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A HOPF-ALGEBRAIC APPROACH TO CUMULANTS–MOMENTS RELATIONS AND WICK POLYNOMIALS

K. EBRAHIMI-FARD, F. PATRAS, N. TAPIA, AND L. ZAMBOTTI

ABSTRACT. We present a different approach to classical definitions and results on cumulant–moment relations and Wick polynomials, which is based on extensive use of convolution products of linear functionals on a coalgebra. It allows, in particular, to understand the construction of Wick polynomials as the result of a Hopf algebra deformation under the action of linear automorphisms induced by multivariate moments associated to an arbitrary family of random variables with moments of all orders.

1. INTRODUCTION

It is notorious that chaos expansions and Wick products have been thought of as key steps in the renormalisation process in perturbative quantum field theory (QFT). The technical reason for this is that they allow to remove contributions to amplitudes (say, probability transitions between two physical states) that come from so-called diagonal terms – from which divergences in the calculation of those amplitudes may originate. Rota and Wallstrom [13] addressed these issues from a strictly combinatorial point of view using, in particular, the structure of the lattice of set partitions. These are the same techniques that are currently used intensively in the approach by Peccati and Taqqu [10] in the context of Wiener chaos and related phenomena. We refer to their book for a detailed study and the classical results on the subject, as well as for a detailed bibliography and historical survey.

Recently, the interest in the fine structure of cumulants as well as Wick products for non-Gaussian variables has been revived, since they both play important roles in M. Hairer’s theory of regularity structures. See, for instance, references [5, 6]. The progress in these works relies essentially on describing the underlying algebraic structures in a transparent way. Indeed, the combinatorial complexity of the corresponding renormalisation process requires the introduction of group-theoretical methods such as, for instance, renormalisation group actions and comodule Hopf algebra structures [3]. Another reference of interest on generalised Wick polynomials in view of the forthcoming developments is the recent paper [9].

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Starting from these remarks, in this article we shall discuss algebraic constructions related to moment–cumulant relations as well as Wick products, using Hopf algebra techniques. A key observation, that seems to be new in spite of being elementary and powerful, relates to the interpretation of multivariate moments of a family of random variables as a linear form on a suitable Hopf algebra. It turns out that the operation of convolution with this linear form happens to encode much of the theory of Wick products and polynomials. On the one hand, this approach enlightens the classical theory, as various structure theorems in the theory of chaos expansions follow immediately from elementary Hopf algebraic constructions, and therefore are given by the latter a group-theoretical meaning. Our methods should be compared with the combinatorial approach in [10]. On the other hand, we show a natural relation with the results and techniques that have been developed in the theory of regularity structures.

Our approach has been partially motivated by similarities with methods that have been developed for bosonic and fermionic Fock spaces by C. Brouder et al. [1, 2] to deal with interacting fields and non-trivial vacua in perturbative QFT. This is not surprising since, whereas the combinatorics of Gaussian families is reflected in the computation of averages of creation and annihilation operators over the vacuum in QFT, combinatorial properties of non-Gaussian families correspond instead to averages over non-trivial vacua.

1.1. Main results. The main results of this paper (Theorems 5.1 and 5.2) are multivariate generalisations of the following statements for single real-valued random variable X with finite moments of all orders.

We denote by $H := \mathbb{R}[x]$ the algebra of polynomials on the variable x , endowed with the standard product

$$x^n \cdot x^m := x^{n+m}, \quad (1)$$

for $n, m \geq 0$. We equip H with the coproduct $\Delta : H \rightarrow H \otimes H$ defined by

$$\Delta x^n := \sum_{k=0}^n \binom{n}{k} x^{n-k} \otimes x^k. \quad (2)$$

This allows to define on the dual space H^* a dual product $\alpha \star \beta \in H^*$:

$$(\alpha \star \beta)(x^n) := (\alpha \otimes \beta)\Delta x^n,$$

for $\alpha, \beta \in H^*$. This product is commutative and associative, and the space $\mathcal{G}(H) := \{\lambda \in H^* : \lambda(1) = 1\}$ forms a group for this multiplication law.

We define the functional $\mu \in H^*$ given by $\mu(x^n) := \mu_n = \mathbb{E}(X^n)$. Then $\mu \in \mathcal{G}(H)$ and therefore its inverse μ^{-1} in $\mathcal{G}(H)$ is well defined.

