A universal adaptive controller for tracking and stabilization control of nonholonomic vehicles

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Abstract

We propose a general class of adaptive controllers for leader-follower simultaneous tracking and stabilization of force-controlled nonholonomic mobile robots, under the hypothesis that the leader velocities are either integrable (*parking problem*), *or* persistently exciting (*tracking problem).* For the first time in the literature, we establish uniform global asymptotic stability for the origin of the closed-loop system (in the kinemtacis state space). We also show that the kinematics controller renders the system robust to perturbations in the sense of integral-to state stability. Then, we show that for the case in which the force dynamics equations are also considered (full model), *any* velocity-tracking controller with the property that the error velocities are square integrable may be used to ensure global tracking or stabilization. This modularity and robustness of our controller, added to the strength of our stability statements, renders direct the extension of our main results to the difficult scenario of control under parametric uncertainty.

Keywords: Formation control, persistency of excitation, Lyapunov design, nonholonomic systems

1 Introduction

There exists a considerable bulk of literature on control of mobile robots; a paradigm studied at least since the landmark article on leader-follower tracking control, [12]. Another problem, which is motivated by the well-known Brockett's condition, is that of stabilization to a *set-point.* As it is well known, nonholonomic robots are not stabilizable to a point via static smooth feedback. As a result, one must employ either discontinuous feedback [3] or time-varying feedback [29]. A somewhat hybrid variant of these problems, which inherits the difficulties of both, is that of *parking* —see [13, Remark 1]. In this scenario, in addition to following the leader reference trajectories, the latter vanish to a point.

A variety of control techniques have been proposed in the literature to solve either of the three problems enunciated above. For tracking control, in [26] were proposed, for the first time, *linear* controllers with persistency of excitation; the key condition to ensure tracking control being that the reference angular velocity is persistently exciting. While this control approach has been used in other works (*e.g.*, [9]) it is clear that it is not fit for the case of following straight paths, stabilization to a point or parking. On the other hand, for instance in [4, 14] nonlinear time-varying controllers are designed to allow for reference velocity trajectories that converge to zero. It is worth to emphasize that [14] covers the case when both the angular and forward velocity may converge to zero.

The simultaneous tracking-stabilization control problem for nonholonomic mobile robots consists in solving, via a *unique* controller, all of the problems considered above: the *tracking scenario*, in which reference trajectories do not vanish, *set-point* stabilization (the reference trajectories are zero), and the *parking problem*, in which case the leader velocities converge to zero. To the best of our knowledge the simultaneous tracking-stabilization has been addressed only in [13, 24, 7, 30].

In [13] a saturated time-varying kinematic controller is proposed to track the leader trajectories under different scenarios determined by the nature of its velocities. In [24] a unified velocity controller is proposed to solve the problem under all possible configurations of the leader velocities using the concept of transverse functions. In [7] and [30] a unified force controller is proposed in order to make the tracking error converging to the origin under the *tracking* and the *parking* scenarios. In [30], an approach that has the advantage to be simple and well structured, is introduced. It consists in using the combination of a tracking controller and a stabilization controller carefully weighted by a function that depends on the leader velocities. This function may be seen as a smoothed version of a supervisor function which is in charge of switching between two controllers.

Inspired by [30], in this paper we extend the class of stabilization controllers. Our contributions are the following. To the best of our knowledge, for the first time in the literature we establish uniform global asymptotic stability (UGAS) in the state space of the kinematic errors. In addition to this, our method of proof is original since we provide, also for the first time, strict Lyapunov functions for the tracking scenario. Furthermore, we establish integral input-to-state stability (integral ISS) for the kinematics model. The significance of this result cannot be over-estimated, it leads directly to a general statement for the case in which the full model (kinematics and dynamics) is considered. Indeed, we show that *any* force controller that guarantees velocity tracking, including under parametric uncertainty, may be easily incorporated. The construction of our Lyapunov functions is based on the techniques developed in [22] as well as some technical results established in [1, 2].

The rest of the paper is organized as follows. In Section 2 we formulate the control problem and we present our main theoretical findings. In Section 3 we present the proofs of our main results. Simulations that illustrate our theoretical findings are presented in Section 4 and concluding remarks are given in Section 5.

2 Problem formulation and its solution

2.1 Problem statement

Let us consider the following dynamical model of a force-controlled nonholonomic vehicle:

$$\left\{ \begin{array}{rcl} \dot{x} & = & v\cos\theta \\ \dot{y} & = & v\sin\theta \\ \dot{\theta} & = & \omega \end{array} \right. \tag{1}$$

$$\begin{cases} \dot{v} &= f_1(t,v,\omega,z) + g_1(t,v,\omega,z)u_1 \\ \dot{\omega} &= f_2(t,v,\omega,z) + g_2(t,v,\omega,z)u_2 \end{cases} \tag{2}$$

where v and ω denote the forward and angular velocities respectively, the first two elements of $z := [x\ y\ \theta]^\top$ correspond to the Cartesian coordinates of a point on the robot with respect to a fixed reference frame, and θ denotes the robot's orientation with respect to the same frame. The two control inputs are the torques u_1 and u_2. The equations (1) correspond to the kinematics model while (2) correspond to the force-balance equations.

The tracking-control problem consists in making the robot follow a fictitious reference vehicle modeled by

$$\dot{x}_r = v_r \cos\theta_r \tag{3a}$$

$$\dot{y}_r = v_r \sin\theta_r \tag{3b}$$

$$\dot{\theta}_r = \omega_r, \tag{3c}$$

that moves about with reference velocities $v_r(t)$ and $\omega_r(t)$. More precisely, it is desired to steer the differences between the Cartesian coordinates to some values d_x, d_y, and to zero the orientation angles and the velocities of the two robots, that is, the quantities

$$p_\theta := \theta_r - \theta, \quad p_x := x_r - x - d_x, \quad p_y := y_r - y - d_y.$$

The distances d_x, d_y define the position of the robot with respect to the (virtual) leader and are assumed to be constant. Then, as it is customary, we transform the error coordinates $[p_\theta\ p_x\ p_y]$ of the leader robot from the global coordinate frame to local coordinates fixed on the robot, that is, we define

$$\begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_\theta \\ p_x \\ p_y \end{bmatrix}. \tag{4}$$

In these new coordinates, the error kinematics equations become

$$\dot{e}_\theta = \omega_r(t) - \omega \tag{5a}$$

$$\dot{e}_x = \omega e_y - v + v_r(t)\cos(e_\theta) \tag{5b}$$

$$\dot{e}_y = -\omega e_x + v_r(t)\sin(e_\theta). \tag{5c}$$

The complete system also includes Eqs (2).

