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**AUTOMORPHISMS OF  $\mathbb{C}^2$  WITH AN INVARIANT  
NON-RECURRENT ATTRACTING FATOU COMPONENT  
BIHOLOMORPHIC TO  $\mathbb{C} \times \mathbb{C}^*$**

FILIPPO BRACCI<sup>◇\*</sup>, JASMIN RAISSY<sup>♣</sup>, AND BERIT STENSØNES<sup>♣\*</sup>

ABSTRACT. We prove the existence of automorphisms of  $\mathbb{C}^2$  having an invariant, non-recurrent Fatou component biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$  which is attracting, in the sense that all the orbits converge to a fixed point on the boundary of the component.

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INTRODUCTION

Let  $F$  be a holomorphic endomorphism of  $\mathbb{C}^k$ ,  $k \geq 1$ . In the study of the dynamics of  $F$ , that is of the behavior of its iterates, a natural dichotomy is given by the division of the space into the *Fatou set* and the *Julia set*. The Fatou set is the largest open set where the family of iterates is locally normal, that is the set formed by all points having an open neighborhood where the restriction of the iterates of the map forms a normal family. The Julia set is the complement of the Fatou set and is the part of the space where chaotic dynamics happens. A *Fatou component* is a connected component of the Fatou set.

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A Fatou component  $\Omega$  for a map  $F$  is called *invariant* if  $F(\Omega) = \Omega$ .

We call *attracting* any invariant Fatou component  $\Omega$  for a map  $F$  such that there exists a point  $p \in \overline{\Omega}$  with  $\lim_{n \rightarrow \infty} F^n(z) = p$  for all  $z \in \Omega$ . Note that, in particular,  $p$  is a fixed point for  $F$ . If  $p \in \Omega$  then  $\Omega$  is called *recurrent*, and it is called *non-recurrent* if  $p \in \partial\Omega$ .

Every attracting recurrent Fatou component of a holomorphic automorphism  $F$  of  $\mathbb{C}^2$  is biholomorphic to  $\mathbb{C}^2$  and it is, in fact, the global basin of attraction of  $F$  at  $p$ , which is an attracting fixed point, that is all eigenvalues of  $dF_p$  have modulus strictly less than 1 (see [9] and [11]).

As a consequence of the results obtained by T. Ueda in [13] and of Theorem 6 in [8] by M. Lyubich and H. Peters, every non-recurrent invariant attracting Fatou component  $\Omega$  of a *polynomial* automorphism of  $\mathbb{C}^2$  is biholomorphic to  $\mathbb{C}^2$ .

For non-polynomial automorphisms of  $\mathbb{C}^2$  the situation is completely different. In fact this paper is devoted to prove the following result:

**Theorem 0.1.** *There exist holomorphic automorphisms of  $\mathbb{C}^2$  having an invariant, non-recurrent, attracting Fatou component biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$ .*

The proof of this result is rather involved and will be outlined in the next section.

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## 1. MOTIVATIONS AND OUTLINE OF THE PROOF

Our result is based on the local behavior of a suitable (family of) germs of biholomorphisms of  $\mathbb{C}^2$  fixing the origin. The dynamics of germs in the attracting/repelling case and when they are tangent to the identity at the origin is quite well understood (see, e.g., [1]), and the cases we know would not be good candidates. In fact, using Hakim's result [7], the third named author together with L. Vivas constructed in [12] an automorphism of  $\mathbb{C}^3$  tangent to the identity at the origin, pointwise fixing an axis, and having an invariant attracting domain biholomorphic to  $\mathbb{C}^2 \times \mathbb{C}^*$ . This construction, however, does not work in dimension 2. Therefore we choose a germ whose linear part at the origin is an irrational rotation in both variables. More precisely, we consider germs of holomorphic diffeomorphisms of  $\mathbb{C}^2$  at the origin of the form

$$(1.1) \quad F_N(z, w) = (z(\lambda + azw + b_1(zw)), w(\bar{\lambda} + azw + b_2(zw))),$$

where  $b_j$  is a germ of a holomorphic function in  $\mathbb{C}$  near 0 such that  $b_j(\zeta) = O(|\zeta|^2)$  for  $j = 1, 2$ ,  $\lambda \in \mathbb{C}$ ,  $|\lambda| = 1$ , is not a root of unity,  $\operatorname{Re} \lambda < 0$  and  $a := -\frac{1}{2\operatorname{Re} \lambda} > 0$ .

Thanks to a result of B. J. Weickert [14] and F. Forstnerič [6], for any large  $l \in \mathbb{N}$  there exists an automorphism  $F$  of  $\mathbb{C}^2$  such that

$$(1.2) \quad F(z, w) - F_N(z, w) = O(\|(z, w)\|^l).$$

The local dynamics of this kind of maps, which are a particular case of the so-called one-resonant germs, has been studied by the first named author with D. Zaitsev in [3] (see also [4]).

Let

$$B := \{(z, w) \in \mathbb{C}^2 : |z| < |zw|^\beta, \quad |w| < |zw|^\beta, zw \in S\},$$

where  $\beta \in (0, \frac{1}{2})$  and  $S$  is a small sector in  $\mathbb{C}$  with vertex at 0 around the positive real axis. In [3] (see also Theorem 2.4) it has been proved that for sufficiently large  $l$  (depending on  $\lambda$  and  $\beta$ ) the domain  $B$  is forward invariant under  $F$ , the origin is on the boundary of  $B$  and  $\lim_{n \rightarrow \infty} F^n(p) = 0$  for all  $p \in B$ .

Setting  $x = zw, y = w$  (which are coordinates on  $B$ ) the domain looks like  $\{(x, y) \in \mathbb{C} \times \mathbb{C}^* : x \in S, |x|^{1-\beta} < |y| < |x|^\beta\}$ . Hence  $B$  is doubly connected.

Now let

$$\Omega := \bigcup_{n \in \mathbb{N}} F^{-n}(B).$$

In Proposition 5.1 we show that  $\Omega$  is connected but not simply connected.

For a point  $(z, w) \in \mathbb{C}^2$ , let  $(z_n, w_n) := F^n(z, w)$ . In Theorem 4.3 we show that

$$\Omega = \{(z, w) \in \mathbb{C}^2 \setminus \{(0, 0)\} : \lim_{n \rightarrow \infty} \|(z_n, w_n)\| = 0, \quad |z_n| \sim |w_n|\},$$

and moreover, if  $(z, w) \in \Omega$  then  $|z_n| \sim |w_n| \sim \frac{1}{\sqrt{n}}$ .

Having a characterization of the behavior of the orbits of a map on a completely invariant domain is however in general not enough to state that such a domain is the whole Fatou component, as this trivial example illustrates: the automorphism  $(z, w) \mapsto (\frac{z}{2}, \frac{w}{2})$  has the completely invariant domain  $\mathbb{C}^* \times \mathbb{C}^*$  which is not a Fatou component but  $|z_n| \sim |w_n|$ .

In order to prove that  $\Omega$  coincides with the Fatou component  $V$  containing it, we assume that  $\lambda$  is also Brjuno (see Section 4 for details). In this case there exist two  $F$ -invariant analytic discs, tangent to the axes, where  $F$  acts as an irrational rotation. In particular, one can choose local coordinates at  $(0, 0)$ , which we may assume to be defined on the unit ball  $\mathbb{B}$  of  $\mathbb{C}^2$  and  $B \subset \mathbb{B}$ , such that  $\{z = 0\}$  and  $\{w = 0\}$  are not contained in  $V \cap \mathbb{B}$ . Now, if  $V \neq \Omega$  one can find a small open set  $U$  such that  $F^n(U) \subset V \cap \mathbb{B}$ ,  $F^n(U) \not\subset B$  for every  $n \in \mathbb{N}$  and  $U \cup B$  is connected. Let  $W := B \cup \bigcup_{n \in \mathbb{N}} F^n(U)$ . Then for every  $\delta > 0$  we can find two points  $p_0 \in B$ ,  $q_0 \in W \setminus B$  such that  $k_W(p_0, q_0) < \delta$ , where  $k_W$  is the Kobayashi (pseudo)distance of  $W$ . Let  $\mathbb{B}_* := \mathbb{B} \setminus \{zw = 0\}$ . By the properties of the Kobayashi distance, for every  $n \in \mathbb{N}$  we have

$$k_{\mathbb{B}_*}(F^n(p_0), F^n(q_0)) \leq k_W(p_0, q_0) < \delta.$$

Also, if  $(z_n, w_n) := F^n(p_0)$ ,  $(x_n, y_n) := F^n(q_0)$ , then

$$k_{\mathbb{D}^*}(z_n, x_n) < \delta, \quad k_{\mathbb{D}^*}(w_n, y_n) < \delta,$$

where  $\mathbb{D}^*$  is the punctured unit disc. Since  $(x_n, y_n) \notin \Omega$  for every  $n \in \mathbb{N}$ , up to passing to a subsequence,  $|x_n| \not\sim |y_n|$ . By the triangle inequality and properties of the Kobayashi distance of  $\mathbb{D}^*$ , the shape of  $B$  forces  $k_{\mathbb{D}^*}(x_n, y_n)$  to be bounded from below by a constant depending only on  $\beta$ , leading to a contradiction (see Theorem 4.8 for details).

Finally, in order to show that  $\Omega$  is biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$  we construct a biholomorphism using the dynamics of the map  $F$ . Note that for this aim we do not need the Brjuno condition on  $\lambda$ .

We first prove the existence of a univalent map  $g$  on  $B$  which intertwines  $F$  on  $B$  with a simple overshear of  $\mathbb{C}^2$ . The first component  $\psi$  of  $g$  is essentially the Fatou coordinate of the projection of  $F$  onto the  $zw$ -plane and satisfies

$$\psi \circ F = \psi + 1.$$

The second component  $\sigma$  is the local uniform limit on  $B$  of the sequence  $\{\sigma_n\}$  defined by

$$\sigma_n(z, w) := \lambda^n \pi_2(F^n(z, w)) \exp\left(-\sum_{j=0}^{n-1} \frac{a\lambda}{\psi(z, w) + j}\right),$$

and satisfies the functional equation

$$\sigma \circ F = \bar{\lambda} e^{\frac{a\lambda}{\psi}} \sigma.$$

See Section 3 for details.

Next, using dynamics, we extend such a map to a univalent map  $G$  defined on  $\Omega$ , and we prove that its image is  $\mathbb{C} \times \mathbb{C}^*$ . Indeed, it turns out that each fiber of the first component of  $G$  contains annuli of the form  $\{(\frac{1}{n})^{1/2-\beta} < |\zeta| < n^{1/2-\beta}\}$  for every  $n \in \mathbb{N}$ . Since  $\beta < 1/2$  it follows that the image of  $\Omega$  under  $G$  contains  $\mathbb{C} \times \mathbb{C}^*$ . But  $\Omega$  is not simply connected, therefore it has to be contained in  $\mathbb{C} \times \mathbb{C}^*$ , hence  $\Omega = \mathbb{C} \times \mathbb{C}^*$  (see Section 5 for details).

## NOTATIONS AND CONVENTIONS

We set up here some notations and conventions we are going to use throughout the paper.

We let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ ,  $\pi_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$ ,  $\pi_2 : \mathbb{C}^2 \rightarrow \mathbb{C}$  be defined by

$$\pi(z, w) = zw, \quad \pi_1(z, w) = z, \quad \pi_2(z, w) = w.$$

If  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is a holomorphic map, we denote by  $F^n$  the  $n$ -th iterate of  $F$ ,  $n \in \mathbb{N}$ , defined by induction as  $F^n = F \circ F^{n-1}$ ,  $F^0 = \text{id}$ . Moreover, for  $(z, w) \in \mathbb{C}^2$  and  $n \in \mathbb{N}$ , we let

$$u_n := \pi(F^n(z, w)), \quad U_n := \frac{1}{u_n}, \quad z_n := \pi_1(F^n(z, w)), \quad w_n := \pi_2(F^n(z, w)).$$

If  $f(n)$  and  $g(n)$  are real positive functions of  $n \in \mathbb{N}$ , we write

$$f(n) \sim g(n),$$

if there exist  $0 < c_1 < c_2$  such that  $c_1 f(n) < g(n) < c_2 f(n)$  for all  $n \in \mathbb{N}$ . Moreover, we use the Landau little/big “O” notations, namely, we write

$$f(n) = O(g(n)),$$

if there exists  $C > 0$  such that  $f(n) \leq Cg(n)$  for all  $n \in \mathbb{N}$ , while we write

$$f(n) = o(g(n)),$$

if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ .

## 2. THE LOCAL BASIN OF ATTRACTION $B$

In this section we recall the construction of the local basin of attraction, and we provide the local characterization that we use in our construction.

Let  $F_N$  be a germ of biholomorphism of  $\mathbb{C}^2$ , fixing the origin, of the form

$$(2.1) \quad F_N(z, w) = (z(\lambda + azw + b_1(zw)), w(\bar{\lambda} + azw + b_2(zw))),$$

where,  $b_j$  is a germ of a holomorphic function in  $\mathbb{C}$  near 0 such that  $b_j(\zeta) = O(|\zeta|^2)$ ,  $j = 1, 2$ ,  $\lambda \in \mathbb{C}$ ,  $|\lambda| = 1$ , is not a root of unity,  $\operatorname{Re} \lambda < 0$  and  $a := -\frac{1}{2\operatorname{Re} \lambda} > 0$ .

