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## MULTIVALUED CONTRACTIONS AND APPLICATIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS

by

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### 1 Preliminaries

Throughout this note, for every couple  $X, Y$  of nonempty sets,  $F(X, Y)$  denotes the family of all mappings  $f : X \rightarrow Y$ . Also,  $N := \{0, 1, \dots\}$  denotes the positive integers,  $R_+ := [0, \infty[$  the positive half-axis, and  $\overline{R}_+ := [0, \infty]$  the extended positive half-axis.

Let  $(X, d)$  be a *generalized metric space* (abbreviated g.m.s.) as in Luxemburg [3]. Denote  $\mathcal{P}(X) = \{Y \subseteq X; Y \neq \emptyset\}$ ; and let  $\mathcal{C}(X)$  be the family of all closed elements in  $\mathcal{P}(X)$ . For every  $x \in X, Y \in \mathcal{P}(X), \varepsilon > 0$ , let us put

$$\begin{aligned} X(x) &= \{y \in X; d(x, y) < \infty\}, \\ d(Y, x) &= \inf\{d(y, x); y \in Y\}, S(Y, \varepsilon) = \{x \in X; d(Y, x) < \varepsilon\}. \end{aligned}$$

Let  $D : (\mathcal{C}(X))^2 \rightarrow \overline{R}_+$  be the mapping defined as: for every  $Y, Z \in \mathcal{C}(X)$ ,

$$\begin{aligned} D(Y, Z) &= \inf\{\varepsilon > 0; Y \subseteq S(Z, \varepsilon); Z \subseteq S(Y, \varepsilon)\}, \\ &\text{if } \{\varepsilon > 0; Y \subseteq S(Z, \varepsilon), Z \subseteq S(Y, \varepsilon)\} \neq \emptyset, \\ D(Y, Z) &= \infty, \text{ if } \{\varepsilon > 0; Y \subseteq S(Z, \varepsilon), Z \subseteq S(Y, \varepsilon)\} = \emptyset. \end{aligned}$$

It is well known (cf. Covitz and Nadler [2]) that  $(\mathcal{C}(X), D)$  is a g.m.s. (the mapping  $D$  being called the *Hausdorff generalized metric* of  $\mathcal{C}(X)$ ).

The following properties of the generalized metric  $D$  are almost evident (see, e.g., the remarks appearing in Nadler [4, Section 3]).

**Lemma 1.1** *Let  $Y, Z \in \mathcal{C}(X)$  and  $\varepsilon > 0$  satisfy  $D(Y, Z) < \varepsilon$ . Then*

*for every  $u \in Y$  (resp.  $Z$ ) there exists  $v \in Z$  (resp.  $Y$ ) with  $d(u, v) < \varepsilon$ .*

**Lemma 1.2** *Let  $Y, Z \in \mathcal{C}(X)$ ,  $\tau > 0$  be such that*

*for every  $u \in Y$  (resp.  $Z$ ) there exists  $v \in Z$  (resp.  $Y$ ) with  $d(u, v) \leq \tau$ .*

*Then, necessarily,  $D(Y, Z) \leq \tau$ .*

By a *multivalued mapping* from  $X$  into itself we shall mean, throughout this note, an element of  $F(X, \mathcal{C}(X))$ . Let  $T : X \rightarrow \mathcal{C}(X)$  be such a multivalued mapping. By an *iterative sequence* generated by some  $x \in X$ , we mean a sequence  $(x_n; n \in N)$  in  $X$ , defined as

$$x_0 = x \text{ and } x_{n+1} \in Tx_n, \text{ for every } n \in N.$$

In the same context,  $z \in X$  is called a *fixed point* of  $T$  iff  $z \in Tz$ .

The main aim of this note is to give some sufficient metric conditions implying existence of fixed points of a multivalued mapping on the one hand, and an approximation of these fixed points by an iterative sequence, on the other hand. The main results (presented in the next section) might be compared with those of Covitz and Nadler [2]; see also Nadler [4]. Finally, in the last section, an application is given to a certain class of "multivalued" functional differential equations with transformed argument.