Generally speaking, the control problem consists in steering the error trajectories $e(t)$, solutions of (5), to zero via the inputs u_1 and u_2 in (2). A natural method consists in designing, first, virtual control laws w^* and v^* so that,

$$\lim_{t\to\infty} e(t) = 0, \qquad e = [e_\theta\ e_x\ e_y]^\top. \tag{6}$$

Then, to design control inputs u_1 and u_2 such that

$$\lim_{t\to\infty} (\tilde{v}, \tilde{w}) = (0,0) \tag{7}$$

where

$$\tilde{v} := v - v^*, \quad \tilde{\omega} := \omega - \omega^*. \tag{8}$$

Depending on the conditions on the reference trajectories v_r and ω_r we identify the following mutually exclusive scenarios:

Tracking scenario (S1): it is assumed that there exist T and $\mu > 0$ such that

$$\int_t^{t+T} \left[|v_r(\tau)|^2 + |\omega_r(\tau)|^2 \right] d\tau \geq \mu \quad \forall t \geq 0. \tag{9}$$

Stabilization scenario (S2): it is assumed that $|v_r(t)| + |\omega_r(t)| \to 0$ and there exists $\beta > 0$ such that, for all $t \geq t_\circ$:

$$\int_{t_\circ}^{t} \left[\, |v_r(\tau)| + |\omega_r(\tau)| \,\right] d\tau < \beta, \quad \forall t \geq t_\circ. \tag{10}$$

2.2 Main result

Under the conditions described above, we design a universal controller that achieves the trajectory tracking objective (6), (7) in either of the two scenarios described above. Our contributions are the following:

- we propose a class of control inputs v^* and ω^* that extends the one proposed in [30] to ensure uniform global asymptotic stability of the origin for (5);
- for the velocity error kinematics in closed loop, we establish integral input-to-state stability with respect to the error velocities $[\tilde{v}, \tilde{\omega}]$;
- for *any* control inputs u_1 and u_2 ensuring that $\tilde{v} \to 0$ and $\tilde{\omega} \to 0$, we establish global attractivity of the origin provided that the error velocities converge sufficiently fast (they are square-integrable).

The control laws that ensure the properties above are:

$$\begin{aligned} v^* &:= v_r(t)\cos(e_\theta) + k_x e_x, & (11)\\ \omega^* &:= \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta) + \rho(t) k_y f(t, e_x, e_y) & (12) \end{aligned}$$

where ϕ is the so-called *sync* function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta},$$

$$\rho(t) := \exp\left(- \int_0^t \left[\, |v_r(\tau)| + |\omega_r(\tau)| \,\right] d\tau \right), \tag{13}$$

and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function defined such that the following hypotheses hold.

A1. There exist a non-decreasing function $\sigma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\sigma_2 > 0$ such that

$$\begin{aligned} \max\left\{ \frac{\partial f}{\partial t}, \frac{\partial f}{\partial e_x}, \frac{\partial f}{\partial e_y} \right\} &\leq \sigma_1(|\,[e_x \; e_y]\,|) & (14)\\ |f(t, e_x, e_y)| &\leq \sigma_2 |\,[e_x \; e_y]\,|. & (15) \end{aligned}$$

A2. For the function

$$f_\circ(t, e_y) := f(t, 0, e_y) \tag{16}$$

we assume that $\partial f_\circ/\partial t$ is uniform $\delta-$persistently exciting with respect to e_y that is,

$$|e_y| \neq 0 \implies \int_t^{t+T} \Big|\frac{\partial f_\circ}{\partial t}(\tau, e_y)\Big| d\tau \geq \mu \quad \forall\, t \geq 0 \tag{17}$$

—*cf.* [17, Definition 3]. Roughly speaking, the purpose of the function f is to excite the e_y–dynamics as long as $|e_y|$ is separated from zero.

The controller (12), which achieves both the tracking and the stabilization control goals, is a weighted sum of the tracking controller of [20],

$$\omega^*_{tra} := \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta),$$

and the stabilization controller in the preliminary version of this paper, [21]. That is,

$$\omega^*_{stab} := \omega_r + k_\theta e_\theta + k_y f(t, e_x, e_y)$$

—*cf.* [23, 30].

The weight function $\rho(t)$ acts as a smoothly-switching supervisor promoting the application of either ω^*_{tra} or ω^*_{stab}, depending on the task scenario **S1** or **S2**. More precisely, from (13) we see that ρ satisfies

$$\dot\rho = -\,\big[\,|v_r(t)| + |\omega_r(t)|\,\big]\rho \tag{18}$$

and $\rho \to 0$ exponentially fast if (9) holds. Hence, the tracking scenario **S1** is promoted. If, instead, (10) holds, the reference velocities converge and $\rho(t) > \exp(-\beta)$. Hence, the action of the stabilization controller is favoured.

Remark 1 The idea of so merging the two controllers for the two scenarios **S1** and **S2** was introduced in [23]. The class of controllers satisfying **A1**–**A2** covers those in [30]; in particular, the function f is not necessarily globally bounded and may depend only on e_y. A more significant contribution with respect to the literature is that we establish uniform global asymptotic stability for (5) in closed-loop with $(v, \omega) = (v^*, \omega^*)$; this is in contrast with [30] and [6] where it is proved that the convergence property (6) holds. In addition, we establish integral ISS of (5) with respect to $[\tilde v, \tilde\omega]$. •

Proposition 1 (Main result) *Consider the system* (5) *with* $v = \tilde v + v^*$, $\omega = \tilde\omega + \omega^*$, *and the virtual inputs* (11) *and* (12). *Let* k_x, k_θ, *and* $k_y > 0$. *Assume that there exist* $\bar\omega_r$, $\bar{\dot\omega}_r$, $\bar v_r$, $\bar{\dot v}_r > 0$ *such that*[1]

$$|\omega_r|_\infty \leq \bar\omega_r, \quad |\dot\omega_r|_\infty \leq \bar{\dot\omega}_r, \quad |v_r|_\infty \leq \bar v_r, \quad |\dot v_r|_\infty \leq \bar{\dot v}_r.$$

Furthermore, assume that ***A1***-***A2*** *hold.*

Then, if either (9) *or* (10) *hold the closed-loop system resulting from* (5), (8), (11), *and* (12) *has the following properties:*

(P1) *if* $\tilde v = \tilde\omega = 0$, *the origin* $\{e = 0\}$ *is uniformly globally asymptotically stable;*

(P2) *the closed-loop system is integral input-to-state stable with respect to* $\eta := [\tilde v\ \tilde\omega]^\top$*;*

(P3) *if* $\eta \to 0$ *and* $\eta \in \mathcal{L}_2$, *then* (6) *holds.* □

The proof is presented in Section 3. Below, we present an example of an adaptive controller that ensures that $\tilde v$, $\tilde\omega \to 0$ for any once continuously differentiable v^*, ω^*.

[1]For a continuous function $t \mapsto \varphi$ we define $|\varphi(t)|_\infty := \sup_{t\geq 0} |\varphi(t)|$.

2.3 Example

As in [5], we consider mobile robots modeled by

$$\dot{z} = J(z)\nu \tag{19a}$$
$$M\dot{\nu} + C(\dot{z})\nu = \tau \tag{19b}$$

where $z := [x\ y\ \theta]$ contains the Cartesian coordinates (x, y) and the orientation θ of the robot, $\tau \in \mathbb{R}^2$ corresponds to the (torque) control input; $\nu := [\nu_1\ \nu_2]$ stands for the angular velocities corresponding to the two robot's wheels, M is the inertia matrix, which is constant, symmetric and positive definite, and $C(\dot{z})$ is the matrix of Coriolis forces, which is skew-symmetric. In addition, we use the coordinate transformation matrix

$$J(z) = \frac{r}{2}\begin{bmatrix} \cos(\theta) & \cos(\theta) \\ \sin(\theta) & \sin(\theta) \\ 1/b & -1/b \end{bmatrix}$$

where r is the radius of either steering wheel and b is the distance from the center of either wheel to the Cartesian point (x, y). The relation between the wheels' velocities, ν, and the robot's velocities in the fixed frame, $\dot{z}$, is given by

$$\begin{bmatrix} v \\ \omega \end{bmatrix} := \frac{r}{2b}\begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} \tag{20}$$

which may be used in (19a) to obtain the familiar model (1).