**Definition 2.1.** For  $\theta \in (0, \frac{\pi}{2})$  and  $R > 0$  we let

$$S(R, \theta) := \left\{ \zeta \in \mathbb{C} : \left| \zeta - \frac{1}{2R} \right| < \frac{1}{2R}, \quad |\operatorname{Arg}(\zeta)| < \theta \right\}.$$

Also, we let

$$H(R, \theta) := \{ \zeta \in \mathbb{C} : \operatorname{Re} \zeta > R, |\operatorname{Arg}(\zeta)| < \theta \}.$$

**Lemma 2.2.** Let  $F_N$  be a germ of biholomorphism at  $(0, 0)$  of the form (2.1). Let

$$\eta_\lambda := \arctan \left( \left| \frac{\operatorname{Re} \lambda}{\operatorname{Im} \lambda} \right| \right).$$

Fix  $\theta \in (0, \eta_\lambda)$  and set

$$c_\theta := \frac{1}{4} \frac{1}{\sqrt{1 + \tan^2 \theta}} \left( 1 - \left| \frac{\operatorname{Im} \lambda}{\operatorname{Re} \lambda} \right| \tan \theta \right) > 0.$$

Then there exists  $R_\theta > 0$  such that for every  $U \in H(R_\theta, \theta)$ ,

$$(2.2) \quad \left| 1 + \frac{a}{\lambda} \frac{1}{U} + \frac{1}{\lambda} b_1 \left( \frac{1}{U} \right) \right| < 1 - \frac{c_\theta}{|U|}$$

and

$$(2.3) \quad \left| 1 + \frac{a}{\bar{\lambda}} \frac{1}{U} + \frac{1}{\bar{\lambda}} b_2 \left( \frac{1}{U} \right) \right| < 1 - \frac{c_\theta}{|U|}.$$

*Proof.* Let  $\theta \in (0, \eta_\lambda)$  and let  $R > 0$ . If  $U \in H(R, \theta)$ , then  $|\operatorname{Im} U| \leq (\tan \theta) \operatorname{Re} U$ , hence,

$$\frac{\operatorname{Re} U}{|U|} - \frac{\operatorname{Im} \lambda \operatorname{Im} U}{\operatorname{Re} \lambda |U|} = \frac{\operatorname{Re} U}{|U|} \left( 1 - \frac{\operatorname{Im} \lambda \operatorname{Im} U}{\operatorname{Re} \lambda \operatorname{Re} U} \right) \geq \frac{\operatorname{Re} U}{|U|} \left( 1 - \left| \frac{\operatorname{Im} \lambda}{\operatorname{Re} \lambda} \right| \tan \theta \right) \geq 4c_\theta.$$

Therefore, for  $U \in H(R, \theta)$ , recalling that  $a = -1/(2\operatorname{Re} \lambda)$ , we have

$$\begin{aligned} \left| 1 + \frac{a}{\lambda} \frac{1}{U} + \frac{1}{\lambda} b_1 \left( \frac{1}{U} \right) \right|^2 &= 1 + 2\operatorname{Re} \left( \frac{a}{\lambda U} \right) + o(|U|^{-1}) = 1 - \frac{1}{\operatorname{Re} \lambda} \frac{\operatorname{Re}(\lambda U)}{|U|^2} + o(|U|^{-1}) \\ &= 1 - \frac{1}{|U|} \left( \frac{\operatorname{Re} U}{|U|} - \frac{\operatorname{Im} \lambda \operatorname{Im} U}{\operatorname{Re} \lambda |U|} \right) + o(|U|^{-1}) \\ &\leq 1 - \frac{4c_\theta}{|U|} + o(|U|^{-1}). \end{aligned}$$

Thus, if  $R > 0$  is sufficiently large, it follows that for every  $U \in H(R, \theta)$ ,

$$(2.4) \quad \left| 1 + \frac{a}{\lambda} \frac{1}{U} + \frac{1}{\lambda} b_1 \left( \frac{1}{U} \right) \right|^2 < 1 - \frac{3c_\theta}{|U|}.$$

On the other hand, if  $R > 0$  is sufficiently large and  $|U| > R$ ,

$$\left( 1 - \frac{c_\theta}{|U|} \right)^2 = 1 - 2\frac{c_\theta}{|U|} + o(|U|^{-1}) \geq 1 - \frac{5}{2} \frac{c_\theta}{|U|}.$$

Thus, from the previous inequality and (2.4), we immediately obtain (2.2) for  $R_\theta > 0$  sufficiently large.

A similar argument holds for (2.3).  $\square$

In [3, Thm. 1.1] it has been proved that any small variation of  $F_N$  admits a local basin of attraction. For the reader's convenience and for the sake of completeness in the present work, we sketch the proof of such a result. To this aim, let us introduce some sets:

**Definition 2.3.** For  $\beta \in (0, \frac{1}{2})$  we let

$$W(\beta) := \{(z, w) \in \mathbb{C}^2 : |z| < |zw|^\beta, \quad |w| < |zw|^\beta\}.$$

For every  $R \geq 0$ ,  $\beta \in (0, \frac{1}{2})$  and  $\theta \in (0, \frac{\pi}{2})$ , we let

$$B(\beta, \theta, R) := \{(z, w) \in W(\beta) : zw \in S(R, \theta)\}.$$

We are now ready to state and prove the existence of local basins of attraction:

**Theorem 2.4.** *Let  $F_N$  be a germ of biholomorphism at  $(0, 0)$  of the form (2.1). Let  $\theta_0 \in (0, \eta_\lambda)$  and let  $c_{\theta_0}, R_{\theta_0}$  be the constants given in Lemma 2.2. Let  $\beta_0 \in (0, 1/2)$  be such that  $\beta_0 < c_{\theta_0}/2$ . Let  $l \in \mathbb{N}$ ,  $l \geq 4$  be such that  $\beta_0(l+1) \geq 4$ . Finally, assume  $F$  is a germ of biholomorphism at  $(0, 0)$  of the form*

$$F(z, w) = F_N(z, w) + O(\|(z, w)\|^l).$$

Then there exists  $R_0 \geq R_{\theta_0} > 0$  such that for every  $R \geq R_0$  the (non-empty) open set  $B_R := B(\beta_0, \theta_0, R)$  is a uniform local basin of attraction for  $F$ . Namely,  $F(B_R) \subseteq B_R$ , and

$$\lim_{n \rightarrow \infty} F^n(z, w) = (0, 0)$$

uniformly in  $(z, w) \in B_R$ .

*Sketch of the Proof.* For  $(z, w) \in \mathbb{C}^2$ , we have

$$u_1 = \pi(F(z, w)) = u - u^2 + O(|u|^3) + O(\|(z, w)\|^{l+1}),$$

and

$$(2.5) \quad U_1 = \frac{1}{u_1} = U \left( \frac{1}{1 - \frac{1}{U} + O\left(\frac{1}{|U|^2}\right) + |U|O(\|(z, w)\|^{l+1})} \right).$$

Let  $R \geq R_{\theta_0}$ . The function  $\zeta \mapsto \frac{1}{\zeta}$  maps the set  $S(R, \theta_0)$  onto the set  $H(R, \theta_0)$ . Hence,  $u_1 \in S(R, \theta_0)$  if and only if  $U_1 \in H(R, \theta_0)$ .

Since  $(z, w) \in W(\beta_0)$  implies that  $\|(z, w)\|^{l+1} \leq |u|^{\beta_0(l+1)}$  and  $\beta_0(l+1) \geq 4$ , we have

$$(2.6) \quad U_1 = U + 1 + f(z, w), \quad |f(z, w)| = O\left(\frac{1}{|U|}\right),$$

for all  $(z, w) \in W(\beta_0)$ . Hence, if  $R \geq R_{\theta_0}$  is sufficiently large so that  $|f(z, w)| < \frac{1}{10}$ , it follows that  $U_1 \in H(R, \theta_0)$  whenever  $U \in H(R, \theta_0)$ , and  $(z, w) \in W(\beta_0)$ .

Now we have to show that if  $(z, w) \in B_R$  then  $F(z, w) \in W(\beta_0)$ .

We prove that  $|\pi_1(F(z, w))| < |u_1|^{\beta_0}$ , a similar argument works for  $\pi_2 \circ F$ . From (2.1), we have

$$\pi_1(F(z, w)) = \lambda \left( 1 + \frac{a}{\lambda} \frac{1}{U} + \frac{1}{\lambda} b_1 \left( \frac{1}{U} \right) \right) z + O(\|(z, w)\|^l).$$

Since  $(z, w) \in W(\beta_0)$  implies  $O(\|(z, w)\|^l) = O(|u|^{l\beta_0})$ , by Lemma 2.2 (setting  $c := c_{\theta_0}$  for simplicity),

$$|\pi_1(F(z, w))| \leq |z| \left( 1 - \frac{c}{|U|} \right) + C|u|^{l\beta_0} \leq \left( 1 - \frac{c}{|U|} + C|u|^{(l-1)\beta_0} \right) |u|^{\beta_0},$$

for some  $C > 0$ .

Taking into account that  $(l-1)\beta_0 = (l+1)\beta_0 - 2\beta_0 \geq 4 - 2\beta_0 > 2$ , we can write

$$|\pi_1(F(z, w))| \leq \left( 1 - \frac{c}{|U|} + \frac{C_1}{|U|^2} \right) |u|^{\beta_0},$$

for some  $C_1 > 0$ .

Hence,  $|\pi_1(F(z, w))| < |u_1|^{\beta_0}$  if we can prove that

$$|u| \leq \left( 1 - \frac{c}{|U|} + \frac{C_1}{|U|^2} \right)^{-1/\beta_0} |u_1|,$$



which is equivalent to

$$(2.7) \quad |U + 1 + f(z, w)| = |U_1| \leq \left(1 - \frac{c}{|U|} + \frac{C_1}{|U|^2}\right)^{-1/\beta_0} |U|.$$

Now, if  $R \geq R_{\theta_0}$  is large enough, bearing in mind that we assumed that  $\beta_0 < c/2$ , then for every  $(z, w) \in W(\beta_0)$  such that  $U \in H(\theta_0, R)$ ,

$$\begin{aligned} \left(1 - \frac{c}{|U|} + \frac{C_1}{|U|^2}\right)^{-1/\beta_0} &\geq \left(1 - \frac{c}{2|U|}\right)^{-1/\beta_0} = 1 + \frac{1}{2\beta_0} \frac{c}{|U|} + o\left(\frac{1}{|U|}\right) \\ &> 1 + \frac{1}{|U|} + \frac{|f(z, w)|}{|U|} = \frac{|U + 1 + f(z, w)|}{|U|} = \frac{|U_1|}{|U|}, \end{aligned}$$

hence, (2.7) follows.

Summing up, we proved that there exists  $R_0 \geq R_{\theta_0}$  such that  $F(B_R) \subset B_R$  for any  $R \geq R_0$ . Moreover, arguing by induction, from (2.6), one can easily prove that

$$|U_n| = \left| \frac{1}{\pi(F^n(z, w))} \right| \geq R_0 + \frac{n}{2},$$

for all  $(z, w) \in B_{R_0}$ , showing that  $\{F^n\}$  converges uniformly to  $(0, 0)$  on  $B_{R_0}$ .

Finally, note that  $B_R$  is non-empty for all  $R \geq R_0$ , since  $(r, r) \in B_R$  for all  $r > 0$  small enough.  $\square$

**Definition 2.5.** Let  $F(z, w) = F_N(z, w) + O(\|(z, w)\|^l)$  be as in Theorem 2.4. We set

$$B := B_{R_0} = B(\beta_0, \theta_0, R_0).$$

In the following, we shall use some properties of  $B$ , that we prove here below. We start with a lemma, which allows us to characterize the pre-images of  $B$ .

**Lemma 2.6.** *Let  $F$  and  $B$  be as in Theorem 2.4. Let  $\beta \in (0, \frac{1}{2})$  be such that  $\beta(l+1) > 2$  and  $(z, w) \in \mathbb{C}^2$  such that  $(z_n, w_n) \rightarrow (0, 0)$  as  $n \rightarrow \infty$ . If there exists  $n_0 \in \mathbb{N}$  such that  $(z_n, w_n) \in W(\beta)$  for all  $n \geq n_0$ , then*

- (1)  $\lim_{n \rightarrow \infty} nu_n = 1$  and  $\lim_{n \rightarrow \infty} \frac{u_n}{|u_n|} = 1$  (in particular,  $|u_n| \sim \frac{1}{n}$ ),
- (2)  $|z_n| \sim n^{-1/2}$  and  $|w_n| \sim n^{-1/2}$ ,
- (3) for every  $\gamma \in (0, 1/2)$  there exists  $n_\gamma \in \mathbb{N}$  such that  $(z_n, w_n) \in W(\gamma)$  for all  $n \geq n_\gamma$ .

In particular,  $(z_n, w_n) \in B$  eventually.

*Proof.* The map  $F$  has the form

$$(2.8) \quad F(z, w) = (z(\lambda + azw + b_1(zw)) + R_1^1(z, w), w(\bar{\lambda} + azw + b_2(zw) + R_1^2(z, w))),$$

where  $b_j(\zeta) = O(|\zeta|^2)$  and  $R_i^j(z, w) = O(\|(z, w)\|^l)$ ,  $j = 1, 2$ .

Since

$$U_{n+1} = \frac{1}{u_{n+1}} = U_n \left( \frac{1}{1 - \frac{1}{U_n} + O\left(\frac{1}{|U_n|^2}\right) + |U_n|O(\|(z_n, w_n)\|^{l+1})} \right),$$

taking into account that  $(z_n, w_n) \rightarrow (0, 0)$ , we have

$$U_{n+1} = U_n \left( 1 + \frac{1}{U_n} + O\left(\frac{1}{|U_n|^2}, |U_n|\|(z_n, w_n)\|^{l+1}\right) \right).$$

For  $n \geq n_0$ , and since  $\beta(l+1) > 2$ , we have that  $O(\|(z_n, w_n)\|^{l+1})$  is at most an  $O(|u_n|^{\beta(l+1)}) = O\left(\frac{1}{|U_n|^{\beta(l+1)}}\right)$ . Hence,

$$(2.9) \quad U_{n+1} = U_n \left( 1 + \frac{1}{U_n} + O\left(\frac{1}{|U_n|^{\beta(l+1)-1}}, \frac{1}{|U_n|^2}\right) \right).$$

Fix  $\epsilon > 0$ . Let  $c := 1 + \epsilon$ . We claim that there exists  $n_c \in \mathbb{N}, n_c \geq n_0$  such that, for all  $n \geq n_c$  we have

$$(2.10) \quad \operatorname{Re} U_n \geq \operatorname{Re} U_{n_c} + \frac{n}{c},$$

and

$$(2.11) \quad |U_n| \leq |U_{n_c}| + cn.$$

In order to prove (2.10) and (2.11), we first notice that, by (2.9), there exists  $n_c \geq n_0$  such that for all  $n \geq n_c$ ,

$$|U_{n+1} - U_n - 1| < \frac{c-1}{c}.$$

Then, starting from (2.9), one can prove easily (2.10) and (2.11) by induction on  $n \geq n_c$ .

From (2.10) and (2.11), letting  $\epsilon \rightarrow 0^+$  (so that  $c \rightarrow 1^+$ ) we obtain immediately that

$$(2.12) \quad \lim_{n \rightarrow \infty} \frac{\operatorname{Re} U_n}{n} = \lim_{n \rightarrow \infty} \frac{|U_n|}{n} = 1.$$

In particular, this means that  $\lim_{n \rightarrow \infty} n \operatorname{Re} u_n = \lim_{n \rightarrow \infty} n |u_n| = 1$ . Hence,  $\lim_{n \rightarrow \infty} \frac{|u_n|}{\operatorname{Re} u_n} = 1$ , which implies at once that

$$(2.13) \quad \lim_{n \rightarrow \infty} \frac{\operatorname{Im} u_n}{\operatorname{Re} u_n} = 0.$$

Hence statement (1) follows.