Theorem 1.1 (Wick polynomials). *We define $W := \mu^{-1} \star \text{id} : H \rightarrow H$, i.e. the linear operator such that*

$$W(x^n) = (\mu^{-1} \otimes \text{id})\Delta x^n = \sum_{k=0}^n \binom{n}{k} \mu^{-1}(x^{n-k}) x^k. \quad (3)$$

Then

- $W : H \rightarrow H$ is the only linear operator such that

$$W(1) = 1, \quad \frac{d}{dx} \circ W = W \circ \frac{d}{dx}, \quad \mu(W(x^n)) = 0 \quad (4)$$

for all $n \geq 1$.

- $W : H \rightarrow H$ is the only linear operator such that for all $n \geq 0$

$$x^n = (\mu \otimes W)\Delta x^n = \sum_{k=0}^n \binom{n}{k} \mu(x^{n-k}) W(x^k)$$

We call $W(x^n)$ the Wick polynomial of degree n associated to the law of X . If X is a standard Gaussian random variable then the recurrence (4) shows that $W(x^n)$ is the Hermite polynomial H_n . Therefore (3) gives an explicit formula for such generalised Wick polynomials in terms of the inverse μ^{-1} of the linear functional μ in the group $\mathcal{G}(H)$.

The Wick product W defines a *deformation* of the Hopf algebra H .

Theorem 1.2. *The linear operator $W : H \rightarrow H$ has an inverse $W^{-1} : H \rightarrow H$ given by $W^{-1} = \mu \star \text{id}$. If we define for $n, m \geq 0$*

$$x^n \cdot_{\mu} x^m := W(W^{-1}(x^n) \cdot W^{-1}(x^m)),$$

and define similarly a twisted coproduct Δ_{μ} and set $\varepsilon_{\mu} := \mu$, then H endowed with \cdot_{μ} , Δ_{μ} and ε_{μ} is a bicommutative Hopf algebra and W is an isomorphism of Hopf algebras. In particular

$$W(x^{n_1+\dots+n_k}) = W(x^{n_1}) \cdot_{\mu} W(x^{n_2}) \cdot_{\mu} \dots \cdot_{\mu} W(x^{n_k})$$

for all $n_1, \dots, n_k \in \mathbb{N}$.

Furthermore, we present Hopf-algebraic versions of classical multivariate formulae on relations between moments and cumulants.

We recall that in the case of a single random variable X with finite moments of all orders, the sequence $(\kappa_n)_{n \geq 0}$ of cumulants of X is defined by the following formal power series relation between exponential generating functions

$$\exp \left(\sum_{n \geq 0} \frac{t^n}{n!} \kappa_n \right) = \sum_{n \geq 0} \frac{t^n}{n!} \mu_n, \quad (5)$$

where t is a formal variable and $\mu_n = \mathbb{E}(X^n)$ is the n th-order moment of X . Note that $\mu_0 = 1$ and $\kappa_0 = 0$. Equation (5) is equivalent to the classical

recursion

$$\mu_n = \sum_{m=1}^n \binom{n-1}{m-1} \kappa_m \mu_{n-m}. \quad (6)$$

In fact, equation (5) together with (6) provide the definition of the classical Bell polynomials, which, in turn, are closely related to the Faà di Bruno formula [11].

Then we show multivariate generalisation of the following formulae

Theorem 1.3. *Setting $\mu, \kappa \in H^*$, $\mu(x^n) := \mu_n$ and $\kappa(x^n) := \kappa_n$, $n \geq 0$, we have the relations*

$$\mu = \exp^*(\kappa) = \varepsilon + \sum_{n \geq 1} \frac{1}{n!} \kappa^{*n}, \quad (7)$$

$$\kappa = \log^*(\mu) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\mu - \varepsilon)^{*n}, \quad (8)$$

where $\varepsilon(x^k) := \mathbb{1}_{(k=0)}$.

The above formulae (7) and (8) are Hopf-algebraic interpretations of the classical *Leonov–Shiryayev relations* [8], see (10) and (11) below.

1.2. Organisation of the paper. In Section 2 we briefly review classical multivariate moment–cumulants relations. Section 3 provides an interpretation of these relations in a Hopf-algebraic context. In Section 4 we extend the previous approach to generalised Wick polynomials. Section 5 describes how to relate Wick polynomials to Hopf algebra deformations. In Section 6 still another interpretation of Wick polynomials in terms of a suitable comodule structure is introduced.

