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► **To cite this version:**

van Hao Can, Viet-Hung Pham. Persistence probability of random weyl polynomial. 2019. hal-01608950v2

HAL Id: hal-01608950

<https://hal.science/hal-01608950v2>

Preprint submitted on 13 Apr 2019

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PERSISTENCE PROBABILITY OF RANDOM WEYL POLYNOMIAL

VAN HAO CAN AND VIET-HUNG PHAM

ABSTRACT. In this paper, we obtain the persistence exponents of random Weyl polynomials in both cases: the nonnegative axis and the whole real axis. Our result confirms the predictions given by Schehr and Majumdar [22]. In the nonnegative axis case, Dembo and Mukherjee [7] gave an upper bound for the persistence exponent by considering the persistence probability on a suitable interval. Our main contribution is to prove this upper bound is the exact exponent and to extend to the whole real axis case.

1. INTRODUCTION

The persistence probability of a stochastic process \mathcal{X} is defined as the probability that the process \mathcal{X} remains positive for a long interval. The study of persistence probability has gained much interest in both mathematical and physical literatures, see more in [1, 2, 4, 5, 12, 20, 21, 22]. However, up to now, most of existed results just deal with stationary Gaussian processes with summable and nonnegative autocorrelation function. In general, understanding the persistence decay is still a big challenge, see [13]. Recently, some new lights have been shed in [8] for processes with non-summable autocorrelation function or in [14] for processes with special spectral properties.

In this paper, we are interested in the persistence probability of random polynomials, a special class of Gaussian processes. Let us consider a polynomial defined as

$$Q_n(x) = c_0 a_0 + c_1 a_1 x + \dots + c_n a_n x^n,$$

where c_i 's are real deterministic coefficients and a_i 's are i.i.d. standard Gaussian random variables. Among various models (choices of the coefficients c_i 's), the three classes being studied most extensively include

- Kac polynomials: $c_0 = c_1 = \dots = c_n = 1$,
- Elliptic polynomials: $c_i = \sqrt{\binom{n}{i}}$, $\forall i = 0, 1, \dots, n$,
- Weyl polynomials: $c_i = \frac{1}{\sqrt{i!}}$.

For more information and results on random polynomial theory, we refer the reader to two standard monographs [4, 12] and recent papers [10, 11, 19, 24].

The first motivation to study the persistence probability for random polynomials arises from Large deviations theory or Extreme value theory to estimate the rare event that the random polynomials have many fewer real zeroes than expected. In particular, the persistence probability is a half of the probability that the random polynomials have no real zero, see [3]. The second motivation comes from some applications in statistical physics to understand the magnetization of spin glass, see [5, 21]. Recently, the connection between persistence probability of random polynomials and the semi-infinite Ising chain with Glauber dynamics has been unveiled, see [20].

2010 *Mathematics Subject Classification.* Primary 60G15, 26C10; secondary 26A12.

Key words and phrases. Random Weyl polynomials, Gaussian processes, Real zeros, Persistence probability.

A very first result given by Littewood and Offord [15, 16] in the 1940's states that for Kac polynomials, if n is even, then the persistence probability is small, in fact $\mathbb{P}(N_{Kac,n} = 0) = O(1/\log n)$. In 2002, the first breakthrough result was given by Dembo et al [9]. They provided a precise asymptotic formula for the persistence probability of Kac polynomials of even degree as

$$(1) \quad \mathbb{P}(Q_{Kac,n}(x) > 0, \forall x \in \mathbb{R}) = n^{-4b_0+o(1)},$$

where $b_0 = -\lim_{t \rightarrow \infty} t^{-1} \log \mathbb{P}(Y_s > 0, \forall s \in [0, t])$, with Y the centered stationary Gaussian process with correlation $\mathbb{E}(Y_0 Y_t) = 1/\cosh(t/2)$. Their main observation is that on some small main intervals the random Kac polynomial can be approximated by the Gaussian process Y above. More strikingly, a universal result for any distribution having finite moments of all orders was achieved by using the strong Komlos-Major-Tusnady approximation. It follows an interesting question to provide good estimates for the constant b_0 above, see [17]. Recently, in [20], the exact value of b_0 is given to be $3/16$.

In [22], Schehr and Majumdar proposed a mean-field approximation (based on the asymptotic behavior of autocorrelation function) to revisit the persistence probability of Kac polynomials and also to *predict* an asymptotic formula for elliptic and Weyl models.

- For elliptic polynomials of even degree,

$$(2) \quad \lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(Q_{Elliptic,n}(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -\pi b,$$

where b is a positive constant defined as

$$(3) \quad b = -\lim_{T \rightarrow \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t \leq T} Z(t) > 0)}{T},$$

with $Z(t)$ a centered stationary Gaussian process with correlation $\mathbb{E}(Z_0 Z_t) = e^{-t^2/2}$.

- For Weyl polynomials of even degree,

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(Q_{Weyl,n}(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2b,$$

with the same constant b as above.

In particular, for Weyl model, Schehr and Majumdar divided the real axis into two parts: the interval $[-\sqrt{n}, \sqrt{n}]$ and its complement. Due to the phase transition of the autocorrelation function on these two parts, the persistence exponents are expected to be different. Indeed, the authors guessed that persistence probability on $[-\sqrt{n}, \sqrt{n}]$ is of order $\exp(-2b\sqrt{n})$ and the one on the complement is of order n^{-2b_0} , where b_0 is the same as in Kac model. Assuming the independence of real zeroes, they proposed the following approximation for the persistence probability of Weyl polynomials on the whole real axis

$$(5) \quad \mathbb{P}(Q_{Weyl,n}(x) > 0, \forall x \in \mathbb{R}) \sim n^{-2b_0} e^{-2b\sqrt{n}}.$$

In [7], Dembo and Mukherjee gave a powerful method to obtain rigorously the persistence exponent of Gaussian processes. Given a sequence of Gaussian processes converging weakly to a stationary Gaussian process, they provided some necessary conditions on the autocorrelation functions of the Gaussian processes (see Lemma 2.1) to ensure the continuity of persistence exponents (defined as (3)). Using the criteria, they gave a rigorous basis to (2). For generalized Kac polynomials where the coefficients c_i 's satisfy some

regularly varying conditions (see [7, Theorem 1.3]), they also proved that the persistence probability in this case has the same form as (1).