We assume that the inertia parameters and the constants contained in $C(\dot{z})$ are unknown while r and b are considered to be known. Let $\hat{M}$ and $\hat{C}$ denote, respectively, the estimates of M and C. Furthermore, let $\nu^* := [\nu_1^*\ \nu_2^*]^\top$,

$$\begin{bmatrix} \nu_1^* \\ \nu_2^* \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v^* \\ \omega^* \end{bmatrix}, \tag{21}$$

and let us introduce the certainty-equivalence control law

$$\tau^* := \hat{M}\dot{\nu}^* + \hat{C}(\dot{z})\nu^* - k_d\tilde{\nu}, \quad k_d > 0 \tag{22}$$

where $\tilde{\nu} := \nu - \nu^*$. Then, let us define $\tilde{M} := \hat{M} - M$ and $\tilde{C} := \hat{C} - C$, so

$$\tau^* := M\dot{\nu}^* + C(\dot{z})\nu^* - k_d\tilde{\nu} + \tilde{M}\dot{\nu}^* + \tilde{C}\nu^* \tag{23}$$

and, setting $\tau = \tau^*$ in (19b), we obtain the closed-loop equation

$$M\dot{\tilde{\nu}} + [C(\dot{z}) + k_d I]\tilde{\nu} = \Psi(\dot{z}, \dot{\nu}^*, \nu^*)^\top\tilde{\Theta} \tag{24}$$

where $\Theta \in \mathbb{R}^m$ is a vector of constant (unknown) lumped parameters in M and C, $\hat{\Theta}$ denotes the estimate of Θ, $\tilde{\Theta} := \hat{\Theta} - \Theta$ is the vector of estimation errors, and $\Psi : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{m\times 2}$ is a continuous known function. To obtain (24), we used the property that (19b) is linear in the constant lumped parameters. In addition, we use the passivity-based adaptation law –*cf.* [25],

$$\dot{\hat{\Theta}} = -\gamma\Psi(\dot{z}, \dot{\nu}^*, \nu^*)\tilde{\nu}, \qquad \gamma > 0. \tag{25}$$

Then, a direct computation shows that the total derivative of

$$V(\tilde{\nu}, \tilde{\Theta}) := \frac{1}{2}\big[|\tilde{\nu}|^2 + \frac{1}{\gamma}|\tilde{\Theta}|^2\big]$$

along the trajectories of (24), (25), yields

$$\dot{V}(\tilde{\nu}, \tilde{\Theta}) \le -k_d|\tilde{\nu}|^2.$$

Integrating the latter from 0 to infinity we obtain that $\tilde{\nu} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\tilde{\Theta} \in \mathcal{L}_\infty$. It follows, *e.g.*, from [11, Lemma 3.2.5], that $\tilde{\nu} \to 0$ and, in view of (20),

$$\lim_{t\to\infty} |\tilde{v}(t)| + |\tilde{\omega}(t)| = 0. \tag{26}$$

3 Proof of the main result

For each scenario, **S1** and **S2** we establish uniform global asymptotic stability for the closed-loop kinematics equation (5) restricted to $\eta = 0$ (**P1**). Then, we establish the iISS with respect to η (**P2**) by showing that the closed-loop trajectories are bounded under the condition that η is square integrable —*cf.* [1].

3.1 Under Scenario S1

The proof of Proposition 1 under condition (9) is constructive; we provide a strict Lyapunov function. To that end, we start by observing that the error system (5), (8), (11) and (12) takes the form

$$\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho(t) + B_2(e)\eta, \tag{27}$$

where $\rho(t)$ is defined in (13),

$$\begin{aligned} A_{v_r}(t,e) &:= \begin{bmatrix} -k_\theta & 0 & -v_r(t)k_y\phi(e_\theta) \\ 0 & -k_x & \omega^*(t,e) \\ v_r(t)\phi(e_\theta) & -\omega^*(t,e) & 0 \end{bmatrix}, \\ B_1(t,e) &:= \begin{bmatrix} -k_y f(t,e_x,e_y) \\ k_y f(t,e_x,e_y)e_y \\ -k_y f(t,e_x,e_y)e_x \end{bmatrix}, \quad B_2(e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}. \end{aligned} \tag{28}$$

Writing the closed-loop dynamics as in (27) is convenient to stress that the "nominal" system $\dot{e} = A_{v_r}(t,e)e$ has a familiar structure encountered in model reference adaptive control. Moreover, defining

$$V_1(e) := \frac{1}{2}\left[e_x^2 + e_y^2 + \frac{1}{k_y}e_\theta^2\right], \tag{29}$$

we obtain, along the trajectories of $\dot{e} = A_{v_r}(t,e)e$,

$$\dot{V}_1(e) \le -k_x e_x^2 - k_\theta e_\theta^2.$$

This is a fundamental first step in the construction of a strict Lyapunov function for the "perturbed" system (27).

Now, to establish the proof in the case of scenario **S1**, we follow the steps $1-3$ below:

1) We build a strict Lyapunov function $V(t,e)$ for the nominal system $\dot{e} = A_{v_r}(t,e)e$. This establishes **P1**.

2) We construct a function $W(t,e)$ for the perturbed system $\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho$.

3) We use $W(t,e)$ to prove integral ISS of (27) with respect to η (*i.e.*, **P2**) as well as the boundedness of the trajectories under the assumption that $\eta \in \mathcal{L}_2$. This and the assumption that $\eta \to 0$ implies (6), *i.e.*, **P3**.

Step 1. We establish UGAS for the nominal system

$$\dot{e} = A_{v_r}(t,e)e \tag{30}$$

via Lyapunov's direct method[2]. Let $F_{[3]}$, $S_{[3]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and $P_{[k]} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be smooth polynomials in V_1 with strictly positive and bounded coefficients of degree 3 and k respectively. After [19, Proposition 1], there exists a positive definite radially unbounded function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as

$$\begin{aligned} V(t,e) \;\; := \;\; & P_{[3]}(t,V_1)V_1(e) - \omega_r(t)e_x e_y \\ & + v_r(t)P_{[1]}(t,V_1)e_\theta e_y, \end{aligned} \tag{31}$$

and such that

$$F_{[3]}(V_1) \leq V(t,e) \leq S_{[3]}(V_1), \tag{32}$$

where V_1 is defined in (29), It is showed in [19] that the total derivative of V along the trajectories of (30) satisfies

$$\dot{V}(t,e) \leq -\frac{\mu}{T}V_1(e) - k_x e_x^2 - k_\theta e_\theta^2. \tag{33}$$

Hence uniform global asymptotic stability of the null solution of (30) follows.

