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PERSISTENCE PROBABILITY OF RANDOM WEYL POLYNOMIAL

VAN HAO CAN AND VIET-HUNG PHAM

ABSTRACT. In this paper, using the method proposed by Dembo and Mukherjee [5], we obtain the persistence exponents of random Weyl polynomials in both cases: half nonnegative axis and the whole real axis. Our result is a confirmation to the predictions of Schehr and Majumdar [22].

1. INTRODUCTION

Let $\{a_i\}_{i=0}^{\infty}$ denote a sequence of i.i.d. random variables of zero mean and unit variance. Consider random algebraic polynomials

$$Q_n(x) = c_0 a_0 + c_1 a_1 x + \dots + c_n a_n x^n,$$

where c_i 's are fixed constants depending on the models.

Study the number of real roots of random polynomials is a topic of much interest, see two standard monographs [3, 11]. A natural question is to calculate the moments and to consider limit theorems. When the common law of a_i 's is nondegenerate, Kac [14] provides the celebrated Kac formula to calculate the moments of the number of real roots expressed in integral forms. Especially, for Gaussian case where $a_i \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$, Edelman and Kostlan [10] give the geometric meaning of the expectation. For Kac polynomials ($c_0 = \dots = c_n = 1$), Ibragimov and Maslova [13] prove the universality of expected number of real roots, and then Maslova [18, 19] provides the universality of the variance and also a universal central limit theorem. For recent results of universality and central limit theorems for other models, we refer to [4, 9, 8, 20, 23].

In this paper, we consider another interesting object that is the persistence probability

$$\mathbb{P}(Q_n(x) > 0, \forall x \in J_n),$$

for some fixed intervals J_n . Such probabilities are extremely important with applications in reliability theory and statistical physics, see [2, 21]. They are well-studied for stationary Gaussian processes with nonnegative covariance function, but for general structure of covariance, it remains a big challenge, see [12]. In connection with studies of the number of real roots, we observe that if a_i 's obey a symmetric law,

$$\mathbb{P}(Q_n(x) \text{ has no real roots}) = 2\mathbb{P}(Q_n(x) > 0, \forall x \in \mathbb{R}).$$

For Kac polynomials ($c_0 = \dots = c_n = 1$), a very early result by Littewood and Offord [16, 17] shows that if n is even, $\mathbb{P}(N_{Kac,n} = 0) = O(1/\log n)$. Notice that if n is odd, the persistence probability is zero. Later on, Dembo et al [7] provide a precise asymptotic formula that

$$(1) \quad \mathbb{P}(Q_{Kac,n}(x) > 0, \forall x \in \mathbb{R}) = n^{-4b_0 + o(1)},$$

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where $b_0 = -\lim_{t \rightarrow \infty} t^{-1} \log \mathbb{P}(Y_s > 0, \forall s \in [0, t])$, with Y the centered stationary Gaussian process with correlation $\mathbb{E}(Y_0 Y_t) = 1/\cosh(t/2)$. They first prove (1) for Gaussian case by approximating the random polynomials in small intervals to the Gaussian process Y above. And then, thanks to strong Komlos-Major-Tusnady approximation, they can generalize (1) to any distribution having finite moments of all orders. See also [15] for some estimates for the constant b_0 above.

In Gaussian case $a_i \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$, Schehr and Majumdar propose in [22] a mean-field approximation to revisit the persistence probability of Kac polynomials and also *predict* an asymptotic formula for elliptic and Weyl models.

- For elliptic polynomials, i.e. $c_i = \sqrt{\binom{n}{i}}$ for $i = 1, \dots, n$,

$$(2) \quad \lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(Q_{\text{Elliptic}, n}(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2\pi b,$$

where b is a positive constant defined as

$$(3) \quad b = -\lim_{T \rightarrow \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t \leq T} Z(t) > 0)}{T},$$

with $Z(t)$ a centered stationary Gaussian process with correlation $\mathbb{E}(Z_0 Z_t) = e^{-t^2/2}$.

- For Weyl polynomials, i.e. $c_i = \sqrt{1/i!}$ for $i = 1, \dots, n$,

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(Q_{\text{Weyl}, n}(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2b,$$

with the same constant b as above.

To confirm these predictions, Dembo and Mukherjee propose in [5] a powerful method for a general setting as follows. Given a sequence of Gaussian processes converging weakly to a stationary Gaussian process, they provide some conditions on the autocorrelation functions of these Gaussian processes to ensure the continuity of persistence exponents (defined as (3)). Using this method, they prove the prediction (2). They also consider the case that the sequence $\{c_i\}$ satisfies the regularly varying condition, see [5, Theorem 1.3]). In this case, their result has the same form as (1).

For Weyl polynomials, by this method again, they obtain the persistence exponent considered in the intervals $J_n = [0, \sqrt{n} - \alpha_n]$ with $n^{-1/2}\alpha_n \rightarrow 0$, see Lemma 3.1 below. They find that the persistence exponent for J_n is $-b$, which is a half of the exponent in (4). Thus they believe that the intervals J_n contribute the main term to persistence probabilities on half nonnegative axis and they also leave an open suggestion.

In this paper, we would like to complete the picture for persistence probability of Weyl polynomials in both cases: half nonnegative axis and the whole real axis. More precisely, we confirm the predictions of Schehr and Majumdar [22], and also of Dembo and Mukherjee [5]. Here is our main result.

Theorem 1.1. *Consider the Weyl random polynomial defined by*

$$f_n(x) = \sum_{i=0}^n \frac{a_i}{\sqrt{i!}} x^i,$$

where the coefficients a_i 's are i.i.d. standard normal random variables. We have

(a) As n tends to infinity,

$$\lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty))}{\sqrt{n}} = -b,$$

where the positive constant b is defined as in (3).

(b) Given that n is even, then

$$\lim_{n \rightarrow \infty} \frac{\log \mathbb{P}(f_n(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2b,$$

with the same constant b as above.

The idea of the proof is as follows. At first we apply the method of Dembo and Mukherjee to obtain the persistence exponents for the interval $[0, \sqrt{n} - \alpha_n]$ and $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$ with a suitable sequence (α_n) . The first interval $[0, \sqrt{n} - \alpha_n]$ has been already mentioned in [5]. But for second interval, we need a careful verification. Then, as suggested by Dembo and Mukherjee, we will prove that the contribution from the complement intervals is negligible.