## 2 The main results

Throughout this section,  $(X, d)$  is a complete g.m.s. We introduce the notations

- (i)  $P = \{f \in F(R_+, R_+); f(0) = 0, f(t) < t, \forall t > 0\}$ ,
- (ii)  $Q = \{f \in F(R_+, R_+); f(0) = 0, f(t) > t, \forall t > 0\}$ ,
- (iii)  $P_1 = \{f \in P; \sum_{n=0}^{\infty} f^{(n)}(t) < \infty, \forall t > 0\}$ .

(Here,  $f^{(n)}$  stands for the  $n$ -th iterate of  $f$ ,  $\forall n \in N$ ).

A multivalued mapping  $T : X \rightarrow \mathcal{C}(X)$  is said to be a *normal multivalued contraction* (abbreviated n.m.c.) with respect to a  $f : R_+ \rightarrow R_+$ , iff it satisfies

$$(2.1) \quad x, y \in X, \tau > 0, d(x, y) \leq \tau \implies D(Tx, Ty) \leq f(\tau).$$

Now, one may state and prove the first main result of this note.

**Theorem 2.1** *Let  $T : X \rightarrow \mathcal{C}(X)$ ,  $f \in P$ ,  $g \in Q$  be such that*

**(2.2)**  *$T$  is a n.m.c. with respect to  $f$*

**(2.3)**  *$g \circ f \in P_1$*

**(2.4)**  *$X(T) := \{x \in X; X(x) \cap Tx \neq \emptyset\}$  is not empty.*

*Then, for every  $x \in X(T)$ , there exist an iterative sequence  $(x_n; n \in N)$  in  $X(x)$  generated by  $x$ , a real number  $\rho > 0$ , and an element  $z \in X(x)$ , such that*

**(a)**  *$z \in Tz$  (i.e.:  $z$  is a fixed point of  $T$ ),*

**(b)**  *$x_n \rightarrow z$  as  $n \rightarrow \infty$ , in the sense that*

$$d(x_n, z) \leq \sum_{m=n}^{\infty} (g \circ f)^{(m)}(\tau), \text{ for any } \tau \geq \rho \text{ and } n \in N.$$

**Proof** Let  $x \in X(T)$  be given and put  $x_0 = x$ ; we may assume  $x_0 \notin Tx_0$ . From (2.4), there is an element  $x_1 \in X(x_0) \cap Tx_0$ ; hence,  $0 < \rho := d(x_0, x_1) < \infty$ ; moreover (as before), we may assume  $x_1 \notin Tx_1$ . Fix  $\alpha \geq \rho (> 0)$ . From (2.2),

$$D(Tx_0, Tx_1) \leq f(\alpha) < (g \circ f)(\alpha) \text{ (because, } x_1 \notin Tx_1 \text{ implies } f(\alpha) > 0).$$

Taking into account Lemma 1.1, there is an  $x_2 \in Tx_1$  with  $d(x_1, x_2) < (g \circ f)(\alpha)$ ; without loss, assume that  $x_2 \notin Tx_2$ . Again invoking (2.2),

$$\begin{aligned} D(Tx_1, Tx_2) &\leq f((g \circ f)(\alpha)) < (g \circ f)^{(2)}(\alpha) \\ &\text{(since, } x_2 \notin Tx_2 \text{ implies } f((g \circ f)(\alpha)) > 0). \end{aligned}$$

Then, by the same procedure as above, there is a  $x_3 \in Tx_2$  with  $d(x_2, x_3) < (g \circ f)^{(2)}(\alpha)$ , etc. By induction, we get an iterative sequence  $(x_n; n \in N)$  generated by  $x_0$ , satisfying

$$d(x_n, x_{n+1}) \leq (g \circ f)^{(n)}(\alpha), \text{ for every } n \in N.$$

From (2.3),  $(x_n; n \in N)$  is a Cauchy sequence; so, by completeness,  $x_n \rightarrow z$  as  $n \rightarrow \infty$ , for some  $z \in X$ . By (2.2),  $T$  is continuous; hence,  $\lim_n Tx_n = Tz$ ; and, since  $(x_{n+1} \in Tx_n, \forall n)$ , conclusion (a) follows by a limit process.