For Weyl polynomials, Dembo and Mukherjee found that the persistence exponent considered on the intervals $J_n = [0, \sqrt{n} - \alpha_n]$ with $\alpha_n = o(\sqrt{n})$ is $-b$, which is a half of the exponent in (4). Naturally, they believed that the interval J_n contributes the main term to persistence probability on the nonnegative axis and they left it as an open question.

In this paper, we would like to complete the picture of the persistence probability of Weyl polynomials in both cases: the nonnegative axis and the whole real axis. Here is our main result which confirms the predictions of Schehr and Majumdar [22], and also of Dembo and Mukherjee [7].

Theorem 1.1. *Consider the Weyl random polynomial defined by*

$$f_n(x) = \sum_{i=0}^n \frac{a_i}{\sqrt{i!}} x^i,$$

where a_i 's are i.i.d. standard normal random variables. The following statements hold.

(a) *As n tends to infinity,*

$$\lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty))}{\sqrt{n}} = -b,$$

where b is the positive constant defined as in (3).

(b) *Assume that n is even, then*

$$\lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(f_n(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2b,$$

with the same constant b as above.

The idea of the proof is as follows. First, using the method of Dembo and Mukherjee, we obtain the persistence exponents for the intervals $[0, \sqrt{n} - \alpha_n]$ (and $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$, respectively) with a suitable sequence (α_n) . While the exponent for $[0, \sqrt{n} - \alpha_n]$ has been already mentioned in [7], the one for $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$ needs a more careful verification, see the proof of Lemma 4.2. Secondly, we show that the contribution from the complement intervals is negligible in the sense that the persistence probability on the complement decays much slower. To do that, instead of considering the persistence of the given process $f_n(x)$, we deal with the persistence of $f_n(x)R_n(x)$ for a suitable positive non random function $R_n(x)$, see Lemma 2.2 and Proof of claim (c2). Finally, to join the persistence probabilities on the intervals, we use the celebrated Slepian's inequality.

It would be interesting to give a rigorous proof for (5). Unfortunately, our method is not sharp enough to do that and we leave the question for future research.

The paper is organized as follows. In Section 2, we recall two key lemmas in [7]. The detailed proofs of Part (a) and Part (b) will be presented respectively in Section 3 and Section 4.

2. PRELIMINARIES

Since a_i 's are i.i.d. random variables of standard normal distribution, the Weyl random polynomial $f_n(x)$ is a Gaussian process with autocorrelation function

$$(6) \quad A_n(x, y) = \text{corr}(f_n(x), f_n(y)) = \frac{\sum_{i=0}^n \frac{(xy)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{x^{2i}}{i!}} \times \sqrt{\sum_{i=0}^n \frac{y^{2i}}{i!}}}.$$

If x and y are fixed, then $A_n(x, y)$ converges to $e^{-(x-y)^2/2}$ as n tends to infinity. It means that the sequence of Weyl random polynomials converges weakly to the centered stationary Gaussian process $Z(t)$ with covariance function $R(t) = e^{-t^2/2}$. Then by heuristic arguments, the persistence probability of Weyl random polynomials might tend to the corresponding one of $Z(t)$. However, in general, the limit implied by heuristic arguments is not true, see [7, p. 89] for a counterexample. To ensure the continuity of persistence exponents, we need some restrictive conditions on the autocorrelation function. The following result combining Theorem 1.6, Remark 1.7 and Lemma 1.8 in [7] gives us such criteria.

Lemma 2.1. *Let \mathcal{S}_+ be the class of all non-negative stationary autocorrelation functions. Then the following assertions hold.*

- (a) *For any centered stationary Gaussian process $\{Z_t\}_{t \geq 0}$ of autocorrelation $A(s, t) = A(0, t - s) \in \mathcal{S}_+$, the nonnegative limit*

$$b(A) = - \lim_{T \rightarrow \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t \leq T} Z(t) > 0)}{T},$$

exists.

- (b) *Let $\{Z_t^{(k)}\}_{t \geq 0}$, $1 \leq k \leq \infty$ be a sequence of centered Gaussian processes of unit variance and nonnegative autocorrelation functions $A_k(s, t)$, such that $A_\infty(s, t) \in \mathcal{S}_+$. Suppose that the following three conditions hold.*

- (b0) *For all $\eta > 0$,*

$$\limsup_{M \rightarrow \infty} \frac{1}{M} \log \mathbb{P} \left(\sup_{t \in [0, M]} Z_t^\infty < M^{-\eta} \right) = -b(A_\infty).$$

- (b1) *We have*

$$\limsup_{k, \tau \rightarrow \infty} \sup_{s \geq 0} \left\{ \frac{\log A_k(s, s + \tau)}{\log \tau} \right\} < -1.$$

- (b2) *There exists a nonnegative autocorrelation function $D \in \mathcal{S}_+$ corresponding to some stationary Gaussian process such that for any finite M , there exist positive $\epsilon_k \rightarrow 0$ satisfying*

$$(1 - \epsilon_k)A_\infty(0, \tau) + \epsilon_k D(0, \tau) \leq A_k(s, s + \tau) \leq (1 - \epsilon_k)A_\infty(0, \tau) + \epsilon_k,$$

for all s, τ such that $\tau \in [0, M]$ and both $s, s + \tau$ belong to the considering interval.

Then

$$\lim_{k, T \rightarrow \infty} \frac{1}{T} \log \mathbb{P} \left(Z_t^{(k)} > 0, \forall t \in [0, T] \right) = -b(A_\infty).$$

The next lemma provides a lower bound on the persistence probability of a differentiable Gaussian process $Z(t)$, assuming a simple condition that the variances of $Z(t)$ and $Z'(t)$ are comparable.