Excepted for the basic definitions in the theory of coalgebras and Hopf algebras, for which we refer the reader to P. Cartier’s *Primer on Hopf algebras* [4], this article aims at being a self-contained reference on cumulants and Wick products both for probabilists and algebraists interested in probability. We have therefore detailed proofs and constructions, even those that may seem obvious for experts from one of these two fields.

For commodity and in view of applications to scalar real-valued random variables, we fix the field of real numbers \mathbb{R} as ground field. Notice however that algebraic results and constructions in the article depend only on the ground field being of characteristic zero.

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2. JOINT CUMULANTS AND MOMENTS

We start by briefly reviewing classical multivariate moment–cumulant relations.

2.1. Cumulants. If we have a finite family of random variables $(X_a, a \in \mathcal{S})$ such that X_a has finite moments of all orders for every $a \in \mathcal{S}$, then the analog of the exponential formula (5) holds

$$\exp\left(\sum_{\underline{n} \in \mathbb{N}^{\mathcal{S}}} \frac{t^{\underline{n}}}{\underline{n}!} \kappa_{\underline{n}}\right) = \sum_{\underline{n} \in \mathbb{N}^{\mathcal{S}}} \frac{t^{\underline{n}}}{\underline{n}!} \mu_{\underline{n}}, \tag{9}$$

where $\mu_{\underline{n}} := \mathbb{E}(X^{\underline{n}})$. Here we use multivariable notation, i.e., with $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ and $(t_a, a \in \mathcal{S})$ commuting variables, we define for $\underline{n} = (n_a, a \in \mathcal{S}) \in \mathbb{N}^{\mathcal{S}}$

$$t^{\underline{n}} := \prod_{a \in \mathcal{S}} (t_a)^{n_a}, \quad X^{\underline{n}} := \prod_{a \in \mathcal{S}} (X_a)^{n_a}, \quad \underline{n}! = \prod_{a \in \mathcal{S}} (n_a)!,$$

and we use the conventions $(t_a)^0 := 1, (X_a)^0 := 1$. This defines in a unique way the family $(\kappa_{\underline{n}}, \underline{n} \in \mathbb{N}^{\mathcal{S}})$ of joint cumulants of $(X_a, a \in \mathcal{S})$ once the family of joint moments $(\mu_{\underline{n}}, \underline{n} \in \mathbb{N}^{\mathcal{S}})$ is given. When it is necessary to specify the dependence of $\kappa_{\underline{n}}$ on $(X_a, a \in \mathcal{S})$ we shall write $\kappa_{\underline{n}}(X)$, and similarly for $\mu_{\underline{n}}$.

Identifying a subset $B \subseteq \mathcal{S}$ with its indicator function $\mathbb{1}_B \in \{0, 1\}^{\mathcal{S}}$, we can use the notation κ_B and μ_B for the corresponding joint cumulants and moments. The families $(\kappa_B, B \subseteq \mathcal{S})$ and $(\mu_B, B \subseteq \mathcal{S})$ satisfy the so-called *Leonov–Shiryayev relations* [8, 14]

$$\mu_B = \sum_{\pi \in \mathcal{P}(B)} \prod_{C \in \pi} \kappa_C \tag{10}$$

$$\kappa_B = \sum_{\pi \in \mathcal{P}(B)} (|\pi| - 1)! (-1)^{|\pi|-1} \prod_{C \in \pi} \mu_C \tag{11}$$

where we write $\mathcal{P}(B)$ for the set of all set partitions of B , namely, all collections π of subsets (blocks) of B such that $\cup_{C \in \pi} C = B$ and elements of π are pairwise disjoint; moreover $|\pi|$ denotes the number of blocks of π , which is finite since B is finite. Formulae (10) and (11) have been intensively studied from a combinatorial perspective, see, e.g., [10, Chapter 2]. Regarding the properties of cumulants we refer the reader to [14].

Formula (10) has in fact been adopted, for instance, in [6] as a recursive definition for $(\kappa_B, B \subseteq \mathcal{S})$. This approach does indeed determine the cumulants uniquely by induction over the cardinal $|B|$ of the finite set B . This follows from the right hand side containing κ_B , which is what we want to define, as well as κ_C for some C with $|C| < |B|$, which have been already defined in lower order.