Now, for  $j \in \mathbb{N}$ , we set

$$A_{1,j} := \frac{a}{\lambda} u_j + \frac{b_1(u_j)}{\lambda} = -\frac{1}{2\lambda \operatorname{Re} \lambda} u_j + \frac{b_1(u_j)}{\lambda},$$

and

$$A_{2,j} := \frac{a}{\bar{\lambda}} u_j + \frac{b_1(u_j)}{\bar{\lambda}} = -\frac{1}{2\bar{\lambda} \operatorname{Re} \lambda} u_j + \frac{b_2(u_j)}{\bar{\lambda}}.$$

Arguing by induction, we have

$$(2.14) \quad \begin{aligned} z_{n+1} &= z_0 \lambda^n \prod_{j=0}^n (1 + A_{1,j}) + \sum_{j=0}^n R_l^1(z_j, w_j) \prod_{k=j+1}^n \lambda(1 + A_{1,k}), \\ w_{n+1} &= w_0 \bar{\lambda}^n \prod_{j=0}^n (1 + A_{2,j}) + \sum_{j=0}^n R_l^2(z_j, w_j) \prod_{k=j+1}^n \bar{\lambda}(1 + A_{2,k}), \end{aligned}$$

Now we concentrate on  $z_n$ , as the argument for  $w_n$  is similar. We have,

$$(2.15) \quad |z_{n+1}| \leq |z_0| \prod_{j=0}^n |1 + A_{1,j}| + \sum_{j=0}^n |R_l^1(z_j, w_j)| \prod_{k=j+1}^n |1 + A_{1,k}|.$$

Taking into account statement (1), we have

$$\begin{aligned} \lim_{j \rightarrow \infty} (-2j) \log |1 + A_{1,j}| &= \lim_{j \rightarrow \infty} (-2j) \left( \frac{1}{2} \log |1 + A_{1,j}|^2 \right) \\ &= \lim_{j \rightarrow \infty} (-2j) \left( \frac{1}{2} \log(1 + |A_{1,j}|^2 + 2\operatorname{Re} A_{1,j}) \right) \\ &= \lim_{j \rightarrow \infty} (-2j) \left( \frac{1}{2} |A_{1,j}|^2 + \operatorname{Re} A_{1,j} \right) = 1. \end{aligned}$$

Therefore,

$$(2.16) \quad \prod_{j=0}^n |1 + A_{1,j}| = \exp \left( \sum_{j=0}^n \log |1 + A_{1,j}| \right) \sim \exp \left( \sum_{j=1}^n -\frac{1}{2j} \right) \sim \frac{1}{\sqrt{n}}.$$

Moreover, since  $(z_n, w_n) \in W(\beta)$  eventually, and  $|R_l^1(z_j, w_j)| = O(\|(z_j, w_j)\|^l)$ , it follows that there exist some constants  $0 < c \leq C$  such that

$$|R_l^1(z_j, w_j)| \leq c |u_j|^{\beta l} \leq C j^{-\beta l}.$$

Hence, by (2.16) we have for  $j > 1$  sufficiently large

$$\begin{aligned} |R_l^1(z_j, w_j)| \prod_{k=j+1}^n |1 + A_{1,k}| &= |R_l^1(z_j, w_j)| \exp \left( \sum_{k=j+1}^n \log |1 + A_{1,k}| \right) \\ &\sim |R_l^1(z_j, w_j)| \exp \left( -\frac{1}{2} \sum_{k=j+1}^n \frac{1}{k} \right) \\ &\sim |R_l^1(z_j, w_j)| \frac{\sqrt{j}}{\sqrt{n}} \leq C \frac{j^{\frac{1}{2} - \beta l}}{\sqrt{n}}. \end{aligned}$$

Since  $\beta l - \frac{1}{2} = \beta(l+1) - \beta - \frac{1}{2} > 2 - \beta - \frac{1}{2} > 1$ , it follows that there exists a constant (which we still denote by)  $C > 0$  such that

$$\sum_{j=0}^n |R_l^1(z_j, w_j)| \prod_{k=j+1}^n |1 + A_{1,k}| \leq C \frac{1}{\sqrt{n}}.$$

Hence, from (2.15), there exists a constant  $C > 0$  such that

$$(2.17) \quad |z_n| \leq C \frac{1}{\sqrt{n}}.$$

A similar argument for  $w_n$ , shows that

$$(2.18) \quad |w_n| \leq C \frac{1}{\sqrt{n}}.$$

By statement (1), it holds  $|z_n| \cdot |w_n| = |u_n| \sim \frac{1}{n}$ . Since  $|z_n| \leq C \frac{1}{\sqrt{n}}$  and  $|w_n| \leq C \frac{1}{\sqrt{n}}$  by (2.17) and (2.18), it follows that, in fact,  $|z_n| \sim \frac{1}{\sqrt{n}}$  and  $|w_n| \sim \frac{1}{\sqrt{n}}$ . Statement (2) is then proved.

Finally, by statement (2), there exist constants  $c, C > 0$  such that  $|z_n| \leq C \frac{1}{\sqrt{n}}$  for all  $n \in \mathbb{N}$  and  $|u_n| \geq c \frac{1}{n}$ . Fix  $\gamma \in (0, 1/2)$ . Hence, for every  $n$  large enough,

$$|z_n| \leq C \frac{1}{\sqrt{n}} \leq \frac{C}{c^{1/2}} |u_n|^{1/2} < |u_n|^\gamma.$$

Similarly, one can prove that  $|w_n| < |u_n|^\gamma$ . As a consequence,  $(z_n, w_n) \in W(\gamma)$  eventually for every  $\gamma \in (0, 1/2)$ .  $\square$

*Remark 2.7.* It follows from the uniform converge of  $\{F^n\}$  to  $(0, 0)$  in  $B$  (see Theorem 2.4) and the proof of the previous lemma, that (1) and (2) in Lemma 2.6 are uniform in  $B$ .

We shall also need the following local result concerning the topology of  $B$ :

**Lemma 2.8.** *Let  $F$  and  $B$  be as in Theorem 2.4. Then  $B$  is a doubly connected domain (i.e.,  $B$  is connected and its fundamental group is  $\mathbb{Z}$ ). Moreover, its fundamental group is generated by the loop  $\gamma: [0, 2\pi] \ni t \mapsto (re^{it}, re^{-it})$  for any fixed  $0 < r < 1/\sqrt{R}$ .*

*Proof.* Let  $\Phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be defined by

$$(2.19) \quad \Phi(z, w) = (zw, w).$$

Note that  $\Phi$  is a biholomorphism from  $\mathbb{C}^2 \setminus \{(z, w) \in \mathbb{C}^2 : w = 0\}$  onto  $\mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 : y = 0\}$ . Since  $B \subset \mathbb{C}^2 \setminus \{(z, w) \in \mathbb{C}^2 : w = 0\}$ , it follows that  $\Phi: B \rightarrow \Phi(B)$  is a biholomorphism. A straightforward computation shows that

$$(2.20) \quad \Phi(B) = \{(x, y) \in \mathbb{C} \times \mathbb{C}^* : x \in S_R(\epsilon), |x|^{1-\beta_0} < |y| < |x|^{\beta_0}\}.$$

In particular,  $\Phi(B)$  is a submersion over  $S_R(\epsilon)$  where each fiber is an annulus, centered at the origin, of internal radius  $|x|^{1-\beta_0}$  and external radius  $|x|^{\beta_0}$ .

Consider the map  $f: \Phi(B) \rightarrow \mathbb{C}^2$  given by

$$(2.21) \quad f(x, y) = (x, \varphi(x, y)),$$

where, writing  $y = \rho e^{i\theta}$  in polar coordinates,  $\rho > 0$  and  $\theta \in \mathbb{R}$ ,

$$\varphi(x, \rho e^{i\theta}) := \left( \frac{\rho - |x|^{1-\beta_0}}{|x|^{\beta_0} - |x|^{1-\beta_0}} + 1 \right) e^{i\theta}.$$

The map  $f$  is a diffeomorphism onto its image, which is given by  $S_R(\epsilon) \times \{\zeta \in \mathbb{C} : 1 < |\zeta| < 2\}$ . Hence  $B$  is doubly connected and clearly the loop  $\gamma$ , which is contained in  $B$ , generates its fundamental group.  $\square$

*Remark 2.9.* Let  $\Gamma(t) := (x(t), y(t)): [0, 2\pi] \rightarrow \Phi(B)$  be a closed continuous curve, where  $\Phi(B)$  is given by (2.20). The curve  $\Gamma$  is contractible in  $\Phi(B)$  if and only if the winding number around 0 of the curve  $[0, 2\pi] \ni t \mapsto y(t)$  is zero. Indeed, by Lemma 2.8,  $\Gamma$  is homotopic to  $\Phi(\gamma^n)$  for some  $n \in \mathbb{Z}$ , where  $\gamma^n(t) = (re^{nit}, re^{-nit})$ . Hence, there exists a homotopy of paths  $H = (H_1, H_2): [0, 1] \times [0, 1] \rightarrow \Phi(B)$  between  $\Gamma$  and  $\Phi(\gamma^n)$ . Clearly,  $H_2$  is a homotopy between  $y$  and  $[0, 2\pi] \ni t \mapsto re^{-nit}$  and thus the winding number of  $y$  is  $-n$ , and the statement is proved.

### 3. LOCAL FATOU COORDINATES ON $B$

In this section we are going to introduce special coordinates on  $B$ , which will be used later on in our construction. The first coordinate was introduced in [4, Prop. 4.3]. Here we shall need more precise information, therefore we are going to prove the following result:

**Proposition 3.1.** *Let  $F$  and  $B$  be as in Theorem 2.4. Then there exists  $\psi: B \rightarrow \mathbb{C}$  holomorphic such that*

$$(3.1) \quad \psi \circ F = \psi + 1.$$

*The map  $\psi$  is the uniform limit in  $B$  of the sequence of functions  $\{\psi_m\}_{m \in \mathbb{N}}$ , where  $\psi_m: B \rightarrow \mathbb{C}$  is defined as*

$$(3.2) \quad \psi_m(z, w) := \frac{1}{\pi(F^m(z, w))} - m - c \log \pi(F^m(z, w)),$$

*where  $c \in \mathbb{C}$  depends only on  $F_N$ . Moreover,  $|\psi(z, w) - \psi_m(z, w)| \sim \frac{1}{m}$  and*

$$(3.3) \quad \psi(z, w) = \frac{1}{zw} + c \log \frac{1}{zw} + v(z, w),$$

*where  $v: B \rightarrow \mathbb{C}$  is a holomorphic function such that for every  $(z, w) \in B$ ,*

$$(3.4) \quad v(z, w) = zw \cdot g(z, w),$$

*where  $g: B \rightarrow \mathbb{C}$  is a bounded holomorphic function.*

*Proof.* From (2.5), we have

$$U_{n+1} = U_n + 1 + \frac{c}{U_n} + O(|U_n|^2)$$

for some  $c \in \mathbb{C}$  and all  $n \in \mathbb{N}$ , where, as usual,  $U_n := \frac{1}{\pi(F^n(z,w))}$ . Then a direct computation as in [4, Prop. 4.3] implies that there exists  $A > 0$  such that for all  $m \in \mathbb{N}$  and all  $(z, w) \in B$ ,

$$(3.5) \quad |\psi_{n+1}(z, w) - \psi_n(z, w)| \leq A|U_n|^{-2}.$$

Therefore, since  $|U_n| = 1/|u_n| \sim n$  uniformly in  $B$  by Lemma 2.6 and Remark 2.7, the sequence  $\sum_{j=0}^n (\psi_{j+1} - \psi_j)$  is uniformly converging in  $B$  to a bounded holomorphic function  $v$ , that is,

$$v(z, w) := \sum_{j=0}^{\infty} (\psi_{j+1}(z, w) - \psi_j(z, w)).$$

Moreover, since  $\psi_n - \psi_0 = \sum_{j=0}^n (\psi_{j+1}(z, w) - \psi_j(z, w))$ , passing to the limit we get (3.3).

From  $\psi_n \circ F = \psi_{n+1} + 1$ , passing to the limit, we obtain (3.1). Also, by (3.5),

$$|\psi - \psi_m| = \left| \sum_{j=m}^{\infty} (\psi_{j+1} - \psi_j) \right| \leq \sum_{j=m}^{\infty} |\psi_{j+1} - \psi_j| \sim \sum_{j=m}^{\infty} \frac{1}{j^2} \sim \frac{1}{m}.$$

Finally, since  $U_n \in H(\theta_0, R_0)$  for all  $n \in \mathbb{N}$ , it follows that there exists  $K \in (0, 1)$  such that  $\operatorname{Re} U_0 > K|U_0|$  for all  $U_0 \in H(\theta_0, R_0)$ . Hence, by (2.10),

$$\begin{aligned} |v(z, w)| &\leq A \sum_{j=0}^{\infty} \frac{1}{|U_j|^2} \leq A \sum_{j=0}^{\infty} \frac{1}{(\operatorname{Re} U_j)^2} \leq A \sum_{j=0}^{\infty} \frac{1}{(\operatorname{Re} U_0 + \frac{j}{2})^2} \\ &\sim A \int_0^{\infty} \frac{dt}{(\operatorname{Re} U_0 + \frac{t}{2})^2} = \frac{A}{\operatorname{Re} U_0} \leq \frac{A}{K|U_0|}, \end{aligned}$$

from which (3.4) follows at once.  $\square$

The map  $\psi : B \rightarrow \mathbb{C}$  is called a *Fatou coordinate* for  $F$ .

**Lemma 3.2.** *Let  $F$  be as in Theorem 2.4. Let  $\psi$  be the Fatou coordinate for  $F$  given by Proposition 3.1. Then there exist  $R_1 \geq R_0$ ,  $\beta_1 \in (\beta_0, \frac{1}{2})$  and  $0 < \theta_1 \leq \theta_0$  such that the holomorphic map*

$$B(\beta_1, \theta_1, R_1) \ni (z, w) \mapsto (\psi(z, w), w)$$

*is injective.*

*Proof.* Recall that by definition of  $B$ , if  $(z, w) \in B$  then  $U \in H(R_0, \theta_0)$ . Thanks to (3.3) and (3.4), we have that  $\psi(z, w) = U + c \log U + v(z, w)$ , where  $v(z, w) = \frac{g(z,w)}{U}$  and  $|g| \leq C$  for some constant  $C > 0$ .

Since the map  $\chi : B \ni (z, w) \mapsto (U, w)$  is univalent, we can consider  $v$  as a function of  $(U, w)$  defined on  $\chi(B) = \{(U, w) : U \in H(R_0, \theta_0), |U|^{\beta_0-1} < |w| < |U|^{-\beta_0}\}$ .

There exists  $\delta_0 > 0$  such that for every  $U \in (H(R_0, \theta_0) + 1)$  (that is  $U = V + 1$  with  $V \in H(R_0, \theta_0)$ ), the distance of  $U$  from  $\partial H(R_0, \theta_0)$  is greater than  $2\delta_0$ . Hence, the disc of center  $U$  and radius  $\delta_0$  is contained in  $H(R_0, \theta_0)$  for all  $U \in (H(R_0, \theta_0) + 1)$ . Let  $\theta_1 \in (0, \theta_0)$  be such that  $H(R_0 + 1, \theta_1) \subset (H(R_0, \theta_0) + 1)$ .

Let  $\tilde{\beta} \in (\beta_0, \frac{1}{2})$ . For  $R \geq R_0$ , we have  $\chi(B(\tilde{\beta}, \theta_1, R)) = \{(U, w) : U \in H(R, \theta_1), |U|^{\tilde{\beta}-1} < |w| < |U|^{-\tilde{\beta}}\}$ . Since

$$\lim_{r \rightarrow \infty} \frac{(r - \delta_0)^{\beta_0 - 1}}{r^{\tilde{\beta} - 1}} = 0, \quad \lim_{r \rightarrow \infty} \frac{(r + \delta_0)^{-\beta_0}}{r^{-\tilde{\beta}}} = \infty,$$

there exists  $\tilde{R} \geq R_0$  such that for all  $(U, w) \in \chi(B(\tilde{\beta}, \theta_1, \tilde{R}))$  and all  $t \in \mathbb{R}$  it holds

$$|U + \delta_0 e^{it}|^{\beta_0 - 1} \leq (|U| - \delta_0)^{\beta_0 - 1} < |U|^{\tilde{\beta} - 1} < |w| < |U|^{-\tilde{\beta}} < (|U| + \delta_0)^{-\beta_0} \leq |U + \delta_0 e^{it}|^{-\beta_0},$$

which, taking into account that  $U + \delta_0 e^{it} \in H(R_0, \theta_0)$  for all  $t \in \mathbb{R}$ , implies that  $(U + \delta_0 e^{it}, w) \in \chi(B)$  for all  $t \in \mathbb{R}$  and all  $(U, w) \in \chi(B(\tilde{\beta}, \theta_1, \tilde{R}))$ .