The paper is organized as follows. In Section 2, we recall two key lemmas in [5]. The detailed proofs of Part (a) and Part (b) will be presented respectively in Section 3 and Section 4.

2. PRELIMINARIES

Since $\{a_i\}$ are i.i.d. random variables of standard normal distribution, the Weyl random polynomial $f_n(x)$ is a Gaussian process with autocorrelation function

$$(5) \quad A_n(x, y) = \text{corr}(f_n(x), f_n(y)) = \frac{\sum_{i=0}^n \frac{(xy)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{x^{2i}}{i!}} \times \sqrt{\sum_{i=0}^n \frac{y^{2i}}{i!}}}.$$

If x and y are fixed, then $A_n(x, y)$ converges to $e^{-(x-y)^2/2}$ as n tends to infinity. It means that the sequence of Weyl random polynomials converges weakly to the centered stationary Gaussian process $Z(t)$ with covariance function $R(t) = e^{-t^2/2}$. Then by heuristic arguments, the persistence probability of Weyl random polynomials might tend to the corresponding one of $Z(t)$. However, in general, the limit implied by heuristic arguments is not true, see [5, p. 89] for a counterexample. To ensure the continuity of persistence exponents, we need some restrictive conditions on the autocorrelation function. The following result which is combination of Theorem 1.6, Remark 1.7 and Lemma 1.8 in [5] gives us such conditions.

Lemma 2.1. *Let \mathcal{S}_+ be the class of all non-negative autocorrelation functions. Then the following statements hold.*

(a) *For centered stationary Gaussian process $\{Z_t\}_{t \geq 0}$ of autocorrelation $A(s, t) = A(0, t - s) \in \mathcal{S}_+$, the nonnegative limit*

$$b(A) = - \lim_{T \rightarrow \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t \leq T} Z(t) > 0)}{T},$$

exists.

(b) *Let $\{Z_t^{(k)}\}_{t \geq 0}$, $1 \leq k \leq \infty$ be a sequence of centered Gaussian processes of unit variance and nonnegative autocorrelation functions $A_k(s, t)$, such that $A_\infty(s, t) \in \mathcal{S}_+$. We consider the following conditions on the sequence of autocorrelation functions.*

(b1) We have

$$\limsup_{k, \tau \rightarrow \infty} \sup_{s \geq 0} \left\{ \frac{\log A_k(s, s + \tau)}{\log \tau} \right\} < -1.$$

(b2) There exists a nonnegative autocorrelation function D corresponding to some stationary Gaussian process such that for any finite M , there exist positive $\epsilon_k \rightarrow 0$ satisfying

$$(1 - \epsilon_k)A_\infty(0, \tau) + \epsilon_k D(0, \tau) \leq A_k(s, s + \tau) \leq (1 - \epsilon_k)A_\infty(0, \tau) + \epsilon_k,$$

for all s, τ such that $\tau \in [0, M]$ and both $s, s + \tau$ belong to the considering interval.

(b3) We have $A_k(s, s + \tau) \rightarrow A_\infty(0, \tau)$ pointwise and for some $\eta > 1$,

$$\limsup_{u \downarrow 0} |\log u|^\eta \sup_{1 \leq k \leq \infty} p_k^2(u) < \infty,$$

where $p_k^2(u) := 2 - 2 \inf_{s \geq 0, \tau \in [0, u]} A_k(s, s + \tau)$.

Assume that either (b1) and (b2) hold or (b1) and (b3) hold. Then

$$\lim_{k, T \rightarrow \infty} \frac{1}{T} \log \mathbb{P} \left(Z_t^{(k)} > 0, \forall t \in [0, T] \right) = -b(A_\infty).$$

While Lemma 2.1 shows the convergence of persistence exponent of general Gaussian process under strict conditions of autocorrelation function, Lemma 2.2 below provides a lower bound on the persistence probability of a differentiable Gaussian process $Z(t)$, assuming a simple condition that the variances of $Z(t)$ and $Z'(t)$ are comparable.

Lemma 2.2. [5, Lemma 4.1] *There is a universal constant $\mu \in (0, 1)$, such that the following statements hold.*

(i) If $(Z_t)_{t \in [a, b]}$ is a \mathcal{C}^1 process satisfying

$$2(b - a)^2 \sup_{t \in [a, b]} \mathbb{E}(Z_t'^2) \leq \sup_{t \in [a, b]} \mathbb{E}(Z_t^2),$$

then

$$\mathbb{P} \left(\inf_{t \in [a, b]} Z_t > 0 \right) \geq \mu.$$

(ii) If $(Z_t)_{t \in [0, \beta_n]}$, with $\beta_n \rightarrow \infty$, is a \mathcal{C}^1 Gaussian process with nonnegative autocorrelation satisfying for all $t \leq \beta_n$

$$2 \Delta^2 \mathbb{E}(Z_t'^2) \leq \mathbb{E}(Z_t^2),$$

for some positive constant Δ , then

$$\mathbb{P} \left(\inf_{t \in [0, \beta_n]} Z_t > 0 \right) \geq \mu^{\lceil \frac{\beta_n}{\Delta} \rceil}.$$

Proof. The part (i) is exactly Lemma 4.1 in [5]. The part (ii) is a direct consequence of (i). Indeed, we divide the interval $[0, \beta_n]$ into $\lceil \frac{\beta_n}{\Delta} \rceil$ small intervals of length Δ . Then the condition of (i) is verified in each small interval. Thus using Slepian lemma and (i), we get (ii). \square

3. PERSISTENCE PROBABILITY ON HALF NONNEGATIVE AXIS

It is clear that Part (a) of Theorem 1.1 follows from the following upper bound and lower bound on the persistence probability

$$(6) \quad \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty)) \leq -b,$$

and

$$(7) \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty)) \geq -b,$$

with b as in (3).

