/2$ . Figure 7 shows the graph of  $f$ . The fixed points of  $f$  are 0,  $\xi := -\frac{3\sqrt{3}}{2}$  and  $\xi' := \frac{3\sqrt{3}}{2}$ . The critical points of

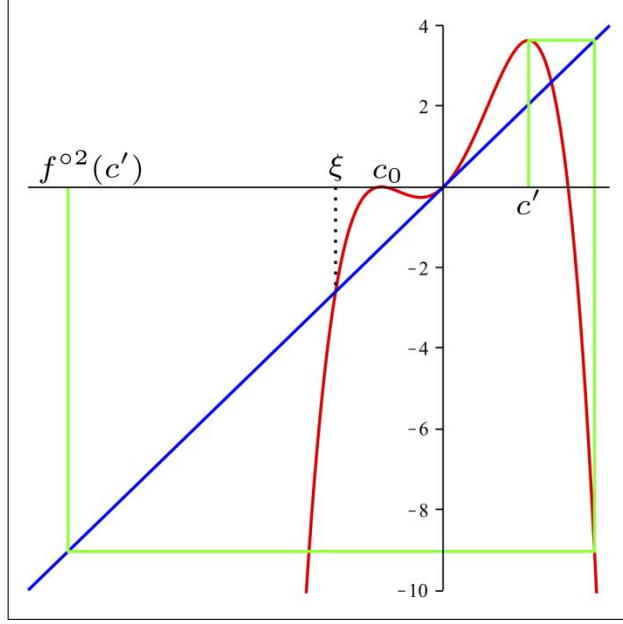


FIGURE 7. The graph of the function  $f(z) := z + z^2 - \frac{4}{27}z^4$  and the first iterates of the critical point  $c' := \frac{3}{4}(1 + \sqrt{3})$ .

$f$  are  $c_0 = -\frac{3}{2}$ ,  $c' := \frac{3}{4}(1 + \sqrt{3})$  and  $c'' := \frac{3}{4}(1 - \sqrt{3})$ . We see that

$$f^{\circ 2}(c') < \xi < c_0 < c'' < 0 < c' < \xi' < f(c').$$

Thus,  $f$  sends the interval  $(-\infty, \xi)$  into itself and the orbit of any point in this interval escapes to  $-\infty$ . In particular, the orbit of  $c'$  escapes to  $-\infty$ . In addition,  $f - \text{id} \geq 0$  on  $[0, c']$ , and  $f$  is increasing on  $[0, c']$ . So, we can define a sequence  $(c'_m)_{m \geq 0}$  recursively by  $c'_0 := c'$  and for  $m \geq 0$ ,

$$c'_{m+1} \in (0, c') \quad \text{and} \quad f(c'_{m+1}) = c'_m.$$

This sequence is decreasing and converges to a fixed point of  $f$ , thus to 0.

Choose  $m_0$  large enough so that  $x := c'_{m_0}$  belongs to the repelling petal  $P_f^{\text{rep}}$ . Hence  $f^{\circ(m_0+2)}(x) < \xi$ . Choose  $\varepsilon > 0$  small enough so that for all  $c \in (c_0, c_0 + \varepsilon)$ , the point  $x$  belongs to the repelling petal of  $f_c$  and  $f_c^{\circ(m_0+2)}(x) < \xi(c)$  where  $\xi(c)$  is the leftmost fixed point of  $f_c$  in  $\mathbb{R}$ . In particular, for all  $m \geq m_0 + 2$ , we have that  $f^{\circ m}(x) < \xi(c) < c$ .

For  $c \in (c_0, c_0 + \varepsilon)$ , set

$$\Psi(c) := \psi_{f_c}^{-1}(x)$$

and

$$\Psi_n(u) := \Psi\left(-\frac{3}{2} + \frac{4}{3(n+u)}\right).$$

Note that  $\Psi(c) = X_0 + O(t)$  since  $x$  lies in the repelling petal of  $f_c$  and  $\psi_c$  varies continuously with  $c$ . Therefore we also have that  $\Psi_n(u) = X_0 + O(1/n)$ .