Therefore, for every  $(U_0, w_0) \in \chi(B(\tilde{\beta}, \theta_1, \tilde{R}))$ , from the Cauchy formula for derivatives we obtain

$$\left| \frac{\partial g}{\partial U}(U_0, w_0) \right| = \frac{1}{2\pi} \left| \int_{|\zeta - U_0| = \delta_0} \frac{g(\zeta, w_0)}{(\zeta - U_0)^2} d\zeta \right| \leq \frac{1}{2\pi\delta_0} \sup_{(U, w_0) \in \chi(B)} |g(U, w_0)| \leq \frac{C}{2\pi\delta_0} =: C_1.$$

Hence, setting  $C_2 := C + C_1$ , for every  $R \geq \min\{\tilde{R}, 1\}$ , we have

$$(3.6) \quad \left| \frac{\partial v}{\partial U}(U_0, w_0) \right| \leq |U_0^{-2}|C + |U_0^{-1}| \left| \frac{\partial g}{\partial U}(U_0, w_0) \right| \leq |U_0^{-2}|C + |U_0^{-1}|C_1 \leq \frac{C_2}{R}$$

for all  $(U_0, w_0) \in \chi(B(\tilde{\beta}, \theta_1, R))$ . Now, there exists  $K \in (0, 1)$  such that  $\operatorname{Re} U > K|U|$  independently of  $R$  for every  $U \in H(\theta_1, R)$ . Fix  $\beta_1 \in (\tilde{\beta}, \frac{1}{2})$  and let  $R \geq \tilde{R}$  be such that

$$(3.7) \quad K^{1-\beta_1} r^{\beta_1-1} > r^{\tilde{\beta}-1} \quad \forall r \geq R.$$

Let  $(U_1, w_0), (U_2, w_0) \in \chi(B(\beta_1, \theta_1, R))$ . Let  $\gamma := \{U \in \mathbb{C} : U = tU_1 + (1-t)U_2, t \in [0, 1]\}$  be the real segment joining  $U_1$  and  $U_2$ . Since  $H(\theta_1, R)$  is convex,  $\gamma \subset H(\theta_1, R)$ . Moreover, for all  $t \in [0, 1]$ , taking into account that  $(U_j, w_0) \in \chi(B(\beta_1, \theta_1, R))$  implies that  $|U_j| > |w_0|^{\frac{1}{\beta_1-1}}$ ,  $j = 1, 2$ , we have

$$\begin{aligned} |tU_1 + (1-t)U_2| &> t\operatorname{Re} U_1 + (1-t)\operatorname{Re} U_2 > tK|U_1| + (1-t)K|U_2| \\ &> K \left( t|w_0|^{\frac{1}{\beta_1-1}} + (1-t)|w_0|^{\frac{1}{\beta_1-1}} \right) = K|w_0|^{\frac{1}{\beta_1-1}}, \end{aligned}$$

hence, by (3.7),

$$|w_0| > \left( \frac{1}{K} \right)^{\beta_1-1} |tU_1 + (1-t)U_2|^{\beta_1-1} > |tU_1 + (1-t)U_2|^{\tilde{\beta}-1}.$$

On the other hand, taking into account that  $(U_j, w_0) \in \chi(B(\beta_1, \theta_1, R))$  implies that  $|U_j| < |w_0|^{-\frac{1}{\beta_1}}$ ,  $j = 1, 2$ , for all  $t \in (0, 1)$  we have,

$$|tU_1 + (1-t)U_2| \leq t|U_1| + (1-t)|U_2| < t|w_0|^{-\frac{1}{\beta_1}} + (1-t)|w_0|^{-\frac{1}{\beta_1}} = |w_0|^{-\frac{1}{\beta_1}},$$

hence,

$$|tU_1 + (1-t)U_2|^{-\tilde{\beta}} > |tU_1 + (1-t)U_2|^{-\beta_1} > |w_0|.$$

Therefore, if  $(U_1, w_0), (U_2, w_0) \in \chi(B(\beta_1, \theta_1, R))$  then  $(tU_1 + (1-t)U_2, w_0) \in \chi(B(\tilde{\beta}, \theta_1, R))$  for all  $t \in [0, 1]$ . Thus, by (3.6),

$$\begin{aligned} |\psi(U_1, w_0) - \psi(U_2, w_0)| &= \left| \int_{\gamma} \frac{\partial \psi}{\partial U}(U, w_0) dU \right| = \left| \int_{\gamma} \left[ 1 + \frac{c}{U} + \frac{\partial v}{\partial U}(U, w_0) \right] dU \right| \\ &\geq \left| \int_{\gamma} dU \right| - \left| \int_{\gamma} \frac{cdU}{U} \right| - \left| \int_{\gamma} \frac{\partial v}{\partial U}(U, w_0) dU \right| \\ &\geq |U_1 - U_0| - \frac{|c|}{R}|U_1 - U_0| - \frac{C_2}{R}|U_1 - U_0| \\ &= \left( 1 - \frac{|c|}{R} - \frac{C_2}{R} \right) |U_1 - U_0|. \end{aligned}$$

If  $R$  is sufficiently large, this implies immediately that  $(U, w) \mapsto (\psi(U, w), w)$  is injective on  $\chi(B(\beta_1, \theta_1, R))$ , hence,  $(z, w) \mapsto (\psi(z, w), w)$  is injective on  $B(\beta_1, \theta_1, R)$ .  $\square$

The next result shows the existence of another ‘‘coordinate’’ on  $B$  defined using the Fatou coordinate.

**Proposition 3.3.** *Let  $F$  and  $B$  be as in Theorem 2.4 and  $\psi$  the Fatou coordinate given by Proposition 3.1. For  $n \in \mathbb{N}$ , we let  $\sigma_n : B \rightarrow \mathbb{C}^*$  be the holomorphic function defined by*

$$(3.8) \quad \sigma_n(z, w) := \lambda^n \pi_2(F^n(z, w)) \exp \left( - \sum_{j=0}^{n-1} \frac{a\lambda}{\psi(z, w) + j} \right).$$

*Then, the sequence  $\{\sigma_n\}$  converges uniformly in  $B$  to a holomorphic function  $\sigma : B \rightarrow \mathbb{C}^*$  of the form  $\sigma(z, w) = w + \eta(z, w)$ , with  $\eta : B \rightarrow \mathbb{C}$  being a holomorphic function such that for every  $(z, w) \in B$*

$$(3.9) \quad \eta(z, w) = (zw)^\alpha \cdot h(z, w),$$

*where  $h : B \rightarrow \mathbb{C}$  is a holomorphic bounded function and  $\alpha \in (1 - \beta_0, 1)$ , in particular,  $\alpha > 1/2$ .*

*Moreover,  $\sigma$  verifies*

$$(3.10) \quad \sigma \circ F = \bar{\lambda} e^{\frac{a\lambda}{\psi}} \sigma.$$



*Proof.* First, assume  $\{\sigma_n\}$  is uniformly convergent on compacta of  $B$ . Then

$$\begin{aligned}\sigma_n \circ F &= \lambda^n w_{n+1} \exp \left( - \sum_{j=0}^{n-1} \frac{a\lambda}{\psi \circ F + j} \right) = \lambda^n w_{n+1} \exp \left( - \sum_{j=0}^{n-1} \frac{a\lambda}{\psi + j + 1} \right) \\ &= \bar{\lambda} \exp \left( \frac{a\lambda}{\psi} \right) \lambda^{n+1} w_{n+1} \exp \left( - \sum_{j=0}^n \frac{a\lambda}{\psi + j} \right) = \bar{\lambda} \exp \left( \frac{a\lambda}{\psi} \right) \sigma_{n+1},\end{aligned}$$

and taking the limit for  $n \rightarrow \infty$ , (3.10) holds.

Now we show that  $\{\sigma_n\}$  is equibounded in  $B$ . By Proposition 3.1,  $|\psi - \psi_j| \sim \frac{1}{j}$  and (3.2) implies then that

$$\frac{1}{j} \sim |\psi - \psi_j| = \left| \psi - \frac{1}{u_j} + j + c \log u_j \right|.$$

By Lemma 2.6,  $|u_j| \sim \frac{1}{j}$  uniformly in  $B$  (see Remark 2.7), hence,

$$(3.11) \quad \frac{1}{\psi + j} = \frac{u_j}{1 - cu_j \log u_j + O(u_j)} = u_j + O(u_j^2 \log u_j).$$

Now, by statement (1) in Lemma 2.6, we have that

$$\lim_{j \rightarrow \infty} j \operatorname{Re}(-a\lambda u_j) = \lim_{j \rightarrow \infty} j |u_j| \operatorname{Re} \left( -a\lambda \frac{u_j}{|u_j|} \right) = \operatorname{Re}(-a\lambda) = \frac{1}{2}.$$

Therefore,

$$\exp \left( \sum_{j=0}^{n-1} \operatorname{Re}(-a\lambda u_j) \right) \sim \exp \left( \sum_{j=1}^{n-1} \frac{1}{2j} \right) = O(n^{1/2}).$$

Moreover, again by Lemma 2.6 and Remark 2.7, there exists  $C > 0$  such that  $\sum_{j=0}^{\infty} |u_j^2 \log u_j| \leq C$ . Hence, there exists  $C' > 0$  such that

$$\begin{aligned}(3.12) \quad \left| \exp \left( - \sum_{j=0}^{n-1} \frac{a\lambda}{\psi(z, w) + j} \right) \right| &= \left| \exp \left( - \sum_{j=0}^{n-1} (a\lambda u_j + O(u_j^2 \log u_j)) \right) \right| \\ &= \exp \left( \sum_{j=0}^{n-1} \operatorname{Re}(-a\lambda u_j) \right) \exp \left( \sum_{j=0}^{n-1} O(u_j^2 \log u_j) \right) \\ &\leq C' \exp \left( \sum_{j=1}^{n-1} \frac{1}{2j} \right) = O(n^{1/2}).\end{aligned}$$

Therefore, taking into account that  $|w_n| \sim n^{-1/2}$  by statement (2) of Lemma 2.6, we have

$$(3.13) \quad |\sigma_n(z, w)| = |w_n| \left| \exp \left( - \sum_{j=0}^{n-1} \frac{a\lambda}{\psi(z, w) + j} \right) \right| = |w_n| O(n^{1/2}) = O(1).$$

This implies in particular that the sequence  $\{\sigma_n\}$  is equibounded on  $B$ .

In order to show that  $\{\sigma_n\}$  is, in fact, converging, let us denote  $\Theta_n := 1 + a\lambda u_n + \lambda b_2(u_n)$ . We have

$$\begin{aligned}
\sigma_{n+1}(z, w) &= \lambda^{n+1} w_{n+1} \exp\left(-\sum_{j=0}^n \frac{a\lambda}{\psi(z, w) + j}\right) \\
&= \lambda^{n+1} [\bar{\lambda} w_n \Theta_n + R_l^2(z_n, w_n)] \exp\left(-\sum_{j=0}^n \frac{a\lambda}{\psi(z, w) + j}\right) \\
&= \lambda^n w_n \exp\left(-\sum_{j=0}^{n-1} \frac{a\lambda}{\psi(z, w) + j}\right) \Theta_n e^{-\frac{a\lambda}{\psi(z, w) + n}} \\
&\quad + \lambda^{n+1} R_l^2(z_n, w_n) \exp\left(-\sum_{j=0}^n \frac{a\lambda}{\psi(z, w) + j}\right) \\
&= \sigma_n(z, w) \Theta_n e^{-\frac{a\lambda}{\psi(z, w) + n}} + \lambda^{n+1} R_l^2(z_n, w_n) \exp\left(-\sum_{j=0}^n \frac{a\lambda}{\psi(z, w) + j}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3.14) \quad \sigma_{n+1}(z, w) - \sigma_n(z, w) &= \sigma_n(z, w) \left[ \Theta_n e^{-\frac{a\lambda}{\psi(z, w) + n}} - 1 \right] \\
&\quad + \lambda^{n+1} R_l^2(z_n, w_n) \exp\left(-\sum_{j=0}^n \frac{a\lambda}{\psi(z, w) + j}\right).
\end{aligned}$$

Now we estimate the terms in the right hand side of (3.14). Fix  $\alpha \in (1 - \beta_0, 1)$ . Note that  $\alpha > \frac{1}{2}$ . By (3.11), and taking into account that  $|u_n| \sim \frac{1}{n}$ , we have

$$\begin{aligned}
(3.15) \quad \Theta_n e^{-\frac{a\lambda}{\psi(z, w) + n}} - 1 &= \Theta_n e^{(-a\lambda u_n + O(u_n^2 \log u_n))} - 1 = \Theta_n (1 - a\lambda u_n + O(u_n^2 \log u_n)) - 1 \\
&= (1 + a\lambda u_n + \lambda b_2(u_n))(1 - a\lambda u_n + O(u_n^2 \log u_n)) - 1 \\
&= (1 + O(u_n^2 \log u_n)) - 1 = O(u_n^2 \log u_n) = |u_n|^\alpha O\left(\frac{\log n}{n^{2-\alpha}}\right).
\end{aligned}$$

Next, since  $(z_n, w_n) \in B$ , we have that  $|R_l^2(z, w)| = O(|u_n|^{\beta_0 l})$ , and by (3.12), we have

$$(3.16) \quad |R_l^2(z_n, w_n)| \left| \exp\left(-\sum_{j=0}^n \frac{a\lambda}{\psi(z, w) + j}\right) \right| \leq C |u_n|^\alpha n^{\frac{1}{2} + \alpha - \beta_0 l},$$

for some  $C > 0$ .

From (3.14), using (3.13), (3.15), (3.16), it follows that there exists a constant  $C' > 0$  such that for all  $(z, w) \in B$ ,

$$(3.17) \quad |\sigma_{n+1}(z, w) - \sigma_n(z, w)| \leq |u_n|^\alpha C' \left( \frac{\log n}{n^{2-\alpha}} + n^{\frac{1}{2} + \alpha - \beta_0 l} \right) =: |u_n|^\alpha C_n.$$

Let  $C := \sum_{n=0}^{\infty} C_n < +\infty$ . Since  $u_n \in S(R_0, \theta_0)$  for all  $n$ , it follows that  $|u_n| \leq 1/R_0$  for all  $n \in \mathbb{N}$ , hence (3.17) implies that  $\sigma_n - \sigma_0 = \sum_{j=0}^n (\sigma_{j+1} - \sigma_j)$  is converging uniformly on  $B$  to a holomorphic function  $\eta$  such that

$$\sigma(z, w) - \sigma_0(z, w) = \sigma(z, w) - w = \sum_{j=0}^{\infty} (\sigma_{j+1}(z, w) - \sigma_j(z, w)) = \eta(z, w).$$

Moreover, for all  $(z, w) \in B$  we have

$$|\eta(z, w)| \leq \sum_{n=0}^{\infty} |\sigma_{n+1}(z, w) - \sigma_n(z, w)| \leq \sum_{n=0}^{\infty} C_n |u_n|^\alpha < |u_0|^\alpha \sum_{n=0}^{\infty} C_n = C |zw|^\alpha.$$

Finally, since  $\sigma_n(z, w) \neq 0$  for all  $n \in \mathbb{N}$  and  $(z, w) \in B$ , it follows that either  $\sigma \equiv 0$  or  $\sigma(z, w) \neq 0$  for all  $(z, w) \in B$ . Since  $(r, r) \in B$  for all  $r > 0$  sufficiently small, recalling that we just proved that  $\sigma(z, w) = w + (zw)^\alpha h(z, w)$ , with  $|h| \leq C$  for all  $(z, w) \in B$ , and  $2\alpha > 1$ , we have

$$|\sigma(r, r)| = |r + r^{2\alpha} h(r, r)| \geq r - r^{2\alpha} C = r(1 - o(r)),$$

proving that  $\sigma \not\equiv 0$ . □

Now we prove that the map  $B \ni (z, w) \mapsto (\psi(z, w), \sigma(z, w))$  restricted to a suitable subset of  $B$  is injective. Such a result is crucial to show that the global basin of attraction which we shall introduce in the next section, is biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$ .