Before proving (6) and (7), we recall a result in [5] that gives us the estimate of persistence probability on the main subinterval of \mathbb{R}_+ . By verifying the conditions (b1) and (b2) in Lemma 2.1, Dembo and Mukherjee obtain in [5, Remark 1.11] the persistence probability of Weyl polynomials on interval $[0, \sqrt{n} - \alpha_n]$, for any sequence (α_n) satisfying $\alpha_n \rightarrow \infty$ but $\alpha_n = o(\sqrt{n})$.

Lemma 3.1. [5, Remark 1.11] *We have*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0, \forall x \in [0, \sqrt{n} - \alpha_n]) = -b,$$

with b as in (3).

In throughout this paper, we always consider the sequence (α_n) defined as

$$\alpha_n = \frac{\sqrt{n}}{\log n}.$$

Proof of (6). The upper bound (6) can be easily deduced from Lemma 3.1 and a simple observation that

$$\mathbb{P}(f_n(x) > 0, \forall x \in [0, \infty)) \leq \mathbb{P}(f_n(x) > 0, \forall x \in [0, \sqrt{n} - \alpha_n]).$$

Proof of (7). By using Slepian inequality (see [1, Theorem 2.2.1]), we get

$$(8) \quad \begin{aligned} P_n &= \mathbb{P}(f_n(x) > 0 \forall x \geq 0) \\ &\geq \mathbb{P}(f_n(x) > 0 \forall 0 \leq x \leq \sqrt{n} - \alpha_n) \times \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} - \alpha_n \leq x \leq \sqrt{n} + \alpha_n) \\ &\quad \times \mathbb{P}(f_n(x) > 0 \forall x \geq \sqrt{n} + \alpha_n) =: A_n \times B_n \times C_n. \end{aligned}$$

Thank to the inequality (8), the lower bound (7) follows from the following claims

(c1)

$$\liminf_{n \rightarrow \infty} \frac{\log A_n}{\sqrt{n}} = -b,$$

(c2)

$$\liminf_{n \rightarrow \infty} \frac{\log B_n}{\sqrt{n}} \geq 0,$$

(c3)

$$\liminf_{n \rightarrow \infty} \frac{\log C_n}{\sqrt{n}} \geq 0.$$

Proof of Claim (c1). This claim is a sequence of Lemma 3.1. □

Proof of Claim (c3). We first observe that for all $0 \leq i \leq n-1$,

$$\frac{x^i}{\sqrt{i!}} = \frac{\sqrt{i+1}}{x} \frac{x^{i+1}}{\sqrt{(i+1)!}} \leq \frac{\sqrt{n}}{x} \frac{x^{i+1}}{\sqrt{(i+1)!}}$$

Hence, for all $0 \leq i \leq n-1$,

$$\frac{x^i}{\sqrt{i!}} \leq \left(\frac{\sqrt{n}}{x}\right)^{n-i} \frac{x^n}{\sqrt{n!}}.$$

Therefore, if $x \geq \sqrt{n} + \alpha_n$ then

$$\sum_{i=0}^{n-1} \frac{x^i}{\sqrt{i!}} \leq \frac{x^n}{\sqrt{n!}} \sum_{k=1}^n \left(\frac{\sqrt{n}}{x}\right)^k \leq \frac{x^n}{\sqrt{n!}} \frac{\sqrt{n}}{x - \sqrt{n}} \leq \frac{x^n}{\sqrt{n!}} \frac{\sqrt{n}}{\alpha_n} = \frac{x^n}{\sqrt{n!}} \log n.$$

Consequently, for all $x \geq \sqrt{n} + \alpha_n$,

$$(9) \quad |f_{n-1}(x)| = \left| \sum_{i=0}^{n-1} a_i \frac{x^i}{\sqrt{i!}} \right| \leq \max_{0 \leq i \leq n-1} |a_i| \times \sum_{i=0}^{n-1} \frac{x^i}{\sqrt{i!}} \leq \max_{0 \leq i \leq n-1} |a_i| \times \log n \frac{x^n}{\sqrt{n!}}.$$

We have $f_n(x) = f_{n-1}(x) + a_n x^n / \sqrt{n!}$. Therefore, using (9),

$$\begin{aligned} \mathbb{P}(f_n(x) > 0 \forall x \geq \sqrt{n} + \alpha_n) &\geq \mathbb{P}\left(a_n \frac{x^n}{\sqrt{n!}} \geq 2|f_{n-1}(x)| \forall x \geq \sqrt{n} + \alpha_n\right) \\ &\geq \mathbb{P}\left(a_n \geq 2 \log n \max_{0 \leq i \leq n-1} |a_i|\right) \\ &\geq \mathbb{P}(a_n \geq 2 \log^2 n) \mathbb{P}\left(\max_{0 \leq i \leq n-1} |a_i| \leq \log n\right) \\ &= \mathbb{P}(a_n \geq 2 \log^2 n) \mathbb{P}(|a_0| \leq \log n)^n \\ &\geq \exp(-4(\log n)^4), \end{aligned}$$

for all n large enough. This estimate implies (c3). □

To prove Claim (c2), we will use Lemma 2.2 to show that the persistence probability of f_n on the interval $[\sqrt{n} - \alpha_n, \sqrt{n} + \alpha_n]$ is greater than $\exp(-c\alpha_n)$ for some $c > 0$. To directly apply Lemma 2.2, we need to verify that the variance of $f_n(x)$ is comparable with the one of $f'_n(x)$ for $x \in [\sqrt{n} - \alpha_n, \sqrt{n} + \alpha_n]$. Unfortunately, this fact is not true. To overcome this difficulty, a natural idea is to multiply f_n by a positive non-random function, say r_n , such that $\mathbb{E}((r_n f_n)^2(x))$ is comparable with $\mathbb{E}((r_n f'_n)^2(x))$. Then applying Lemma 2.2 for the process $r_n f_n$, we obtain the lower bound on the persistence probability of $r_n f_n$, and of f_n also. By direct calculations, we observe that the variance of $f_n(x)$ behaves differently when x crosses the value \sqrt{n} . Hence, we will choose the function r_n differently in two intervals $[\sqrt{n} - \alpha_n, \sqrt{n}]$ and $[\sqrt{n}, \sqrt{n} + \alpha_n]$. The detailed computations are carried out as follows.