Now, let  $\tau \geq \rho$  be arbitrary fixed. As  $\tau \geq d(x_0, x_1)$ , the above evaluations are clearly valid, and hence,

$$d(x_n, x_{n+p}) \leq \sum_{m=n}^{n+p-1} (g \circ f)^{(m)}(\tau) \leq \sum_{m=n}^{\infty} (g \circ f)^{(m)}(\tau), \forall n, p \in N, p \geq 1.$$

Performing the limit as  $p \rightarrow \infty$ , conclusion (b) follows as well; and so, the proof is complete.  $\blacksquare$

As an important particular case, let the mapping  $f \in P$  be defined by

$$f(t) = \lambda t, \text{ for any } t \geq 0 \text{ and some } \lambda \in ]0, 1[.$$

Then, the mapping  $g \in Q$ , defined by

$$g(t) = \mu t, \text{ for any } t \geq 0 \text{ and some } \mu \in ]1, 1/\lambda[$$

has, the property  $g \circ f \in P_1$ . Note that, in this way, Theorem 2.1 above reduces to Theorem 5 of Nadler [4]; see also Covitz and Nadler [2].

A multivalued mapping  $T : X \rightarrow \mathcal{C}(X)$  is said to be a *strong multivalued contraction* (abbreviated s.m.c.) with respect to an  $f : R_+ \rightarrow R_+$ , provided

$$\begin{aligned} (2.1') \text{ if } x, y \in X, \tau > 0 \text{ satisfy } d(x, y) \leq \tau, \text{ then, for every} \\ u \in Tx \text{ (resp. } Ty) \text{ there is a } v \in Ty \text{ (resp. } Tx) \text{ with } d(u, v) \leq f(\tau). \end{aligned}$$

It is important to remark that, from Lemma 1.2, every s.m.c. is a n.m.c.

Now, the second main result of this note is

**Theorem 2.2** *Let  $T : X \rightarrow \mathcal{C}(X)$  and  $f \in P$  be such that*

(2.2')  *$T$  is a s.m.c. with respect to  $f$*

(2.3')  *$f \in P_1$*

(2.4')  *$X(T) = \{x \in X; X(x) \cap Tx \neq \emptyset\}$  is not empty.*

*Then, for every  $x \in X$ , there exist an iterative sequence  $(x_n; n \in \mathbb{N})$  in  $X(x)$  generated by  $x$ , a  $\rho > 0$ , and a  $z \in X(x)$ , such that*

(a')  *$z \in Tz$  ( $z$  is a fixed point of  $T$ )*

(b')  *$x_n \rightarrow z$  as  $n \rightarrow \infty$ , in the sense that*

$$d(x_n, z) \leq \sum_{m=n}^{\infty} f^{(m)}(\tau), \text{ for any } \tau \geq \rho \text{ and any } n \in \mathbb{N}.$$

**Proof** Let  $x \in X(T)$  be a given element and put  $x_0 = x$ ; without loss, one may assume  $x_0 \notin Tx_0$ . From the definition of  $X(T)$ , there is an  $x_1 \in Tx_0$  with  $0 < \rho := d(x_0, x_1) < \infty$ . Let  $\alpha \geq d(x_0, x_1)$  be arbitrary fixed. From (2.2'), there exist an  $x_2 \in Tx_1$  with  $d(x_1, x_2) \leq f(\alpha)$ ; then, an  $x_3 \in Tx_2$  with  $d(x_2, x_3) \leq f^{(2)}(\alpha)$ ; and so on. By induction, we get an iterative sequence  $(x_n; n \in \mathbb{N})$  generated by  $x$ , satisfying

$$d(x_n, x_{n+1}) \leq f^{(n)}(\alpha), \text{ for every } n \in \mathbb{N}.$$

The remaining part of the argument is the same as in Theorem 2.1; and so, we may consider that the proof is complete.  $\blacksquare$