Together with the Intermediate Value Theorem, this implies that for large enough  $n$ , the equation

$$\Phi_n(u) = \Psi_n(u) + \{\log n\}$$

admits at least one solution  $u_n \in (X_0 - 1, X_0 + 2)$ , where  $\{\log n\}$  denotes the fractional part of  $\log n$ .

Now set

$$c_n := -\frac{3}{2} + \frac{4}{3(n+u_n)} \quad \text{and} \quad s_n := \lfloor \log n \rfloor.$$

We have that

$$\begin{aligned} \phi_{f_{c_n}}(c_n) &= \Phi_n(u_n) + n - \log n - 1 \\ &= \Psi_n(u_n) + \{\log n\} + n - \log n - 1 \\ &= \Psi_n(u_n) + n - s_n - 1. \end{aligned}$$

Thus,

$$\begin{aligned} \mathfrak{L}(c_n) &= \psi_{f_{c_n}} \circ \phi_{f_{c_n}}(c_n) \\ &= \psi_{f_{c_n}}(\Psi_n(u_n) + n - s_n - 1) \\ &= \psi_{f_{c_n}}(\psi_{f_{c_n}}^{-1}(x) + n - s_n - 1) = f_{c_n}^{\circ(n-s_n-1)}(x). \end{aligned}$$

Finally, since  $n - s_n - 1 > m_0$  for  $n$  large enough and since  $f^{\circ m}(x) < c$  for all  $m \geq m_0 + 2$  and all  $c \in (c_0, c_0 + \varepsilon)$ , we deduce that  $\mathfrak{L}(c_n) = f_{c_n}^{\circ(n-s_n-1)}(x) < c_n$  for  $n$  large enough. This completes the proof of Claim 4.  $\square$

#### APPENDIX A. FATOU COORDINATES

Throughout this Section  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a polynomial of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + O(z^4) \quad \text{with} \quad a_2 \in \mathbb{C} \setminus \{0\}.$$

In the coordinate  $Z = -1/(a_2 z)$ , the expression of  $f$  becomes

$$F(Z) = Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right) \quad \text{with} \quad b = 1 - \frac{a_3}{a_2^2}.$$

Choose  $R > 0$  sufficiently large so that  $F$  is univalent on  $\mathbb{C} \setminus \overline{\mathbb{D}}(0, R)$ , and the estimates (15) hold. Denote by  $\mathbb{H}_R$  the right half-plane  $\mathbb{H}_R := \{Z \in \mathbb{C}; \Re(Z) > R\}$  and by  $-\mathbb{H}_R$  the corresponding left half-plane.

Finally, denote by  $\log : \mathbb{C} \setminus \mathbb{R}^- \rightarrow \mathbb{C}$  be the principal branch of logarithm.

**A.1. Attracting Fatou coordinate.** As  $\Re(Z) \rightarrow +\infty$ ,

$$\begin{aligned} F(Z) - b \log(F(Z)) &= Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right) - b \log\left(Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right)\right) \\ &= Z - b \log Z + 1 + O\left(\frac{1}{Z^2}\right). \end{aligned}$$

The map  $F$  is univalent on the right half-plane  $\mathbb{H}_R$  and if  $Z \in \mathbb{H}_R$ , then

$$F^{\circ m}(Z) - b \log(F^{\circ m}(Z)) = Z - b \log Z + m + O(1) \quad \text{as} \quad m \rightarrow +\infty.$$

