Long-time behaviour of generalised Zig-Zag process

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Abstract

We study the long-time behaviour of an extension of some recent works on bacterial chemotaxis. Indeed, we consider a d-dimensional piecewise-deterministic Markov process, $d \geq 1$, that models the motion of a bacterium in presence of a chemo-attractant, with parameters depending both on the position and the velocity of the bacterium. We show that under some good assumptions on the parameters of the model, the process converges exponentially fast towards its invariant measure.

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Contents

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1 Introduction

Piecewise-deterministic Markov processes (PDMPs) have been introduced by Davis, [9], to distinguish these particular processes from diffusive processes. They are the subject of much current work in various domains, since they are a simple alternative to diffusions to model stochastic systems (see [2] for an overview of recent results on PDMPs). In this paper, we study a PDMP that comes from biological modeling for the motion of flagellated bacteria which are in presence of a chemo-attractant. The motion of such a bacterium has been described as run-and-tumble, which means that the bacterium alternates sequences of linear runs with periods of random reorientation (tumbling). The tumble-periods being typically much shorter than the run-periods, we can suppose them to be instantaneous. Moreover, the presence of a chemo-attractant in the environment of the bacterium influences the rate at which the bacterium changes its direction, and also the new direction it takes (see [21] for more details on the model).
More precisely, we consider the PDMP $((X_t, V_t))_{t\geq 0}$ with values in $E = \mathbb{R}^d \times \mathcal{B}(1)$, where $\mathcal{B}(1) = \left\{v \in \mathbb{R}^d, |v| \leq 1\right\}$ is the Euclidian ball of radius 1, with infinitesimal generator given by

$$Lh(x,v) = v \cdot \nabla_x h(x,v) + \lambda(x,v) \int_{\mathcal{B}(1)} (h(x,v') - h(x,v))\, Q(x,v,\mathrm{d}v'), \tag{1}$$

where $Q(x,v,\cdot)$ is a probability kernel on $\mathcal{B}(1)$. We call this process "generalised Zig-Zag process" since it is a generalisation of the process called Zig-Zag process in [4] and [3], in the sense that we do not any more have a velocity with values in $\{-1,1\}^d$, but in $\mathcal{B}(1)$.
In our model, X_t represents the position of a bacterium at instant t, and V_t its velocity. The form of the generator indicates that the first component X is continuous and evolves according to $\frac{\mathrm{d}X_t}{\mathrm{d}t} = V_t$, whereas V is constant during a random time, and jumps according to the kernel Q at rate $\lambda(x,v)$ when $(X_t, V_t) = (x,v)$. The fact that the motion of the bacterium is biaised by the presence of a chemo-attractant will be taken into account in the assumptions we will make on the jump rate λ and the velocity kernel Q.
In this paper, we are interested in the long-time behaviour of the process driven by (1) under some good assumptions.

This process, driven by (1), has already been studied in different ways and under different assumptions. Let first mention some works on the process in dimension 1 with a modeling point of view: in [13], Fontbona, Guérin and Malrieu have shown the exponential ergodicity of the telegraph process, namely the process with a jump rate equal to $a\mathbf{1}_{xv\leq 0} + b\mathbf{1}_{xv>0}$ with $b > a > 0$, and the velocity taking its values in $\{-1,+1\}$. For this, they give an exact description of the excursions of the process away from the origin and give an explicit construction of a coalescent coupling for both velocity and position. In [14] and [4], the previous results have been extended by considering a more general jump rate, depending on the position and the velocity of the bacterium. Calvez, Raoul and Schmeiser have shown in [6] by an analytical method the exponential ergodicity of the process driven by (1) in the particular case where the kernel Q is the uniform kernel on $[-1,1]$, and under similar assumptions to the ones introduced here (see Section 5).
Furthermore, there exists also results for this process in high dimension. We can for instance cite [3], in which Bierkens, Fearnhead and Roberts study the Ziz-Zag process, that is the process with values in $\mathbb{R}^d \times \{-1,+1\}^d$. They prove its ergodicity in the case where it can be seen as a product of independant one-dimensional Zig-Zag processes. In [5], [10] and [19], the authors are

interested in the ergodicity of the bouncy particle sampler, with values in $\mathbb{R}^d \times \mathbb{R}^d$ or $\mathbb{R}^d \times \mathcal{S}^{d-1}$, where $\mathcal{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$. This PDMP is a particular case of the process driven by (1): for instance in [19], the jump rate is given by $\lambda(x,v) = (v \cdot \nabla_x U(x))_+$, where U is a potential, and at each jump, the velocity is reflected according to optical laws on the level set of U it has reached.
Finally, let us mention that the study of this kind of processes has an interest not only for biological modelling, but also for simulating a target distribution.Let us refer to [4, 3, 5, 10, 19, 12], where the authors want to sample a distribution with a density proportional to e^{-U}, where U is a potential on $\mathbb{R}^d$. For a jump rate λ and a jump kernel Q well chosen, the PDMP converges towards the targeted distribution. An example is the bouncy particle sampler from [19] described in the previous paragraph. The estimation of the speed of convergence to equilibrium of these processes gives then informations on the efficiency of the corresponding algorithms to sample the target distribution. An interest of considering PDMPs to catch a distribution is the irreversibility of PDMPs. Indeed, while many Markov chain Monte Carlo (MCMC) methods rely on realisations from a discrete reversible ergodic Markov chain, it has been observed that non-reversibility often implies favourable convergence properties (see for instance [15, 16]). Moreover, PDMPs have the advantage to be easy to sample, and can even in some cases being simulated without discretisation error.

Framework

Let us now introduce the framework of the paper. Denoting by $x \cdot v$ the scalar product of $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$, and $|x|$ the Euclidian norm of the vector x, the assumptions made on the model are the following:

$(\mathcal{H}_1)$: There exists $\lambda_{\min} > 0$ such that for all $(x,v) \in E$, $\lambda(x,v) \geq \lambda_{\min}$;

$(\mathcal{H}_2)$: The quantity $\lambda_{\max} = \sup_{\{(x,v)\in E \,:\, x\cdot v \leq 0\}} \lambda(x,v)$ is finite;

$(\mathcal{H}_3)$: There exists $p > 0, \theta_0 \in (0,1]$ such that for all $(x,v) \in E$, $\int_{\{v' \in \mathcal{V} \,:\, \frac{x\cdot v'}{|x|} \leq -\theta_0\}} Q(x,v,\mathrm{d}v') \geq p$;

$(\mathcal{H}_4)$: There exists $\theta_* \in \left[0, (p\theta_0)^2 \frac{\lambda_{\min}}{\lambda_{\max}}\right)$, $\beta > \frac{1}{(p\theta_0)^2}$ and $\Delta > 0$ such that

$$\inf_{\left\{\frac{x\cdot v}{|x|} \geq \theta_*, |x| \geq \Delta\right\}} \lambda(x,v) \geq \beta \lambda_{\max}.$$

Assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ mean that the jump rate is uniformly bounded by below, and that it is bounded from above when the bacterium is moving towards 0, where the chemo-attractant is assumed to be.
Assumptions $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ reflect the attraction of the bacterium to the origin. Indeed, in Assumption $(\mathcal{H}_3)$, we suppose that wherever the bacterium is, and whatever its velocity is, when a jump happens, the bacterium always has a chance to go towards the origin. Moreover, in Assumption $(\mathcal{H}_4)$, we assume a kind of monotony of the jump rate. Roughly speaking, we suppose that when the bacterium is far from the origin, and goes in a too bad direction, its jump rate is strictly bigger than $\lambda_{\max}$, which is the maximum of the jump rate when the bacterium is coming closer the origin.

Under these assumptions, we can show the exponential ergodicity (see Section 2 for the definition) of the process (X,V) driven by (1).

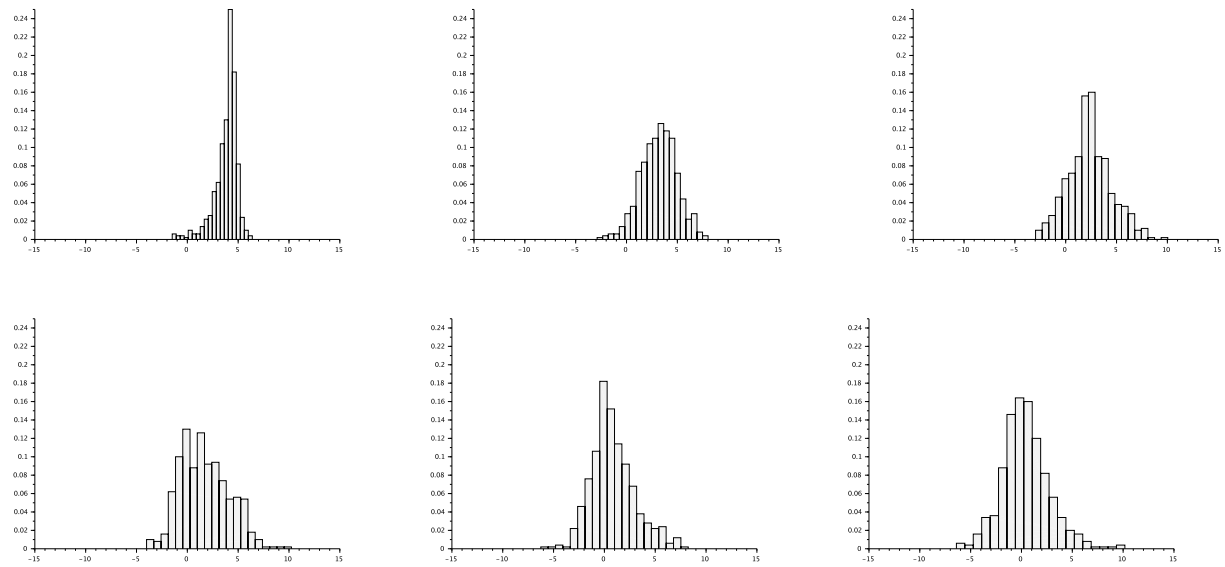

Figure 1: Empirical distribution of X_t starting at $(5,-1)$ for $t \in \{0.5, 5, 10, 17, 30, 40\}$ with $\lambda(x,v) = \mathbf{1}_{xv<0} + 2\mathbf{1}_{xv\geq 0}$ and $Q(x,v,\mathrm{d}v') = \frac{1}{2}\mathbf{1}_{[-1,1]}(v')\mathrm{d}v'$.