**Proposition 3.4.** *Let  $F$  and  $B$  be as in Theorem 2.4, let  $\psi : B \rightarrow \mathbb{C}$  be the Fatou coordinate given by Proposition 3.1 and let  $\sigma : B \rightarrow \mathbb{C}$  be the second local coordinate defined in Proposition 3.3. Then there exist  $R_1 \geq R_0$ ,  $\beta_1 \in (\beta_0, \frac{1}{2})$  and  $0 < \theta_1 \leq \theta_0$  such that the holomorphic map*

$$B(\beta_1, \theta_1, R_1) \ni (z, w) \mapsto Q(z, w) := (\psi(z, w), \sigma(z, w))$$

is injective.

Moreover, there exist  $\tilde{R} > 1$ ,  $\tilde{\theta} \in (0, \frac{\pi}{2})$  and  $\tilde{\beta} \in (0, \frac{1}{2})$  such that

$$(3.18) \quad \left\{ (U, w) \in \mathbb{C}^2 : U \in H(\tilde{R}, \tilde{\theta}), |U|^{\tilde{\beta}-1} < |w| < |U|^{-\tilde{\beta}} \right\} \subset Q(B).$$

*Proof.* We first show that  $Q$  is injective. Let  $R_1 \geq R_0$ ,  $\beta_1 \in (\beta_0, \frac{1}{2})$  and  $0 < \theta_1 \leq \theta$  be given by Lemma 3.2. Recall that  $\sigma$  is the uniform limit of the sequence  $\{\sigma_n\}$  defined in

(3.3) for  $n \in \mathbb{N}$ . Thanks to the injectivity of  $B(\beta_1, \theta_1, R_1) \ni (z, w) \mapsto (\psi(z, w), w)$  showed in Lemma 3.2, it follows easily that the map

$$B(\beta_1, \theta_1, R_1) \ni (z, w) \mapsto (\psi(z, w), \sigma_n(z, w))$$

is injective for all  $n \in \mathbb{N}$ . Therefore, it follows that either the Jacobian of  $(\psi, \sigma)$  is identically zero on  $B(\beta_1, \theta_1, R_1)$ , or  $(z, w) \mapsto Q(z, w)$  is injective on  $B(\beta_1, \theta_1, R_1)$ .

We are going to compute the Jacobian of  $Q$  at  $(r, r)$ , for  $r > 0$ ,  $r$  sufficiently small (note that  $(r, r) \in B(\beta_1, \theta_1, R_1)$  for all  $r$  sufficiently small). To simplify computation, we consider the holomorphic change of coordinates  $\chi : B(\beta_1, \theta_1, R_1) \rightarrow \mathbb{C}^2$  given by  $\chi(z, w) = (U, w)$ , where, as usual,  $U = \frac{1}{zw}$ . Therefore, we compute the Jacobian of  $Q(U, w)$  at  $(\frac{1}{r^2}, r)$ .

By Proposition 3.1 and Proposition 3.3, we have

$$(3.19) \quad Q(U, w) = (U + c \log U + v(U, w), w + \eta(U, w)),$$

where  $v(U, w) = \frac{1}{U^\alpha} g(U, w)$  and  $\eta(U, w) = \frac{1}{U^\alpha} h(U, w)$ ,  $\alpha \in (1 - \beta_0, 1)$ , with  $|g|, |h| \leq C$  for some  $C > 0$  on  $B$ . Hence,

$$\begin{aligned} \text{Jac}_{(\frac{1}{r^2}, r)} Q &= \det \begin{pmatrix} 1 + cr^2 + \frac{\partial v}{\partial U} \left( \frac{1}{r^2}, r \right) & \frac{\partial v}{\partial w} \left( \frac{1}{r^2}, r \right) \\ \frac{\partial \eta}{\partial U} \left( \frac{1}{r^2}, r \right) & 1 + \frac{\partial \eta}{\partial w} \left( \frac{1}{r^2}, r \right) \end{pmatrix} \\ &= \left( 1 + cr^2 + \frac{\partial v}{\partial U} \left( \frac{1}{r^2}, r \right) \right) \left( 1 + \frac{\partial \eta}{\partial w} \left( \frac{1}{r^2}, r \right) \right) - \frac{\partial v}{\partial w} \left( \frac{1}{r^2}, r \right) \frac{\partial \eta}{\partial U} \left( \frac{1}{r^2}, r \right) \end{aligned}$$

First of all, note that for  $\gamma \in (0, \frac{1}{2})$ ,  $\tilde{R} > 1$  and  $\tilde{\theta} \in (0, \frac{\pi}{2})$  there exists  $r_0 > 0$  such that  $(\frac{1}{r^2}, r) \in \chi((B(\gamma, \tilde{\theta}, \tilde{R})))$  for all  $r \in (0, r_0)$ . Hence, by (3.6), there exists  $C_2 > 0$  such that for  $r$  sufficiently small,

$$\left| \frac{\partial v}{\partial U} \left( \frac{1}{r^2}, r \right) \right| \leq r^2 C_2.$$

A similar argument as in (3.6) for  $\eta$  instead of  $v$ , shows that for  $r$  sufficiently small,

$$\left| \frac{\partial \eta}{\partial U} \left( \frac{1}{r^2}, r \right) \right| \leq r^{2\alpha} C_3,$$

for some  $C_3 > 0$ .

On the other end, it is easy to see that for every  $t \in \mathbb{R}$  it holds  $(\frac{1}{r^2}, r(1 + e^{it})) \in \chi(B)$  whenever  $r$  positive and small enough. Hence, by the Cauchy formula for derivatives

$$\begin{aligned} \left| \frac{\partial v}{\partial w} \left( \frac{1}{r^2}, r \right) \right| &= \frac{1}{2\pi} \left| \int_{|\zeta-r|=r} \frac{v(\frac{1}{r^2}, \zeta)}{(\zeta-r)^2} d\zeta \right| \leq \frac{\max_{|\zeta-r|=r} |v(\frac{1}{r^2}, \zeta)|}{r} \\ &= \frac{r^2 \max_{|\zeta-r|=r} |g(\frac{1}{r^2}, \zeta)|}{r} \leq Cr. \end{aligned}$$

Similarly,

$$\left| \frac{\partial \eta}{\partial w} \left( \frac{1}{r^2}, r \right) \right| \leq Cr^{2\alpha-1}.$$

Therefore,

$$\text{Jac}_{(\frac{1}{r^2}, r)} Q = 1 + O(r^{2\alpha-1}),$$

showing that the Jacobian is not zero for  $r$  sufficiently small since  $\alpha > 1/2$ . Hence  $Q$  is injective on  $B(\beta_1, \theta_1, R_1)$ .

Now we prove there exist  $\tilde{R} > 1$ ,  $\tilde{\theta} \in (0, \frac{\pi}{2})$  and  $\tilde{\beta} \in (0, \frac{1}{2})$  such that (3.18) holds. The idea is that  $Q|_B$  is “very close” to the map  $(z, w) \mapsto (\frac{1}{zw} - c \log(zw), w)$ , for which the statement is true, and hence (3.18) follows by the Rouché theorem. In the rest of the proof we give details of this argument.

As recalled before, by Lemma 3.2, there exist  $R_1 \geq R_0$ ,  $\beta_1 \in (\beta_0, \frac{1}{2})$  and  $0 < \theta_1 \leq \theta_0$  such that the holomorphic map

$$\tilde{B} := B(\beta_1, \theta_1, R_1) \ni (z, w) \mapsto (\psi(z, w), w)$$

is injective. We consider the holomorphic change of coordinates on  $\tilde{B}$  given by  $(U, w) = \chi(z, w) = (\frac{1}{zw}, w)$ . Then  $\chi(\tilde{B}) = \{(U, w) : U \in H(R_1, \theta_1), |U|^{\beta_1-1} < |w| < |U|^{-\beta_1}\}$ .

The map  $\chi(\tilde{B}) \ni (U, w) \mapsto Q(U, w) = (\psi(U, w), \sigma(U, w))$  is given by (3.19). In particular

$$(3.20) \quad \psi(U, w) = U(1 + \tau(U, w)),$$

where  $|\tau| < C$  on  $\chi(\tilde{B})$  for some  $C > 0$ , and  $\lim_{|U| \rightarrow \infty} \tau(U, w) = 0$ .

This implies immediately that there exist  $\tilde{R}_1 > 0$  and  $\tilde{\theta} \in (0, \frac{\theta_0}{2})$  such that

$$H(\tilde{R}_1, 2\tilde{\theta}) \subset \psi(\tilde{B}) \subset \psi(B).$$

We claim that there exist  $\tilde{R} \geq \tilde{R}_1$  and  $\tilde{\beta} \in (\beta_1, \frac{1}{2})$  such that for every  $\zeta_0 \in H(\tilde{R}, \tilde{\theta})$ ,

$$(3.21) \quad \{\xi \in \mathbb{C} : |\zeta_0|^{\tilde{\beta}-1} < |\xi| < |\zeta_0|^{-\tilde{\beta}}\} \subset \sigma(\psi^{-1}(\zeta_0)).$$

It is clear that (3.18) follows from (3.21).

In order to prove (3.21), we first prove that there exist  $\tilde{R}_2 \geq \tilde{R}_1$  and  $\tilde{\beta}_2 \in (\beta_1, \frac{1}{2})$  such that for every  $\zeta_0 \in H(\tilde{R}_2, \tilde{\theta})$  it holds

$$(3.22) \quad \{\xi \in \mathbb{C} : |\zeta_0|^{\tilde{\beta}_2-1} < |\xi| < |\zeta_0|^{-\tilde{\beta}_2}\} \subset \pi_2(\psi^{-1}(\zeta_0)).$$

Indeed, by (3.20),  $\zeta_0 = \psi(U, w) = U(1 + \tau(U, w))$  with  $|\tau| < C$  and  $\lim_{|U| \rightarrow \infty} \tau(U, w) = 0$ . Hence, if  $\zeta_0 \in H(\tilde{R}_2, \tilde{\theta})$  for some  $\tilde{R}_2 \geq \tilde{R}_1$ ,

$$|U| \geq \frac{|\zeta_0|}{1 + |\tau(U, w)|} \geq \frac{\tilde{R}_2}{1 + C}.$$

Therefore, fixed  $c' \in (0, 1)$ , we can choose  $\tilde{R}_2 \geq \tilde{R}_1$  large enough so that for every  $(U, w) \in \chi(\tilde{B})$  such that  $\psi(U, w) = \zeta_0$  and  $\zeta_0 \in H(\tilde{R}_2, \tilde{\theta})$ , the modulus  $|U|$  is so large that  $|\tau(U, w)| < c'$ . This implies that

$$(3.23) \quad (1 - c')|U| < |\zeta_0| < (1 + c')|U|$$

for every  $U \in \mathbb{C}$  such that there exists  $w \in \mathbb{C}$  so that  $(U, w) \in \chi(\tilde{B})$  and  $\psi(U, w) = \zeta_0 \in H(\tilde{R}_2, \tilde{\theta})$ .

Let  $\tilde{\beta}_2 \in (\beta_1, \frac{1}{2})$ . Let  $r_0 > 0$  be such that

$$\frac{1}{[(1+c')t]^{1-\beta_1}} < \frac{1}{t^{1-\tilde{\beta}_2}} < \frac{1}{[(1-c')t]^{\tilde{\beta}_2}} < \frac{1}{t^{\beta_1}}, \quad \forall t \geq r_0.$$

Up to choosing  $\tilde{R}_2 \geq r_0$ , (3.23) implies that

$$(3.24) \quad |U_0|^{\beta_1-1} < |\zeta_0|^{\tilde{\beta}_2-1} < |\zeta_0|^{-\tilde{\beta}_2} < |U_0|^{-\beta_1}$$

for every  $U_0 \in \mathbb{C}$  such that there exists  $w \in \mathbb{C}$  so that  $(U_0, w) \in \chi(\tilde{B})$  and  $\psi(U_0, w) = \zeta_0 \in H(\tilde{R}_2, \tilde{\theta})$ .

Fix  $\zeta_0 \in H(\tilde{R}_2, \tilde{\theta})$  and fix  $\xi_0 \in \mathbb{C}$  such that  $|\zeta_0|^{\tilde{\beta}_2-1} < |\xi_0| < |\zeta_0|^{-\tilde{\beta}_2}$ . Since there exists  $(U_0, w_0) \in \chi(\tilde{B})$  such that  $\psi(U_0, w_0) = \zeta_0$ , it follows from (3.24) that  $(U_0, \xi_0) \in \chi(\tilde{B})$ . In particular,  $\chi(\tilde{B}) \cap \{w = \xi_0\} \neq \emptyset$ . We let

$$A(\xi_0) := \left\{ U \in H(R_1, \theta_1) : \frac{1}{|\xi_0|^{\frac{1}{1-\beta_1}}} < |U| < \frac{1}{|\xi_0|^{\frac{1}{\beta_1}}} \right\} = \chi(\tilde{B}) \cap \{w = \xi_0\}.$$

Then,

$$A(\xi_0) \ni U \mapsto \psi_{\xi_0}(U) := \psi(U, \xi_0) = U + c \log U + \frac{g(U, \xi_0)}{U} \in \mathbb{C}$$

is well defined and holomorphic. Moreover, up to take  $\tilde{R}_2$  larger and  $\tilde{\theta}$  smaller if necessary, we can assume that the set  $H(\tilde{R}_2, \tilde{\theta})$  is contained in the image of the map  $\chi(\tilde{B}) \ni (U, w) \mapsto U + c \log U$ . Hence, there exists  $(U_0, w_0) \in \chi(\tilde{B})$  such that  $U_0 + c \log U_0 = \zeta_0$ . Since  $\zeta_0 = U_0(1 + c \frac{\log U_0}{U_0})$ , it follows that

$$|U_0|(1 - \epsilon) \leq |\zeta_0| \leq |U_0|(1 + \epsilon)$$

for some  $\epsilon \in (0, 1)$ , provided that  $\tilde{R}_2$  is sufficiently large. Taking into account that  $|\zeta_0|^{\tilde{\beta}_2-1} < |\xi_0| < |\zeta_0|^{-\tilde{\beta}_2}$ , we have

$$|U_0| \geq \frac{|\zeta_0|}{1 + \epsilon} > \frac{1}{(1 + \epsilon)|\xi_0|^{1/(1-\tilde{\beta}_2)}} > \frac{1}{|\xi_0|^{\frac{1}{1-\beta_1}}},$$

where the last inequality holds provided  $\tilde{R}_2$  is sufficiently large. Similarly, one can show that  $|U_0| < \frac{1}{|\xi_0|^{\frac{1}{\beta_1}}}$ , namely,  $U_0 \in A(\xi_0)$ .