It follows that the sequence of univalent maps

$$\mathbb{H}_R \ni Z \mapsto F^{\circ m}(Z) - m - b \log m \in \mathbb{C}$$

is normal and converges locally uniformly to a univalent map  $\Phi_F : \mathbb{H}_R \rightarrow \mathbb{C}$  satisfying

$$\Phi_F \circ F = T_1 \circ \Phi_F \quad \text{with} \quad T_1(Z) = Z + 1.$$

In addition,

$$\Phi_F(Z) = Z - b \log Z + o(1) \quad \text{as} \quad \Re(Z) \rightarrow +\infty.$$

Transferring this to the initial coordinate, we see that the sequence of mappings  $\mathcal{B}_f \rightarrow \mathbb{C}$  defined by

$$(22) \quad z \mapsto -\frac{1}{a_2 \cdot f^{\circ m}(z)} - m - b \log m$$

converges locally uniformly to an *attracting Fatou coordinate*  $\phi_f : \mathcal{B}_f \rightarrow \mathbb{C}$  which semi-conjugates  $f : \mathcal{B}_f \rightarrow \mathcal{B}_f$  to  $T_1 : \mathbb{C} \rightarrow \mathbb{C}$ , that is  $\phi_f \circ f = T_1 \circ \phi_f$ , and satisfies

$$\phi_f(z) = -\frac{1}{a_2 z} - b \log \left( -\frac{1}{a_2 z} \right) + o(1) \quad \text{as} \quad \Re(-1/z) \rightarrow +\infty.$$

The restriction of  $\phi_f$  to the *attracting petal*

$$P_f^{\text{att}} := \left\{ z \in \mathbb{C} ; \Re \left( -\frac{1}{a_2 z} \right) > R \right\}$$

coincides with  $z \mapsto \Phi_F(-1/(a_2 z))$ , hence it is univalent.

In addition, the convergence in (22) is locally uniform with respect to  $f$  in the open set  $\mathcal{B} = \{(f, z), z \in \mathcal{B}_f\}$ , which yields the following result.

**Proposition A.1.** *The map  $\phi_f$  depends holomorphically on  $f$ .*

Figure 8 illustrates the behavior of the extended Fatou coordinate for the cubic polynomial  $f_1(z) = z + z^2 + z^3$  which has two critical points  $c^\pm := (-1 \pm i\sqrt{2})/3$ . The basin of attraction  $\mathcal{B}_f$  is colored according to the following scheme:

- blue if  $\Im(\phi_f(z)) > \Im(\phi_f(c^+))$ ,
- red if  $\Im(\phi_f(z)) < \Im(\phi_f(c^-))$ ,
- green if  $\Im(\phi_f(c^-)) < \Im(\phi_f(z)) < \Im(\phi_f(c^+))$ .

**A.2. Repelling Fatou coordinate.** As  $\Re(Z) \rightarrow -\infty$ ,

$$\begin{aligned} F(Z + b \log(-Z)) &= Z + b \log(-Z) + 1 + \frac{b}{Z + b \log(-Z)} + O\left(\frac{1}{Z^2}\right) \\ &= (Z + 1) + b \log(-Z - 1) + O\left(\frac{1}{Z^2}\right). \end{aligned}$$

It follows that if  $R > 0$  is sufficiently large and  $\Re(Z) < -R$ , then

$$F^{\circ m}((Z - m) + b \log(m - Z)) = O(1) \quad \text{as} \quad m \rightarrow +\infty.$$

In that case, the sequence of univalent maps

$$-\mathbb{H}_R \ni Z \mapsto F^{\circ m}(Z - m + b \log m) \in \mathbb{C}$$

converges locally uniformly to a map  $\Psi_F : -\mathbb{H}_R \rightarrow \mathbb{C}$  satisfying