Theorem 1.1. *Let $(X_t, V_t)_{t\geq 0}$ be a PDMP on $E = \mathbb{R}^d \times \mathcal{B}(1)$ with infinitesimal generator given by (1).*
If λ and Q satisfy Assumption $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, then the process is exponentially ergodic.

In Figure 1, we can observe the convergence of the empirical law of $(X_t)_{t\geq 0}$ in the case $\lambda(x,v) = \mathbf{1}_{xv<0} + 2\mathbf{1}_{xv\geq 0}$ and $Q(x,v,\mathrm{d}v') = \frac{1}{2}\mathbf{1}_{[-1,1]}(v')\mathrm{d}v'$.

The approach we will carry out in Section 3 to prove Theorem 1.1 is the method of Meyn and Tweedie ([11, 17, 18]), by showing the existence of a Lyapunov function for the process, and the existence of petite sets. In Section 2, we thus first briefly recall the generalities about ergodicity of Markov processes.
Then, we will study in Section 4 the existence of exponential moments for the invariant measure. For this purpose, the method we use is based on a convergence result of switching processes, a notion we will therefore introduce.
Finally, we will go back to the particular case of the dimension 1 in Section 5, in which we study our process with a different approach than in Section 3. In particular, in this section, we will assume that the jump kernel does not depend on the position of the bacterium. Thus, only the jump rate will favour the return towards the origin, and therefore the ergodicity of the process. The interest of this alternative approach in dimension 1 is that it will give more quantitative results on the exponential convergence of the process, and the exponential moments of the invariant measure, whereas the results obtained for the d-dimensional process by the method of Meyn and Tweedie are only qualitative. Moreover, the results obtained in this section can be linked with the previous works [6], [13] and [14], since they are in fact a generalisation of these latter. However, the disadvantage of this approach is that it can not be adapted in higher dimension.

2 Preliminaries

2.1 About ergodicity

In this paper, we will study the convergence of our process with the total variation distance. Let us recall its definition. Let ν and $\widetilde{\nu}$ be two probability measures on a measurable space E. We say that a probability measure on $E \times E$ is a coupling of ν and $\widetilde{\nu}$ if its marginals are ν and $\widetilde{\nu}$. Denoting by $\Gamma(\nu, \widetilde{\nu})$ the set of all the couplings of ν and $\widetilde{\nu}$, we say that two random variables X and $\widetilde{X}$ satisfy $(X, \widetilde{X}) \in \Gamma(\nu, \widetilde{\nu})$ if ν and $\widetilde{\nu}$ are the respective laws of X and $\widetilde{X}$. The total variation distance between these two probability measures is then defined by

$$\|\nu - \widetilde{\nu}\|_{TV} = \inf_{(X,\widetilde{X})\in\Gamma(\nu,\widetilde{\nu})} \mathbb{P}(X \neq \widetilde{X}). \tag{2}$$

For other definitions of the total variation distance and its properties, see for instance [20].

Let us now introduce some usefull results to study the convergence of a Markov process towards its invariant measure (see [17, 18, 11]).
Let $(Y_t)_{t\geq 0}$ be a Markov process on the state space E, and denote by $\mathtt{P}$ its semi-group and $\mathtt{L}$ its infinitesimal generator. We say that the Markov process Y is exponentially ergodic if there exists a probability measure π, a function $M : E \longrightarrow \mathbb{R}_+$ and a constant $0 < \rho < 1$ such that

$$\|\mathtt{P}_t(x, \cdot) - \pi\|_{TV} \leq M(x)\rho^t, \text{ for all } t \geq 0, \tag{3}$$

where $\mathtt{P}_t(x, \cdot) = \mathbb{P}_x(Y_t \in \cdot)$.
A set K is said to be petite for the process Y if there exists a probability measure ν on $\mathbb{R}_+$ and a nontrivial measure μ on E such that, for all $x \in K$,

$$\int_0^\infty \mathtt{P}_t(x, \cdot)\nu(\mathrm{d}t) \geq \mu(\cdot). \tag{4}$$

This notion is weaker than the notion of small sets: K is said to be a small set for the process Y if there exists $t_0 > 0$ and a non trivial measure μ on E such that, for all $x \in K$,

$$\mathtt{P}_{t_0}(x, \cdot) \geq \mu(\cdot). \tag{5}$$

Let $K \subset E$ be a compact set, and let $H : E \longrightarrow \mathbb{R}$ be a function. We say that H is a Lyapunov function (associated to the set K) for the process Y if $H(x) \geq 1$ for all $x \in E$ and if there exists some constants $\alpha > 0$ and $\beta \geq 0$ such that for all $x \in E$,

$$\mathtt{L}H(x) \leq -\alpha H(x) + \beta \mathbf{1}_K(x). \tag{6}$$

Finally, we recall that the Markov process Y is called φ-irreducible if there exists a σ-finite measure φ such that for all measurable set A with $\varphi(A) > 0$ we have, for all $x \in E$, $\mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{Y_t\in A}\mathrm{d}t\right] > 0$. In that case, there exists a maximal irreducibility measure ψ such that for any other irreducibility measure ν, ν is absolutely continuous with respect to ψ. We then write $\mathcal{B}^+(E) = \{A \subset E \text{ measurable}, \psi(A) > 0\}$. The process Y is said to be aperiodic if for some small set $C \in \mathcal{B}^+(E)$, there exists $T > 0$ such that for all $t \geq T$ and all $x \in C$ we have $\mathtt{P}_t(x, C) > 0$.

We can now recall the main result we will use in Section 3.2 to prove Theorem 1.1.

Theorem 2.1. *([17]) Let Y be an irreducible and aperiodic Markov process. If there exists a petite set K for the process Y and a Lyapunov function associated to this set K (i.e. (4) and (6) hold) , then the process Y is exponentially ergodic.*

Remark 2.2. *This theorem is a qualitative result and does not give a quantitative speed of convergence. However, in Section 5, we will use an alternative way to show the exponential ergodicity of our process in dimension 1, without needing to find a Lyapunov function, and this would give more quantitative results.*

2.2 Description of the process

Let us now describe the dynamic of the process.
The variables $0 = T_0, T_1, T_2, \dots$ refer to the successive jumping times of the process, and for $n \geq 1$ we denote by τ_n the inter-jump time: $\tau_n = T_n - T_{n-1}$.
In order to make the paper easier to read, we note V_i the velocity on the time interval $[T_i, T_{i+1})$, instead of V_{T_i}.
Finally, we denote by $\overline{F}((\cdot, x, v)$ the survival function of T_1 with initial data $(X_0, V_0) = (x, v)$: $\overline{F}(\cdot, x, v) = \mathbb{P}_{x,v}(T_1 > \cdot)$.
The time T_1 satisfies, when $(X_0, V_0) = (x, v)$:

$$T_1 = \inf\left\{t \geq 0, \int_0^t \lambda(X_s, V_s)\mathrm{d}s \geq E\right\} = \inf\left\{t \geq 0, \int_0^t \lambda(x + vs, v)\mathrm{d}s \geq E\right\}$$

where E is an exponential variable with unit mean, because the process is deterministic between jump times. We then get:

$$\begin{aligned}\overline{F}(t, x, v) &= \mathbb{P}_{x,v}(T_1 > t) \\ &= \mathbb{P}_{x,v}\left(\int_0^t \lambda(x + vs, v)\mathrm{d}s \leq E\right) \\ &= \exp\left(-\int_0^t \lambda(x + vs, v)\mathrm{d}s\right).\end{aligned}$$

The conditional distribution of the inter-jump times is then given by, for all $n \geq 0$:

$$\mathbb{P}\left(\tau_{n+1} \geq t \,|X_{T_n}, V_n\right) = \exp\left(-\int_0^t \lambda(X_{T_n} + V_n s, V_n)\mathrm{d}s\right).$$

3 Main result

In this section we prove our main result, Theorem 1.1, using Theorem 2.1. We first need to find a Lyapunov function for the process.

3.1 A Lyapunov function

Let us introduce some constants which will appear in the definition of our Lyapunov function. Thanks to Assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, the following quantities are well defined:

$$\theta_1 \in \left(\max\left\{ \frac{\theta_* \lambda_{\max}}{p\theta_0 \lambda_{\min}}; \frac{1}{p\theta_0 \beta} \right\} \ , \ p\theta_0 \right);$$
$$\alpha \in \left(0 \ , \ \min\left\{ \frac{p\theta_0 \lambda_{\min}}{\theta_*} - \frac{\lambda_{\max}}{\theta_1}; p\theta_0 \beta \lambda_{\max} - \frac{\lambda_{\max}}{\theta_1} \right\} \right);$$
$$a \in \left(1 + \frac{\lambda_{\max}}{\alpha\theta_1} \ , \ \min\left\{ \frac{p\theta_0 \lambda_{\min}}{\alpha\theta_*}; \frac{p\theta_0 \beta \lambda_{\max}}{\alpha} \right\} \right).$$

We then consider the function H defined on $E = \mathbb{R}^d \times \mathcal{B}(1)$ by

$$H(x,v) = \mathrm{e}^{\alpha|x|} \left(a + \varphi\left(\frac{x \cdot v}{|x|} \right) \right),$$

where φ is a non-decreasing function of class C^1 on $[-1,1]$, with $\varphi(\theta) = \theta$ if $\theta \in [-1, -\theta_1]$, and $\varphi(\theta) = 0$ if $\theta \in [0,1]$. We introduce m the supremum norm of the derivative of φ, that is:

$$m = \sup_{\theta \in [-1,1]} |\varphi'(\theta)|.$$

Then, we have the following result:

Proposition 3.1. *There exist some constants $R > 0$, $\eta > 0$ such that for all $(x,v) \in (\mathbb{R}^d \backslash \mathcal{B}(R)) \times \mathcal{B}(1)$,*

$$LH(x,v) \leq -\eta H(x,v).$$

The function H is then a Lyapunov function associated to the set $\mathcal{B}(R) \times \mathcal{B}(1)$ for the process (X,V) driven by the generator L.