Lemma 2.2. *There is an universal constant $\mu \in (0, 1)$, such that the following statements hold.*

(i) *Suppose that $(Z_t)_{t \in [a,b]}$ is a centered \mathcal{C}^1 Gaussian process satisfying*

$$(7) \quad 2(b-a)^2 \sup_{t \in [a,b]} \mathbb{E}(Z_t'^2) \leq \sup_{t \in [a,b]} \mathbb{E}(Z_t^2).$$

Then

$$\mathbb{P} \left(\inf_{t \in [a,b]} Z_t > 0 \right) \geq \mu.$$

(ii) *Let $I \subset \mathbb{R}$ be an interval and $r : I \mapsto \mathbb{R}$ be a continuous function. Suppose that $(Z_t)_{t \in I}$ is a centered \mathcal{C}^1 Gaussian process with nonnegative autocorrelation satisfying for all $t \in I$,*

$$(8) \quad 2\Delta^2 \mathbb{E} [(Z_t' - r(t)Z_t)^2] \leq \mathbb{E} [Z_t^2],$$

for some positive constant Δ . Then

$$\mathbb{P} \left(\inf_{t \in I} Z_t > 0 \right) \geq \mu^{\lceil \frac{|I|}{\Delta} \rceil}.$$

Proof. Part (i) is exactly Lemma 4.1 in [7]. We now prove (ii). Let us define a new centered Gaussian process as

$$X_t = R(t)Z_t,$$

where

$$R(t) = \exp \left(- \int_0^t r(s) ds \right).$$

Since $R(t) > 0$ for all $t \in I$,

$$(9) \quad \mathbb{P} \left(\inf_{t \in I} Z_t > 0 \right) = \mathbb{P} \left(\inf_{t \in I} X_t > 0 \right).$$

Since $R(t)$ is differentiable, $(X_t)_{t \in I}$ is also a centered \mathcal{C}^1 Gaussian process. Moreover, $X_t' = R(t)[Z_t' - r(t)Z_t]$. Hence, the assumption (8) implies that for all $t \in I$,

$$2\Delta^2 \mathbb{E}(X_t'^2) \leq \mathbb{E}(X_t^2).$$

Now, we divide I into $\lceil \frac{|I|}{\Delta} \rceil$ small intervals of length Δ . Then the condition (7) holds for all the small intervals. Thus using Slepian's inequality and (i), we obtain

$$(10) \quad \mathbb{P} \left(\inf_{t \in I} X_t > 0 \right) \geq \mu^{\lceil \frac{|I|}{\Delta} \rceil}.$$

Combining this estimate with (9), we get the desired result. \square

3. PERSISTENCE PROBABILITY ON THE NONNEGATIVE AXIS

Before proving Part (a) of Theorem 1.1, we recall the following lemma given by Dembo and Mukherjee [7] that provides the persistence exponent on suitable intervals.

Lemma 3.1. [7, Remark 1.11] *For any sequence (α_n) such that the ratio $\frac{\alpha_n}{\sqrt{n}}$ tends to 0, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} (f_n(x) > 0, \forall x \in [0, \sqrt{n} - \alpha_n]) = -b,$$

with the constant b defined as in (3).

Throughout this paper, we always consider the sequence (α_n) defined as

$$\alpha_n = \frac{\sqrt{n}}{\log n}.$$

Part (a) of Theorem 1.1 follows from the following upper bound and lower bound on the persistence probability

$$(11) \quad \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty)) \leq -b,$$

and

$$(12) \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty)) \geq -b,$$

with b as in (3).

Proof of (11). The upper bound (11) follows from Lemma 3.1 and a simple observation that

$$\mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty)) \leq \mathbb{P}(f_n(x) > 0, \forall x \in [0, \sqrt{n} - \alpha_n]).$$

Proof of (12). By using Slepian's inequality (see [1, Theorem 2.2.1]),

$$(13) \quad \begin{aligned} P_n &= \mathbb{P}(f_n(x) > 0 \forall x \geq 0) \\ &\geq \mathbb{P}(f_n(x) > 0 \forall 0 \leq x \leq \sqrt{n} - \alpha_n) \times \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} - \alpha_n < x < \sqrt{n} + \alpha_n) \\ &\quad \times \mathbb{P}(f_n(x) > 0 \forall x \geq \sqrt{n} + \alpha_n) =: A_n \times B_n \times C_n. \end{aligned}$$

Thanks to the inequality (13), the lower bound (12) follows from the following claims

(c1)

$$\liminf_{n \rightarrow \infty} \frac{\log A_n}{\sqrt{n}} = -b,$$

(c2)

$$\liminf_{n \rightarrow \infty} \frac{\log B_n}{\sqrt{n}} \geq 0,$$

(c3)

$$\liminf_{n \rightarrow \infty} \frac{\log C_n}{\sqrt{n}} \geq 0.$$

Proof of Claim (c1). This claim is a consequence of Lemma 3.1. □

Proof of Claim (c3). We first notice that for all $0 \leq i \leq n-1$,

$$\frac{x^i}{\sqrt{i!}} = \frac{\sqrt{i+1}}{x} \frac{x^{i+1}}{\sqrt{(i+1)!}} \leq \frac{\sqrt{n}}{x} \frac{x^{i+1}}{\sqrt{(i+1)!}}$$

Hence, for all $0 \leq i \leq n-1$,

$$\frac{x^i}{\sqrt{i!}} \leq \left(\frac{\sqrt{n}}{x} \right)^{n-i} \frac{x^n}{\sqrt{n!}}.$$

Therefore, if $x \geq \sqrt{n} + \alpha_n$ then

$$\sum_{i=0}^{n-1} \frac{x^i}{\sqrt{i!}} \leq \frac{x^n}{\sqrt{n!}} \sum_{k=1}^n \left(\frac{\sqrt{n}}{x} \right)^k \leq \frac{x^n}{\sqrt{n!}} \frac{\sqrt{n}}{x - \sqrt{n}} \leq \frac{x^n}{\sqrt{n!}} \frac{\sqrt{n}}{\alpha_n} = \frac{x^n}{\sqrt{n!}} \log n.$$

Consequently, for all $x \geq \sqrt{n} + \alpha_n$,

$$(14) \quad |f_{n-1}(x)| = \left| \sum_{i=0}^{n-1} a_i \frac{x^i}{\sqrt{i!}} \right| \leq \max_{0 \leq i \leq n-1} |a_i| \times \sum_{i=0}^{n-1} \frac{x^i}{\sqrt{i!}} \leq \max_{0 \leq i \leq n-1} |a_i| \times \log n \frac{x^n}{\sqrt{n!}}.$$