Step 2. Now we construct a strict Lyapunov function for the system

$$\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho(t). \tag{34}$$

To that end, we start by "reshaping" the function V defined in (31) to obtain a particular negative bound on its time derivative. Let

$$Z(t,e) := Q_{[3]}(V_1)V_1(e) + V(t,e) \tag{35}$$

where $Q_{[3]}(V_1)$ is a third order polynomial of V_1, with a strictly positive coefficients. Then, in view of (33), the total derivative of Z along the trajectories of (30) satisfies

$$\dot{Z}(t,e) \leq -\frac{\mu}{T}V_1(e) - Q_{[3]}(V_1)\left[k_x e_x^2 + k_\theta e_\theta^2\right]. \tag{36}$$

Next, we recall that in view of (9) $\rho(t)$, which staisfies (18) is uniformly integrable. Therefore, for any $\gamma > 0$, there exists $c > 0$ such that

$$G(t) := exp\left(-\gamma \int_0^t \rho(s)ds\right) \;\geq\; c > 0 \quad \forall t \geq 0 \tag{37}$$

and, since $Z(t,e)$ and $V(t,e)$ are positive definite radially unbounded —see (32) and (35), so is the function

$$W(t,e) \;\; := \;\; G(t)Z(t,e); \tag{38}$$

[2]This proof of uniform stability replaces the corresponding one proposed in [21], which is incorrect.

indeed, we have

$$exp\left(-\gamma\int_0^\infty \rho(s)ds\right) Z(t,e)\le W(t,e)\le Z(t,e).$$

Now, the time derivative of W along trajectories of (34) verifies

$$\begin{aligned}\dot W(t,e) &\le -Y(t,e)+\dot G(t)Z(t,e)\\ &\quad + G(t)\frac{\partial\left(V+Q_{[3]}(V_1)V_1\right)}{\partial e}B_1(t,e)\rho(t) \qquad (39)\\ Y(t,e) &:= G(t)\Big[\frac{\mu}{T}V_1(e)+Q_{[3]}(V_1)\big[k_xe_x^2+k_\theta e_\theta^2\big]\Big]. \qquad (40)\end{aligned}$$

Note that, in view of (37), $Y(t,e)$ is positive definite. We proceed to show that the rest of the terms bounding $\dot W$ are negative semi-definite. To that end, we develop (dropping the arguments of $f(t,e_x,e_y)$)

$$\begin{aligned}\frac{\partial V}{\partial e}B_1(t,e) &= \frac{\partial V}{\partial V_1}\frac{\partial V_1}{\partial e}B_1(t,e)-\omega_rk_yf(\cdot)\left[e_x+e_y^2\right]\\ &\quad - v_rP_{[1]}(t,V_1)k_yf(\cdot)\left[e_\theta e_x+e_y\right] \qquad (41)\end{aligned}$$

and

$$\frac{\partial\left(Q_{[3]}(V_1)V_1\right)}{\partial e}B_1(t,e) = \frac{\partial\left(Q_{[3]}(V_1)V_1\right)}{\partial V_1}\frac{\partial V_1}{\partial e}B_1(t,e), \qquad (42)$$

and we decompose $B_1(t,e)$ into

$$B_1(t,e)=\begin{bmatrix}-k_yf(\cdot)\\0\\0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&k_yf(\cdot)\\0&-k_yf(\cdot)&0\end{bmatrix}e.$$

Then, since

$$\frac{\partial V_1}{\partial e}\begin{bmatrix}0&0&0\\0&0&k_yf(\cdot)\\0&-k_yf(\cdot)&0\end{bmatrix}e=0,$$

it follows that it follows that

$$\frac{\partial V_1}{\partial e}B_1(t,e)=-\frac{\partial V_1}{\partial e_\theta}k_yf(\cdot)=-e_\theta f(\cdot).$$

Thus, using the latter equation, we obtain

$$\begin{aligned}\dot W(t,e)\le& -Y(t,e)+\dot G(t)Z(t,e)\\ &-G(t)\rho(t)f(\cdot)\frac{\partial\left(V+Q_{[3]}(V_1)V_1\right)}{\partial V_1}e_\theta\\ &+v_rf(\cdot)G(t)\rho(t)P_{[1]}(t,V_1)\left[-k_ye_\theta e_x-k_ye_y\right]\\ &+\omega_rG(t)\rho(t)f(\cdot)\left[-k_ye_x+k_ye_y^2\right]. \qquad (43)\end{aligned}$$

In view of (15) and the boundedness of v_r and ω_r, there exists a polynomial $R_{[3]}(V_1)$ with non-negative coefficients, such that

$$R_{[3]}(V_1)V_1\ge -f(\cdot)\frac{\partial\left(V+Q_{[3]}(V_1)V_1\right)}{\partial V_1}e_\theta$$

$$
\begin{aligned}
&+ \omega_r f(\cdot)\left[-k_y e_x + k_y e_y^2\right] \\
&+ v_r f(\cdot) P_{[1]}(t, V_1)\left[-k_y e_\theta e_x - k_y e_y\right].
\end{aligned} \tag{44}
$$

Hence, since $V(t,e) \geq F_{[3]}(V_1)V_1$ —see (32), we obtain

$$
\dot{W} \quad \leq \quad -Y(t,e) + \dot{G}(t)F_{[3]}(V_1)V_1 + G(t)\rho(t)R_{[3]}(V_1)V_1.
$$

On the other hand, in view of (37), $\dot{G}(t) \leq -\gamma G(t)\rho(t)$ for any $\gamma > 0$ and the coefficients of $F_{[3]}(V_1)$ are strictly positive. Therefore, there exists $\gamma > 0$ such that

$$
\gamma F_{[3]}(V_1) \geq R_{[3]}(V_1)
$$

and, consequently, $\dot{W}(t,e) \leq -Y(t,e)$ for all $t \geq 0$ and all $e \in \mathbb{R}^3$. Uniform global asymptotic stability of the null solution of (34) follows.

Step 3. In order to establish iISS with respect to η and boundedness of the closed-loop trajectories subject to $\eta \in \mathcal{L}_2$, we proceed as in [19, Proposition 4]. Let

$$
W_1(t,e) := \ln\left(1 + W(t,e)\right). \tag{45}
$$

The derivative of W_1 along trajectories of (27) satisfies

$$
\dot{W}_1 \quad \leq \quad -G_m \frac{\frac{\mu}{T}V_1(e) + Q_{[3]}\left[k_x e_x^2 + k_\theta e_\theta^2\right]}{1 + W(t,e)} + \frac{\left|\frac{\partial W}{\partial e} B_2 \eta\right|}{1 + W(t,e)} \tag{46}
$$

with $G_m := exp\left(-\gamma \int_0^\infty \rho(t)dt\right)$.

Next, we decompose $B_2(e)\eta$ introduced in (27) into

$$
B_2(e)\eta := B_{21}(\eta) + B_{22}(\eta)e
$$

where

$$
B_{21}(\eta) := \begin{bmatrix} -\tilde{\omega} \\ -\tilde{v} \\ 0 \end{bmatrix}, \quad B_{22}(\eta) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\omega} \\ 0 & -\tilde{\omega} & 0 \end{bmatrix}.
$$

Then, using the fact that $\frac{\partial V_1}{\partial e} B_{22}(\eta)e = 0$, defining

$$
H(e, V_1) := Q_{[3]} + P_{[3]} + \frac{\partial Q_{[3]}}{\partial V_1}V_1 + \frac{\partial P_{[3]}}{\partial V_1}V_1 + \bar{v}_r\,|e_\theta|\,|e_y|\,\frac{\partial P_{[1]}}{\partial V_1},
$$