Proof. For $(x,v) \in E$ we have:

$$LH(x,v) = \mathrm{e}^{\alpha|x|} \left(A_1 + A_2 + A_3 \right) \tag{7}$$

with

$$A_1 = \alpha \frac{x \cdot v}{|x|} \left(a + \varphi\left(\frac{x \cdot v}{|x|} \right) \right)$$
$$A_2 = \varphi'\left(\frac{x \cdot v}{|x|} \right) \frac{1}{|x|} \left(|v|^2 - \left(\frac{x \cdot v}{|x|} \right)^2 \right)$$
$$A_3 = \lambda(x,v) \int_{\mathcal{B}(1)} \left(\varphi\left(\frac{x \cdot v'}{|x|} \right) - \varphi\left(\frac{x \cdot v}{|x|} \right) \right) Q(x,v,\mathrm{d}v').$$

Let first remark that since the derivative of φ is bounded by m, we always have

$$A_2 \leq \frac{m}{|x|}.$$

Moreover, thanks to Assumption $(\mathcal{H}_3)$ and the definition of the function φ, we can bound A_3 as follows:

$$\begin{aligned} A_3 &\leq \lambda(x,v)\left[\int_{\{v'\in\mathcal{B}(1)/\frac{x\cdot v'}{|x|}\leq-\theta_0\}}\varphi\left(\frac{x\cdot v'}{|x|}\right)Q(x,v,\mathrm{d}v') - \varphi\left(\frac{x\cdot v}{|x|}\right)\right] \\ &\leq \lambda(x,v)\left[-\theta_0\int_{\{v'\in\mathcal{B}(1)/\frac{x\cdot v'}{|x|}\leq-\theta_0\}}Q(x,v,\mathrm{d}v') - \varphi\left(\frac{x\cdot v}{|x|}\right)\right] \\ &\leq -\lambda(x,v)\left[p\theta_0 + \varphi\left(\frac{x\cdot v}{|x|}\right)\right]. \end{aligned}$$

We can now study Equation (7) depending on the different values taken by $\frac{x\cdot v}{|x|}$.

- If $\frac{x\cdot v}{|x|}\in[-1,-\theta_1]$:

$$LH(x,v) \leq \mathrm{e}^{\alpha|x|}\left[-\alpha\theta_1(a-1) + \frac{m}{|x|} + \lambda_{\max}\right].$$

- If $\frac{x\cdot v}{|x|}\in(-\theta_1,0)$:

$$\begin{aligned} LH(x,v) &\leq \mathrm{e}^{\alpha|x|}\left[\frac{m}{|x|} - \lambda(x,v)\left(p\theta_0-\theta_1\right)\right] \\ &\leq \mathrm{e}^{|x|}\left[\frac{m}{|x|} - \lambda_{\min}\left(p\theta_0-\theta_1\right)\right]. \end{aligned}$$

- If $\frac{x\cdot v}{|x|}\in[0,\theta_*]$:

$$LH(x,v) \leq \mathrm{e}^{\alpha|x|}\left[\alpha\theta_* a + \frac{m}{|x|} - \lambda_{\min}p\theta_0\right].$$

- If $\frac{x\cdot v}{|x|}\in(\theta_*,1]$, and if $|x|\geq\Delta$, with Δ defined in Assumption $(\mathcal{H}_4)$:

$$LH(x,v) \leq \mathrm{e}^{\alpha|x|}\left[\alpha a + \frac{m}{|x|} - p\theta_0\beta\lambda_{\max}\right].$$

Let us now define

$$R = \max\left\{\frac{m}{\alpha\theta_1(a-1)-\lambda_{\max}}; \frac{m}{\lambda_{\min}\left(p\theta_0-\theta_1\right)}; \frac{m}{p\theta_0\lambda_{\min}-\alpha\theta_* a}; \Delta; \frac{m}{p\theta_0\beta\lambda_{\max}-\alpha a}\right\}.$$

Thanks to the assumptions made on each parameter, the constant R is well defined and finite. Therefore, the previous calculations give the existence of a constant $\eta>0$ such that for all $(x,v)\in(\mathbb{R}^d\backslash\mathcal{B}(R))\times\mathcal{B}(1)$:

$$LH(x,v) \leq -\eta\mathrm{e}^{\alpha|x|}.$$

Finally, the function $(y,w)\longmapsto a+\varphi\left(\frac{y\cdot w}{|y|}\right)$ being bounded from above by a, we get:

$$LH(x,v) \leq -\eta a H(x,v),$$

which ends the proof. □

3.2 Proof of Theorem 1.1

Since we have already found a Lyapunov function, we still have to review three points in order to use Theorem 2.1 : the irreducibility and the aperiodicity of the process, and the existence of petite sets. These are the object of the following proposition.

Proposition 3.2. *All compact sets of the form $K \times \mathcal{B}(1)$, with K a compact set of $\mathbb{R}^d$, are petite for the process (X, V).*
Moreover, the process is irreducible and aperiodic.

Proof. We do not detail the proof since we just have follow the ideas of the proof of Lemma 2 in [10]. It is indeed enough to show that for all $M > 0$, for all $(x, v) \in \mathcal{B}(M) \times \mathcal{B}(1)$, for all positive bounded function $\varphi : E \to \mathbb{R}$, there exist a constant $C > 0$ and a compact set $A \subset \mathbb{R}^d \times \mathcal{B}(1)$, both independent of M and φ, such that:

$$\int_0^\infty \mathrm{e}^{-t}\mathbb{E}_{(x,v)}\left[\varphi(X_t, V_t)\right] \mathrm{d}t \geq C \int\int_A \varphi(y, w)\mathrm{d}y\mathrm{d}w.$$

to deduce the first result of the proposition. Let us mention that the proof is based only on Assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.
Then, the irreducibility of the process follows easily, and we refer once more to [10], Lemma 3, for the aperiodicity. □

Remark 3.3. *Let us mention different proofs of this result in some particular cases.*
In dimension 1, a proof has been given in [4] for the process with a velocity equal to -1 or $+1$, with the same main ideas that in [10] and [19].
For the process in dimension 2, if the jump kernel has a density with respect to the Lebesgue measure which is bounded from below by a strictly positive constant, and if the jump rate is bounded, the previous proposition can be proved by geometric considerations, as in the proof of Lemma 4.5 in [7].

Proof of Theorem 1.1. By Proposition 3.1 and 3.2, all conditions of 2.1 are satisfied, so that the stated result follows. □

4 Exponential moments for the invariant measure

In this section, we want to study the existence of exponential moments for the unique invariant probabilty measure of the process (X, V), that we denote by π, whose existence follows from Theorem 1.1.

Let first say that if the lengths of the consecutive excursions away from 0 were independent and identically distributed, the process (X, V) would be a regenerative process, and we could then apply standard results on regenerative processes (see [1] or [8] for instance) to collect some informations on the invariant measure of the process. However, the excursions away from 0 do not satisfy this assumption in our case, because of the dependence in speed between two consecutive excursions. We thus introduce the notion of switching process (see [8]), which is adapted to our process.

Let $\zeta = (\zeta_t)_{t\geq 0}$ be a càdlàg process with values in some state space F.
Let $M(\zeta; dz, dt)$ be a stochastic kernel on F which depends on ζ and such that we have in some sense the good measurabilities. We suppose moreover that $M(\zeta; F \times \{0\}) = 0$.

Definition 4.1. A switching process (SP) on F associated with ζ and M, with initial state $y_0 \in F$ is a process $(\Psi_t)_{t\geq 0}$ that can be constructed as follows :

1. let $\zeta^{(1)}$ be a process such that $\mathcal{L}(\zeta^{(1)}) = \mathcal{L}(\zeta|\zeta_0 = y_0)$;
2. let (Y_1, T_1) whose law conditionally to $\zeta^{(1)}$ is $M(\zeta^{(1)}; dz, dt)$;
3. we define $\Psi_t = \zeta_t^{(1)}$ for $t < T_1$ and $\Psi_{T_1} = Y_1$;

we now suppose constructed $\zeta^{(1)}, \ldots, \zeta^{(n)}, Y_1, T_1, \ldots, Y_n, T_n$ $(n \geq 1)$;

5. let $\zeta^{(n+1)}$ such that $\mathcal{L}(\zeta^{(n+1)}|\zeta^{(1)}, \ldots, \zeta^{(n)}, Y_1, T_1, \ldots, Y_n, T_n) = \mathcal{L}(\zeta|\zeta_0 = Y_n)$;
6. let (Y_{n+1}, T_{n+1}) such that the law of $(Y_{n+1}, T_{n+1} - T_n)$ conditionally to $(\zeta^{(1)}, \ldots, \zeta^{(n)}, \zeta^{(n+1)}, Y_1, T_1, \ldots, Y_n, T_n)$ is $M(\zeta^{(n+1)}; dz, dt)$;
7. we define $\Psi_t = \zeta_t^{(n+1)}$ if $T_n \leq t < T_{n+1}$ and $\Psi_{T_{n+1}} = Y_{n+1}$.

The times $(T_n)_{n\geq 1}$ are called the jump times of the SP Ψ. We define $T_0 = 0$ and $Y_0 = y_0$ and suppose that $\lim_{n\to+\infty} T_n = +\infty$.