Proof of Claim (c2). Using Slepian inequality again, we have

$$\begin{aligned} B_n &= \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} - \alpha_n \leq x \leq \sqrt{n} + \alpha_n) \\ &\geq \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} - \alpha_n \leq x \leq \sqrt{n}) \times \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} \leq x \leq \sqrt{n} + \alpha_n). \end{aligned}$$

Therefore, to prove the claim (c2), it suffices to show that

$$(10) \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} - \alpha_n \leq x \leq \sqrt{n}) \geq 0,$$

and

$$(11) \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(f_n(x) > 0 \forall \sqrt{n} \leq x \leq \sqrt{n} + \alpha_n) \geq 0.$$

Proof of (10). We define for $x \in [\sqrt{n} - \alpha_n, \sqrt{n}]$

$$g_n(x) = e^{-x^2/2} f_n(x).$$

We first show that

$$(12) \quad 2 \Delta^2 \mathbb{E}(g'_n(x)^2) \leq \mathbb{E}(g_n(x)^2),$$

for some positive constant Δ . Then applying Lemma 2.2 (ii), we get

$$\mathbb{P}\left(\inf_{x \in [\sqrt{n} - \alpha_n, \sqrt{n}]} f_n(x) > 0\right) = \mathbb{P}\left(\inf_{x \in [\sqrt{n} - \alpha_n, \sqrt{n}]} g_n(x) > 0\right) \geq \mu^{\lceil \alpha_n / \Delta \rceil},$$

which implies (10) by using that $\alpha_n = o(\sqrt{n})$. Now it remains to prove (12).

We observe that

$$\begin{aligned} g'_n(x) &= e^{-x^2/2} (f'_n(x) - x f_n(x)) \\ &= e^{-x^2/2} \left(\sum_{i=1}^n i a_i \frac{x^{i-1}}{\sqrt{i!}} - \sum_{i=0}^n a_i \frac{x^{i+1}}{\sqrt{i!}} \right) \\ &= e^{-x^2/2} \sum_{i=0}^n a_i \left(\frac{i}{x} - x \right) \frac{x^i}{\sqrt{i!}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}(g_n(x)^2) &= e^{-x^2} \sum_{i=0}^n \frac{x^{2i}}{i!}, \\ \mathbb{E}(g'_n(x)^2) &= e^{-x^2} \sum_{i=0}^n \left(\frac{i}{x} - x \right)^2 \frac{x^{2i}}{i!}. \end{aligned}$$

We denote

$$k = \lceil x^2 \rceil \in (n - 2\sqrt{n}\alpha_n, n).$$

Then we have

$$\begin{aligned} \mathbb{E}(g_n(x)^2) &\geq e^{-x^2} \sum_{i=k-\lfloor \sqrt{k} \rfloor}^k \frac{x^{2i}}{i!} = \frac{e^{-x^2} x^{2k}}{k!} \sum_{i=k-\lfloor \sqrt{k} \rfloor}^k \frac{x^{2i-2k} k!}{i!} \\ &= \frac{e^{-x^2} x^{2k}}{k!} \sum_{j=0}^{\lfloor \sqrt{k} \rfloor} \frac{x^{-2j} k!}{(k-j)!} \geq \frac{e^{-x^2} x^{2k}}{k!} \sum_{j=0}^{\lfloor \sqrt{k} \rfloor} x^{-2j} (k-j)^j \\ \text{(use } k = \lceil x^2 \rceil) &= \frac{e^{-x^2} x^{2k}}{k!} \sum_{j=0}^{\lfloor \sqrt{k} \rfloor} \left(\frac{k-j}{x^2} \right)^j \geq \frac{e^{-x^2} x^{2k}}{k!} \sum_{j=0}^{\lfloor \sqrt{k} \rfloor} \left(1 - \frac{j+1}{x^2} \right)^j \\ (13) &\geq \frac{e^{-x^2} x^{2k}}{k!} \frac{\lfloor \sqrt{k} \rfloor + 1}{3}, \end{aligned}$$

where we have used that

$$\left(1 - \frac{j+1}{x^2}\right)^j \geq 1/3,$$

as $(j+1)j \leq ([\sqrt{k}] + 1)[\sqrt{k}] \leq x^2$. Similarly,

$$\begin{aligned} \mathbb{E}(g'_n(x)^2) &= e^{-x^2} \sum_{i=0}^n \left(\frac{i-x^2}{x}\right)^2 \frac{x^{2i}}{i!} \\ (\text{use } k = [x^2]) &\leq 2e^{-x^2} \sum_{i=0}^n \frac{(i-k)^2 + 1}{x^2} \frac{x^{2i}}{i!} = \frac{e^{-x^2} x^{2k}}{k!} \frac{2}{x^2} \sum_{i=0}^n [(i-k)^2 + 1] \frac{x^{2i-2k} k!}{i!} \\ &= \frac{e^{-x^2} x^{2k}}{k!} \frac{2}{x^2} \sum_{j=-k}^{n-k} (j^2 + 1) \frac{x^{-2j} k!}{(k-j)!} \leq \frac{e^{-x^2} x^{2k}}{k!} \frac{2}{x^2} \sum_{j=-k}^{n-k} (j^2 + 1) \frac{k^{-j} k!}{(k-j)!} \\ (14) \quad &= \frac{e^{-x^2} x^{2k}}{k!} \frac{2}{x^2} \left[\sum_{j=0}^{n-k} (j^2 + 1) \frac{k^j k!}{(k+j)!} + \sum_{j=1}^k (j^2 + 1) \frac{k^{-j} k!}{(k-j)!} \right]. \end{aligned}$$

For the first sum, we observe that

$$\frac{(k+j)!}{k!} \geq k^{j/2} \left(k + \frac{j}{2}\right)^{j/2}.$$

Thus

$$\frac{k! k^j}{(k+j)!} \leq \left(\frac{k}{k+j/2}\right)^{j/2} = \left(1 - \frac{j}{2k+j}\right)^{j/2} \leq \exp\left(\frac{-j^2}{2(2k+j)}\right) \leq e^{-j^2/2n}.$$