Let  $\delta \in (0, 1)$  be such that  $D(\delta) := \{U \in \mathbb{C} : |U - U_0| < \delta\} \subset A(\xi_0)$ . Since  $|g(U, \xi_0)|/|U| < c'$ , up to choosing  $\tilde{R}_2$  so large that  $c' + \delta < |c| \max_{|U-U_0|=\delta} |\log U - \log U_0|$ , it

follows that for all  $U \in \partial D(\delta)$ ,

$$\begin{aligned} |\psi_{\xi_0}(U) - U - c \log U| &< c' < |c| \left| \log \frac{U}{U_0} \right| - \delta \leq |U + c \log U - \zeta_0| \\ &\leq |U + c \log U - \zeta_0| + |\psi_{\xi_0}(U) - \zeta_0|. \end{aligned}$$

Hence, Rouché's theorem implies that there exists  $U_1 \in D(\delta) \subset A(\xi_0)$  such that  $\psi(U_1, \xi_0) = \psi_{\xi_0}(U_1) = \zeta_0$ , proving (3.22).

Let  $H : \chi(\tilde{B}) \rightarrow \mathbb{C}^2$  be defined by  $H(U, w) := (\psi(U, w), w)$ . Then the map  $H$  is injective and from (3.22), we obtain that

$$(3.25) \quad \chi(B(\tilde{\beta}_2, \tilde{\theta}, \tilde{R}_2)) \subset H(\chi(\tilde{B})).$$

Let  $\tilde{R} \geq \tilde{R}_2$ , and let  $\zeta_0 \in H(\tilde{R}, \tilde{\theta})$ . Let

$$J(\zeta_0) := \{w \in \mathbb{C} : |\zeta_0|^{\tilde{\beta}_2-1} < |w| < |\zeta_0|^{-\tilde{\beta}_2}\}.$$

Then  $(\zeta_0, w) \in H(\chi(\tilde{B}))$  for every  $w \in J(\zeta_0)$ , by (3.25).

Let  $\tilde{\beta} \in (\tilde{\beta}_2, \frac{1}{2})$ . Let  $\xi_0 \in \mathbb{C}$  be such that  $|\zeta_0|^{\tilde{\beta}-1} < |\xi_0| < |\zeta_0|^{-\tilde{\beta}}$ . Then  $\xi_0 \in J(\zeta_0)$ . Moreover, let

$$r := \min\{|\zeta_0|^{\tilde{\beta}-1} - |\zeta_0|^{\tilde{\beta}_2-1}, |\zeta_0|^{-\tilde{\beta}_2} - |\zeta_0|^{-\tilde{\beta}}\} > 0.$$

The disc  $D(\xi_0, r) := \{\xi \in \mathbb{C} : |\xi - \xi_0| < r\}$  is contained in  $J(\zeta_0)$ . Moreover, if  $\tilde{R}$  is sufficiently large,

$$(3.26) \quad r > \frac{1}{2} \min\{|\zeta_0|^{\tilde{\beta}-1}, |\zeta_0|^{-\tilde{\beta}_2}\}.$$

Let  $(\tilde{U}, w) := H(U, w)$ . Hence, for every  $(\tilde{U}, w) \in H(\chi(\tilde{B}))$ , we can write

$$\tilde{\sigma}(\tilde{U}, w) := (\sigma \circ H^{-1})(\tilde{U}, w) = w + \eta(\tilde{U}, w),$$

where  $\eta(\tilde{U}, w) = \frac{1}{\tilde{U}^\alpha} h(\tilde{U}, w)$ , with  $\alpha \in (1 - \beta_0, 1)$ , and  $|h| \leq C$  for some  $C > 0$ .

By (3.26), bearing in mind that  $\alpha > 1 - \beta_0 > 1/2$ , if  $\tilde{R}$  is sufficiently large, then

$$|\eta(\zeta_0, w)| < r,$$

for every  $w \in J(\zeta_0)$ . Therefore, for all  $w \in \partial D(\xi_0, r)$ ,

$$|w - \tilde{\sigma}(\zeta_0, w)| = |\eta(\zeta_0, w)| < r = |w - \xi_0| \leq |w - \xi_0| + |\tilde{\sigma}(\zeta_0, w) - \xi_0|.$$

Hence, by Rouché's theorem, there exists  $w_0 \in D(\xi_0, r)$  such that  $\tilde{\sigma}(\zeta_0, w_0) = \xi_0$ . By the arbitrariness of  $\xi_0$ , this implies that for every  $\zeta_0 \in H(\tilde{R}, \tilde{\theta})$

$$\{\xi \in \mathbb{C} : |\zeta_0|^{\tilde{\beta}-1} < |\xi| < |\zeta_0|^{-\tilde{\beta}}\} \subset \tilde{\sigma}(\zeta_0, \cdot)(J(\zeta_0)) \subset \sigma(\psi^{-1}(\zeta_0)),$$

which finally proves (3.21).  $\square$

4. THE GLOBAL BASIN  $\Omega$  AND THE FATOU COMPONENT CONTAINING  $B$ 

Let  $F_N$  be a germ of biholomorphism of  $\mathbb{C}^2$  at  $(0,0)$  of the form (2.1). By Forstnerič's extension result (see [6, Corollary 2.2]), given any  $l \geq 2$  there exists an automorphism  $F$  of  $\mathbb{C}^2$  such that  $\|F(z, w) - F_N(z, w)\| = O(\|(z, w)\|^l)$ .

In particular, given  $\lambda$  a unimodular number, not a root of unit, with negative real part, and setting  $a := -1/(2\operatorname{Re} \lambda)$  as in Section 2, we take  $l \geq 4$  such that  $\beta_0(l+1) \geq 4$ , where  $0 < \beta_0 < 1/2$  is given by Theorem 2.4, and we consider automorphisms of  $\mathbb{C}^2$  of the form

$$(4.1) \quad F(z, w) = (z(\lambda + azw + b_1(zw)) + R_l^1(z, w), w(\bar{\lambda} + azw + b_2(zw)) + R_l^2(z, w)),$$

where  $b_j(\zeta) = O(|\zeta|^2)$  and  $R_l^j(z, w) = O(\|(z, w)\|^l)$ ,  $j = 1, 2$ .

**Definition 4.1.** Let  $F$  be an automorphism of  $\mathbb{C}^2$  of the form (4.1). Let  $B$  be the local basin of attraction of  $F$  given by Theorem 2.4. We define the global attracting basin

$$\Omega := \bigcup_{n \in \mathbb{N}} F^{-n}(B).$$

Since  $B$  is connected by Lemma 2.8, and  $\{F^n\}$  converges to  $(0,0)$  uniformly on  $B$ , there exists an invariant Fatou component, which we denote by  $V(B)$ , containing  $B$ , and we clearly have

$$\Omega \subseteq V(B).$$

The aim of this section is to characterize  $\Omega$  in terms of orbits behavior, and to prove that  $\Omega = V(B)$  under a generic condition on  $\lambda$ .

We use the same notations introduced in the previous sections. We start with the following corollary of Lemma 2.6.

**Corollary 4.2.** *Let  $F$  be an automorphism of  $\mathbb{C}^2$  of the form (4.1). Suppose that  $\{(z_n, w_n) := F^n(z_0, w_0)\}$ , the orbit under  $F$  of a point  $(z_0, w_0)$ , converges to  $(0,0)$ . Then  $(z_0, w_0) \in \Omega$  if and only if  $(z_n, w_n)$  is eventually contained in  $W(\beta)$  for some—and hence any— $\beta \in (0, 1/2)$  such that  $\beta(l+1) > 2$ .*

*Proof.* If  $(z_n, w_n) \in W(\beta)$  eventually for some  $\beta \in (0, 1/2)$  with  $\beta(l+1) > 2$  then, by Lemma 2.6,  $(z_n, w_n) \in B$  eventually, and hence,  $(z_0, w_0) \in \Omega$ . Conversely, if  $(z_0, w_0) \in \Omega$ , then  $(z_n, w_n) \in W(\beta_0)$  eventually and  $\beta_0(l+1) \geq 4$ , and hence Lemma 2.6 implies that  $(z_n, w_n) \in W(\beta)$  eventually for any  $\beta \in (0, 1/2)$  such that  $\beta(l+1) > 2$ .  $\square$

We can now prove the following characterization of  $\Omega$ .

**Theorem 4.3.** *Let  $F$  be an automorphism of  $\mathbb{C}^2$  of the form (4.1). Then,*

$$\Omega = \{(z, w) \in \mathbb{C}^2 \setminus \{(0,0)\} : \lim_{n \rightarrow \infty} \|(z_n, w_n)\| = 0, \quad |z_n| \sim |w_n|\},$$

where  $(z_n, w_n) = F^n(z, w)$ .



*Proof.* If  $(z, w) \in \Omega$ , then eventually  $(z_n, w_n) \in W(\beta_0)$  and, hence,  $|z_n| \sim |w_n|$  by Lemma 2.6. On the other hand, if  $(z_n, w_n) \rightarrow (0, 0)$  and  $|z_n| \sim |w_n|$ , it follows that for every  $\beta \in (0, 1/2)$ ,  $(z_n, w_n) \in W(\beta)$ . Indeed, let  $0 < c_1 < c_2$  be such that  $c_1|z_n| < |w_n| < c_2|z_n|$  eventually. Let  $\beta \in (0, 1/2)$ . Then for  $n$  large,

$$|z_n|^{\frac{1-\beta}{\beta}} < c_1|z_n| < |w_n|,$$

that is,  $|z_n| < |w_n|^\beta$ , and similarly it can be proved that  $|w_n| < |u_n|^\beta$ . Hence, by Corollary 4.2,  $(z, w) \in \Omega$ .  $\square$

In order to show that, under some generic arithmetic assumptions on  $\lambda$ ,  $\Omega$  coincides with the Fatou component which contains it, we need to prove some preliminary lemmas.

**Lemma 4.4.** *Let  $\chi$  be a germ of biholomorphism of  $\mathbb{C}^2$  at  $(0, 0)$  given by*

$$\chi(z, w) = (z + A(z, w), w + B(z, w)),$$

where  $A$  and  $B$  are germs of holomorphic functions at  $(0, 0)$  with  $A(z, w) = O(\|(z, w)\|^h)$  and  $B(z, w) = O(\|(z, w)\|^h)$  for some  $h \geq 2$ . Let  $\beta \in (0, 1/2)$ . Assume that  $\beta(h+1) > 1$ . Then for any  $\beta' \in (0, \beta)$  there exists  $\epsilon > 0$  such that for every  $(z, w) \in W(\beta)$  with  $\|(z, w)\| < \epsilon$  it holds  $\chi(z, w) \in W(\beta')$ .

*Proof.* Let us write  $(\tilde{z}, \tilde{w}) = \chi(z, w)$ . Then we have  $\tilde{z} = z + A(z, w)$  and  $\tilde{w} = w + B(z, w)$ .

Fix  $r > 0$ ,  $\beta \in (0, 1/2)$  such that  $\beta(h+1) > 1$ , and  $\beta' \in (0, \beta)$ . By definition, for  $\|(z, w)\| < r$ , if  $(z, w) \in W(\beta)$ , then there exists a constant  $C > 0$  such that  $|A(z, w)| \leq C|zw|^{\beta h}$  and  $|B(z, w)| \leq C|zw|^{\beta h}$ . Hence, for all  $(z, w) \in W(\beta)$  with  $\|(z, w)\| < r$ ,

$$|\tilde{z}| \leq |z| + |A(z, w)| < |zw|^\beta + C|zw|^{\beta h} = |zw|^\beta(1 + o(|zw|^{\beta(h-1)})),$$

Similarly,

$$|\tilde{w}| \leq |w| + |B(z, w)| < |zw|^\beta + C|zw|^{\beta h} = |zw|^\beta(1 + o(|zw|^{\beta(h-1)})).$$

Therefore, since  $\beta(h+1) > 1$ ,

$$\begin{aligned} |\tilde{z}\tilde{w}| &\geq |zw| - |z||B| - |w||A| - |AB| \\ &\geq |zw| - 2C|zw|^{\beta(h+1)} - C^2|zw|^{2h\beta} \\ &= |zw|(1 + o(|zw|^{\beta(h+1)-1})). \end{aligned}$$

It thus follows that, for  $(z, w) \in W(\beta)$  sufficiently close to  $(0, 0)$ , we have

$$|\tilde{z}| < |zw|^\beta(1 + o(|zw|^{\beta(h-1)})) \leq |\tilde{z}\tilde{w}|^\beta \frac{1 + o(|zw|^{\beta(h-1)})}{1 + o(|zw|^{\beta(h+1)-1})} \leq |\tilde{z}\tilde{w}|^\beta(1 + o(1)) < |\tilde{z}\tilde{w}|^{\beta'}.$$

A similar argument holding for  $\tilde{w}$ , the statement is proved.  $\square$

**Example 4.5.** In general, without the hypothesis  $\beta(h+1) > 1$  the previous lemma does not hold true. For instance, consider  $\chi(z, w) = (z + w^2, w)$ . Then the points of the form  $(-w^2, w)$  belong to  $W(\beta)$  for all  $\beta < 1/3$  but  $\chi(-w^2, w) = (0, w) \notin W(\beta')$  for any  $\beta' \in (0, 1/2)$ .

The following lemma is probably a well known fact, but since we could not find a reference and we shall need it to prove Theorem 4.8, we give here a proof of it.