Since $f_n(x) = f_{n-1}(x) + a_n x^n / \sqrt{n!}$, using (14) we get

$$\begin{aligned} \mathbb{P}(f_n(x) > 0 \forall x \geq \sqrt{n} + \alpha_n) &\geq \mathbb{P}\left(a_n \frac{x^n}{\sqrt{n!}} \geq 2|f_{n-1}(x)| \forall x \geq \sqrt{n} + \alpha_n\right) \\ &\geq \mathbb{P}\left(a_n \geq 2 \log n \max_{0 \leq i \leq n-1} |a_i|\right) \\ &\geq \mathbb{P}\left(a_n \geq 2 \log^2 n\right) \mathbb{P}\left(\max_{0 \leq i \leq n-1} |a_i| \leq \log n\right) \\ &= \mathbb{P}\left(a_n \geq 2 \log^2 n\right) \mathbb{P}\left(|a_0| \leq \log n\right)^n \\ &\geq \exp(-4 \log^4 n), \end{aligned}$$

for all n large enough. This estimate implies (c3). \square

For the proof of Claim (c2), to apply Lemma 2.2 (ii), we will find a non random function r_n such that $\mathbb{E}[(f'_n(x) - r_n(x)f_n(x))^2]$ is comparable to $\mathbb{E}[f_n(x)^2]$ for all $x \in (\sqrt{n} - \alpha_n, \sqrt{n} + \alpha_n)$. To minimize $\mathbb{E}[(f'_n(x) - r_n(x)f_n(x))^2]$, we should choose r_n as close to f'_n/f_n as possible. Since f_n (also f'_n) is random and its expectation is zero, a natural and computable candidate for r_n is the following

$$(15) \quad r_n(x) \approx \left(\frac{\mathbb{E}[f'_n(x)^2]}{\mathbb{E}[f_n(x)^2]} \right)^{1/2}.$$

We observe that $\mathbb{E}[f_n(x)^2] = \sum_0^n \frac{x^{2i}}{i!}$ and $\mathbb{E}[f'_n(x)^2] = \sum_0^n \frac{i^2 x^{2i-2}}{i!}$. Moreover, the sequences $(\frac{x^{2i}}{i!})_{i \geq 0}$ and $(\frac{i^2 x^{2i-2}}{i!})_{i \geq 0}$ are increasing when $i \leq x^2$ and decreasing when $i > x^2$. Thus the two sequences $(\frac{x^{2i}}{i!})_{0 \leq i \leq n}$ and $(\frac{i^2 x^{2i-2}}{i!})_{0 \leq i \leq n}$ attain the maximum values at $i_{\max} = [x^2] \wedge n$. Furthermore, we will see later that

$$\left(\frac{\mathbb{E}[f'_n(x)^2]}{\mathbb{E}[f_n(x)^2]} \right)^{1/2} \approx \left(\frac{i_{\max}^2 x^{2i_{\max}-2}}{i_{\max}!} / \frac{x^{2i_{\max}}}{i_{\max}!} \right)^{1/2} \approx \frac{i_{\max}}{x}.$$

Hence a promising candidate for $r_n(x)$ is $(x^2 \wedge n)/x$. The detail computations are carried below. The choice of $r_n(x)$ as in (15) is quite natural and robust. This strategy may be applied to other persistence problems, specially for the case of centered \mathcal{C}^1 Gaussian processes, to discard intervals which do not contribute to the persistence exponent. In particular, we use the same method to study the persistence probability of a random polynomial arising from evolutionary game theory, see [6].

Proof of Claim (c2). Let us define a continuous function

$$r_n(x) = x \wedge \frac{n}{x}.$$

We claim that there exists a positive constant Δ , such that for all $x \in (\sqrt{n} - \alpha_n, \sqrt{n} + \alpha_n)$,

$$(16) \quad 2\Delta^2 \mathbb{E}[(f'_n(x) - r_n(x)f_n(x))^2] \leq \mathbb{E}[f_n(x)^2].$$

Combining this claim with Lemma 2.2 (ii), we get

$$\mathbb{P}(f_n(x) > 0 \forall x \in (\sqrt{n} - \alpha_n, \sqrt{n} + \alpha_n)) \geq \mu^{\lceil 2\alpha_n/\Delta \rceil},$$

which implies (c2), by noting that $\alpha_n = o(\sqrt{n})$.

Now it remains to prove (16). Let us define

$$k = [x^2] \wedge n \in [n - 2\sqrt{n}\alpha_n, n].$$

We observe that

$$\begin{aligned} \mathbb{E}[f_n(x)^2] &= \sum_{i=0}^n \frac{x^{2i}}{i!} \geq \sum_{i=k-\lfloor\sqrt{k}\rfloor}^k \frac{x^{2i}}{i!} = \frac{x^{2k}}{k!} \sum_{i=k-\lfloor\sqrt{k}\rfloor}^k \frac{x^{2i-2k}k!}{i!} \\ &= \frac{x^{2k}}{k!} \sum_{j=0}^{\lfloor\sqrt{k}\rfloor} \frac{x^{-2j}k!}{(k-j)!} \geq \frac{x^{2k}}{k!} \sum_{j=0}^{\lfloor\sqrt{k}\rfloor} x^{-2j}(k-j)^j \\ &= \frac{x^{2k}}{k!} \sum_{j=0}^{\lfloor\sqrt{k}\rfloor} \left(\frac{k}{x^2}\right)^j \left(\frac{k-j}{k}\right)^j \\ (17) \quad &\geq \frac{x^{2k}}{3k!} \sum_{j=0}^{\lfloor\sqrt{k}\rfloor} \left(\frac{k}{x^2}\right)^j, \end{aligned}$$

where we used the inequality that $(1 - \frac{j}{k})^j \geq (1 - \frac{1}{\sqrt{k}})^{\sqrt{k}} \geq 1/3$, for k large enough and $j \leq \sqrt{k}$.