Here, we used that $(1-x)^y \leq e^{-xy}$ for all $x \in (0, 1)$ and $y > 0$, and $2k \geq 2x^2 - 2 \geq n$. Therefore, by integral approximation,

$$(15) \quad \sum_{j=0}^{n-k} (j^2 + 1) \frac{k^j k!}{(k+j)!} \leq \sum_{j=0}^{n-k} (j^2 + 1) e^{-j^2/2n} = O(n\sqrt{n}).$$

To estimate the second sum of (14), by using Cauchy inequality we get

$$\begin{aligned} \frac{k!}{(k-j)! k^j} &= \frac{k \dots (k-j+1)}{k^j} \leq \left(\frac{kj - \frac{j(j-1)}{2}}{kj}\right)^j = \left(1 - \frac{j-1}{2k}\right)^j \\ &\leq \exp\left(-\frac{j(j-1)}{2k}\right) \leq \sqrt{e} \exp\left(-\frac{j^2}{2k}\right) \leq \sqrt{e} \exp\left(-\frac{j^2}{2n}\right). \end{aligned}$$

Hence, using integral approximation again,

$$(16) \quad \sum_{j=1}^k (j^2 + 1) \frac{k^{-j} k!}{(k-j)!} \leq \sqrt{e} \sum_{j=1}^k (j^2 + 1) \exp\left(-\frac{j^2}{2n}\right) = O(n\sqrt{n}).$$

Combining (14), (15) and (16), we have

$$(17) \quad \mathbb{E}(g'_n(x)^2) \leq C \frac{e^{-x^2} x^{2k}}{k!} n\sqrt{n},$$

for some positive constant C . Now, we can deduce (12) from (13) and (17).

Proof of (11). We use the same arguments as for (10). Define for $x \in [\sqrt{n}, \sqrt{n} + \alpha_n]$

$$h_n(x) = x^{-n} f_n(x).$$

Then

$$\begin{aligned}
h'_n(x) &= x^{-n} (f'_n(x) - (n/x)f_n(x)) \\
&= x^{-n} \left(\sum_{i=1}^n i a_i \frac{x^{i-1}}{\sqrt{i!}} - \sum_{i=0}^n a_i \frac{nx^{i-1}}{\sqrt{i!}} \right) \\
&= x^{-n} \sum_{i=0}^n a_i \frac{(i-n)}{x} \frac{x^i}{\sqrt{i!}}.
\end{aligned}$$

Therefore,

$$(18) \quad \mathbb{E}(h_n(x)^2) = x^{-2n} \sum_{i=0}^n \frac{x^{2i}}{i!}$$

$$(19) \quad \mathbb{E}(h'_n(x)^2) = x^{-2n} \sum_{i=0}^n \frac{(i-n)^2 x^{2i}}{x^2 i!}.$$

To estimate $\mathbb{E}(h_n(x)^2)$, we observe that

$$\begin{aligned}
\sum_{i=0}^n \frac{x^{2i}}{i!} &\geq \sum_{i=n-\lfloor\sqrt{n}\rfloor}^n \frac{x^{2i}}{i!} = \frac{x^{2n}}{n!} \sum_{i=n-\lfloor\sqrt{n}\rfloor}^n \frac{x^{2i-2n} n!}{i!} \\
&= \frac{x^{2n}}{n!} \sum_{j=0}^{\lfloor\sqrt{n}\rfloor} \frac{x^{-2j} n!}{(n-j)!} \geq \frac{x^{2n}}{n!} \sum_{j=0}^{\lfloor\sqrt{n}\rfloor} x^{-2j} (n-j)^j \\
&= \frac{x^{2n}}{n!} \sum_{j=0}^{\lfloor\sqrt{n}\rfloor} \left(\frac{n}{x^2}\right)^j \left(\frac{n-j}{n}\right)^j \\
(20) \quad &\geq \frac{x^{2n}}{3n!} \sum_{j=0}^{\lfloor\sqrt{n}\rfloor} \left(\frac{n}{x^2}\right)^j,
\end{aligned}$$

where we used that for n large enough and $j \leq n$,

$$\left(1 - \frac{j}{n}\right)^j \geq 1/3.$$

On the other hand, to estimate $\mathbb{E}(h'_n(x)^2)$, we have

$$\begin{aligned}
\sum_{i=0}^n \left(\frac{i-n}{x}\right)^2 \frac{x^{2i}}{i!} &= \frac{x^{2n}}{x^2 n!} \sum_{i=0}^n (i-n)^2 \frac{x^{2i-2n} n!}{i!} = \frac{x^{2n}}{x^2 n!} \sum_{j=0}^n j^2 \frac{x^{-2j} n!}{(n-j)!} \\
(21) \quad &= \frac{x^{2n}}{x^2 n!} \sum_{j=0}^n j^2 \left(\frac{n}{x^2}\right)^j \frac{n!}{(n-j)! n^j}.
\end{aligned}$$

It follows from Cauchy inequality that

$$\begin{aligned}
\frac{n!}{(n-j)! n^j} &\leq \left(\frac{nj - \frac{j(j-1)}{2}}{nj}\right)^j = \left(1 - \frac{j-1}{2n}\right)^j \\
&\leq \exp\left(-\frac{j(j-1)}{2n}\right) \leq \sqrt{e} \exp\left(-\frac{j^2}{2n}\right).
\end{aligned}$$

Combining this inequality with (21), we get

$$(22) \quad \begin{aligned} & \sum_{i=0}^n \left(\frac{i-n}{x} \right)^2 \frac{x^{2i}}{i!} \leq \frac{\sqrt{e}x^{2n}}{x^{2n}n!} \sum_{j=0}^n j^2 \left(\frac{n}{x^2} \right)^j \exp\left(-\frac{j^2}{2n}\right) \\ & \leq \frac{\sqrt{e}x^{2n}}{x^{2n}n!} \left[\sum_{j=0}^{\lfloor \sqrt{n} \rfloor} j^2 \left(\frac{n}{x^2} \right)^j + \sum_{j=\lfloor \sqrt{n} \rfloor}^n j^2 \left(\frac{n}{x^2} \right)^j \exp\left(-\frac{j^2}{2n}\right) \right]. \end{aligned}$$