and

$$
\xi = \begin{bmatrix} \frac{e_\theta}{k_y} \\ e_x \end{bmatrix}, \tag{47}
$$

we obtain

$$
\begin{aligned}
\left|\frac{\partial W}{\partial e} B_2 \eta\right| &\leq H(e, V_1)|\xi||\eta| + \bar{\omega}_r|e_y||\eta| + \bar{v}_r P_{[1]}|e_y||\eta| \\
&\quad + \bar{\omega}_r V_1|\eta| + \bar{v}_r P_{[1]}|e_\theta||e_x||\eta| \\
&\leq H(e, V_1)\left[\frac{1}{2\epsilon}|\xi|^2 + \frac{\epsilon}{2}|\eta|^2\right] + \bar{\omega}_r\left[\frac{1}{2\epsilon}V_1 + \frac{\epsilon}{2}V_1|\eta|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
& + \bar{\omega}_r \left[\frac{1}{2\epsilon} V_1 + \frac{\epsilon}{2} |\eta|^2 \right] + \bar{v}_r \left[\frac{1}{2\epsilon} V_1 + \frac{\epsilon}{2} P_{[1]}^2 |\eta|^2 \right] \\
& + \bar{v}_r P_{[1]} \left[\frac{1}{2\epsilon} V_1 |e_\theta|^2 + \frac{\epsilon}{2} |\eta|^2 \right] \\
\leq & \left[H(e, V_1) + \bar{v}_r P_{[1]} k_y^2 V_1 \right] \frac{1}{2\epsilon} |\xi|^2 + [2\bar{\omega}_r + \bar{v}_r] \frac{1}{2\epsilon} V_1 \\
& + \frac{\epsilon}{2} |\eta|^2 \left[H(e, V_1) + \bar{\omega}_r V_1 + \bar{\omega}_r + \bar{v}_r P_{[1]}^2 + \bar{v}_r P_{[1]} \right].
\end{aligned}
$$

Next, we choose $\epsilon > 0$ such that

$$
\begin{aligned}
\frac{H + \bar{v}_r P_{[1]} k_y^2 V_1}{\epsilon} |\xi|^2 & \leq G_m Q_{[3]} \left[k_x e_x^2 + k_\theta e_\theta^2 \right], \\
\frac{2\bar{\omega}_r + \bar{v}_r}{\epsilon} & \leq G_m \frac{\mu}{T}.
\end{aligned}
$$

Such $\epsilon > 0$ exists because $Q_{[3]}$ is a third order polynomial of V_1 with strictly positive coefficients. So (46) becomes

$$
\begin{aligned}
\dot{W}_1 \quad \leq \quad & -\frac{G_m}{2} \frac{\frac{\mu}{T} V_1(e) + Q_{[3]} \left[k_x e_x^2 + k_\theta e_\theta^2 \right]}{1 + W(t, e)} \\
& + \frac{D_{[3]}(V_1)}{1 + W(t, e)} \frac{\epsilon}{2} |\eta|^2
\end{aligned} \tag{48}
$$

where $D_{[3]}(V_1)$ is a third order polynomial satisfying

$$
H + \bar{\omega}_r V_1 + \bar{\omega}_r + \bar{v}_r P_{[1]}^2 + \bar{v}_r P_{[1]} \leq D_{[3]}.
$$

From the positivity of V, (32), and the definition of W in (38), we have

$$
G_m Q_{[3]}(V_1) V_1 \leq W(t, e) \leq S_{[3]}(V_1) V_1 \tag{49}
$$

hence,

$$
\begin{aligned}
\dot{W}_1 \quad \leq \quad & -\frac{G_m}{2} \frac{\frac{\mu}{T} V_1 + Q_{[3]}(V_1) \left[k_x e_x^2 + k_\theta e_\theta^2 \right]}{1 + S_{[3]}(V_1)} \\
& + \frac{D_{[3]}(V_1)}{1 + G_m Q_{[3]}(V_1)} \frac{\epsilon}{2} |\eta|^2 .
\end{aligned} \tag{50}
$$

This implies the existence of a positive constant $c > 0$ and a positive definite function $e \mapsto \alpha$ such that

$$
\dot{W}_1 \leq -\alpha(e) + c |\eta|^2 . \tag{51}
$$

The result follows from [2].

3.2 Under the scenario S2:

The proof of Proposition 1 under condition (10) relies on arguments for stability of cascaded systems as well as on tools tailored for systems with persistency of excitation; it is inspired by the preliminary version of this paper, [21].

We start by rewriting the closed-loop equations in a convenient form for the analysis under the conditions of Scenario 2. To that end, to compact the notation, let us introduce

$$f_\rho(t, e_x, e_y) := \rho(t) f(t, e_x, e_y) \tag{52}$$

$$\Phi(t, e_\theta, e_x, e_y) = k_\theta e_\theta + k_y f_\rho(t, e_x, e_y) \tag{53}$$

Then, the closed-loop equations become

$$\dot{e} = f_e(t, e) + g(t, e)\eta, \quad \eta = [\tilde{v}\ \tilde{\omega}]^\top, \tag{54}$$

where

$$f_e(t,e) := \begin{bmatrix} -k_\theta e_\theta - k_y f_\rho - k_y v_r \phi(e_\theta) e_y \\ -k_x e_x + \Phi e_y + \big[\omega_r + k_y v_r \phi(e_\theta) e_y\big] e_y \\ -\Phi e_x - \big[\omega_r + k_y v_r \phi(e_\theta) e_y\big] e_x + v_r \sin e_\theta \end{bmatrix},$$

$$g(t,e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}.$$

Following the proof-lines of [27, Lemma 1] for cascaded systems, we establish the following for the system (54):

Claim 1. The solutions are uniformly globally bounded subject to $\eta \in \mathcal{L}_2$,

Claim 2. The origin of $\dot{e} = f_e(t, e)$ is uniformly globally asymptotically stable (*i.e.*, **P1**).

After [1] the last two claims together imply integral ISS with respect to η (*i.e.*, **P2**). Moreover, Claim 1 implies the convergence of the closed-loop trajectories to the origin provided that the input η tends to zero and is square integrable (*i.e.*, **P3**).

3.2.1 Proof of Claim 1

Let

$$W(e) := \ln(1 + V_1(e)), \quad V_1(e) := \frac{1}{2}\big[e_x^2 + e_y^2\big]. \tag{55}$$

The total derivative of V_1 along the trajectories of (54) yields

$$\dot{V}_1(e) \leq -k_x e_x^2 + |e_x||\tilde{v}| + |v_r||\sin(e_\theta)||e_y| \tag{56}$$

hence,

$$\dot{W}(e) \leq \frac{1}{1+V_1}\Big[-\frac{k_x}{2}e_x^2 + |v_r||e_y| + \frac{\tilde{v}^2}{2k_x}\Big] \tag{57}$$

$$\leq \frac{|e_y|}{1+V_1}|v_r| + \frac{1}{2k_x[1+V_1]}\tilde{v}^2. \tag{58}$$

Integrating on both sides of (58) along the trajectories, from 0 to t, and invoking the integrability of v_r and the square integrability of η we see that $W(e(t))$ is bounded for all $t \geq 0$. Boundedness of $e_x(t)$ and $e_y(t)$ follows since W is positive definite and radially unbounded in (e_x, e_y).

Next, we observe that the $\dot{e}_\theta$–equation in (54) corresponds to an exponentially stable system with bounded input $u(t) = -k_y v_r(t)\phi(e_\theta(t))e_y(t) - k_y f_\rho(t, e_x(t), e_y(t)) - \tilde{\omega}(t)$ hence, we also have $e_\theta \in \mathcal{L}_\infty$.