Remark 4.2. *A PDMP is clearly a SP, with ζ a deterministic process, T_n the jump times of the PDMP, and Y_n the positions of the PDMP at times T_n.*

Before stating the result we will use in this part, let us first briefly speak about Harris-recurrence and non-arithmetic process (see for instance [8]). The Markov chain Y with state space F is called Harris-recurrent if there exists a measurable set $A \subset F$, $c > 0$, $m \geq 1$, and a distribution φ such that

1. for all $y \in F$, $\mathbb{P}_y(\tau_A < \infty) = 1$, where $\tau_A = \inf\{n \geq 0, Y_n \in A\}$;
2. for all $y \in A$, $\mathbb{P}_y(Y_m \in \cdot) \geq c\varphi(\cdot)$.

Then, we recall that the process $(Y_n, T_n)_{n\geq 0}$ is non-arithmetic if the laws of the variables $T_n - T_{n-1}$, $n \geq 1$ have a part which is absolutely continuous with respect to the Lebesgue measure (see [8] for the definition of an arithmetic process).

We have the following result concerning the convergence of SP:

Theorem 4.3 ([8])**.** *Let $(\Psi_t)_{t\geq 0}$ be a SP as is the definition.*
We suppose that (Y, T) is non-arithmetic and that Y is Harris-recurrent, and let ν be an invariant measure for this chain.
Let $f : F \longrightarrow \mathbb{R}$ be a measurable positive function such that $(z, t) \longmapsto \mathbb{E}[f(\Psi_t)|Y_0 = z]$ is bounded on $F \times [0, t]$ for all t. We define $g : F \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by $g(z, t) = \mathbb{E}[f(\Psi_t)\mathbf{1}_{t<T_1}|Y_0 = z]$.
We suppose :

1. *for ν-almost all $z \in F$, $t \mapsto g(z, t)$ is almost everywhere continuous with respect to the Lebesgue measure;*
2. *there exists $\Delta > 0$ such that*

$$\int_F \sum_{n\in\mathbb{N}} \sup_{n\Delta\leq t<(n+1)\Delta} |g(z, t)|\nu(dz) < \infty.$$

Then for ν-almost all $y \in F$:

$$\mathbb{E}_y\left[f(\Psi_t)\right] \underset{t\to\infty}{\longrightarrow} \frac{\int_F \mathbb{E}\left[\int_0^{T_1} f(\zeta_s)ds|\zeta_0 = z\right]\nu(dz)}{\int_F \mathbb{E}\left[T_1|\zeta_0 = z\right]\nu(dz)}. \tag{8}$$

The natural idea to apply this theorem in our context is to see our PDMP (X, V) as a SP associated to the chain $((X_{T_n}, V_n), T_n)_{n\geq 0}$. However, we are not able to control the right quantity of Equation (8), because in that case, we do not know enough about the measure ν which is the invariant measure of the chain (X_{T_n}, V_n) .
We will therefore see our process (X, V) as a SP associated to other "jump times". Let $Z_0 = 0$ and $(Z_n)_{n\geq 1}$ be the successive times at which the process X enters the ball $\mathcal{B}(1)$. We thus have $|X_{Z_n}| = 1$ for all $n \geq 1$. Moreover, just before the time Z_n, $|X_t| \geq 1$, and just after the time Z_n, $|X_t| \leq 1$. The PDMP (X, V) we are looking at is a SP associated to the chain $((X_{Z_n}, V_{Z_n}), Z_n)_{n\geq 0}$.
Indeed, let $\zeta = (X, V)$ be our initial process. Let $(x_0, v_0) \in \mathcal{S}^{d-1} \times \mathcal{B}(1)$, where $\mathcal{S}^{d-1}$ names the unit sphere of $\mathbb{R}^d$, such that $|x_0 + v_0 t| \leq 1$ for small t. Let $\zeta^{(1)}, \zeta^{(2)}, \ldots$ be processes such that $\mathcal{L}(\zeta^{(1)}) = \mathcal{L}(\zeta|\zeta_0 = (x_0, v_0))$, and
$\mathcal{L}(\zeta^{(n+1)}|\zeta^{(1)}, \ldots, \zeta^{(n)}, (X_{Z_1}, V_1), Z_1, \ldots, (X_{Z_n}, V_{Z_n}), Z_n) = \mathcal{L}(\zeta|\zeta_0 = (X_{Z_n}, V_{Z_n}))$ for all $n \geq 1$.
Let us write $M(\zeta^{(1)}; dz, dt)$ the law of $((X_{Z_1}, V_{Z_1}), Z_1)$ conditionally to $\zeta^{(1)}$.
The laws of the couples $((X_{Z_{n+1}}, V_{Z_{n+1}}), Z_{n+1} - Z_n)$ differ only on the value of (X_{Z_n}, V_{Z_n}), which is also the only data that changes in the laws of the $\zeta^{(n)}$. Therefore, for all $n \geq 1$, the couple $((X_{Z_{n+1}}, V_{Z_{n+1}}), Z_{n+1} - Z_n)$ has for law $M(\zeta^{(n)}; dz, dt)$.
According to the definition 4.1, we have defined by this way a SP.

Consequently, the previous theorem can be applied to our process, and it implies the existence of exponential moments for the invariant measure π of the PDMP (X, V). More precisely we have the following result:

Theorem 4.4. *Let η be as in Proposition 3.1. Then for all $0 < \beta < \eta$ and all $\gamma > 0$, we have :*

$$\int_E \mathrm{e}^{\beta|x|+\gamma|v|}\pi(dx, dv) < \infty,$$

where we recall that π is the unique invariant probability measure of the process (X, V).

This result is a consequence of Proposition 3.1, which implies the existence of exponential moments for the hitting time of the compact set $\mathcal{B}(1)$ associated to the Lyapunov function. Let us precise this fact:

Proposition 4.5. *Let us note $\tau_{\mathcal{B}(1)} = \inf\{t \geq 0, X_t \in \mathcal{B}(1)\}$ the hitting time of $\mathcal{B}(1)$. For all $(x, v) \in E$,*

$$\mathbb{E}_{x,v}\left[\mathrm{e}^{\eta\tau_{\mathcal{B}(1)}}\right] \leq H(x, v),$$

where η and H are defined in Proposition 3.1.

Proof. In order to make the proof easier to read, we note τ instead of $\tau_{\mathcal{B}(1)}$.
For $(x, v) \in E$, Dynkin's formula gives:

$$\begin{aligned}
\mathbb{E}_{x,v}\left[\mathrm{e}^{\eta(t\wedge\tau)}\right] &\leq \mathbb{E}_{x,v}\left[H\left(X_{t\wedge\tau}, V_{t\wedge\tau}\right)\mathrm{e}^{\eta(t\wedge\tau)}\right] \\
&= H(x, v) + \mathbb{E}_{x,v}\left[\int_0^{t\wedge\tau} (\eta + L)H\left(X_{s\wedge\tau}, V_{s\wedge\tau}\right)\mathrm{e}^{\eta(s\wedge\tau)}\mathrm{d}s\right] \\
&\leq H(x, v),
\end{aligned}$$

the last inequality resulting from Proposition 3.1.
Then, when t goes to infinity, the monotone convergence theorem gives:

$$\mathbb{E}_{x,v}\left[\mathrm{e}^{\eta\tau}\right] \leq H(x,v).$$

□

Proof of Theorem 4.4. Let first remark that the chain $(X_{Z_n}, V_{Z_n})_{n\geq 0}$ is Harris-recurrent. Indeed, let $A = \mathcal{S}^{d-1} \times \mathcal{B}(1)$ be the state space of $(X_{Z_n}, V_{Z_n})_{n\geq 0}$. We obviously have, for all $(x,v) \in A$, $\mathbb{P}(\tau_A < \infty) = 1$. The second point in the definition of the Harris-recurrence can be proved with φ the Lebesgue measure on $\mathbb{R}^d$, using in particular the fact that Z_m is almost-surely finite for each $m \geq 1$ thanks to Proposition 4.5.
Moreover, the chain $((X_{Z_n}, V_{Z_n}), Z_n)_{n\geq 0}$ is non-arithmetic because the law of the times $Z_{n+1} - Z_n$ has a part which is absolutely continuous with respect to the Lebesgue measure.
Let us now introduce $f : (x,v) \in \mathbb{R}^d \times \mathcal{B}(1) \longmapsto \mathrm{e}^{\beta|x|+\gamma|v|}$ for $0 \leq \beta < \eta$ and $\gamma \geq 0$.
We first observe that f is a positive measurable function, and that

$$((x,v),t) \longrightarrow \mathbb{E}\left[f(X_t,V_t)|(X_{Z_n},V_{Z_n}) = (x,v)\right]$$

is bounded on $\left(\mathcal{S}^{d-1} \times \mathcal{B}(1)\right) \times [0,t]$ for all t.
Let ν be the unique invariant probability measure of the chain $(X_{Z_n}, V_{Z_n})_{n\geq 0}$, which exists since the chain is positive Harris-recurrent.
Let us define the function g on $\left(\mathbb{R}^d \times \mathcal{B}(1)\right) \times \mathbb{R}_+$ by

$$g\left((x,v),t\right) = \mathbb{E}_{x,v}\left[\mathrm{e}^{\beta|X_t|+\gamma|V_t|}\mathbf{1}_{t<Z_1}\right],$$

and let us check if g satisfies the assumption of Theorem 4.3:

1. For ν-almost all $(x,v) \in \mathcal{S}^{d-1} \times \mathcal{B}(1)$, the function $t \longmapsto g((x,v),t)$ is almost everywhere continuous with respect to the Lebesgue measure since it is right-continuous and has thus an at most countable set of discontinuities.

2. Let $\Delta > 0$. According to Markov inequality we have

$$\mathbb{P}_{x,v}\left(Z_1 > t\right) \leq \mathrm{e}^{-\eta t}\mathbb{E}_{x,v}\left[\mathrm{e}^{\eta Z_1}\right].$$

   Using Proposition 4.5 and the fact that the Lyapunov function H is uniformly bounded on $\mathcal{S}^{d-1} \times \mathcal{B}(1)$, we get the existence of a finite constant M such that for all $(x,v) \in \mathcal{S}^{d-1} \times \mathcal{B}(1)$,

$$\mathbb{P}_{x,v}\left(Z_1 > t\right) \leq M\mathrm{e}^{-\eta t}.$$

   Moreover, let remark that if $(X_0,V_0) \in \mathcal{S}^{d-1} \times \mathcal{B}(1)$, since $|X_0| = 1$ and the velocity of the process lives in $\mathcal{B}(1)$, then for all $t \geq 0$, $|X_t| \leq 1 + t$.