We also have

$$\begin{aligned} \mathbb{E}[(f'_n(x) - r_n(x)f_n(x))^2] &= \sum_{i=0}^n \left(\frac{i}{x} - r_n(x)\right)^2 \frac{x^{2i}}{i!} = \sum_{i=0}^n \left(\frac{i - (n \wedge x^2)}{x}\right)^2 \frac{x^{2i}}{i!} \\ (\text{use } k = [x^2] \wedge n) \quad &\leq 2 \sum_{i=0}^n \frac{(i-k)^2 + 1}{x^2} \frac{x^{2i}}{i!} = \frac{2x^{2k-2}}{k!} \sum_{i=0}^n [(i-k)^2 + 1] \frac{x^{2i-2k}k!}{i!} \\ &= \frac{2x^{2k-2}}{k!} \left[\sum_{j=1}^k (j^2 + 1) \frac{x^{-2j}k!}{(k-j)!} + \sum_{j=0}^{n-k} (j^2 + 1) \frac{x^{2j}k!}{(k+j)!} \right] \\ (18) \quad &= \frac{2x^{2k-2}}{k!} [S_1 + S_2]. \end{aligned}$$

The first term can be written as

$$(19) \quad S_1 = \sum_{j=1}^k (j^2 + 1) \left(\frac{k}{x^2}\right)^j \frac{k!}{(k-j)!k^j}.$$

Moreover,

$$\begin{aligned} \frac{k!}{(k-j)!k^j} &= \prod_{i=1}^j \left(1 - \frac{j-i}{k}\right) \leq \prod_{i=1}^{\lfloor j/2 \rfloor} \left(1 - \frac{j-i}{k}\right) \leq \left(1 - \frac{\lfloor j/2 \rfloor}{2k}\right)^{\lfloor j/2 \rfloor} \\ &\leq \exp\left(-\frac{\lfloor j/2 \rfloor \lfloor j/2 \rfloor}{k}\right) \leq \exp\left(1 - \frac{j^2}{4k}\right). \end{aligned}$$

Therefore,

$$(20) \quad \begin{aligned} S_1 &\leq e \sum_{j=1}^k (j^2 + 1) \left(\frac{k}{x^2}\right)^j \exp\left(-\frac{j^2}{4k}\right) \\ &\leq e \left[\sum_{j=1}^{[\sqrt{k}]} (j^2 + 1) \left(\frac{k}{x^2}\right)^j + \sum_{j=[\sqrt{k}]}^k (j^2 + 1) \left(\frac{k}{x^2}\right)^j \exp\left(-\frac{j^2}{4k}\right) \right]. \end{aligned}$$

For the first sum,

$$(21) \quad \sum_{j=1}^{[\sqrt{k}]} (j^2 + 1) \left(\frac{k}{x^2}\right)^j \leq 2k \sum_{j=1}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j.$$

For the second sum, using $k \leq x^2$ we obtain that

$$(22) \quad \begin{aligned} &\sum_{j=[\sqrt{k}]}^k (j^2 + 1) \left(\frac{k}{x^2}\right)^j \exp\left(-\frac{j^2}{4k}\right) \leq 2 \left(\frac{k}{x^2}\right)^{[\sqrt{k}]} \sum_{j=[\sqrt{k}]}^k j^2 \exp\left(-\frac{j^2}{4k}\right) \\ &\leq 2 \left(\frac{k}{x^2}\right)^{[\sqrt{k}]} \sum_{t=1}^{[\sqrt{k}]} \sqrt{k} [(t+1)\sqrt{k}]^2 \exp\left(-\frac{t^2}{4}\right) \\ &= 2 \left(\frac{k}{x^2}\right)^{[\sqrt{k}]} k \sqrt{k} \sum_{t=1}^{\infty} (t+1)^2 \exp(-t^2/4) \\ &\leq C \left(\frac{k}{x^2}\right)^{[\sqrt{k}]} k \sqrt{k} \leq Ck \times \sum_{j=1}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j, \end{aligned}$$

for some $C > 0$. Combining (20), (21) and (22), we get

$$(23) \quad S_1 \leq Ck \sum_{j=1}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j,$$

for some positive constant C . We now estimate the second term S_2 . If $x \geq \sqrt{n}$, then $k = n$ and thus $S_2 = 1$. Assume that $x \leq \sqrt{n}$, then $k = [x^2] \geq x^2 - 1$. Hence $x^{2j} \leq (k+1)^j \leq 3k^j$ for all $j \leq n - k \leq k$. Using the fact that

$$\frac{k!k^j}{(k+j)!} = \prod_{i=1}^j \frac{k}{k+i} \leq \left(\frac{k}{k+j/2}\right)^{j/2} = \left(1 - \frac{j}{2k+j}\right)^{j/2} \leq \exp\left(\frac{-j^2}{2(2k+j)}\right) \leq e^{-j^2/6k}$$

and the same argument for (22), we have

$$S_2 = \sum_{j=0}^{n-k} (j^2 + 1) \frac{x^{2j} k!}{(k+j)!} \leq C' \sum_{j=0}^{n-k} j^2 \frac{k!k^j}{(k+j)!} \leq C' \sum_{j=0}^{n-k} j^2 e^{-j^2/6k} \leq Ck\sqrt{k},$$

for some constants $C > C' > 0$. Since $k \leq x^2 \leq k+1$,

$$\sum_{j=0}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j \geq [\sqrt{k}] \left(\frac{k}{x^2}\right)^{[\sqrt{k}]} \geq [\sqrt{k}] \left(\frac{k}{k+1}\right)^{[\sqrt{k}]} \geq \frac{[\sqrt{k}]}{2}.$$

It follows from the last two estimates that

$$(24) \quad S_2 \leq Ck \sum_{j=0}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j,$$

for some positive constant C .