### 3 An application

In what follows,  $n \geq 1$  is a given positive integer. Let  $(\mathbb{R}^n, \|\cdot\|)$  be the Euclidean  $n$ -dimensional space endowed with a given norm  $\|\cdot\|$ . Let  $X$  (resp.  $A$ ) denote the set of all continuous  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  (resp.,  $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ). For every  $x \in X$ , define  $\|x\| \in A$  by

$$\|x\|(t) = \|x(t)\|, \text{ for every } t \in \mathbb{R}_+;$$

and for every  $g \in A$ , let  $\|\cdot\|_g : X \rightarrow \overline{\mathbb{R}_+}$  be defined as: for every  $x \in X$ ,

$$\begin{aligned} \|x\|_g &= \inf\{\lambda \in \mathbb{R}_+; \|x\| \leq \lambda g\}, \text{ if } \{\lambda \in \mathbb{R}_+; \|x\| \leq \lambda g\} \neq \emptyset; \\ \|x\|_g &= \infty, \text{ if } \{\lambda \in \mathbb{R}_+; \|x\| \leq \lambda g\} = \emptyset. \end{aligned}$$

It is simple to verify that  $(X, \|\cdot\|_g)$  is a generalized Banach space; or, equivalently:  $(X, d_g)$  is a complete g.m.s., where

$$d_g(x, y) = \|x - y\|_g, \text{ } x, y \in X \text{ (the associated metric).}$$

For every  $g \in A$ , denote also

$$X_g = \{x \in X; \|x\|_g < \infty\}, \mathcal{C}_g(X) = \{Y \in \mathcal{P}(X); Y \text{ is } \|\cdot\|_g\text{-closed}\}.$$

Now, let  $x \mapsto k(x)$  be a mapping from  $X$  to  $\mathcal{P}(X)$ ; and  $x^0 \in R^n$  be a fixed element. Consider the "multivalued" Cauchy problem

$$(C.P.) \quad x'(t) \in k(x)(t), \quad t \in R_+, \quad x(0) = x^0.$$

The following existence and approximation result concerning the solutions of (C.P.) may be stated.

**Theorem 3.1** *Suppose that there exist a mapping  $a \mapsto h(a)$  from  $A$  into itself, and a couple  $g \in A$ ,  $f \in P_1$  such that*

(3.1) *for every  $x \in X$ , the set  $K(x)$  of all  $\bar{y} \in X$  with*

$\bar{y}(t) = x^0 + \int_0^t y(s)ds$ ,  $t \in R_+$  (for some  $y \in k(x)$ ) *is  $\|\cdot\|_g$ -closed*

(3.2) *if  $x, y \in X$ ,  $a \in A$  satisfy  $\|x - y\| \leq a$ , then for any  $u \in k(x)$  (resp.  $k(y)$ ) there is a  $v \in k(y)$  (resp.  $k(x)$ ) with  $\|u - v\| \leq h(a)$*

(3.3)  $\int_0^t h(g\tau)(s)ds \leq f(\tau)g(t)$ ,  $\forall \tau > 0$ ,  $\forall t \in R_+$

(3.4) *the set  $X^0 \subseteq X$  of all  $y^0 \in X$  with  $\|y^0(t) - x^0 - \int_0^t u(s)ds\| \leq \mu g(t)$ , for all  $t \in R_+$ , and some  $\mu > 0$ ,  $u \in k(y^0)$ , is not empty.*