Therefore we have

$$\begin{aligned}
&\int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)} \sum_{n\in\mathbb{N}} \sup_{n\Delta\le t<(n+1)\Delta} |g((x,v),t)|\nu(\mathrm{d}x,\mathrm{d}v) \\
&\quad= \int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)} \sum_{n\in\mathbb{N}} \sup_{n\Delta\le t<(n+1)\Delta} \mathbb{E}_{x,v}\left[\mathrm{e}^{\beta|X_t|+\gamma|V_t|}\mathbf{1}_{t<Z_1}\right]\nu(\mathrm{d}x,\mathrm{d}v) \\
&\quad\le \int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)} \sum_{n\in\mathbb{N}} \sup_{n\Delta\le t<(n+1)\Delta} \mathrm{e}^{\beta(1+t)+\gamma}\mathbb{P}_{x,v}(Z_1>t)\,\nu(\mathrm{d}x,\mathrm{d}v) \\
&\quad\le \int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)} \sum_{n\in\mathbb{N}} \sup_{n\Delta\le t<(n+1)\Delta} \mathrm{e}^{\beta(1+t)+\gamma}M\mathrm{e}^{-\eta t}\nu(\mathrm{d}x,\mathrm{d}v) \\
&\quad\le M\mathrm{e}^{\beta+\gamma}\sum_{n\in\mathbb{N}}\mathrm{e}^{\beta(n+1)\Delta-\eta n\Delta} \\
&\quad= M\mathrm{e}^{\beta(1+\Delta)+\gamma}\sum_{n\in\mathbb{N}}\left(\mathrm{e}^{\Delta(\beta-\eta)}\right)^n.
\end{aligned}$$

This quantity is finite since $\beta<\eta$.

The function g satisfying all the required assumptions, we can apply Theorem 4.3: for ν-almost all $(x_0,v_0)\in\mathcal{S}^{d-1}\times\mathcal{B}(1)$ we have

$$\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\beta|X_t|+\gamma|V_t|}\right] \underset{t\to\infty}{\longrightarrow} \frac{\int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)}\mathbb{E}_{x,v}\left[\int_0^{Z_1}\mathrm{e}^{\beta|X_s|+\gamma|V_s|}\mathrm{d}s\right]\nu(\mathrm{d}x,\mathrm{d}v)}{\int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)}\mathbb{E}_{x,v}\left[Z_1\right]\nu(\mathrm{d}x,\mathrm{d}v)}.$$

The quantity on the right is clearly finite thanks to the previous calculations. Moreover, the ergodic theorem gives:

$$\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\beta|X_t|+\gamma|V_t|}\right] \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d\times\mathcal{B}(1)}\mathrm{e}^{\beta|x|+\gamma|v|}\pi(\mathrm{d}x,\mathrm{d}v).$$

We then deduce that

$$\int_{\mathbb{R}^d\times\mathcal{B}(1)}\mathrm{e}^{\beta|x|+\gamma|v|}\pi(\mathrm{d}x,\mathrm{d}v) = \frac{\int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)}\mathbb{E}_{x,v}\left[\int_0^{Z_1}\mathrm{e}^{\beta|X_s|+\gamma|V_s|}\mathrm{d}s\right]\nu(\mathrm{d}x,\mathrm{d}v)}{\int_{\mathcal{S}^{d-1}\times\mathcal{B}(1)}\mathbb{E}_{x,v}\left[Z_1\right]\nu(\mathrm{d}x,\mathrm{d}v)},$$

and we have proved the theorem. □

5 The particular case of dimension 1

In this section, we will study PDMPs in dimension 1, but in a different way, and under some different assumptions (see Section 5.4 for a comparison between the two approaches in dimension 1). This section is thus complementary to the previous study applied at the one-dimension. An interest of this section is that the approach carried out gives more quantitative results. Moreover, we recover with this approach the previous works on this kind of PDMPs ([4], [13], [14]).

The main difference with the previous study in higher dimension is that, instead of giving a Lyapunov function to prove the exponential ergodicity, we are going to estimate more precisely

the hitting time of a compact set.

We consider here the PDMP $((X_t, V_t))_{t\geq 0}$ with values in $\mathbb{R}\times[-1,1]$ with infinitesimal generator given by

$$Lh(x,v) = v\partial_x h(x,v) + \lambda(x,v)\int_{-1}^{1}(h(x,v') - h(x,v))\,Q(x,v,\mathrm{d}v'), \tag{9}$$

where $Q(x,v,\cdot)$ is a probability kernel on $[-1,1]$. This process is the one-dimensional version of the process driven by (1).

The framework of this section is the following:

($\mathcal{A}_1$): $Q(x,v,\mathrm{d}v') = q(v,v')(\nu(\mathrm{d}v') + \mu(\mathrm{d}v'))$ with ν a discrete measure and μ a restriction of the Lebesgue measure. We denote by $\mathcal{V}$ the support of Q and suppose that there exists a constant $q_{\min} > 0$ such that $q(v,v') \geq q_{\min}$ for all $v, v' \in \mathcal{V}$;

($\mathcal{A}_2$): The process is symmetric: $\mathcal{V}$ is symmetric with respect to 0 and $\lambda(x,v) = \lambda(-x,-v)$ for all $(x,v)\in\mathbb{R}\times\mathcal{V}$;

($\mathcal{A}_3$): There exists $0 < \lambda_{\min}$ such that for all $(x,v)\in\mathbb{R}\times\mathcal{V}$, $\lambda_{\min}\leq\lambda(x,v)$, and the quantity $\sup_{x\geq 0, v\leq 0}\lambda(x,v)$ is finite.

The importance of the existence of q_{min} in Assumption ($\mathcal{A}_1$) will not explicitly appear in the following since we are not going to give all the proofs in detail, but we note that it is useful in Proposition 5.8 and Theorem 5.9.
Assumption ($\mathcal{A}_2$) implies that the process is spatially symmetric with respect to the origin, which will allow us to reduce the number of computations. Nevertheless, the following results are still available without this symmetry.
The fact that we suppose the kernel Q to be independent of the position of the bacterium implies that the chain composed by the velocities at the jump times is a Markov chain with kernel Q. In particular, contrary to what is assumed in higher dimension, the attraction of the bacterium by the origin is not favoured by the jump kernel since it does not take care of the position of the bacterium with respect to the origin.
We make thus an additional assumption ($\mathcal{A}_4$) that takes into account this attraction to the origin, because of the presence of a chemo-attractant there. This assumption is the one that makes the process ergodic.

$$(\mathcal{A}_4): \exists I_* \subset \left[0, \inf_{x\geq 0, v\in(0,1]}\frac{\lambda(x,v)}{v}\right) \text{ an interval } I_*, \exists 0 < J_* < 1, \forall\alpha\in I_*, \forall v'\in\mathcal{V},$$
$$\int_{-1}^{1} G(\alpha,v)Q(v',\mathrm{d}v) \leq J_*,$$

where

$$G(\alpha,v) = \frac{\sup_{x\geq 0}\lambda(x,v)}{\sup_{x\geq 0}\lambda(x,v) - \alpha v}\mathbf{1}_{v<0} + \frac{\inf_{x\geq 0}\lambda(x,v)}{\inf_{x\geq 0}\lambda(x,v) - \alpha v}\mathbf{1}_{v\geq 0} \tag{10}$$

for $\alpha\geq 0$ and $v\in\mathcal{V}$.

Before stating the theorem that gives the exponential ergodicity of the process, we make some remarks on Assumption $(\mathcal{A}_4)$.

Remark 5.1. *Let us give a sufficient condition to satisfy Assumption $(\mathcal{A}_4)$ when the jump kernel Q does not depend on the previous velocity.*
Let us write $J_{v'}(\alpha) = \int_{-1}^{1} G(\alpha, v) Q(v', \mathrm{d}v)$.
For all $v' \in [-1, 1]$, the function $J_{v'}$ is C^1 on $[0, \inf\limits_{x\geq 0, v\in]0,1]} \frac{\lambda(x,v)}{v})$, is convex and

$$J_{v'}(0) = 1 \ , \ J'_{v'}(0) = \int_{-1}^{1} \left(\frac{v}{\sup\limits_{x\geq 0} \lambda(x, v)} \mathbf{1}_{v<0} + \frac{v}{\inf\limits_{x\geq 0} \lambda(x, v)} \mathbf{1}_{v\geq 0} \right) Q(v', \mathrm{d}v)$$

$$\textit{and} \lim_{\alpha \to \inf\limits_{x\geq 0, v\in]0,1]} \frac{\lambda(x,v)}{v}} J_{v'}(\alpha) = +\infty.$$

If we assume that $J'_{v'}(0) < 0$, we get that there exists a unique $\hat{\alpha} \in (0, \inf\limits_{x\geq 0, v\in]0,1]} \frac{\lambda(x,v)}{v})$ such that $J(\hat{\alpha}) = 1$, and then there exists an interval $\hat{I} \subset (0, \alpha)$ such that for all $\alpha \in \hat{I}$, $J_{v'}(\alpha)$ is uniformly bounded by a constant strictly smaller than 1 on the interval $\hat{I}$.
Consequently, for all $v' \in [-1, 1]$,

$$\exists \hat{I} \subset (0, \inf_{x\geq 0, v\in (0,1]} \frac{\lambda(x, v)}{v}), \exists 0 < \hat{K} < 1, \forall \alpha \in \hat{I}, \int_{-1}^{1} G(\alpha, v) Q(v', \mathrm{d}v) \leq \hat{K}.$$

In Assumption $(\mathcal{A}_4)$, we suppose in addition that the interval $\hat{I}$ and the constant $\hat{K}$ are uniform in v'. In particular, if $Q(v', \mathrm{d}v) = Q(\mathrm{d}v)$, the assumption $J'(0) < 0$, that is

$$\int_{-1}^{1} \left(\frac{v}{\sup\limits_{x\geq 0} \lambda(x, v)} \mathbf{1}_{v<0} + \frac{v}{\inf\limits_{x\geq 0} \lambda(x, v)} \mathbf{1}_{v\geq 0} \right) Q(\mathrm{d}v) < 0$$

is sufficient for Assumption $(\mathcal{A}_4)$ to be satisfied.
This assumption is the equivalent of Assumption $(H3)$ made by Calvez, Raoul and Schmeiser in [6]. The probability study carried out in this section covers thus the framework of [6].