Remark 2 For further development, we also emphasize that proceeding as above from Inequality (57) we conclude that $e_x \in \mathcal{L}_2$, uniformly in the initial conditions. •

3.2.2 Proof of Claim 2

We split the drift of the nominal system $\dot{e} = f_e(t,e)$ into the output injection form:

$$f_e(t,e) = F(t,e) + K(t,e) \tag{59}$$

where

$$F(t,e) := \begin{bmatrix} -k_\theta e_\theta - k_y f_\rho(t, e_x, e_y) \\ -k_x e_x + \Phi(t, e_\theta, e_x, e_y) e_y \\ -\Phi(t, e_\theta, e_x, e_y) e_x \end{bmatrix}$$

and

$$K(t,e) := \begin{bmatrix} -k_y v_r \phi(e_\theta) e_y \\ \big[\omega_r + k_y v_r \phi(e_\theta) e_y\big] e_y \\ -\big[\omega_r + k_y v_r \phi(e_\theta) e_y\big] e_x + v_r \sin e_\theta \end{bmatrix}. \tag{60}$$

Then, to establish UGAS for the origin of $\dot{e} = f_e(t,e)$ we invoke the output-injection statement [28, Proposition 3]. According to the latter, UGAS follows if:

a) there exist: an "output" y, non decreasing functions k_1, k_2, and β: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a class $\mathcal{K}_\infty$ function k, and a positive definite function γ such that, for all $t \geq 0$ and all $e \in \mathbb{R}^3$,

$$|K(t,e)| \leq k_1(|e|) k(|y|) \tag{61}$$

$$|y(t,e)| \leq k_2(|e|) \tag{62}$$

$$\int_\circ^\infty \gamma\big(|y(t)|\big) dt \leq \beta(|e(0)|); \tag{63}$$

b) the origin of $\dot{e} = f_e(t,e)$ is uniformly globally stable;

c) the origin of $\dot{e} = F(t,e)$ is UGAS.

Condition a. Using (60), a direct computation shows that there exists $c > 0$ such that

$$|K(t,e)| \leq c\big[\,|e|^2 + |e|\,\big]\,|\,[v_r\ \omega_r]\,|, \tag{64}$$

so (61) holds with $k_1(s) := c(s^2+s)$, $k(s) := s$, and $y := [v_r\ \omega_r]$. Moreover, (62) and (63) hold with $\gamma(s) = s$, since $[v_r\ \omega_r] \in \mathcal{L}_1$, for a constant functions β and k_2 which, moreover, are independent of the initial state.

Condition b. Uniform global stability is tantamount to uniform stability and uniform global boundedness of the solutions —see [10]. The latter was established already for the closed-loop system under the action of the "perturbation" η hence, it holds all the more in this case, where $\eta = 0$.

In order to establish uniform stability, we use Lyapunov's direct method. Let $R > 0$ be arbitrary but fixed.

We claim that, for the system $\dot{e} = F(t,e)$, there exists a Lyapunov function candidate $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ and positive constants α_1, α_2, and α_3 such that

$$\alpha_1 |e|^2 \leq V(t,e) \leq \alpha_2 |e|^2 \qquad \forall t \geq 0,\ e \in \mathbb{R}^3 \tag{65}$$

$$\left|\frac{\partial V(t,e)}{\partial e}\right| \leq \alpha_3 |e| \qquad \forall t \geq 0,\ e \in \mathbb{R}^3 \tag{66}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e} F(t,e) \leq 0 \qquad \forall t \geq 0,\ e \in B_R \tag{67}$$

where $B_R := \{e \in \mathbb{R}^3 \,:\, |e| \leq R\}$. Furthermore, from (64) it follows that

$$|K(t,e)| \leq c(R+1)\big[|v_r| + |\omega_r|\big]|e| \quad \forall\, t \geq 0,\ e \in B_R.$$

Then, evaluating the time derivative of V along the trajectories of (59), we obtain

$$\begin{aligned}\dot V(t,e) &\le \frac{\partial V(t,e)}{\partial e}K(t,e) \le \alpha_3 c(R+1)\big[|v_r|+|\omega_r|\big]|e|^2\\ &\le \frac{\alpha_3 c[R+1]}{\alpha_1}\big[|v_r|+|\omega_r|\big]V(t,e) \quad \forall\, t\ge 0,\ e\in B_R.\end{aligned}$$

Defining $v(t) := V(t,e(t))$ and invoking the comparison lemma, we conclude that

$$v(t) \le \exp\left(\frac{c\alpha_3[R+1]}{\alpha_1}\int_{t_\circ}^{\infty}\big[\,|v_r(s)|+|\omega_r(s)|\,\big]ds\right)v(t_\circ)$$

for all initial conditions $t_\circ \ge 0$ and $e(t_\circ)$ generating trajectories $e(t)\in B_R$. In view of (10), we obtain

$$|e(t)|^2 \le \frac{\alpha_2}{\alpha_1}\exp\left(\frac{\alpha_3 c[R+1]}{\alpha_1}\beta\right)|e(t_\circ)|^2$$

so uniform stability of (59) follows.

It is left to construct a Lyapunov function candidate V for the system $\dot e = F(t,e)$, that satisfies the conditions (65)-(67). To that end, consider the coordinates

$$e_z = e_\theta + g(t,e_y) \tag{68}$$

where $g : \mathbb{R}_{\ge 0}\times\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ defined by

$$g(t,e_y) := e^{-k_\theta(t-t_\circ)}g(t_\circ,e_y) + \int_{t_\circ}^{t} k_y e^{-k_\theta(t-s)}f(s,0,e_y)ds$$

and, for further development we observe that

$$\frac{\partial g}{\partial t}(t,e_y) = -k_\theta g(t,e_y) + k_y f_\rho(t,0,e_y). \tag{69}$$

Let $g(t_\circ,e_y)$ be such that $|g(t_\circ,e_y)| \le |e_y|$ which implies, using Assumption **A1**, that

$$|g(t,e_y)| \le (1+k_y\sigma_2)\,|e_y|\,. \tag{70}$$

In the new coordinates, we obtain

$$\dot e_z = -k_\theta e_z - \frac{\partial g}{\partial e_y}\Phi e_x - k_y\tilde f(t,e_x,e_y)$$

where $\tilde f(t,e_x,e_y) := f_\rho(t,e_x,e_y) - \tilde f_\rho(t,0,e_y)$. Then, Assumption **A1** implies that for any $R>0$ there exists a positive constant $c_R>0$ such that

$$\max_{e\in B_R}\left\{\sup_{t\ge 0}\left|\tilde f_\rho(t,e_x,e_y)\right|,\quad \sup_{t\ge 0}\left|\frac{\partial g}{\partial e_y}\Phi e_x\right|\right\} \le c_R\,|e_x|\,.$$

Thus, consider the following Lyapunov function candidate

$$V(t,e) := \left[\frac{1}{2}\frac{c_R^2}{k_\theta k_x} + (1+k_y\sigma_2)^2\right]\left[e_x^2+e_y^2\right] + \frac{1}{2}e_z^2 \tag{71}$$

which trivially satisfies (66). Its total time derivative is

$$\begin{aligned}\dot{V}(t,e) =& -\frac{c_R^2}{k_\theta}e_x^2 - e_z\left[k_\theta e_z + \frac{\partial g}{\partial e_y}\Phi e_x + k_y\tilde{f}(t,e_x,e_y)\right]\\ \le& -\frac{c_R^2}{k_\theta}e_x^2 - k_\theta e_z^2 - c_R|e_z||e_x| \le 0, \quad \forall e \in B_R,\end{aligned}$$

so (67) holds. Using (70) and the inequalities

$$e_z^2 \ge e_\theta^2 - 2|e_\theta||g(t,e_y)| + |g(t,e_y)|^2 \ge \frac{1}{2}e_\theta^2 - (1+k_y\sigma_2)^2|e_y|^2.$$

$$e_z^2 \le e_\theta^2 + 2|e_\theta||g(t,e_y)| + |g(t,e_y)|^2 \le 2e_\theta^2 + 2(1+k_y\sigma_2)^2|e_y|^2,$$

we see that the following bounds on V follow

$$V(t,e) \ge \frac{1}{2}\frac{c_R^2}{k_\theta k_x}\left[e_x^2+e_y^2\right] + \frac{1}{4}e_\theta^2$$

$$V(t,e) \le \left[\frac{1}{2}\frac{c_R^2}{k_\theta k_x} + 2(1+k_y\sigma_2)^2\right]\left[e_x^2+e_y^2\right] + e_\theta^2.$$

Thus the inequalities in (65) also hold.