**Lemma 4.6.** *Let  $\mathbb{D}^* = \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$ . Let  $k_{\mathbb{D}^*}$  denote the hyperbolic distance in  $\mathbb{D}^*$ . Let*

$$g(\zeta, \xi) := 2\pi \max \left\{ -\frac{1}{\log |\zeta|}, -\frac{1}{\log |\xi|} \right\}.$$

*Then for all  $\zeta, \xi \in \mathbb{D}^*$  it holds*

$$\left| \log \frac{\log |\zeta|}{\log |\xi|} \right| - g(\zeta, \xi) \leq k_{\mathbb{D}^*}(\zeta, \xi) \leq \left| \log \frac{\log |\zeta|}{\log |\xi|} \right| + g(\zeta, \xi).$$

*Proof.* It is well known that the hyperbolic metric  $\kappa_{\mathbb{D}^*}$  is given for  $\zeta \in \mathbb{D}^*$  and  $v \in \mathbb{C}$  by

$$\kappa_{\mathbb{D}^*}(\zeta; v) := -\frac{|v|}{|\zeta| \log |\zeta|}.$$

We first notice that if  $\Gamma_\rho$  is a circle of radius  $0 < \rho < 1$  and center 0 then its hyperbolic length is

$$\text{hyper-length}(\Gamma_\rho) := \int_{\Gamma_\rho} -\frac{|d\zeta|}{|\zeta| \log |\zeta|} = -\frac{2\pi}{\log \rho}.$$

In particular, for every  $\xi \in \mathbb{D}^*$ ,

$$(4.2) \quad k_{\mathbb{D}^*}(\xi, |\xi|) \leq \text{hyper-length}(\Gamma_{|\xi|}) = -\frac{2\pi}{\log |\xi|}.$$

On the other hand, if  $0 < r_1 < r_2 < 1$ , it is easy to check by direct computation (see for example [2, Section 12.1]) that

$$(4.3) \quad k_{\mathbb{D}^*}(r_1, r_2) = \log \left( \frac{\log r_2}{\log r_1} \right).$$

Now, let  $\zeta, \xi \in \mathbb{D}^*$ . We can assume  $|\zeta| \leq |\xi|$ . Since  $\kappa_{\mathbb{D}^*}$  is invariant under rotation, we can assume that  $\zeta = |\zeta| = r_1$ . Hence, on the one hand, by the triangle inequality, (4.3) and (4.2)

$$\begin{aligned} k_{\mathbb{D}^*}(\zeta, \xi) &\geq k_{\mathbb{D}^*}(r_1, |\xi|) - k_{\mathbb{D}^*}(|\xi|, \xi) \\ &\geq \log \left( \frac{\log |\xi|}{\log r_1} \right) - \text{hyper-length}(\Gamma_{|\xi|}) = \log \left( \frac{\log |\xi|}{\log r_1} \right) + \frac{2\pi}{\log |\xi|} \\ &= \log \left( \frac{\log |\xi|}{\log |\zeta|} \right) - g(\zeta, \xi), \end{aligned}$$

while, on the other hand,

$$\begin{aligned} k_{\mathbb{D}^*}(\zeta, \xi) &\leq k_{\mathbb{D}^*}(r_1, |\xi|) + k_{\mathbb{D}^*}(|\xi|, \xi) \\ &\leq \log \left( \frac{\log |\xi|}{\log r_1} \right) + \text{hyper-length}(\Gamma_{|\xi|}) = \log \left( \frac{\log |\xi|}{\log |\zeta|} \right) + g(\zeta, \xi), \end{aligned}$$

and the statement follows.  $\square$

To state and prove Theorem 4.8 we need one last ingredient, namely a arithmetic condition on the eigenvalue  $\lambda$ .

Let  $\lambda \in \mathbb{C}$  be such that  $|\lambda| = 1$ . We say that  $\lambda$  is *Siegel* if there exist  $c > 0$  and  $N \in \mathbb{N}$  such that  $|\lambda^k - 1| \geq ck^{-N}$  for all  $k \in \mathbb{N}$ ,  $k \geq 1$  (such a condition holds for  $\theta$  in a full Lebesgue measure subset of the unit circle, see, e.g., [10]). More generally, one says that a number  $\lambda$  is *Brjuno* if

$$(4.4) \quad \sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\omega(2^{k+1})} < +\infty ,$$

where  $\omega(m) = \min_{2 \leq k \leq m} |\lambda^k - \lambda|$  for any  $m \geq 2$ . Roughly speaking, the logarithm of a Brjuno number is badly approximated by rationals (see [5] or [10] for more details). Siegel numbers are examples of Brjuno numbers.

**Lemma 4.7.** *Let  $F$  be given by (2.8). If  $\lambda$  is Brjuno then there exists a germ of biholomorphism  $\chi$  of  $\mathbb{C}^2$  at  $(0, 0)$  of the form*

$$\chi(z, w) = (z, w) + O(\|(z, w)\|^l),$$

such that

$$(4.5) \quad \tilde{F}(\tilde{z}, \tilde{w}) := (\chi \circ F \circ \chi^{-1})(\tilde{z}, \tilde{w}) = (\lambda\tilde{z} + \tilde{z}\tilde{w}A(\tilde{z}, \tilde{w}), \bar{\lambda}\tilde{w} + \tilde{z}\tilde{w}B(\tilde{z}, \tilde{w})),$$

where  $A, B$  are germs of holomorphic functions at  $(0, 0)$ .

*Proof.* Thanks to the fact that  $\lambda$  is Brjuno, the divisors  $\lambda^k - \lambda$  and  $\lambda^k - \bar{\lambda}$  are “admissible” in the sense of Pöschel [10] for all  $k \in \mathbb{N}$ ,  $k \geq 2$ . Hence, by [10, Theorem 1], there exist  $\delta > 0$  and an injective holomorphic map  $\varphi_1 : \mathbb{D}_\delta \rightarrow \mathbb{C}^2$ , where  $\mathbb{D}_\delta := \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$ , such that  $\varphi_1(0) = (0, 0)$ ,  $\varphi_1'(0) = (1, 0)$  and

$$(4.6) \quad F(\varphi_1(\zeta)) = \varphi_1(\lambda\zeta),$$

for all  $\zeta \in \mathbb{D}_\delta$ . Since  $F$  is tangent to  $\{w = 0\}$  up to order  $l$ , it follows from the proof of [10, Theorem 1] that  $\varphi_1$  can be chosen of the form  $\varphi_1(\zeta) = (\zeta, 0) + O(|\zeta|^l)$ . In particular, we can write  $\varphi_1(\mathbb{D}_\delta)$  (possibly taking a smaller  $\delta$  if necessary) implicitly as  $w = \psi_1(z)$  for some holomorphic function  $\psi_1$  defined on  $\mathbb{D}_\delta$  and such that  $\psi_1(\zeta) = O(|\zeta|^l)$ .

Similarly,  $\bar{\lambda}^k - \lambda$  and  $\bar{\lambda}^k - \bar{\lambda}$  are admissible divisors in the sense of Pöschel [10] for all  $k \in \mathbb{N}$ ,  $k \geq 2$  and hence there exist  $\delta' > 0$  and a holomorphic function  $\psi_2 : \mathbb{D}_{\delta'} \rightarrow \mathbb{C}$  with  $\psi_2(\zeta) = O(|\zeta|^l)$ , such that  $F$  leaves invariant the local curve  $C := \{(z, w) : z = \psi_2(w)\}$  and the restriction of  $F$  to  $C$  is a  $\bar{\lambda}$ -rotation.

We can therefore define  $(\tilde{z}, \tilde{w}) := \chi(z, w) = (z - \psi_2(w), w - \psi_1(z))$ . By construction,  $\chi$  is a germ of biholomorphism at  $(0, 0)$  and  $\chi(z, w) = (z, w) + O(\|(z, w)\|^l)$ . Moreover, the conjugate germ  $\tilde{F}(\tilde{z}, \tilde{w}) := (\chi \circ F \circ \chi^{-1})(\tilde{z}, \tilde{w})$  satisfies our thesis, that is  $\tilde{F}(\tilde{z}, \tilde{w}) - (\lambda\tilde{z}, \bar{\lambda}\tilde{w})$  is divisible by  $\tilde{z}\tilde{w}$ , and (4.5) holds. Indeed,  $\tilde{z} = 0$  corresponds to  $z - \psi_2(w) = 0$ , and since  $F$

leaves such a curve invariant and it is a  $\bar{\lambda}$ -rotation on it, it follows that  $\tilde{F}(0, \tilde{w}) = (0, \bar{\lambda}\tilde{w})$ . A similar argument proves that  $\tilde{F}(\tilde{z}, 0) = (\lambda\tilde{z}, 0)$ .  $\square$

**Theorem 4.8.** *Suppose  $F$  is an automorphism of the form (4.1). Assume  $\lambda$  is Brjuno. Then  $\Omega = V(B)$ .*

*Proof.* Assume by contradiction the statement is not true. Hence, there exists  $q_0 \in V(B) \setminus \Omega$ . Let  $p_0 \in \Omega$ , and let  $Z$  be an open connected set containing  $p_0$  and  $q_0$  and such that  $\bar{Z} \subset V(B)$ .

By Lemma 4.7, since  $\lambda$  is Brjuno, there exists an open neighborhood  $U$  of  $(0, 0)$  and a biholomorphism  $\chi : U \rightarrow \chi(U)$ , such that (4.5) holds for all  $(\tilde{z}, \tilde{w}) \in \chi(U)$ . Up to rescaling, we can assume that

$$\mathbb{B}^2 := \{(\tilde{z}, \tilde{w}) \in \mathbb{C}^2 : |\tilde{z}|^2 + |\tilde{w}|^2 < 1\} \subset \chi(U).$$

Since  $\{F^n\}$  converges uniformly to  $(0, 0)$  on  $\bar{Z}$ , up to replace  $F$  with  $F^m$  for some fixed  $m \in \mathbb{N}$ , we can assume that  $Q := \cup_{n \in \mathbb{N}} F^n(Z)$  satisfies  $\tilde{Q} := \chi(Q) \subset \mathbb{B}^2$ .

Since the axes  $\tilde{z}$  and  $\tilde{w}$  are  $\tilde{F}$ -invariant and  $\tilde{F}$  is a rotation once restricted to the axes, it follows that, in fact,

$$\tilde{Q} \subset \mathbb{B}_*^2 := \mathbb{B}^2 \setminus (\{\tilde{z} = 0\} \cup \{\tilde{w} = 0\}).$$

For a complex manifold  $M$ , let us denote by  $k_M$  its Kobayashi distance. By construction, for every  $\delta > 0$ , one can find  $p \in Z \cap \Omega$  and  $q \in Z \cap (V(B) \setminus \Omega)$  such that  $k_Q(p, q) \leq k_Z(p, q) < \delta$ . Let  $\tilde{p} := \chi(p)$  and  $\tilde{q} := \chi(q)$ . Hence,  $k_{\tilde{Q}}(\tilde{p}, \tilde{q}) < \delta$ . Thus, since  $\tilde{F}(\tilde{Q}) \subset \tilde{Q}$  by construction, and  $\tilde{Q} \subset \mathbb{B}_*^2$ , it follows that for all  $n \in \mathbb{N}$ ,

$$(4.7) \quad k_{\mathbb{B}_*^2}(\tilde{F}^n(\tilde{p}), \tilde{F}^n(\tilde{q})) \leq k_{\tilde{Q}}(\tilde{F}^n(\tilde{p}), \tilde{F}^n(\tilde{q})) < \delta.$$

Now, since  $q \notin \Omega$ , by Lemma 2.6, there is no  $\beta \in (0, 1/2)$  with  $\beta(l+1) > 2$  such that  $\{F^n(q)\} \subset W(\beta)$  eventually. We claim that the same happens to  $\{\tilde{F}^n(\tilde{q})\}$ . Indeed, if there existed  $\beta \in (0, 1/2)$  with  $\beta(l+1) > 2$  such that  $\{\tilde{F}^n(\tilde{q})\} \subset W(\beta)$  eventually, taking  $\beta' \in (0, \beta)$  in such a way that  $\beta'(l+1) > 2$ , Lemma 4.4 applied to  $\chi^{-1}(\tilde{z}, \tilde{w}) = (\tilde{z}, \tilde{w}) + O(\|(\tilde{z}, \tilde{w})\|^l)$  would imply that  $\{F^n(q)\} \subset W(\beta')$  eventually, a contradiction.

Therefore, fixing  $\beta \in (0, 1/2)$  with  $\beta(l+1) > 2$ , we can assume, without loss of generality, that there exists an increasing subsequence  $\{n_k\} \subset \mathbb{N}$  tending to  $\infty$  such that, setting  $(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{q})) := \tilde{F}^{n_k}(\tilde{q})$ , for all  $n_k$  it holds  $|\tilde{z}_{n_k}(\tilde{q})| \geq |\tilde{z}_{n_k}(\tilde{q})\tilde{w}_{n_k}(\tilde{q})|^\beta$ , that is

$$(4.8) \quad |\tilde{w}_{n_k}(\tilde{q})| \leq |\tilde{z}_{n_k}(\tilde{q})|^{\frac{1-\beta}{\beta}}.$$

On the other hand, by Lemma 2.6,  $\{F^n(p)\} \subset W(\beta)$  eventually for all  $\beta \in (0, 1/2)$  such that  $\beta(l+1) > 2$ . Hence, by Lemma 4.4, it follows that  $\{\tilde{F}^n(\tilde{p})\} \subset W(\beta)$  eventually for all  $\beta \in (0, 1/2)$  such that  $\beta(l+1) > 2$ . Therefore, again by Lemma 2.6, it follows that there exist  $0 < c < C$  and  $\tilde{n} > 0$  such that for all  $n \geq \tilde{n}$

$$(4.9) \quad c|\tilde{z}_n(\tilde{p})| \leq |\tilde{w}_n(\tilde{p})| \leq C|\tilde{z}_n(\tilde{p})|.$$

Consider the holomorphic projections  $\pi_1 : \mathbb{B}_*^2 \rightarrow \mathbb{D}^*$  given by  $\pi_1(\tilde{z}, \tilde{w}) = \tilde{z}$ , and  $\pi_2 : \mathbb{B}_*^2 \rightarrow \mathbb{D}^*$  given by  $\pi_2(\tilde{z}, \tilde{w}) = \tilde{w}$ . By the properties of the Kobayashi distance,  $k_{\mathbb{D}^*}(\pi_j(A), \pi_j(B)) \leq k_{\mathbb{B}_*^2}(A, B)$  for every  $A, B \in \mathbb{B}_*^2$ . Hence, by (4.7), for all  $n_k$ ,

$$(4.10) \quad k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{p}), \tilde{z}_{n_k}(\tilde{q})) < \delta, \quad k_{\mathbb{D}^*}(\tilde{w}_{n_k}(\tilde{p}), \tilde{w}_{n_k}(\tilde{q})) < \delta.$$

Thanks to (4.9) and Lemma 4.6, since the orbit of  $\tilde{p}$  converges to the origin, there exists  $k_0 \in \mathbb{N}$  such that for all  $n_k \geq k_0$ ,

$$k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{p}), \tilde{w}_{n_k}(\tilde{p})) \leq \left| \log \frac{\log |\tilde{z}_{n_k}(\tilde{p})|}{\log |\tilde{w}_{n_k}(\tilde{p})|} \right| + g(\tilde{z}_{n_k}(\tilde{p}), \tilde{w}_{n_k}(\tilde{p})) < \delta.$$

Hence, by (4.10) and the triangle inequality, for all  $n_k \geq k_0$ ,

$$(4.11) \quad k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{p})) \leq k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{q}), \tilde{z}_{n_k}(\tilde{p})) + k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{p}), \tilde{w}_{n_k}(\tilde{p})) < 2\delta.$$

On the other hand, let  $k_1 \in \mathbb{N}$  be such that, for all  $n_k \geq k_1$ ,

$$g(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{q})) < \delta,$$

where  $g$  is the function defined in Lemma 4.6. Hence, by the same lemma and (4.8)

$$(4.12) \quad \begin{aligned} k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{q})) &\geq \left| \log \frac{\log |\tilde{z}_{n_k}(\tilde{q})|}{\log |\tilde{w}_{n_k}(\tilde{q})|} \right| - g(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{q})) \\ &= \log \left( \frac{\log |\tilde{w}_{n_k}(\tilde{q})|}{\log |\tilde{z}_{n_k}(\tilde{q})|} \right) - g(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{q})) \\ &\geq \log \left( \frac{\log |\tilde{z}_{n_k}(\tilde{q})|^{\frac{1-\beta}{\beta}}}{\log |\tilde{z}_{n_k}(\tilde{q})|} \right) - \delta = \log \frac{1-\beta}{\beta} - \delta. \end{aligned}$$

By the triangle inequality and (4.11), (4.12), for  $n_k \geq \max\{k_0, k_1\}$ ,

$$\begin{aligned} k_{\mathbb{D}^*}(\tilde{w}_{n_k}(\tilde{p}), \tilde{w}_{n_k}(\tilde{q})) &\geq k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{q})) - k_{\mathbb{D}^*}(\tilde{z}_{n_k}(\tilde{q}), \tilde{w}_{n_k}(\tilde{p})) \\ &\geq \log \frac{1-\beta}{\beta} - 3\delta. \end{aligned}$$

Therefore, by (4.10),

$$4\delta \geq \log \frac{1-\beta}{\beta}.$$

Since this should hold for every choice of  $\delta > 0$ , while  $\frac{1-\beta}{\beta} > 1$  is fixed, we get a contradiction, and the theorem holds.  $\square$

## 5. THE TOPOLOGY OF $\Omega$

In all this section,  $F$  is an automorphism of the form (4.1),  $B$  is the local basin of attraction given by Theorem 2.4 and  $\Omega$  is the associated global basin of attraction.