Remark 5.2. *Let see that in the case where the kernel Q does not depend on the previous velocity and is symmetric in the sense that $q(v) = q(-v)$ for all $v \in \mathcal{V}$, Assumption $(\mathcal{A}_4)$ holds in the particular case where $\inf\limits_{x\geq 0} \lambda(x, v) > \sup\limits_{x\leq 0} \lambda(x, v)$ for all $v \in [0, 1]$, i.e. when the velocity tends to jump even more when the bacterium goes away from 0 than when it moves towards the origin. This case is the one considered for instance in [4], [13] and [14], with a velocity taking its values in $\{-1, +1\}$*
Let us prove this fact. Under this assumption on the jump rate, with the same notations as in

the previous remark, and using the symmetries of the process we have:

$$\begin{aligned}
J'_{v'}(0) &= \int_{-1}^{1} \left(\frac{v}{\sup_{x\geq 0}\lambda(x,v)} \mathbf{1}_{v<0} + \frac{v}{\inf_{x\geq 0}\lambda(x,v)} \mathbf{1}_{v\geq 0} \right) q(v)\mathrm{d}v \\
&= \int_{-1}^{0} \frac{v}{\sup_{x\geq 0}\lambda(x,v)} q(v)\mathrm{d}v + \int_{0}^{1} \frac{v}{\inf_{x\geq 0}\lambda(x,v)} q(v)\mathrm{d}v \\
&= -\int_{0}^{1} \frac{v}{\sup_{x\geq 0}\lambda(x,-v)} q(v)\mathrm{d}v + \int_{0}^{1} \frac{v}{\inf_{x\geq 0}\lambda(x,v)} q(v)\mathrm{d}v \\
&= -\int_{0}^{1} \frac{v}{\sup_{x\leq 0}\lambda(x,v)} q(v)\mathrm{d}v + \int_{0}^{1} \frac{v}{\inf_{x\geq 0}\lambda(x,v)} q(v)\mathrm{d}v \\
&< 0.
\end{aligned}$$

And the end of Remark 5.1 ensures that Assumption $(\mathcal{A}_4)$ is satisfied.

Let us now give the main theorem of this section.

Theorem 5.3. *Let $(X_t, V_t)_{t\geq 0}$ be the PDMP on $\mathbb{R}\times[-1,1]$ whose infinitesimal generator is given by*

$$Lh(x,v) = v\partial_x h(x,v) + \lambda(x,v)\int_{-1}^{1} (h(x,v') - h(x,v))\, Q(x,v,\mathrm{d}v').$$

Under Assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$, the process is exponentially ergodic.

5.1 The hitting time of the origin

As mentioned before, we are going to estimate the exponential moments of the hitting times of compact sets in order to prove Theorem 5.3. We introduce two new notations.
We denote by Z the first hitting time of 0, i.e.

$$Z = \inf\{t > 0, X_t = 0\}, \tag{11}$$

and S is the index of the first jump-time at which the position of the process has changed its sign: $S = \inf\{n \geq 1, X_{T_n}X_0 \leq 0\}$.
We have the following result concerning the random variable Z:

Proposition 5.4. *For every $(x_0, v_0) \in \mathbb{R}\times\mathcal{V}$, there exists a constant $K_{x_0,v_0} < \infty$ such that for every $0 < \rho < \lambda_{\min}(1 - J_*)$, where J_* is given by $(\mathcal{A}_4)$,*

$$\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha Z}\right] \leq K_{x_0,v_0} \frac{1}{1 - \frac{\lambda_{\min}}{\lambda_{\min}-\rho} J_*}.$$

Moreover, K_{x_0,v_0} is uniformly bounded from above for $(x_0, v_0) \in [-x_1, x_1]\times\mathcal{V}$ for all $x_1 > 0$.

To prove this proposition, we will first study the random variable S. Then, the inequality $Z \leq \sum_{i=1}^{S} \tau_i$ a.s. will allow us to come back to Z.

Proposition 5.5. *For all $(x_0, v_0) \in \mathbb{R} \times \mathcal{V}$, there exist a constant $\kappa_{x_0,v_0} < \infty$ such that for all $n \geq 1$,*

$$\mathbb{P}_{x_0,v_0}(S > n) \leq \kappa_{x_0,v_0}(J_*)^{n-1},$$

where J_* is given by $(\mathcal{A}_4)$.

Proof. In order to make the proof easier to read, we distinguish the cases where x_0 and v_0 are positive or negative. We first look at the case $x_0 > 0$ and $v_0 \in \mathcal{V} \cap [0,1]$, and the other cases are similar because of the symmetry of the process.

Let $x_0 > 0$ and $v_0 \in \mathcal{V} \cap [0,1]$.
We have $S = \inf\{n \geq 1, X_{T_n} \leq 0\}$, and $X_{T_n} = X_0 + \sum_{i=0}^{n-1} V_i\tau_{i+1} = x_0 + v_0\tau_1 + \sum_{i=1}^{n-1} V_i\tau_{i+1}$.
Let $\alpha \in [0, \inf\limits_{x\geq 0, v\in(0,1]} \frac{\lambda(x,v)}{v})$.
Since x_0 is positive, on the event $\{S > n\}$, X_{T_n} is also positive, and the sign of α implies:

$$\begin{aligned}
\mathbb{P}_{x_0,v_0}(S > n) &\leq \mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha X_{T_n}}\mathbf{1}_{S>n}\right] \\
&= \mathrm{e}^{\alpha x_0}\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha\left(v_0\tau_1 + \sum_{i=1}^{n-1} V_i\tau_{i+1}\right)}\mathbf{1}_{S>n}\right] \\
&= \mathrm{e}^{\alpha x_0}\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha v_0\tau_1}\prod_{i=1}^{n-1}\mathrm{e}^{V_i\tau_{i+1}}\mathbf{1}_{S>n}\right] \\
&= \mathrm{e}^{\alpha x_0}\mathbb{E}_{x_0,v_0}\left[\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha v_0\tau_1}\prod_{i=1}^{n-1}\mathrm{e}^{V_i\tau_{i+1}}\mathbf{1}_{S>n}|X_{T_1}, V_1, \ldots, X_{T_{n-1}}, V_{n-1}\right]\right].
\end{aligned}$$

Since the inter-jump times $\tau_1, \ldots, \tau_n$ are independent conditionally to the couples $(X_0, V_0), (X_{T_1}, V_1), \ldots, (X_{T_{n-1}}, V_{n-1})$, and since $\{S > n\} = \{X_{T_1} > 0, \cdots, X_{T_n} > 0\}$ we have:

$$\mathbb{P}_{x_0,v_0}(S > n) \leq \mathrm{e}^{\alpha x_0}\mathbb{E}_{x_0,v_0}\left[\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha v_0\tau_1}\right]\prod_{i=1}^{n-1}\mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\alpha V_i\tau_{i+1}}|X_{T_i}, V_i\right]\mathbf{1}_{X_{T_1}>0,\cdots,X_{T_n}>0}\right]. \tag{12}$$

Moreover, we know that

$$\mathbb{P}\left(\tau_{i+1} \geq t|X_{T_i}, V_i\right) = \overline{F}(t, X_{T_i}, V_i),$$

which gives, for $1 \leq i \leq n-1$:

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\alpha V_i\tau_{i+1}}|X_{T_i}, V_i\right] \\
&= \int_0^{+\infty} \mathrm{e}^{\alpha V_i t}\lambda(X_{T_i} + V_i t, V_i)\mathrm{e}^{-\int_0^t \lambda(X_{T_i}+V_i s, V_i)\mathrm{d}s}\mathrm{d}t \\
&= \int_0^{+\infty} \frac{\lambda(X_{T_i} + V_i t, V_i)}{\lambda(X_{T_i} + V_i t, V_i) - \alpha V_i}\left(\lambda(X_{T_i} + V_i t, V_i) - \alpha V_i\right)\mathrm{e}^{-\int_0^t\left(\lambda(X_{T_i}+V_i t, V_i) - \alpha V_i\right)\mathrm{d}s}\mathrm{d}t.
\end{aligned}$$

Furthermore, for $\alpha \in \mathbb{R}$ and $v \in [-1,1]$ the function $\lambda \in \mathbb{R}_+ \longmapsto \frac{\lambda}{\lambda - \alpha v}$ is increasing if $\alpha v \leq 0$ and decreasing otherwise. Therefore, if $V_i \geq 0$, since on $\{S > n\}$ the position of the bacterium

is positive, we get:

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\alpha V_i \tau_{i+1}} | X_{T_i}, V_i\right] \mathbf{1}_{X_{T_1>0},\cdots,X_{T_n>0}} \\
&\leq \frac{\inf\limits_{x\geq 0} \lambda(x, V_i)}{\inf\limits_{x\geq 0} \lambda(x, V_i) - \alpha V_i} \int_0^{+\infty} \left(\lambda(X_{T_i} + V_i t, V_i) - \alpha V_i\right) \mathrm{e}^{-\int_0^t \left(\lambda(X_{T_i}+V_i t, V_i) - \alpha V_i\right) \mathrm{d}s} \mathrm{d}t \\
&= \frac{\inf\limits_{x\geq 0} \lambda(x, V_i)}{\inf\limits_{x\geq 0} \lambda(x, V_i) - \alpha V_i}.
\end{aligned}$$