Condition c. Since the solutions are uniformly globally bounded, for any $r>0$, there exists $R>0$ such that $|e(t)| \le R := \{|e| \le R\}$ for all $t \ge t_\circ$, all $e_\circ \in B_r$, and all $t_\circ \ge 0$. It is only left to establish uniform global attractivity. To that end, we observe that the nominal $\dot{e} = F(t,e)$ has the form

$$\dot{e}_\theta = -k_\theta e_\theta - k_y f_\rho(t,e_x,e_y) \tag{72a}$$

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} = \begin{bmatrix}-k_x & \Phi_\theta(t,e_x,e_y)\\ -\Phi_\theta(t,e_x,e_y) & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} \tag{72b}$$

where, for each $e_\theta \in B_R$, we define the smooth *parameterised* function $\Phi_\theta : \mathbb{R}_{\ge 0}\times\mathbb{R}^2 \to \mathbb{R}$ as

$$\Phi_\theta(t,e_x,e_y) := \Phi(t,e_\theta,e_x,e_y).$$

Then, the system (72) may be regarded as a cascaded system —*cf.* [15]. Moreover, the system (72a) is input-to-state stable and the perturbation term $k_y f_\rho(t,e_x(t),e_y(t))$ is uniformly bounded. Therefore, in order to apply a statement for cascaded systems, we must establish that the origin of (72b) is globally asymptotically stable, uniformly in the initial conditions $(t_\circ, e_{x\circ}, e_{y\circ}) \in \mathbb{R}_{\ge 0}\times\mathbb{R}^2$ and in the "parameter" $e_\theta \in B_R$. For this, we invoke [17, Theorem 3] as follows. Since $k_x > 0$ there is only left to show that $\Phi_\theta^\circ(t,e_y)e_y$, where

$$\Phi_\theta^\circ(t,e_y) := \Phi_\theta(t,0,e_y),$$

is uniformly δ-persistently exciting with respect to e_y, uniformly for any $\theta \in B_R$ —*cf.* [17, Definition 3], [16]. Since Φ_θ° is smooth, it suffices to show that for any $|e_y| \ne 0$ and r, there exist T and μ such that

$$|e_y| \ne 0 \implies \int_t^{t+T}\left|\tilde{\Phi}_\theta^\circ(\tau,e_y)\right|d\tau \ge \mu \quad \forall t \ge 0 \tag{73}$$

—see [17, Lemma 1].

Remark 3 In general, μ depends both on e_θ and on e_y, but since $e_\theta \in B_R$ and B_R is compact, by continuity, one can always choose the smallest qualifying μ, for each fixed e_y. Therefore, as in [17], μ may be chosen as a class $\mathcal{K}$ function dependent of $|e_y|$ only. •

Now, we show that (73) holds under Assumption **A2**. To that end, we remark that

$$\Phi_\theta^\circ(t, e_y) = k_\theta e_\theta + k_y \rho(t) f_\circ(t, e_y)$$

–*cf.* Eq. (53), satisfies

$$\dot{\Phi}_\theta^\circ \;\; = \;\; -k_\theta \Phi + k_y \dot{\rho} f_\circ + k_y \rho \frac{\partial f_\circ}{\partial t} - k_y \rho \frac{\partial f_\circ}{\partial e_y} \Phi e_x$$

where we used $\dot{e}_\theta = -\Phi$ and $\dot{e}_y = \Phi e_x$. Therefore, defining

$$K_\Phi(t, e) \;\; := \;\; k_\theta[\Phi_\theta^\circ - \Phi] - k_y \rho \frac{\partial f_\circ}{\partial e_y} \Phi e_x$$

we obtain

$$\dot{\Phi}_\theta^\circ \;\; = \;\; -k_\theta \Phi_\theta^\circ - k_y \rho \frac{\partial f_\circ}{\partial t} + k_y \dot{\rho} f_\circ + K_\Phi(t, e).$$

The latter equation corresponds to that of a linear filter with state Φ_θ° and input

$$\Psi(t, e_y) := -k_y \rho(t) \frac{\partial f_\circ}{\partial t}(t, e_y) + k_y \dot{\rho}(t) f_\circ(t, e_y) + K_\Phi(t, e(t))$$

therefore, after [18, Property 4], Φ_θ° is uniformly δ-PE with respect to e_y, if so is Ψ. Now, from Assumption **A1** and uniform global boundedness of the solutions, for any r there exists $c > 0$ such that

$$\left| k_y \dot{\rho}(t) f_\circ(t, e_y(t)) + K_\Phi(t, e(t)) \right| \leq c(r) \left[\, |e_x(t)| + |\dot{\rho}(t)| \, \right]$$

Therefore, uniform δ-PE with respect to e_y of Ψ follows from Assumption **A2** and the fact that $\dot{\rho}$ and e_x are uniformly square integrable. That $\dot{\rho} \in \mathcal{L}_2$, with a bound uniform in the initial times, follows from (18) because v_r, ω_r, and ρ are bounded and $|v_r| + |\omega_r|$ is uniformly integrable. That e_x is uniformly $\mathcal{L}_2$ follows from (57) —see Remark 2.

This concludes the proof of UGAS for the nominal system $\dot{e} = f_e(t, e)$ hence, **Claim 2.** is proved.

This completes the proof of Proposition 1. ■

4 Simulations

To illustrate our main theoretical results we performed some simulation tests under Simulink™ of Matlab™. Firstly, we define the reference velocities v_r and ω_r as a functions that converge to zero exponentially, according to the scenario **S2** — see Figure 1. Then, in order to illustrate the performance of our controller under the first scenario **S1** we design the leader reference velocities such that their norm is persistently exciting –see Figure 2.

The robot's physical parameters are taken from [8]:

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}, \quad C(\dot{z}) = \begin{bmatrix} 0 & c\omega \\ -c\omega & 0 \end{bmatrix},$$

with $m_1 = 0.6227$, $m_2 = -0.2577$, $c = 0.2025$, $r = 0.15$, and $b = 0.5$. The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$ for the reference vehicle and to $[x(0), y(0), \theta(0)] = [1, 1, 1]$ for the actual robot. The control gains are set to $k_x = k_y = k_\theta = 1$ and the function f which verifies the assumptions **A1** and **A2** is designed as $f(t, e_x, e_y) := p(t)e_y$ with $p(t) = 180\sin(0.5t) + 0.5$, we notice that both $p(t)$ and $\dot{p}$ are persistently exciting signals. Therefore, the conditions (14), (15) and (17) hold.

The desired distance between the leader and the follower robots is obtained by setting the desired orientation offset to zero and defining $[d_{x_r}, d_{y_r}] := [0, 0]$. The control gains (γ, k_d) are taken equal to $(10^{-5}, 10)$ and the initial conditions for the adaptation law (25) are set to $\hat{\Theta}(0) = (\hat{m}_1, \hat{m}_2, \hat{c}) = (0, 0, 0)$.