For the first sum, using monotone inequality, we obtain

$$(23) \quad \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} j^2 \left(\frac{n}{x^2} \right)^j \leq \frac{1}{\lfloor \sqrt{n} \rfloor + 1} \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} j^2 \times \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} \left(\frac{n}{x^2} \right)^j \leq n \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} \left(\frac{n}{x^2} \right)^j.$$

On the other hand, using the fact that $x^2 \geq n$ and the integral approximation,

$$(24) \quad \begin{aligned} & \sum_{j=\lfloor \sqrt{n} \rfloor}^n j^2 \left(\frac{n}{x^2} \right)^j \exp\left(-\frac{j^2}{2n}\right) \leq \left(\frac{n}{x^2} \right)^{\lfloor \sqrt{n} \rfloor} \sum_{j=\lfloor \sqrt{n} \rfloor}^n j^2 \exp\left(-\frac{j^2}{2n}\right) \\ & \leq Cn\sqrt{n} \left(\frac{n}{x^2} \right)^{\lfloor \sqrt{n} \rfloor} \leq Cn \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} \left(\frac{n}{x^2} \right)^j, \end{aligned}$$

for some positive constant C . Combining (22), (23) and (24),

$$(25) \quad \sum_{i=0}^n \left(\frac{i-n}{x} \right)^2 \frac{x^{2i}}{i!} \leq \sqrt{e}(1+C) \frac{x^{2n}}{n!} \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} \left(\frac{n}{x^2} \right)^j.$$

Using (18), (19), (20) and (25), we have

$$c\mathbb{E}(h'_n(x)^2) \leq \mathbb{E}(h_n(x)^2),$$

for some positive constant c . Now, we can use the same arguments for (10) to handle with (11). \square

4. PERSISTENCE PROBABILITY ON THE WHOLE REAL AXIS

We start this section with an elementary lemma helping us control the autocorrelation function

$$A_n(x, y) = \text{corr}(f_n(x), f_n(y)) = \frac{\sum_{i=0}^n \frac{(xy)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{x^{2i}}{i!}} \times \sqrt{\sum_{i=0}^n \frac{y^{2i}}{i!}}}.$$

Lemma 4.1. *The following statements hold.*

- (i) *If n is even, the autocorrelation $A_n(x, y)$ is nonnegative for all x, y .*
- (ii) *Let (α_n) be a sequence satisfying $\alpha_n \rightarrow \infty$ and $\alpha_n = o(\sqrt{n})$. Then for all n large enough and $0 \leq x \leq n - \sqrt{n}\alpha_n$,*

$$1 - e^{-\alpha_n^2/4} \leq e^{-x} \sum_{i=0}^n \frac{x^i}{i!} \leq 1.$$

(iii) Let n be an even number. Assume that (α_n) is a sequence satisfying $\alpha_n \rightarrow \infty$ and $\alpha_n = o(\sqrt{n})$. Then for all n large enough and $0 \leq x \leq n - \sqrt{n}\alpha_n$,

$$1 - e^{2x} \frac{x^{n+1}}{(n+1)!} \leq e^x \sum_{i=0}^n \frac{(-x)^i}{i!} \leq 1 + e^{2x} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}}.$$

Proof. To prove (i), it suffices to show that if n is even, then for all $x \in \mathbb{R}$,

$$(26) \quad 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \geq 0.$$

It is clear that (26) holds for $x \geq 0$. For $x \leq 0$, we consider

$$q_n(x) = e^{-x} \sum_{i=0}^n \frac{x^i}{i!}.$$

Since n is even,

$$q'_n(x) = -e^{-x} \frac{x^n}{n!} \leq 0.$$

Thus the function $q_n(x)$ is decreasing, so $q_n(x) \geq q_n(0) = 1$ for all $x \leq 0$. Therefore, (26) holds for $x \leq 0$.

We now prove (ii). The upper bound is trivial, so we only need to show the lower bound. We observe that for all $0 \leq x \leq n - \sqrt{n}\alpha_n$,

$$(27) \quad 1 - e^{-x} \sum_{i=0}^n \frac{x^i}{i!} = \mathbb{P}(Poi(x) > n) \leq \mathbb{P}(Poi(k) > n),$$

where $Poi(\lambda)$ stands for the Poisson distribution with intensity λ and

$$k = \lceil n - \sqrt{n}\alpha_n \rceil.$$

Let (X_i) be a sequence of i.i.d random variables with Poisson distribution of density 1. Then by large deviation principles (see e.g. [6]),

$$(28) \quad \mathbb{P}(Poi(k) > n) = \mathbb{P}\left(\sum_{i=1}^k X_i > n\right) = \mathbb{P}\left(\frac{\sum_{i=1}^k X_i}{k} > \frac{n}{k}\right) \leq C e^{-kI(n/k)},$$

where C is some positive constant and $I(\cdot)$ is the rate function of $Poi(1)$ defined as

$$I(\theta) = \theta \log \theta - (\theta - 1).$$

We have

$$I\left(\frac{n}{k}\right) = \frac{n}{k} \log\left(\frac{n}{k}\right) - \left(\frac{n}{k} - 1\right) \geq \frac{n}{k} \left[\left(\frac{n}{k} - 1\right) - \frac{1}{2} \left(\frac{n}{k} - 1\right)^2\right] - \left(\frac{n}{k} - 1\right) \geq \frac{1}{3} \left(\frac{n}{k} - 1\right)^2.$$

Therefore,

$$(29) \quad kI\left(\frac{n}{k}\right) \geq \frac{k}{3} \left(\frac{n}{k} - 1\right)^2 \geq \frac{\alpha_n^2}{3}.$$

Combining (27), (28) and (29), we get

$$1 - e^{-x} \sum_{i=0}^n \frac{x^i}{i!} \leq C e^{-kI(n/k)} \leq e^{-\alpha_n^2/4},$$

for n large enough.