$$\Psi_F \circ T_1 = F \circ \Psi_F.$$

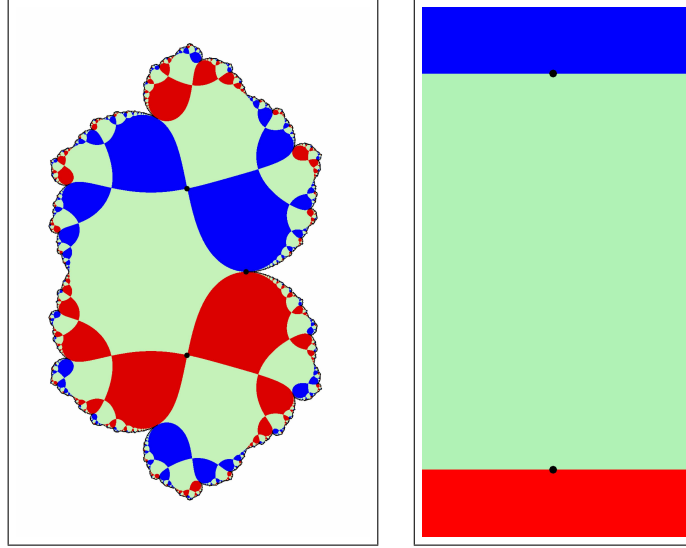


FIGURE 8. Left: the basin of attraction  $\mathcal{B}_f$ . The attracting Fatou coordinate  $\phi_f$  is univalent on each tile. It sends each blue tile to an upper half-plane, each red tile to a lower half-plane and each green tile to an horizontal strip. The parabolic point at 0 and the critical points  $c^\pm$  are marked. Right: the range of  $\phi_f$ . The points  $\phi_f(c^\pm)$  are marked.

In addition,

$$\Psi_F(Z) = Z + b \log(-Z) + o(1) \quad \text{as } \Re(Z) \rightarrow -\infty.$$

Transferring this to the initial coordinate, we see that the sequence of maps

$$(23) \quad \mathbb{C} \ni Z \mapsto f^{\circ m} \left( -\frac{1}{a_2 \cdot (Z - m + b \log m)} \right) \in \mathbb{C}$$

converges locally uniformly to an *repelling Fatou parameterization*  $\psi_f : \mathbb{C} \rightarrow \mathbb{C}$  which semi-conjugates  $T_1 : \mathbb{C} \rightarrow \mathbb{C}$  to  $f : \mathbb{C} \rightarrow \mathbb{C}$ , that is  $\psi_f \circ T_1 = f \circ \psi_f$ , and satisfies

$$-\frac{1}{\psi_f(Z)} = Z + b \log(-Z) + o(1) \quad \text{as } \Re(Z) \rightarrow -\infty.$$

The restriction of  $\psi_f$  to the left half-plane  $-\mathbb{H}_R$  coincides with  $Z \mapsto -1/(a_2 \Psi_F(Z))$ , whence is univalent. The image  $P_f^{\text{rep}} := \psi_f(-\mathbb{H}_R)$  is called a *repelling petal*.

The following proposition holds for the same reasons as in the attracting case (see [BEE, Section 5] for details).

**Proposition A.2.** *The map  $\psi_f$  depends holomorphically on  $f$ .*

**A.3. Lavaurs maps.** For  $\sigma \in \mathbb{C}$ , the Lavaurs map with phase  $\sigma$  is the map

$$L_{f,\sigma} := \psi_f \circ T_\sigma \circ \phi_f : \mathcal{B}_f \rightarrow \mathbb{C} \quad \text{with } T_\sigma(Z) = Z + \sigma.$$

In this article, we are only concerned by the Lavaurs map  $\mathcal{L}_f := \mathcal{L}_{f,0} := \psi_f \circ \phi_f$  with phase 0. The relevance of Lavaurs maps is justified by the following result due to Pierre Lavaurs [La].