*Then, for every  $y^0 \in X^0$ , there exists a sequence  $(y^{[m]}; m \in N)$  in  $y^0 + X_g$  and a  $z \in y^0 + X_g$ , such that*

(a)  *$z$  is a solution of (C.P.)*

(b)  $y^{[0]} = y^0$ , and  $(y^{[m+1]})' \in k(y^{[m]})$ , for any  $m \in N$

(c)  $(y^{[m]}; m \in N)$  converges to  $z$ , in the sense that

$$\|y^{[m]} - z\|_g \leq \sum_{p=m}^{\infty} f^{(p)}(\tau), \quad \forall \tau \geq \sup\{\|y^{[i]} - y^{[j]}\|_g; i, j \in N\}, \quad \forall m \in N.$$

**Proof** Let  $T : X \rightarrow \mathcal{C}_g(X)$  be the multivalued mapping defined by

$$(3.5) \quad Tx = K(x), \quad \text{for every } x \in X.$$

Let  $x, y \in X$ ,  $\tau > 0$  be such that  $\|x - y\|_g \leq \tau$ . From the definition of  $\|\cdot\|_g$ ,  $\|x - y\| \leq g\tau$ . Let  $\bar{u} \in Tx$  (resp.  $Ty$ ). From (3.5),

$$\bar{u}(t) = x^0 + \int_0^t u(s)ds, \quad t \in R_+, \quad \text{for some } u \in k(x) \text{ (resp. } k(y)).$$

Let  $v \in k(y)$  (resp.  $k(x)$ ) be the associated element given by (3.2), and let  $\bar{v} \in X$  be defined by  $\bar{v}(t) = x^0 + \int_0^t v(s)ds$ ,  $t \in R_+$ . Clearly,  $\bar{v} \in Ty$  (resp.  $Tx$ ) on the one hand and (again by (3.2))  $\|u - v\| \leq h(g\tau)$ , on the other hand. But then, from (3.3),

$$\|\bar{u}(t) - \bar{v}(t)\| \leq \int_0^t \|u(s) - v(s)\|ds \leq \int_0^t h(g\tau)(s)ds \leq f(\tau)g(t), \quad t \in R_+;$$

hence,  $\|\bar{u} - \bar{v}\|_g \leq f(\tau)$ ; showing that  $T$  is a s.m.c, with respect to  $f$ . Finally, (3.4) says that for every  $y^0 \in X^0$  there is a  $\bar{u} \in Ty^0$  with  $\|y^0 - \bar{u}\|_g \leq \mu < \infty$ . Thus, Theorem 2.2 applies to  $(X, d_g)$ ,  $T$  and  $f$ ; hence the conclusion.  $\blacksquare$

At this point, we make a useful remark. Namely, let  $t \mapsto \widehat{t}$  be a mapping in  $F(R_+, \mathcal{P}(R_+))$ . In this case, for every  $x \in X$ ,  $t \in R_+$ ,  $k(x)(t)$  may be considered as being defined only by the values of  $x$  in the set  $\widehat{t}$ . Concerning the choice of the above mapping, two cases are of a great interest in the study of a (C.P.)

$$(3.6) \quad \widehat{t} \subseteq [0, t], \text{ for any } t \in R_+,$$

$$(3.7) \quad \widehat{t} \cap ]t, \infty[ \neq \emptyset, \text{ for any } t \in R_+.$$

**A.** Suppose that (3.6) holds. Then, (C.P.) is called a *nonanticipative* multivalued functional differential equation with transformed argument. Particular cases of (3.6) are, of course,

$$\widehat{t} = \{t\}, t \in R_+ \text{ (the "ordinary" case),}$$

$$\widehat{t} = [0, t], t \in R_+ \text{ (the "functional" case; cf. Corduneanu [1]).}$$

**B.** Suppose (3.7) holds. Then, (C.P.) is called an *anticipative* multivalued functional differential equation with transformed argument. From such a perspective, the main result of this section may be compared with the related ones in Oberg [5] and Turinici [6].

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