The lower bound in (iii) follows from the following estimate

$$\left| 1 - e^x \sum_{i=0}^n \frac{(-x)^i}{i!} \right| = \left| e^x \sum_{i=n+1}^{\infty} \frac{(-x)^i}{i!} \right| \leq \frac{x^{n+1}}{(n+1)!} e^x \sum_{i=0}^{\infty} \frac{x^i (n+1)!}{(n+1+i)!} \leq \frac{x^{n+1}}{(n+1)!} e^{2x}.$$

To prove the upper bound, we define

$$\ell_n(t) = e^t \sum_{i=0}^n \frac{(-t)^i}{i!}.$$

Since n is even, we have

$$\ell'_n(t) = \frac{e^{t^n}}{n!}$$

By Stirling formula,

$$\begin{aligned} \frac{e^{t^n}}{n!} &\leq \frac{e^{t+n}}{\sqrt{2\pi n}} \left(\frac{t}{n}\right)^n = \frac{e^{t+n}}{\sqrt{2\pi n}} \exp \left[n \log \left(1 - \frac{n-t}{n} \right) \right] \\ &\leq \frac{e^{t+n}}{\sqrt{2\pi n}} \exp \left[n \left(\frac{t-n}{n} - \frac{(t-n)^2}{4n^2} \right) \right] \leq \frac{e^{2t}}{\sqrt{2\pi n}} e^{-\alpha_n^2/4}, \end{aligned}$$

for $t \leq n - \alpha_n \sqrt{n}$. Therefore,

$$\ell_n(x) = 1 + \int_0^x \frac{e^{t^n}}{n!} dt \leq 1 + \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}} \int_0^x e^{2t} dt \leq 1 + e^{2x} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}},$$

thus (iii) follows. \square

The part (i) of Lemma 4.1 guarantees that the autocorrelation function $A_n(x, y)$ of Weyl random polynomials of even degree is always nonnegative. Therefore, we can apply Lemma 2.1 and Slepian inequality. While the part (ii) leads to a tight estimate on $A_n(x, y)$, the part (iii) only gives us rough estimates. In Lemma 4.2 below, we will use these estimates to verify the conditions of Lemma 2.1 and deduce the persistence exponent.

Lemma 4.2. *Given that n is even. Then as $n \rightarrow \infty$,*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} (f_n(x) > 0, \forall x \in [-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]) = -2b.$$

Proof of Part (b) of Theorem 1.1. Using Lemma 4.2 and the same arguments for (6), we immediately get the upper bound

$$(30) \quad \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} (f_n(x) > 0, \forall x \in \mathbb{R}) \leq -2b.$$

On the other hand, by analogous arguments for (7), the lower bound

$$(31) \quad \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} (f_n(x) > 0, \forall x \in \mathbb{R}) \geq -2b.$$

follows from Lemma 4.2 and the following claims

(d1)

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} (f_n(x) > 0, \forall x \in (-\infty, -\sqrt{n} + \alpha_n]) \geq 0,$$

(d2)

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} (f_n(x) > 0, \forall x \in [\sqrt{n} - \alpha_n, \infty)) \geq 0.$$

By symmetry, the law of the Weyl random polynomial on the interval $(-\infty, -\sqrt{n} + \alpha_n]$ is as on the interval $[\sqrt{n} - \alpha_n, \infty)$. Thus the claim (d1) is equivalent to the claim (d2). The claim (d2) follows from the claims (c2) and (c3) by using Slepian inequality. Now, it remains to show Lemma 4.2. \square

The strategy of the proof of Lemma 4.2 is to verify the conditions on the correlation function given in Lemma 2.1. We notice that to prove Lemma 3.1 (which deals with the persistence probability in \mathbb{R}_+), Dembo and Mukherjee verify conditions (b1) and (b2). In particular, to check (b2), they use the estimate in Lemma 4.1 (ii) to show that the autocorrelation $A_n(x, y)$ converges uniformly to $e^{-(x-y)^2}$ for $x, y \in [0, \sqrt{n} - \alpha_n]$. However, this convergence does not hold for $xy < 0$. Thus in our proof, we will use the condition (b3) instead of (b2).

For any given even number n , we consider the Weyl polynomial as a Gaussian process on the interval $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$ of size $2(\sqrt{n} - \alpha_n)$. To recover the exact form in the statement of Lemma 2.1, one should make the change of variable to transform the interval $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$ into the interval $[0, 2(\sqrt{n} - \alpha_n)]$. However, for convenience in the sequel, we keep the interval $[-\sqrt{n} + \alpha_n, \sqrt{n} - \alpha_n]$.

Proof of Lemma 4.2. We now verify two conditions (b1) and (b3).

Verification of (b1). We first observe that by Lemma 4.1 (ii), if $|s|, |s + \tau| \leq \sqrt{n} - \alpha_n$,

$$(32) \quad A_n(s, s + \tau) \leq \frac{4 \sum_{i=0}^n \frac{(s(s+\tau))^i}{i!}}{\sqrt{e^{s^2}} \sqrt{e^{(s+\tau)^2}}}.$$

In case when $s(s + \tau) \geq 0$, we have

$$A_n(s, s + \tau) \leq \frac{4e^{s(s+\tau)}}{\sqrt{e^{s^2}} \sqrt{e^{(s+\tau)^2}}} = 4e^{-\tau^2/2},$$

and thus (b1) is verified. The next lemma deals with the remaining case $s(s + \tau) < 0$.

Lemma 4.3. *Given that n is even. For all n and τ large enough, if $s(s + \tau) < 0$ and $0 < |s|, |s + \tau| < \sqrt{n} - \alpha_n$ then*

$$A_n(s, s + \tau) \leq \tau^{-2}.$$

In particular, the condition (b1) is verified.