**Theorem (Lavaurs).** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a polynomial such that  $f(z) = z + z^2 + O(z^3)$  as  $z \rightarrow 0$ . For  $\varepsilon \in \mathbb{C}$ , set  $f_\varepsilon(z) := f(z) + \varepsilon^2$ . Let  $(\varepsilon_n)_{n \geq 0}$  be a sequence of complex numbers and  $(m_n)_{n \geq 0}$  be a sequence of integers, such that*

$$\frac{\pi}{\varepsilon_n} - m_n \rightarrow \sigma \in \mathbb{C} \quad \text{as } n \rightarrow +\infty.$$

*Then, the sequence of polynomials  $f_{\varepsilon_n}^{m_n}$  converges locally uniformly on  $\mathcal{B}_f$  to  $\mathcal{L}_{f,\sigma}$ .*

It is also relevant to consider the map

$$\mathcal{E}_f := \phi_f \circ \psi_f : \mathcal{U}_f \rightarrow \mathbb{C} \quad \text{with } \mathcal{U}_f := \psi_f^{-1}(\mathcal{B}_f).$$

The repelling parameterization  $\psi_f$  semi-conjugates  $\mathcal{E}_f : \mathcal{U}_f \rightarrow \mathbb{C}$  to  $\mathcal{L}_f : \mathcal{B}_f \rightarrow \mathbb{C}$ . Figure 9 illustrates the behavior of the map  $\mathcal{E}_f$  for  $f(z) = z + z^2 + z^3$ .

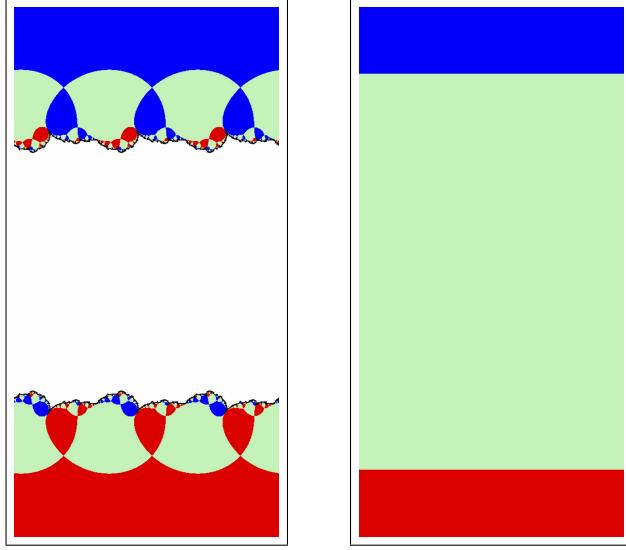


FIGURE 9. Behavior of the map  $\mathcal{E}_f$  for  $f(z) = z + z^2 + z^3$ . The domain  $\mathcal{U}_f$  has two connected components, one containing an upper half-plane and the other containing a lower half-plane. The domain is tiled according to the behavior of  $\mathcal{E}_f$ . The restriction of  $\mathcal{E}_f$  to each tile is univalent. The image of blue tiles is the blue upper half-plane on the right. The image of red tiles is the red lower half-plane on the right. The image of green tiles is the horizontal green strip on the right.

Propositions A.1 and A.2 imply that  $\mathcal{E}_f$  and  $\mathcal{L}_f$  vary nicely with  $f$ .

**Proposition A.3.** *The mappings  $\mathcal{E}_f$  and  $\mathcal{L}_f$  depend holomorphically on  $f$ .*

Note that  $\mathcal{E}_f$  commutes with  $T_1$ :

$$\mathcal{E}_f \circ T_1 = \phi_f \circ \psi_f \circ T_1 = \phi_f \circ f \circ \psi_f = T_1 \circ \phi_f \circ \psi_f = T_1 \circ \mathcal{E}_f.$$

So, the universal cover  $\exp : \mathbb{C} \ni Z \mapsto e^{2\pi i Z} \in \mathbb{C} \setminus \{0\}$  semi-conjugates  $\mathcal{E}_f : \mathcal{U}_f \rightarrow \mathbb{C}$  to a map

$$e_f : \mathcal{U}_f \rightarrow \mathbb{C} \setminus \{0\} \quad \text{with } \mathcal{U}_f := \exp(\mathcal{U}_f) \subset \mathbb{C} \setminus \{0\}.$$

The map  $e_f$  has removable singularities at 0 and  $\infty$ , thus it extends as a map  $e_f : \widehat{\mathcal{U}}_f \rightarrow \widehat{\mathbb{C}}$ , where  $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  is the Riemann sphere and  $\widehat{\mathcal{U}}_f := \mathcal{U}_f \cup \{0, \infty\} \subset \widehat{\mathbb{C}}$ .