In the case $V_i < 0$ we get in the same way:

$$\mathbb{E}\left[\mathrm{e}^{\alpha V_i \tau_{i+1}} | X_{T_i}, V_i\right] \mathbf{1}_{X_{T_1>0},\cdots,X_{T_n>0}} \leq \frac{\sup\limits_{x\geq 0} \lambda(x, V_i)}{\sup\limits_{x\geq 0} \lambda(x, V_i) - \alpha V_i}.$$

We have thus obtained the following inequality:

$$\mathbb{E}\left[\mathrm{e}^{\alpha V_i \tau_{i+1}} | X_{T_i}, V_i, S\right] \mathbf{1}_{X_{T_1>0},\cdots,X_{T_n>0}} \leq G(\alpha, V_i),$$

where G is given by (10).
Getting back to (12), we get:

$$\mathbb{P}_{x_0,v_0}\left(S > n\right) \leq \mathrm{e}^{\alpha x_0} C(\alpha, v_0) \mathbb{E}_{x_0,v_0}\left[\prod_{i=1}^{n-1} G(\alpha, V_i)\right],$$

where we write $C(\alpha, v_0) = \frac{\inf\limits_{x\geq 0} \lambda(x,v_0)}{\inf\limits_{x\geq 0} \lambda(x,v_0) - \alpha v_0}$.
Using now the fact that $(V_i)_{i\geq 0}$ is a Markov chain with kernel Q, we have:

$$\mathbb{E}_{x_0,v_0}\left[\prod_{i=1}^{n-1} G(\alpha, V_i)\right] \leq \int_{-1}^{1} G(\alpha, v_1) \times \ldots \times \int_{-1}^{1} G(\alpha, v_{n-1}) Q(v_{n-2}, \mathrm{d}v_{n-1}) \ldots Q(v_0, \mathrm{d}v_1).$$

We deduce from Assumption $(\mathcal{A}_4)$, that for all $\alpha \in I_*$ we have $\mathbb{E}_{x_0,v_0}\left[\prod_{i=1}^{n-1} G(\alpha, V_i)\right] \leq (J_*)^{n-1}$.
And finally:

$$\mathbb{P}_{x_0,v_0}\left(S > n\right) \leq \mathrm{e}^{\alpha x_0} C(\alpha, v_0) (J_*)^{n-1}, \tag{13}$$

for all $\alpha \in I_*$, $x_0 > 0$ and $v_0 \in \mathcal{V} \cap [0, 1]$.

For $x_0 > 0$ and $v_0 \in \mathcal{V} \cap [-1, 0)$, the calculations are exactly the same, only the constant $C(\alpha, v_0)$ changes into $\frac{\sup\limits_{x\geq 0} \lambda(x,v_0)}{\sup\limits_{x\geq 0} \lambda(x,v_0) - \alpha v_0}$. In the following, we thus write

$$C(\alpha, v_0) = \frac{\sup\limits_{x\geq 0} \lambda(x, v_0)}{\sup\limits_{x\geq 0} \lambda(x, v_0) - \alpha v_0} \mathbf{1}_{v_0<0} + \frac{\inf\limits_{x\geq 0} \lambda(x, v_0)}{\inf\limits_{x\geq 0} \lambda(x, v_0) - \alpha v_0} \mathbf{1}_{v_0\geq 0} \text{ , for } \alpha \geq 0.$$

One more time, for $x_0 \leq 0$, the calculations are made in exactly the same way. The symmetry of the process and Assumption $(\mathcal{A}_4)$ ensure that, for all α such that $-\alpha \in I_*$, for all $x_0 \leq 0$ and all $v_0 \in \mathcal{V}$,

$$\mathbb{P}_{x_0,v_0}(S > n) \leq \mathrm{e}^{\alpha x_0} C(-\alpha, v_0)(J_*)^{n-1}. \tag{14}$$

Finally, we have proved in (13) and (14) that for all $(x_0, v_0) \in \mathbb{R} \times \mathcal{V}$ and all α such that $|\alpha| \in I_*$,

$$\mathbb{P}_{x_0,v_0}(S > n) \leq \mathrm{e}^{|\alpha x_0|} C(|\alpha|, v_0)(J_*)^{n-1}. \tag{15}$$

The proposition is then proved with $\kappa_{x_0,v_0} = \mathrm{e}^{|\alpha x_0|} C(|\alpha|, v_0)$.

□

Proof of Proposition 5.4. As mentioned before, we are going to use the inequality $Z \leq \sum_{i=1}^{S} \tau_i$ a.s. to get some informations on the hitting time of the origin.
Let us notice the variables $\tau_i, i \geq 1$ are stochastically smaller than i.i.d. exponential variables $E_i, i \geq 1$ with mean $\frac{1}{\lambda_{\min}}$. Indeed, the survival function of the variable τ_{i+1}, conditionally to X_{T_i} and V_i, is $\overline{F}(\cdot, X_{T_i}, V_i) = \exp\left(\int_0^t \lambda(X_{T_i} + V_i s, V_i)\mathrm{d}s\right)$, and the jump rate λ is uniformly bounded from below by $\lambda_{\min}$. The variables $(E_i)_{i\geq 1}$ can be taken as independent because of the independence of the variables $(\tau_i)_{i\geq 1}$ conditionally to $(X_{T_i}, V_i)_{i\geq 0}$, and can also be taken as independent of the variables $(\tau_i)_{i\geq 1}$.
Thanks to this comment and the previous proposition, we get, for $\eta > 0$ and α such that $|\alpha| \in I_*$:

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{\eta Z}\right] &\leq \mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\eta \sum_{i=1}^{S} \tau_i}\right] \\
&\leq \mathbb{E}_{x_0,v_0}\left[\mathrm{e}^{\eta \sum_{i=1}^{S} \tilde{E}_i}\right] \\
&= \mathbb{E}_{x_0,v_0}\left[\mathbb{E}\left[\mathrm{e}^{\eta \sum_{i=1}^{S} \tilde{E}_i} | S\right]\right] \\
&= \mathbb{E}_{x_0,v_0}\left[\prod_{i=1}^{S} \mathbb{E}\left[\mathrm{e}^{\eta \tilde{E}_i}\right]\right] \\
&= \mathbb{E}_{x_0,v_0}\left[\left(\frac{\lambda_{min}}{\lambda_{min} - \eta}\right)^{S}\right] \\
&= \sum_{n=0}^{+\infty} \left(\frac{\lambda_{min}}{\lambda_{min} - \eta}\right)^{n} \mathbb{P}_{x_0,v_0}(S = n) \\
&= \sum_{n=0}^{+\infty} \left(\frac{\lambda_{min}}{\lambda_{min} - \eta}\right)^{n} \left(\mathbb{P}(S_{x_0,v_0} > n-1) - \mathbb{P}(S_{x_0,v_0} > n)\right) \\
&\leq \mathrm{e}^{|\alpha x_0|} C(|\alpha|, v_0) \frac{(1 - J_*)}{J_*^2} \sum_{n=0}^{+\infty} \left(\frac{\lambda_{min}}{\lambda_{min} - \eta} J_*\right)^{n}.
\end{aligned}$$

Finally, for all $\eta > 0$ such that $\frac{\lambda_{min}}{\lambda_{min} - \eta} J_* < 1$, we get :

$$\mathbb{E}\left[\mathrm{e}^{\eta Z}\right] \leq \mathrm{e}^{|\alpha x_0|} C(|\alpha|, v_0) \frac{(1 - J_*)}{J_*^2} \frac{1}{1 - \frac{\lambda_{min}}{\lambda_{min} - \eta} J_*} < \infty, \tag{16}$$

which ends the proof of the proposition. □

5.2 Exponential ergodicity of the process

As mentioned before, we are going to give a proof of the exponential ergodicity of our process based on Proposition 5.4, without using the existence of a Lyapunov function. For this purpose, we recall another result on the exponential ergodicity of a Markov process.

Theorem 5.6. *(Theorem 6.2 in [11]) Let Y be an irreducible and aperiodic Markov process. Suppose that there exists a function $f \geq 1$, a closed measurable set $C \in E$ and some constants $\delta, \eta > 0$, $M < \infty$ such that*

$$\mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \mathrm{e}^{\eta t} f(Y_t) \mathrm{d}t \right] < \infty, \quad \textit{for all } x \notin C$$

and

$$\sup_{x \in C} \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \mathrm{e}^{\eta t} f(Y_t) \mathrm{d}t \right] \leq M$$

where $\tau_C(\delta) = \inf\{t \geq \delta, Y_t \in C\}$.
If the set C is petite for Y, then the process is exponentially ergodic.

Remark 5.7. *The proof of this theorem relies on the introduction of the function $H_0(x) := 1 + \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \mathrm{e}^{\eta t} f(Y_t) \mathrm{d}t \right]$, which plays the role of a Lyapunov function for a Markov chain linked to the process Y.*

To apply this theorem, we need then, as previously, to find a petite set. The following proposition states that all compact sets are petite for the process (X, V), and is just a consequence of Proposition 3.2.

Proposition 5.8. *For all $x_1 > 0$, the set $C = [-x_1, x_1] \times \mathcal{V}$ is petite for the process (X, V).*

We can now prove Theorem 5.3:

Proof of Theorem 5.3. Let $x_1 > 0$. Proposition 5.8 ensures that the closed set $C = [-x_1, x_1] \times \mathcal{V}$ is petite for the process (X, V).
Moreover, let us consider the quantity $\mathbb{E}_{x,v} \left[\mathrm{e}^{\eta \tau_C(\delta)} \right]$ with $\eta, \delta > 0$. It satisfies the assumptions of Theorem 5.6 (we have taken $f \equiv 1$) for $0 < \eta < \lambda_{\min}(1 - J_*)$ thanks to Proposition 5.4.
Finally, let *Leb* be the Lebesgue measure on $\mathbb{R}$. The process (X, V) is $Leb \otimes Q$-irreducible, and aperiodic. The proof of these facts can be handled as mentioned previously in the general case.
The conclusion of Theorem 5.6 gives then the exponential ergodicity of (X, V). □

5.3 Exponential moments of the invariant measure

As in the general case, we can show that the invariant measure, that we still denote by π, has exponential moments. The interest of the current study of the process is that it is quantitatively more precise, since the estimation of the exponential moments of Z have been carried out explicitly in the previous section.