The aim of this section is to prove that the global basin  $\Omega$  is biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$ . We start first by proving that  $\Omega$  is not simply connected:

**Proposition 5.1.** *The open set  $\Omega$  is connected but not simply connected.*

*Proof.* For  $n \in \mathbb{N}$ , we have

$$F^{-n}(B) = F^{-(n+1)}(F(B)) \subset F^{-(n+1)}(B).$$

Hence,  $\Omega$  is the growing union of images biholomorphic to  $B$  which is connected by Lemma 2.8. Therefore,  $\Omega$  is connected.

In order to prove that  $\Omega$  is doubly connected, let us consider  $\gamma_r^n(t) = (re^{nit}, re^{-nit})$  for  $0 < r < 1/\sqrt{R}$  and  $n \in \mathbb{Z} \setminus \{0\}$  fixed and  $t \in [0, 2\pi]$ . By Lemma 2.8, the loop  $\gamma_r^n$  is not contractible in  $B$ .

We first claim that  $F(\gamma_r^n)$  is not contractible in  $B$ .

To prove the claim, it is enough to show that  $\Phi(F(\gamma_r^n))$  is not contractible in  $\Phi(B)$ , where  $\Phi$  is given by (2.19). By Remark 2.9, this is equivalent to showing that the second component of  $\Phi(F(\gamma_r^n(t)))$  has non-zero winding number around 0.

Let  $\alpha_r(t) := \pi_2(\Phi(F(\gamma_r^n(t))))$ ,  $t \in [0, 2\pi]$ , where  $\pi_2: \mathbb{C}^2 \rightarrow \mathbb{C}$  is the projection on the second coordinate  $\pi_2(z, w) = w$ . Writing  $R_i^2(\gamma_r^n(t)) = r^l k(t, r)$ , with  $k: [0, 2\pi] \times (0, 1/\sqrt{R}) \rightarrow \mathbb{C}$  a smooth function, we have

$$\alpha_r(t) = e^{-nit} \bar{\lambda} r (1 + ar^2 \lambda + O(r^4)) + r^l k(t, r).$$

Note that there exists  $C > 0$  such that for every  $(t, r) \in [0, 2\pi] \times (0, 1/\sqrt{R})$  it holds

$$(5.1) \quad |k(t, r)| \leq C, \quad \left| \frac{\partial k(t, r)}{\partial t} \right| \leq C.$$

In order to simplify notations, let  $\Theta := (1 + ar^2 \lambda + O(r^4))$ . We have

$$\begin{aligned} \frac{\alpha_r'(t)}{\alpha_r(t)} &= \frac{-nie^{-nit} r \bar{\lambda} \Theta + r^l \frac{\partial k(t, r)}{\partial t}}{e^{-nit} r \bar{\lambda} \Theta + r^l k(t, r)} = -ni + ni + \frac{-nie^{-nit} r \bar{\lambda} \Theta + r^l \frac{\partial k(t, r)}{\partial t}}{e^{-nit} r \bar{\lambda} \Theta + r^l k(t, r)} \\ &= -ni + r^{l-1} e^{nit} \lambda \frac{\frac{\partial k(t, r)}{\partial t} + nik(t, r)}{\Theta + r^{l-1} e^{nit} \lambda k(t, r)} = -ni + r^{l-1} m(t, r), \end{aligned}$$

where  $m: [0, 2\pi] \times (0, 1/\sqrt{R}) \rightarrow \mathbb{C}$  is a smooth function. The winding number  $I(\alpha_r, 0)$  of  $\alpha_r$  around 0 is thus given by

$$(5.2) \quad \begin{aligned} I(\alpha_r, 0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{\alpha_r'(t)}{\alpha_r(t)} dt = \frac{1}{2\pi i} \int_0^{2\pi} (-ni + r^{l-1} m(t, r)) dt \\ &= -n + \frac{r^{l-1}}{2\pi i} \int_0^{2\pi} m(t, r) dt. \end{aligned}$$

Since  $\Theta + r^{l-1} e^{nit} \lambda k(t, r) = 1 + O(r^2)$ , there exists  $r_1 \in (0, 1/\sqrt{R})$  such that  $|\Theta + r^{l-1} e^{nit} \lambda k(t, r)| \geq 1/2$  for all  $r \in (0, r_1)$ . Then for all  $r \in (0, r_1)$ , we have

$$(5.3) \quad |m(t, r)| = \left| \frac{\frac{\partial k(t, r)}{\partial t} + nik(t, r)}{\Theta + r^{l-1} e^{nit} \lambda k(t, r)} \right| \leq 2(1 + |n|)C.$$

Let  $r_2 \in (0, r_1)$  be such that  $|n| - r_2^{l-1} 2(1 + |n|)C > 0$ . By (5.2) and (5.3), for all  $r \in (0, r_2)$  it holds

$$|I(\alpha_r, 0)| \geq |n| - r^{l-1} 2(1 + |n|)C > 0.$$

Therefore, for all  $r \in (0, r_2)$  the curve  $F(\gamma_r^n)$  is not contractible in  $B$ . On the other hand, by Lemma 2.8, for every  $r, s \in (0, 1/\sqrt{R})$  the curves  $\gamma_r^n$  and  $\gamma_s^n$  are homotopic, and so are  $F(\gamma_r^n)$  and  $F(\gamma_s^n)$ , whence  $F(\gamma_r^n)$  is not contractible for every  $r \in (0, 1/\sqrt{R})$  and  $n \in \mathbb{Z} \setminus \{0\}$ . In particular, for  $n \neq 0$ , the curve  $F(\gamma_r^n)$  is homotopic equivalent in  $B$  to  $\gamma_r^{\tilde{n}}$  for some  $\tilde{n} \neq 0$ . It follows inductively that for every  $N \in \mathbb{N}$  the loop  $F^N \circ \gamma_r^1$  is not contractible in  $B$  for every  $r \in (0, 1/\sqrt{R})$ .

Now, we show that the loop  $\gamma_r^1$  is not contractible in  $\Omega$  for all  $r \in (0, 1/\sqrt{R})$ . Indeed, if  $H: [0, 2\pi] \times [0, 1] \rightarrow \Omega$  were a homotopy of paths between  $\gamma_r^1$  and a constant path, then there would exist  $N \in \mathbb{N}$  such that  $H([0, 2\pi] \times [0, 1]) \subset F^{-N}(B)$ . Hence,  $F^N \circ H$  would be a homotopy in  $B$  between the loop  $F^N \circ \gamma_r^1$  and a constant path, and thus,  $F^N \circ \gamma_r^1$  would be contractible in  $B$ , which is a contradiction.

Therefore,  $\Omega$  is not simply connected.  $\square$

We will now prove that  $\Omega$  is biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$ , by providing an explicit biholomorphism.

Let  $\psi$  be the Fatou coordinate for  $F$  given by Proposition 3.1 and let  $\sigma$  be the holomorphic function given by Proposition 3.3. We can use the functional equations (3.1) and (3.10) to extend  $\psi, \sigma$  to all  $\Omega$ . Indeed, let  $p \in \Omega$ . Then there exists  $n \in \mathbb{N}$  such that  $F^n(p) \in B$ . We define

$$g_1(p) := \psi(F^n(p)) - n,$$

and

$$\begin{aligned} g_2(p) &:= \lambda^n \exp \left( - \sum_{j=0}^{n-1} \frac{a\lambda}{g_1(p) + j} \right) \sigma(F^n(p)) \\ &= \lambda^n \exp \left( - \sum_{j=1}^n \frac{a\lambda}{\psi(F^n(p)) + j - n} \right) \sigma(F^n(p)). \end{aligned}$$

The next lemma shows that the map  $G := (g_1, g_2): \Omega \rightarrow \mathbb{C}^2$  is well defined and holomorphic:

**Lemma 5.2.** *The holomorphic map  $G := (g_1, g_2): \Omega \rightarrow \mathbb{C}^2$  is well-defined and injective.*

*Proof.* The definition of  $G$  does not depend on the  $n$ . Indeed, if  $n$  and  $m$  are both integers so that  $F^n(p)$  and  $F^m(p)$  belong to  $B$ , and  $n < m$ , then  $F^m(p) = F^{m-n}(F^n(p))$ . Therefore  $\psi(F^m(p)) = \psi(F^{m-n}(F^n(p))) = \psi(F^n(p)) + m - n$ , whence  $\psi(F^m(p)) - m = \psi(F^n(p)) - n$ .

Analogously,  $\sigma(F^m(p)) = \bar{\lambda}^{m-n} \exp(\sum_{j=1}^{m-n} a\lambda/(\psi(F^n(p)) + j))\sigma(F^n(p))$ , and so

$$\begin{aligned} & \lambda^m \exp\left(-\sum_{j=1}^m \frac{a\lambda}{\psi(F^m(p)) + j - m}\right) \sigma(F^m(p)) \\ &= \lambda^m \exp\left(-\sum_{j=1}^m \frac{a\lambda}{\psi(F^n(p)) + j - n}\right) \bar{\lambda}^{m-n} \exp\left(\sum_{j=1}^{m-n} \frac{a\lambda}{\psi(F^n(p)) + j}\right) \sigma(F^n(p)) \\ &= \lambda^n \exp\left(-\sum_{j=1}^n \frac{a\lambda}{\psi(F^n(p)) + j - n}\right) \sigma(F^n(p)), \end{aligned}$$

and we are done.

Let us now prove the injectivity of  $G$ . Let  $p, q \in \Omega$ . By the very definition of  $G$ ,  $G(p) = G(q)$  if and only if

$$(\psi(F^n(p)), \sigma(F^n(p))) = (\psi(F^n(q)), \sigma(F^n(q)))$$

for all  $n \in \mathbb{N}$  such that  $F^n(p)$  and  $F^n(q)$  are contained in  $B$ .

By Proposition 3.4, there exist  $R_1 \geq R_0$ ,  $\beta_1 \in (\beta_0, \frac{1}{2})$  and  $0 < \theta_1 \leq \theta_0$  such that  $Q := (\psi, \sigma)$  is injective on  $B(\beta_1, \theta_1, R_1)$ . Also, by Lemma 2.6, there exists  $n \in \mathbb{N}$  such that  $F^n(p), F^n(q) \in B(\beta_1, \theta_1, R_1)$ . Therefore,  $G(p) = G(q)$  if and only if  $p = q$ .  $\square$

**Proposition 5.3.**  $G(\Omega) = \mathbb{C} \times \mathbb{C}^*$ .

*Proof.* Let  $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be the holomorphic map defined by

$$T(\zeta, \xi) := (\zeta + 1, \bar{\lambda} e^{\frac{a\lambda}{\zeta}} \xi).$$

Clearly,  $T$  is an automorphism. Moreover,

$$G \circ F = T \circ G,$$

hence,  $T^{-1} \circ G = G \circ F^{-1}$ . Therefore,

$$G(\Omega) = G(\cup_{n \in \mathbb{N}} F^{-n}(B)) = \cup_{n \in \mathbb{N}} T^{-n}(G(B)).$$

Hence, a point  $(\zeta, \xi)$  belongs to  $G(\Omega)$  if and only if there exists  $n \in \mathbb{N}$  such that  $T^n(\zeta, \xi) \in G(B)$ .

Let  $(\zeta_0, \xi_0) \in \mathbb{C} \times \mathbb{C}^*$ . By induction, for  $n \in \mathbb{N}$ , we have

$$(\zeta_n, \xi_n) := T^n(\zeta_0, \xi_0) = \left( \zeta_0 + n, \bar{\lambda}^n \exp\left(\sum_{j=0}^{n-1} \frac{a\lambda}{\zeta_0 + j}\right) \xi_0 \right).$$



Now,

$$\begin{aligned}
|\xi_n| &= \exp \left( \sum_{j=0}^{n-1} \operatorname{Re} \left( \frac{a\lambda}{\zeta_0 + j} \right) \right) |\xi_0| \\
&= \exp \left( -\frac{1}{2} \sum_{j=0}^{n-1} \operatorname{Re} \left( \frac{1}{\zeta_0 + j} \right) + \frac{\operatorname{Im} \lambda}{2\operatorname{Re} \lambda} \sum_{j=0}^{n-1} \operatorname{Im} \left( \frac{1}{\zeta_0 + j} \right) \right) |\xi_0| \\
&= \exp \left( -\frac{1}{2} \sum_{j=0}^{n-1} \frac{1}{j} \left( \frac{1 + j^{-1} \operatorname{Re} \zeta_0}{|j^{-1} \zeta_0 + 1|^2} \right) + \frac{\operatorname{Im} \lambda}{2\operatorname{Re} \lambda} \sum_{j=0}^{n-1} \frac{1}{j^2} \frac{\operatorname{Im} \zeta_0}{|j^{-1} \zeta_0 + 1|^2} \right) |\xi_0|,
\end{aligned}$$

which implies that

$$|\zeta_n| \sim n, \quad |\xi_n| \sim \frac{1}{\sqrt{n}}.$$

Note that this implies that, given  $\tilde{\beta} \in (0, \frac{1}{2})$ , for all  $n$  sufficiently large,

$$(5.4) \quad |\zeta_n|^{\tilde{\beta}-1} < |\xi_n| < |\zeta_n|^{-\tilde{\beta}}.$$

Moreover, since  $\zeta_n = \zeta_0 + n$ , it follows that, given  $\tilde{R} > 0$  and  $\tilde{\theta} \in (0, \frac{\pi}{2})$ , for all  $n$  sufficiently large,

$$(5.5) \quad \zeta_n \in H(\tilde{R}, \tilde{\theta}).$$

Note that  $G(z, w) = Q(z, w) = (\psi(z, w), \sigma(z, w))$  for all  $(z, w) \in B$ . Hence, by Proposition 3.4, there exist  $\tilde{\beta} \in (0, \frac{1}{2})$ ,  $\tilde{\theta} \in (0, \pi/2)$  and  $\tilde{R} > 1$  such that  $\{(U, w) \in \mathbb{C}^2 : U \in H(\tilde{R}, \tilde{\theta}), |U|^{\tilde{\beta}-1} < |w| < |U|^{-\tilde{\beta}}\} \subset G(B)$ . Therefore, from (5.4) and (5.5), it follows at once that  $\mathbb{C} \times \mathbb{C}^* \subseteq G(\Omega)$ , and, in fact, equality holds since  $G(\Omega)$  cannot be simply connected by Proposition 5.1.  $\square$

## 6. THE PROOF OF THEOREM 0.1

Let  $F$  be an automorphism of the form (4.1), and assume that  $\lambda$  is Brjuno. By Theorem 4.8,  $\Omega$  is an invariant attracting Fatou component at  $(0, 0)$  and  $\Omega$  is biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$  by Proposition 5.3.

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