The map  $e_f : \widehat{U}_f \rightarrow \widehat{\mathbb{C}}$  is called the *horn map* associated to  $f$ . As observed by Adam Epstein in his PhD thesis [Ep], this horn map is a finite type analytic map (see Definition A.4 below).

**A.4. Finite type analytic maps.** Let  $h : W \rightarrow X$  be an analytic map of complex 1-manifolds, possibly disconnected. An open set  $U \subseteq X$  is *evenly covered* by  $h$  if  $h|_V : V \rightarrow U$  is a homeomorphism for each component  $V$  of  $h^{-1}(U)$ ; we say that  $x \in X$  is a *regular value* for  $h$  if some neighborhood  $U$  of  $x$  is evenly covered, and a *singular value* for  $h$  otherwise. Note that the set  $S(h)$  of singular values is closed. Recall that  $w \in W$  is a *critical point* if the derivative of  $h$  at  $w$  vanishes, and then  $h(w) \in X$  is a *critical value*. We say that  $x \in X$  is an *asymptotic value* if  $h$  approaches  $x$  along some path tending to infinity relative to  $W$ . It follows from elementary covering space theory that the critical values together with the asymptotic values form a dense subset of  $S(h)$ . In particular, every isolated point of  $S(h)$  is a critical or asymptotic value.

**Definition A.4.** *An analytic map  $h : W \rightarrow X$  of complex 1-manifolds is of finite type if*

- $h$  is nowhere locally constant,
- $h$  has no isolated removable singularities,
- $X$  is a finite union of compact Riemann surfaces, and
- $S(h)$  is finite.

When  $h : W \rightarrow X$  is a finite type analytic map with  $W \subseteq X$ , we say that  $h$  is a finite type analytic map on  $X$ . The reason why finite type analytic maps are relevant when studying Lavaurs maps is the following.

Let  $f : \mathbb{P} \rightarrow \mathbb{P}$  be a rational map, let  $\phi_f : \mathcal{B}_f \rightarrow \mathbb{C}$  be an attracting Fatou coordinate defined on the parabolic basin of some fixed point of  $f$  with multiplier 1 and let  $\psi_f : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C})$  be a repelling Fatou parameterization associated to some fixed point of  $f$  with multiplier 1. Define

$$\mathcal{E}_f = \phi_f \circ \psi_f : \mathcal{U}_f \rightarrow \mathbb{C} \quad \text{with} \quad \mathcal{U}_f = (\psi_f)^{-1}(\mathcal{B}_f).$$

Finally set  $\widehat{U}_f = \exp(\mathcal{U}_f) \cup \{0, \infty\}$  and let  $e_f : \widehat{U}_f \rightarrow \widehat{\mathbb{C}}$  be defined by

$$\exp \circ \mathcal{E}_f = e_f \circ \exp.$$

The following result is stated as [BEE, Prop. 7.3].

**Proposition A.5.** *The map  $e_f : \widehat{U}_f \rightarrow \widehat{\mathbb{C}}$  is a finite type analytic map on  $\widehat{\mathbb{C}}$ . The singular values are:*

- $0$  and  $\infty$ , which are fixed asymptotic values of  $e_f$ , and
- the images by  $\exp \circ \phi_f$  of the critical orbits of  $f$  contained in  $\mathcal{B}_f$ , which are critical values of  $e_f$ .

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