Proof. Let us define $a = -s(s + \tau)$. Then by Lemma 4 (ii),

$$A_n(s, s + \tau) \leq \frac{4 \sum_{i=0}^n \frac{(-a)^i}{i!}}{\sqrt{e^{s^2}} \sqrt{e^{(s+\tau)^2}}} = \frac{4e^a \sum_{i=0}^n \frac{(-a)^i}{i!}}{e^{\tau^2/2}}.$$

By assumptions of the lemma, $\alpha_n - \sqrt{n} \leq s < 0 < \tau \leq 2(\sqrt{n} - \alpha_n)$. Thus

$$0 < a = (\tau - |s|)|s| \leq \tau^2/4 \leq n - \sqrt{n}\alpha_n.$$

Therefore, by Lemma 4.1 (iii), for n and τ large enough

$$A_n(s, s + \tau) \leq 4e^{-\tau^2/2} \left(1 + e^{2a} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}} \right) \leq 4e^{-\tau^2/2} \left(1 + e^{\tau^2/2} \frac{e^{-\alpha_n^2/4}}{\sqrt{2\pi n}} \right) \leq \tau^{-2},$$

since $0 < \tau \leq 2(\sqrt{n} - \alpha_n)$. \square

Verification of (b3). To show (b3), it suffices to prove that for all u small enough

$$(33) \quad \sup_{n \in 2\mathbb{N}} \left(1 - \inf_{\substack{st < 0, \\ 0 \leq t-s \leq u}} A_n(s, t) \right) \leq \frac{1}{|\log u|^2}.$$

We will verify (33) in three regimes $st < 0$; $st > 0$ with $n \leq \sqrt{|\log u|}$; and $st > 0$ with $n \geq \sqrt{|\log u|}$ by three Lemmas 4.4, 4.5 and 4.6 respectively.

Lemma 4.4. *If $st < 0$ and $0 \leq t - s \leq u$ then for all u small enough,*

$$1 - \inf_{\substack{st < 0, \\ 0 \leq t-s \leq u}} A_n(s, t) \leq u^2.$$

Proof. By assumptions, we have $-u \leq s < 0$ and $0 < t \leq u$, so $-u^2 \leq st < 0$. Hence, by Lemma 4.1 (iii)

$$e^{-st} \sum_{i=0}^n \frac{(st)^i}{i!} \geq 1 - \frac{|st|^{n+1}}{(n+1)!} e^{-2st} \geq 1 - \frac{u^{2(n+1)}}{(n+1)!} e^{2u^2}.$$

Thus

$$\sum_{i=0}^n \frac{(st)^i}{i!} \geq e^{st} \left(1 - \frac{u^{2(n+1)}}{(n+1)!} e^{2u^2} \right).$$

Therefore,

$$\begin{aligned} A_n(s, t) &= \frac{\sum_{i=0}^n \frac{(st)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{s^{2i}}{i!}} \sqrt{\sum_{i=0}^n \frac{t^{2i}}{i!}}} \\ &\geq e^{-(t-s)^2/2} \left(1 - \frac{u^{2(n+1)}}{(n+1)!} e^{2u^2} \right) \geq e^{-u^2/2} \left(1 - \frac{u^{2(n+1)}}{(n+1)!} e^{2u^2} \right), \end{aligned}$$

where for the last inequality we used that $0 \leq t - s \leq u$. In conclusion,

$$1 - \inf_{\substack{st < 0, \\ 0 \leq t-s \leq u}} A_n(s, t) \leq 1 - e^{-u^2/2} + \frac{u^{2(n+1)}}{(n+1)!} e^{u^2} \leq u^2,$$

for all u small enough. □

Lemma 4.5. *For all u small enough and $n \leq \sqrt{|\log u|}$, we have*

$$1 - \inf_{\substack{0 \leq t-s \leq u, \\ 0 \leq s, t \leq \sqrt{n} - \alpha_n}} A_n(s, t) \leq u.$$

Proof. We have

$$\begin{aligned}
1 - A_n(s, t) &= \frac{\sqrt{\sum_{i=0}^n \frac{s^{2i}}{i!}} \sqrt{\sum_{i=0}^n \frac{t^{2i}}{i!}} - \sum_{i=0}^n \frac{(st)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{s^{2i}}{i!}} \sqrt{\sum_{i=0}^n \frac{t^{2i}}{i!}}} \\
&\leq \left(\sum_{i=0}^n \frac{s^{2i}}{i!} \right) \left(\sum_{i=0}^n \frac{t^{2i}}{i!} \right) - \left(\sum_{i=0}^n \frac{(st)^i}{i!} \right)^2 \\
&= \sum_{1 \leq i < j \leq n} \left(\frac{s^i t^j}{i! j!} - \frac{s^j t^i}{i! j!} \right)^2 \\
&\leq \sum_{1 \leq i < j \leq n} (st)^{2i} (t^{j-i} - s^{j-i})^2 \\
&\leq \sum_{1 \leq i < j \leq n} (st)^{2i} [t^{j-i-1} (j-i) (t-s)]^2 \\
&\leq (t-s)^2 \sum_{1 \leq i < j \leq n} n^{2i} (n^{(n-1)/2} n)^2 \leq n^{3n+3} (t-s)^2.
\end{aligned}$$

Therefore, for all $n \leq \sqrt{|\log u|}$ and u small enough,

$$1 - \inf_{\substack{0 \leq t-s \leq u, \\ 0 \leq s, t \leq \sqrt{n} - \alpha_n}} A_n(s, t) \leq n^{3n+3} u^2 \leq \exp\left(2\sqrt{|\log u|} \log |\log u| - 2|\log u|\right) \leq u.$$

□

Lemma 4.6. For u small enough and $n \geq \sqrt{|\log u|}$, we have

$$1 - \inf_{\substack{0 \leq t-s \leq u, \\ 0 \leq s, t \leq \sqrt{n} - \alpha_n}} A_n(s, t) \leq \frac{1}{|\log u|^2}.$$

Proof. By Lemma 4 (ii), we have

$$A_n(s, t) = \frac{\sum_{i=0}^n \frac{(st)^i}{i!}}{\sqrt{\sum_{i=0}^n \frac{s^{2i}}{i!}} \sqrt{\sum_{i=0}^n \frac{t^{2i}}{i!}}} \geq e^{-(t-s)^2/2} \left(1 - e^{-\alpha_n^2/4}\right).$$

Therefore, if $n \geq \sqrt{|\log u|}$ then

$$\begin{aligned}
1 - \inf_{\substack{0 \leq t-s \leq u, \\ 0 \leq s, t \leq \sqrt{n} - \alpha_n}} A_n(s, t) &\leq 1 - e^{-u^2/2} + e^{-\alpha_n^2/4} e^{-u^2} \\
&\leq u^2/2 + e^{-(\sqrt{n}/\log n)^2/4} \leq \frac{1}{|\log u|^2},
\end{aligned}$$

for u small enough. □

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