Combining (18), (23) and (24), we obtain

$$(25) \quad \mathbb{E} [(f'_n(x) - r_n(x)f_n(x))^2] \leq \frac{4Cx^{2k-2}}{k!} k \sum_{j=0}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j \leq \frac{4Cx^{2k}}{k!} \sum_{j=0}^{[\sqrt{k}]} \left(\frac{k}{x^2}\right)^j,$$

since $k \leq x^2$. In conclusion, (16) follows from (17) and (25) and this completes the proof of Part (a). \square

4. PERSISTENCE PROBABILITY ON THE WHOLE REAL AXIS

We first recall the autocorrelation function

$$A_n(x, y) = \text{corr}(f_n(x), f_n(y)) = \frac{\sum_{i=0}^n \frac{(xy)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{x^{2i}}{i!}} \times \sqrt{\sum_{i=0}^n \frac{y^{2i}}{i!}}}.$$

Let us define

$$g_n(x) = e^{-x} \sum_{i=0}^n \frac{x^i}{i!}.$$

Then

$$(26) \quad A_n(x, y) = \frac{g_n(xy)}{\sqrt{g_n(x^2)g_n(y^2)}} e^{-(x-y)^2/2}.$$

In the following lemma, we study the asymptotic behavior of $g_n(x)$.

Lemma 4.1. *The following statements hold.*

- (i) *If $n \in 2\mathbb{N}$, then $g_n(x)$ is nonnegative for all x , and thus $A_n(x, y)$ is nonnegative for all x, y .*
- (ii) *For any fixed $M \geq 0$,*

$$\lim_{n \rightarrow \infty} \sup_{-M^2 \leq x \leq n - \sqrt{n}\alpha_n} |g_n(x) - 1| = 0.$$

- (iii) *For all $n \in 2\mathbb{N}$ large enough and $\sqrt{n}\alpha_n - n \leq x \leq 0$,*

$$g_n(x) \leq 1 + e^{2|x|} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}}.$$

- (iv) *For any fixed $M \geq 0$, define*

$$\begin{aligned} \theta_n(M) := & \sup_{2M \leq s \leq \sqrt{n} - \alpha_n} \left[(s^2 + s + 1) \sup_{s(s-2M) \leq x \leq s(s+2M)} (|g'_n(x)| + |g''_n(x)|) \right] \\ & + (4M^2 + 2M + 1) \sup_{-M^2 \leq x \leq 9M^2} (|g''_n(x)| + |g'_n(x)|). \end{aligned}$$

Then $\theta_n(M) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. It is obvious that $g_n(x) \geq 0$ for $x \geq 0$. Let us consider $x \leq 0$. We have

$$(27) \quad g'_n(x) = -e^{-x} \frac{x^n}{n!}.$$

If n is even, $g'_n(x) \leq 0$ for all $x \in \mathbb{R}$. Thus the function $g_n(x)$ is decreasing, so $g_n(x) \geq g_n(0) = 1$ for all $x \leq 0$. Therefore, $g_n(x) \geq 0$ for all $x \in \mathbb{R}$, and (i) holds.

We turn to prove (ii). Since $g_n(x)$ converges uniformly to 1 for $-M^2 \leq x \leq 0$,

$$(28) \quad \lim_{n \rightarrow \infty} \sup_{-M^2 \leq x \leq 0} |g_n(x) - 1| = 0.$$

For $0 \leq x \leq n - \sqrt{n}\alpha_n$, we observe that

$$0 \leq 1 - g_n(x) = \mathbb{P}(\text{Poi}(x) > n) \leq 1 - g_n(k) = \mathbb{P}(\text{Poi}(k) > n),$$

where $\text{Poi}(\mu)$ stands for the Poisson distribution with intensity μ and

$$k = \lceil n - \sqrt{n}\alpha_n \rceil.$$

Let $\lambda_n = \alpha_n/\sqrt{n}$. Using Markov's inequality,

$$\begin{aligned} \mathbb{P}(\text{Poi}(k) > n) &= \mathbb{P}(e^{\lambda_n \text{Poi}(k)} > e^{\lambda_n n}) \\ &\leq e^{-\lambda_n n} \mathbb{E}(e^{\lambda_n \text{Poi}(k)}) = \exp(-\lambda_n n + k(e^{\lambda_n} - 1)). \end{aligned}$$

Since $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, for all n large enough $e^{\lambda_n} \leq 1 + \lambda_n + 3\lambda_n^2/4$. Therefore,

$$\log \mathbb{P}(\text{Poi}(k) > n) \leq -\lambda_n(n - k) + \frac{3n\lambda_n^2}{4} \leq -\frac{\alpha_n^2}{4}.$$

Hence as $n \rightarrow \infty$,

$$(29) \quad \sup_{0 \leq x \leq n - \sqrt{n}\alpha_n} |g_n(x) - 1| \leq \exp(-\alpha_n^2/4) \rightarrow 0.$$

Combining (28) and (29) yields (ii).

We now prove (iii). Define for $t \geq 0$,

$$\ell_n(t) = e^t \sum_{i=0}^n \frac{(-t)^i}{i!}.$$

Since n is even, $\ell'_n(t) = \frac{e^t t^n}{n!}$. By Stirling formula and noting that $\log(1 - x) \leq -x - \frac{x^2}{4}$ for positive x small enough,

$$(30) \quad \begin{aligned} \frac{e^t t^n}{n!} &\leq \frac{e^{t+n}}{\sqrt{2\pi n}} \left(\frac{t}{n}\right)^n = \frac{e^{t+n}}{\sqrt{2\pi n}} \exp\left[n \log\left(1 - \frac{n-t}{n}\right)\right] \\ &\leq \frac{e^{t+n}}{\sqrt{2\pi n}} \exp\left[n\left(\frac{t-n}{n} - \frac{(t-n)^2}{4n^2}\right)\right] \leq \frac{e^{2t}}{\sqrt{2\pi n}} e^{-\alpha_n^2/4}, \end{aligned}$$

for $t \leq n - \alpha_n\sqrt{n}$ and n large enough. Hence, for $\sqrt{n}\alpha_n - n \leq x \leq 0$,

$$g_n(x) = \ell_n(|x|) = 1 + \int_0^{|x|} \frac{e^t t^n}{n!} dt \leq 1 + \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}} \int_0^{|x|} e^{2t} dt \leq 1 + e^{2|x|} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}},$$

thus (iii) follows.