Theorem 5.9. *Let $\gamma > 0$ such that $\frac{\lambda_{\min}}{\lambda_{\min}-\gamma} J_* < 1$.*
For all $0 < \alpha < \gamma$ and all $\beta > 0$, we have :

$$\int_{\mathbb{R}\times[-1,1]} \mathrm{e}^{\alpha|x|+\beta|v|}\pi(dx,dv) < \infty.$$

Proof. The proof relies, as in Section 4, on Theorem 4.3. We see the process (X,V) as a SP between the successive hitting times of 0. Then, using the upper bound of the exponential moments of the hitting time of 0 obtained in Proposition 5.4, we can verify that the assumptions of Theorem 4.3 are satisfied, and the result follows. □

5.4 Comparison between the two studies

First, let us recall that the proof of the exponential ergodicity carried out in any dimension is obviously relevant in dimension 1, whereas the converse is not possible, or at least not directly. Indeed, the process in dimension 1 is quite simple since it goes towards the origin or not in terms of the sign of $X_t \cdot V_t$. In higher dimension, if we want to estimate the hitting time of a compact set, let say a ball, we can not see if the process is evolving towards the ball just in terms of the sign of $X_t \cdot V_t$, and the computations do not proceed as well as in dimension 1.
Nevertheless, even if the approach handled in dimension 1 can not be expanded in higher dimension, it has the advantage to give more quantitative results than those obtained with the Lyapunov function.
Finally, let see that the two approaches carried out in dimension 1 and in higher dimension cover different types of PDMPs.

- Let look at the process studied in [13], [14] and [4], whose generator has the following form :

  $$Lf(x,v) = v\partial_x f(x,v) + \lambda(x,v)\left(f(x,-v) - f(x,v)\right),$$

  for $(x,v) \in \mathbb{R}\times\{-1,+1\}$.
  This process does not satisfy Assumption $(\mathcal{A}_4)$, because for $v' = -1$ we have

  $$\int_{-1}^{1} G(\tilde{\alpha},v)Q(-1,\mathrm{d}v) = G(\tilde{\alpha},1) = \frac{\inf\limits_{x\geq 0}\lambda(x,1)}{\inf\limits_{x\geq 0}\lambda(x,1) - \alpha} > 1,$$

  whereas it is still ergodic. Indeed, even if it does not satisfy Assumption $(\mathcal{H}_3)$, the Lyapunov function obtained in 2.1 can be adapted (see [4] or [14] for an explicit formula).

- In the d-dimensional case, Assumption $(\mathcal{H}_4)$ assume a kind of monotony of the jump rate, which is not necessary in Assumption $(\mathcal{A}_4)$ in the one-dimensional case. We can thus construct a particular jump rate which, associated to the kernel Q, satisfies Assumption $(\mathcal{A}_4)$, but does not verify Assumption $(\mathcal{H}_4)$. Let for instance consider the case

  $$Lf(x,v) = v\partial_x f(x,v) + (2\mathbf{1}_{\mathrm{sgn}(x)v\geq -\frac{1}{2}} + \frac{1}{2}\mathbf{1}_{\mathrm{sgn}(x)v<-\frac{1}{2}})\frac{1}{2}\int_{-1}^{1}\left(f(x,v') - f(x,v)\right)\mathrm{d}v',$$

  for $(x,v) \in \mathbb{R}\times[-1,1]$.
  This process obviously does not satisfies Assumption $(\mathcal{H}_4)$ since for all $\theta_* \in [0,1]$, and all

$\Delta > 0$, $\inf_{\{\mathrm{sgn}(x)v \geq \theta_*, |x| \geq \Delta\}} \lambda(x,v) = 2 = \sup_{\{\mathrm{sgn}(x)v \leq 0\}} \lambda(x,v)$.

Nevertheless, Assumption $(\mathcal{A}_4)$ is verified for this process. Indeed, let $\alpha \in \left(0, \inf_{x \geq 0, v \in (0,1]} \frac{\lambda(x,v)}{v}\right)$, we have:

$$\begin{aligned}
&\frac{1}{2}\int_{-1}^{1} G(\alpha, v)\mathrm{d}v \\
&= \frac{1}{2}\int_{-1}^{-\frac{1}{2}} \frac{\sup_{x \geq 0}\lambda(x,v)}{\sup_{x \geq 0}\lambda(x,v) - \alpha v}\mathrm{d}v + \frac{1}{2}\int_{-\frac{1}{2}}^{-0} \frac{\sup_{x \geq 0}\lambda(x,v)}{\sup_{x \geq 0}\lambda(x,v) - \alpha v}\mathrm{d}v + \frac{1}{2}\int_{0}^{1} \frac{\inf_{x \geq 0}\lambda(x,v)}{\inf_{x \geq 0}\lambda(x,v) - \alpha v}\mathrm{d}v \\
&= \frac{1}{2}\int_{-1}^{-\frac{1}{2}} \frac{\frac{1}{2}}{\frac{1}{2} - \alpha v}\mathrm{d}v + \frac{1}{2}\int_{-\frac{1}{2}}^{1} \frac{2}{2 - \alpha v}\mathrm{d}v \\
&= \frac{1}{4\alpha}\log\left(\frac{\frac{1}{2} + \alpha}{\frac{1}{2} + \frac{\alpha}{2}}\right) + \frac{1}{\alpha}\log\left(\frac{2 + \frac{\alpha}{2}}{2 - \alpha}\right).
\end{aligned}$$

A curve sketching shows that there exists an interval $I_* \subset \left[0, \inf_{x \geq 0, v \in (0,1]} \frac{\lambda(x,v)}{v}\right)$ and a constant $J_* \in (0,1)$ such that for all $\alpha \in I_*$, $\frac{1}{2}\int_{-1}^{1} G(\alpha, v)\mathrm{d}v \leq J_*$, i.e. Assumption $(\mathcal{A}_4)$ is satisfied.

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References

[1] S. Asmussen. *Applied Probability and Queues*. Applications of mathematics : stochastic modelling and applied probability. Springer, 2003.

[2] R. Azaïs, J.-B. Bardet, A. Genadot, N. Krell, and P.-A. Zitt. Piecewise deterministic Markov process - recent results. *ESAIM: Proceedings*, 44:276–290, 2014. Clermont-Ferrand, France, 29-31 August 2012.

[3] J. Bierkens, P. Fearnhead, and G. Roberts. The Zig-Zag Process and Super-Efficient Sampling for Bayesian Analysis of Big Data. *ArXiv e-prints*, July 2016.

[4] J. Bierkens and G. Roberts. A piecewise deterministic scaling limit of lifted Metropolis–Hastings in the Curie–Weiss model. *Ann. Appl. Probab.*, 27(2):846–882, 2017.

[5] A. Bouchard-Côté, S. J. Vollmer, and A. Doucet. The Bouncy Particle Sampler: A Non-Reversible Rejection-Free Markov Chain Monte Carlo Method. *ArXiv e-prints*, October 2015.

[6] V. Calvez, G. Raoul, and C. Schmeiser. Confinement by biased velocity jumps: aggregation of *escherichia coli*. *Kinet. Relat. Models*, 8(4):651–666, 2015.

[7] Nicolas Champagnat and Denis Villemonais. Exponential convergence to quasi-stationary distribution for absorbed one-dimensional diffusions with killing. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(1):177–199, 2017.

[8] C. Cocozza-Thivent. Processus de renouvellement markovien processus de markov déterministes par morceaux. http://perso-math.univ-mlv.fr/users/cocozza.christiane/recherche-page-perso/livreRMetPDMPv2.pdf, 2016.

[9] M. H. A. Davis. Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 46(3), 1984.

[10] G. Deligiannidis, A. Bouchard-Côté, and A. Doucet. Exponential Ergodicity of the Bouncy Particle Sampler. *ArXiv e-prints*, May 2017.

[11] D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of markov processes. *The Annals of Probability*, 23(4):1671–1691, 1995.

[12] A. Durmus, A. Guillin, and P. Monmarché. Geometric ergodicity for the bouncy particle sampler. *Work in progress.*

[13] J. Fontbona, H. Guérin, and F. Malrieu. Quantitative estimates for the long-time behavior of an ergodic variant of the telegraph process. *Adv. in Appl. Probab.*, 44(4):977–994, 12 2012.

[14] J. Fontbona, H. Guérin, and F. Malrieu. Long time behavior of telegraph processes under convex potentials. *Stochastic Processes and their Applications*, 126(10):3077–3101, 2016.

[15] C.-R. Hwang, S.-Y. Hwang-Ma, and S.-J. Sheu. Accelerating diffusions. *Ann. Appl. Probab.*, 15(2):1433–1444, 2005.

[16] T. Lelièvre, F. Nier, and G. A. Pavliotis. Optimal non-reversible linear drift for the convergence to equilibrium of a diffusion. *J. Stat. Phys.*, 152(2):237–274, 2013.

[17] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability.* Springer-Verlarg, London, 1993.

[18] S. P. Meyn and R. L. Tweedie. Stability of markovian processes iii: Foster-Lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, 25(3):518–548, 1993.

[19] P. Monmarché. Piecewise deterministic simulated annealing. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(1):357–398, 2016.

[20] N. O'Connell. Review: Torgny Lindvall, lectures on the coupling method. *Ann. Probab.*, 23(3):1456–1460, 07 1995.

[21] H. G. Othmer, S. R. Dunbar, and W. Alt. Models of dispersal in biological systems. *Journal of Mathematical Biology*, 26:263–298, 1988.