For the stabilization scenario **S2**, in Figure 3 we depict the system's response in terms of the tracking errors between the leader and the follower; in Figure 4 we depict the torque inputs for the follower robot. The simulation results under the Scenario **S1** are shown in Figures 5, 6.

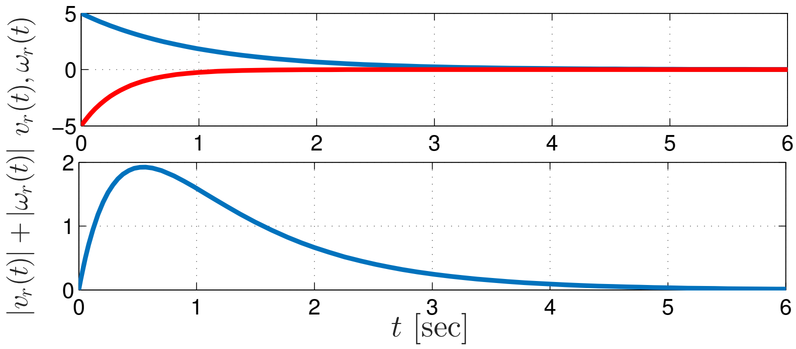

Figure 1: Reference velocities v_r and ω_r for the scenario **S2**

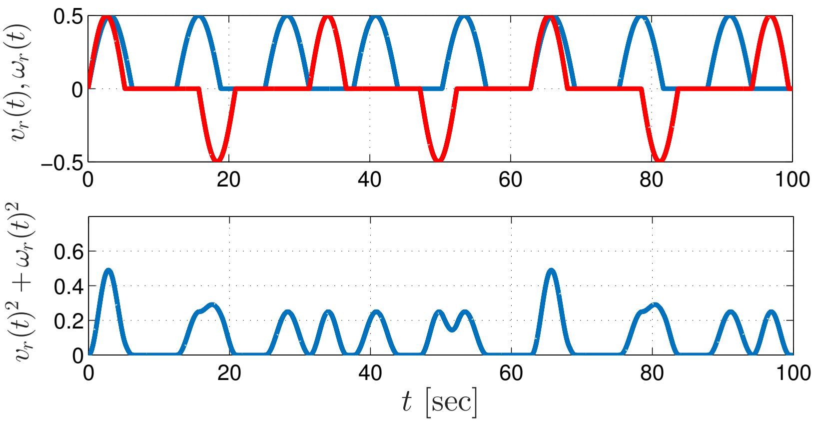

Figure 2: Reference velocities v_r and ω_r for the scenario **S1**

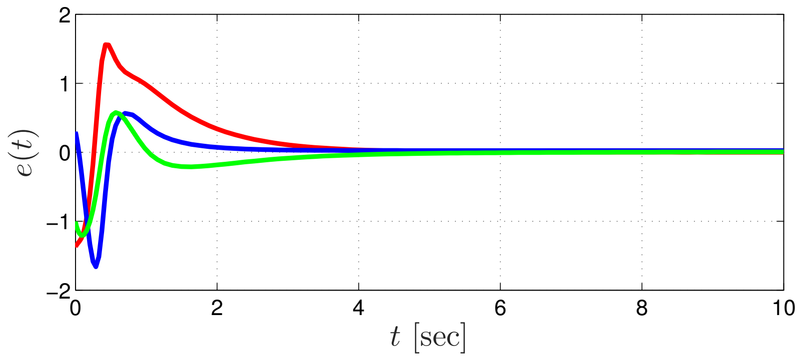

Figure 3: Relative errors (in norm) for each pair leader-follower under **S2**

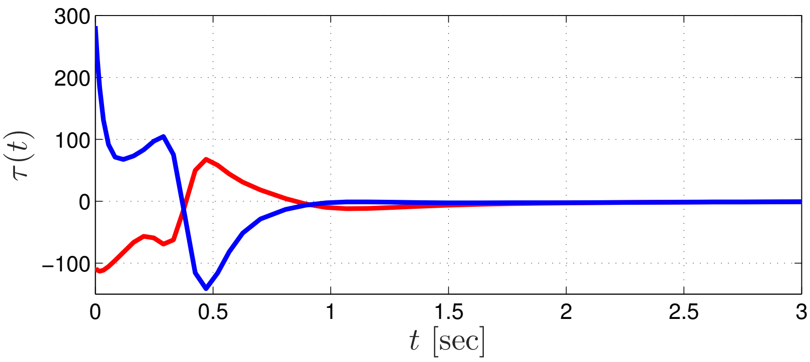

Figure 4: Illustration of the torque inputs for each agent under **S2**

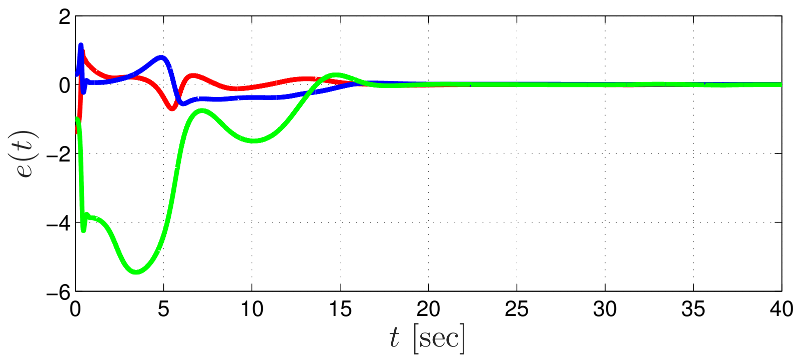

Figure 5: Relative errors (in norm) for each pair leader-follower under **S1**

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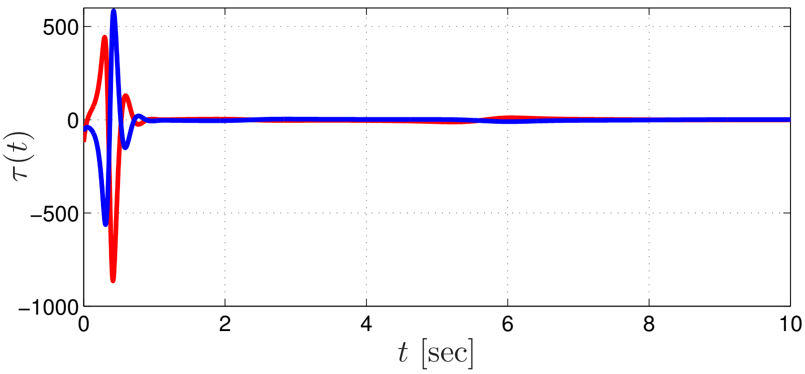

Figure 6: Illustration of the torque inputs for each agent under **S1**

5 Conclusion

We have provided formal proof of uniform global asymptotic stability for the control of nonholonomic vehicles with generic dynamics. Our approach applies to the mutually exclusive scenarios of tracking control and stabilization (parking). The controller, which acts as a smooth supervisor choosing between two smooth time-varying control laws, covers others proposed in the literature. The simplicity and modularity of our design seems promising to broach other scenarios such as control under input constraints.

Our proofs are constructive for the tracking-control scenario and, moreover, the construction of strict Lyapunov functions makes it possible to extend our designs to the cases of output feedback and parametric uncertainty. While an example of the latter is given, the former is under study. Furthermore, current research is being carried out to relax the standing assumption of integrability of the reference velocities in the stabilization scenario, to allow for slowly-converging reference velocities.

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