For (iv), we observe that by (27),

$$(31) \quad g''_n(x) = -e^{-x} \frac{x^{n-1}}{(n-1)!} \left(\frac{n-x}{n}\right).$$

Using (27) and (31), for all $|x| \leq n$,

$$(32) \quad |g'_n(x)| + |g''_n(x)| \leq 3e^{-x} \frac{|x|^{n-1}}{(n-1)!}.$$

Therefore, by Stirling's approximation,

$$(33) \quad \lim_{n \rightarrow \infty} (4M^2 + 2M + 1) \sup_{-M^2 \leq x \leq 9M^2} (|g''_n(x)| + |g'_n(x)|) = 0.$$

For $x \geq 0$, define

$$G_n(x) = e^{-x} \frac{x^{n-1}}{(n-1)!}.$$

Then $G'_n(x) = e^{-x} x^{n-2} (n-1-x)/(n-1)!$. Hence the function G_n is increasing on $(0, n-1)$. Therefore, for all $2M \leq s \leq \sqrt{n} - \alpha_n$,

$$(34) \quad \sup_{s(s-2M) \leq x \leq s(s+2M)} G_n(x) \leq G_n(s(s+2M)),$$

and following the same argument as in (30), we can prove that

$$(35) \quad \lim_{n \rightarrow \infty} \sup_{2M \leq s \leq \sqrt{n} - \alpha_n} (s^2 + s + 1) G_n(s(s+2M)) = 0.$$

Combining (32) – (35), we get (iv). \square

As in Lemma 3.1, the following lemma shows the interval providing the main contribution to the persistence probability on the whole real axis.

Lemma 4.2. *As $n \in 2\mathbb{N}$ tends to infinity,*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]) = -2b.$$

To prove Lemma 4.2, we shall verify the conditions on the sequence of correlation functions given in Lemma 2.1. Assume that n is even. We consider the Weyl polynomial as a Gaussian process on the interval $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$, whose size is equal to $2(\sqrt{n} - \alpha_n)$. To recover the exact form in the statements of Lemma 2.1, one should make the change of variable to transform the interval $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$ to $[0, 2(\sqrt{n} - \alpha_n)]$. However, for the simplicity of notation, we work directly with the interval $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$.

Proof. By Lemma 2.1, it is sufficient to prove that Conditions (b0)–(b2) hold.

Verification of (b0). This condition is relatively mild and is shown to hold for $A_\infty(0, t) = e^{-t^2/2}$ in [7, Lemma 1.11].

Verification of (b1). Let us denote

$$h_n(s, \tau) = \frac{g_n(s(s+\tau))}{\sqrt{g_n(s^2)g_n((s+\tau)^2)}}.$$

Then $A_n(s, s+\tau) = h_n(s, \tau)e^{-\tau^2/2}$. Using Lemma 4.1 (ii), we have $\frac{1}{2} \leq g_n(x) \leq 2$ for all n large enough and $0 \leq x \leq n - \sqrt{n}\alpha_n$. Therefore,

$$(36) \quad A_n(s, s+\tau) \leq 2g_n(s(s+\tau))e^{-\tau^2/2}.$$

In addition, if $s(s+\tau) \geq 0$ then $g_n(s(s+\tau)) \leq 2$, and thus

$$A_n(s, s+\tau) \leq 4e^{-\tau^2/2},$$

so (b1) is verified.

Assume that $s(s + \tau) < 0$. Then $s < 0$ and $s + \tau > 0$, so

$$(37) \quad a := |s(s + \tau)| = |s|(\tau - |s|) \leq \tau^2/4.$$

Moreover, $\tau \leq 2(\sqrt{n} - \alpha_n)$, since $|s|, |s + \tau| \leq \sqrt{n} - \alpha_n$. Using (36), (37) and Lemma 4.1 (iii), for n and τ large enough

$$A_n(s, s + \tau) \leq 2e^{-\tau^2/2} \left(1 + e^{2a} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}} \right) \leq 4e^{-\tau^2/2} \left(1 + e^{\tau^2/2} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}} \right) \leq \tau^{-2},$$

since $0 < \tau \leq 2(\sqrt{n} - \alpha_n)$. Hence, (b1) is verified.

Verification of (b2). We claim that for any fixed number M as $n \rightarrow \infty$,

$$(38) \quad \delta_n = \delta_n(M) := \sup_{\tau \in [0, M]} \sup_{s: |s|, |s + \tau| \leq \sqrt{n} - \alpha_n} \frac{2|h_n(s, \tau) - 1|}{\tau^2} \rightarrow 0.$$

Assuming (38), we can prove (b2). Indeed, let (D_t) be the centered Gaussian process corresponding to the autocorrelation e^{-t^2} and $\epsilon_n = \delta_n/\kappa$, where

$$\kappa = \kappa(M) = \inf_{|x| \leq M^2/2} \frac{|e^x - 1|}{|x|}.$$

Then for all $\tau \in [0, M]$ and s such that $|s|, |s + \tau| \leq \sqrt{n} - \alpha_n$, we have

$$\begin{aligned} A_n(s, s + \tau) &= h_n(s, \tau)e^{-\tau^2/2} \\ &= (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n + e^{-\tau^2/2} \left[(h_n(s, \tau) - 1) - \epsilon_n(e^{\tau^2/2} - 1) \right] \\ &= (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n + \frac{\tau^2 e^{-\tau^2/2}}{2} \left[\frac{2(h_n(s, \tau) - 1)}{\tau^2} - \epsilon_n \frac{(e^{\tau^2/2} - 1)}{\tau^2/2} \right] \\ &\leq (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n + \frac{\tau^2 e^{-\tau^2/2}}{2} [\delta_n - \epsilon_n \kappa] \\ (39) \quad &= (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n, \end{aligned}$$

and

$$\begin{aligned} A_n(s, s + \tau) &= h_n(s, \tau)e^{-\tau^2/2} \\ &= (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n e^{-\tau^2} + e^{-\tau^2/2} \left[(h_n(s, \tau) - 1) + \epsilon_n(1 - e^{-\tau^2/2}) \right] \\ &= (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n e^{-\tau^2} + \frac{\tau^2 e^{-\tau^2/2}}{2} \left[\frac{2(h_n(s, \tau) - 1)}{\tau^2} + \epsilon_n \frac{(1 - e^{-\tau^2/2})}{\tau^2/2} \right] \\ &\geq (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n e^{-\tau^2} + \frac{\tau^2 e^{-\tau^2/2}}{2} [-\delta_n + \epsilon_n \kappa] \\ (40) \quad &= (1 - \epsilon_n)e^{-\tau^2/2} + \epsilon_n e^{-\tau^2}. \end{aligned}$$

Hence Condition (b2) is verified and it remains to show (38).

We first notice that if $\frac{1}{2} \leq A, B, C \leq 2$, then

$$\begin{aligned}
\left| \frac{B}{\sqrt{AC}} - 1 \right| &= \frac{|B^2 - AC|}{\sqrt{AC}(B + \sqrt{AC})} \leq 2|B^2 - AC| \\
&= 2|(B - A)(C - B) + B[(B - A) - (C - B)]| \\
(41) \quad &\leq 4[|(B - A)(C - B)| + |(B - A) - (C - B)|].
\end{aligned}$$

Let us consider $A = g_n(s^2)$, $B = g_n(s(s + \tau))$ and $C = g_n((s + \tau)^2)$. Observe that $s(s + \tau) \geq -\tau^2/4 \geq -M^2/4$. Therefore, $-M^2 < s^2, s(s + \tau), (s + \tau)^2 \leq n - \sqrt{n}\alpha_n$. Hence by Lemma (4.1) (ii), $\frac{1}{2} \leq A, B, C \leq 2$ for all n large enough. Now, we notice that

$$B - A = \int_0^{s\tau} g'_n(u + s^2)du, \quad C - B = \int_0^{s\tau + \tau^2} g'_n(u + s(s + \tau))du.$$

Therefore,

$$\begin{aligned}
&|(B - A) - (C - B)| \\
&= \left| \int_0^{s\tau} [g'_n(u + s^2) - g'_n(u + s(s + \tau))]du + \int_{s\tau}^{s\tau + \tau^2} g'_n(u + s(s + \tau))du \right| \\
&\leq (s\tau)^2 \sup_{x \in [s^2, s(s + 2\tau)]} |g''_n(x)| + \tau^2 \sup_{x \in [s(s + 2\tau), (s + \tau)^2]} |g'_n(x)| \\
(42) \quad &= \tau^2 \left[s^2 \sup_{x \in [s^2, s(s + 2\tau)]} |g''_n(x)| + \sup_{x \in [s(s + 2\tau), (s + \tau)^2]} |g'_n(x)| \right] =: \tau^2 F_n(s).
\end{aligned}$$

Observe that $s(s + 2\tau) \geq -\tau^2 \geq -M^2$ for all $s \in \mathbb{R}$ and $\tau \in [0, M]$. Therefore, if $|s| \leq 2M$ then $-M^2 \leq s^2, s(s + 2\tau), (s + \tau)^2 \leq 9M^2$. Thus

$$(43) \quad \sup_{|s| \leq 2M} F_n(s) \leq (4M^2 + 1) \sup_{-M^2 \leq x \leq 9M^2} (|g''_n(x)| + |g'_n(x)|).$$

If $|s| \geq 2M$, then $s(s + 2\tau) \geq 0$, since $\tau \in [0, M]$. In addition, we can prove that

$$|s|(|s| - 2M) \leq s^2, s(s + 2\tau), (s + \tau)^2 \leq |s|(|s| + 2M).$$

Hence for $|s| \geq 2M$,

$$(44) \quad F_n(s) \leq (s^2 + 1) \sup_{|s|(|s| - 2M) \leq x \leq |s|(|s| + 2M)} (|g''_n(x)| + |g'_n(x)|).$$

Combining (43) and (44), we get

$$\sup_{s: |s|, |s + \tau| \leq \sqrt{n} - \alpha_n} F_n(s) \leq \theta_n(M),$$

with $\theta_n(M)$ as in Lemma 4.1 (iv). Combining this with (42), we obtain

$$(45) \quad |(B - A) - (C - B)| \leq \tau^2 \theta_n(M).$$

Similarly, we can prove that

$$\begin{aligned}
|(B - A)(C - B)| &\leq \tau^2 \left(|s| \sup_{x \in [s^2, s(s + \tau)]} |g'_n(x)| \right) \left(|s + \tau| \sup_{x \in [s(s + \tau), (s + \tau)^2]} |g'_n(x)| \right) \\
(46) \quad &\leq \theta_n^2(M) \tau^2.
\end{aligned}$$

It follows from (41), (45) and (46) that $|h_n(s, \tau) - 1| \leq 4[\theta_n^2(M) + \theta_n(M)]\tau^2$. Hence (38) holds, since $\theta_n(M) \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 4.1 (iv). \square

Proof of Theorem 1.1 (b). Using Lemma 4.2 and the same arguments for (11), we get

$$(47) \quad \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in \mathbb{R}) \leq -2b.$$

On the other hand, by analogous arguments for (12), the lower bound that

$$(48) \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in \mathbb{R}) \geq -2b$$

follows from Lemma 4.2 and the following claims

(d1)

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in (-\infty, -\sqrt{n} + \alpha_n]) \geq 0,$$

(d2)

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [\sqrt{n} - \alpha_n, \infty)) \geq 0.$$

By symmetry, the law of the random Weyl polynomial on the interval $(-\infty, -\sqrt{n} + \alpha_n]$ is as on the interval $[\sqrt{n} - \alpha_n, \infty)$. Thus Claim (d1) is equivalent to Claim (d2). On the other hand, (d2) follows from Claims (c2) and (c3) as in Section 3. \square

Acknowledgments. We would like to thank anonymous referees for carefully reading the manuscript and many valuable comments. The first author is supported by the fellowship of the Japan Society for the Promotion of Science and the Grant-in-Aid for JSPS fellows Number 17F17319. The second author is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.03-2017.316.

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VAN HAO CAN, RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, 606–8502 KYOTO, JAPAN & INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET, 10307 HANOI, VIETNAM

VIET-HUNG PHAM, INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET, 10307 HANOI, VIETNAM