Although this recursive approach seems less general than the one via exponential generating functions as in (9), since it forces to consider only

$\underline{n} \in \{0, 1\}^{\mathcal{S}}$, it turns out that they are equivalent. Indeed, replacing $(X_a, a \in \mathcal{S})$ with $(Y_b, b \in \mathcal{S} \times \mathbb{N})$, where $Y_b := X_a$ for $b = (a, k) \in \mathcal{S} \times \mathbb{N}$, then for $\underline{n} \in \mathbb{N}^{\mathcal{S}}$ we have

$$\kappa_{\underline{n}}(X) = \kappa_B(Y), \quad B = \{(a, k) : a \in \mathcal{S}, 1 \leq k \leq \underline{n}(a)\}.$$

In this paper we show that the Leonov–Shiryaev relations (10)-(11) have an elegant Hopf-algebraic interpretation which also extends to Wick polynomials. Notice that a different algebraic interpretation of (10)-(11) has been given in terms of Möbius calculus [10, 14]. Moreover, the idea of writing moment–cumulant relations in terms of convolution products is closely related to Rota’s Umbral calculus [7, 12].

3. FROM CUMULANTS TO HOPF ALGEBRAS

In this section we explain how classical moment–cumulant relations can be encoded using Hopf algebra techniques. These results may be folklore among followers of Rota’s combinatorial approach to probability, and, as we already alluded at, there exist actually in the literature already various other algebraic descriptions of moment–cumulant relations (via generating series as well as more sophisticated approaches in terms of umbral calculus, tensor algebras and set partitions). Our approach is most suited regarding our later applications, i.e., the Hopf algebraic study of Wick products. Since these ideas do not seem to be well-known to probabilists, we believe that they deserve a detailed presentation.

3.1. Moment–cumulant relations via multisets. Throughout the paper we consider a fixed collection of real-valued random variables $\mathcal{X} = \{X_a\}_{a \in \mathcal{A}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for an index set \mathcal{A} . We suppose that X_a has finite moments of all orders for every $a \in \mathcal{A}$.

We do not assume that \mathcal{A} is finite, but moments and cumulants will be defined only for finite subfamilies. We extend the setting of (9), where \mathcal{S} was a finite set, by defining $\mathcal{M}(\mathcal{A}) \subset \mathbb{N}^{\mathcal{A}}$ as the set of all *finitely supported* functions $B : \mathcal{A} \rightarrow \mathbb{N}$. In the case of $B \in \mathcal{M}(\mathcal{A}) \cap \{0, 1\}^{\mathcal{A}}$, we have that B is the indicator function of a finite set $\mathcal{S}(B)$, namely the support of B . For a general $C \in \mathcal{M}(\mathcal{A})$, we can identify the finite set $\mathcal{S}(C)$ given by the support of C , and then $C(a) \geq 1$ can be interpreted as the multiplicity of $a \in \mathcal{S}(C)$ in C viewed as a multiset.

This multiset context is motivated by the following natural definition for $B \in \mathcal{M}(\mathcal{A})$:

$$X^\emptyset = 1, \quad X^B := \prod_{\substack{a \in \mathcal{A} \\ B(a) > 0}} (X_a)^{B(a)}. \quad (12)$$

For all $B \in \mathcal{M}(\mathcal{A})$ we also set

$$|B| := \sum_{a \in \mathcal{A}} B(a) < +\infty.$$

The set $\mathcal{M}(\mathcal{A})$ is a poset for the partial order defined by $B \leq B'$ if and only if $B(a) \leq B'(a)$ for all a in \mathcal{A} . Moreover, it is a commutative semigroup for the product

$$(A \cdot B)(a) := A(a) + B(a), \quad a \in \mathcal{A}, \quad (13)$$

for A, B in $\mathcal{M}(\mathcal{A})$, i.e. the map $(A, B) \rightarrow A \cdot B$ is associative and commutative. The set $\mathcal{M}(\mathcal{A})$ is actually the free commutative semigroup generated by the indicator functions of the one-element sets $\{a\}$, $a \in \mathcal{A}$. Its linear span, denoted H , is therefore isomorphic as a vector space to the algebra of polynomials over the set of (commuting) variables x_a , $a \in \mathcal{A}$ (the isomorphism given by mapping $B \in \mathcal{M}(\mathcal{A})$ to monomials $\prod_{a \in \mathcal{A}} x_a^{B(a)}$). Moreover the product (13) is motivated, in the notation (12), by

$$X^{A \cdot B} = X^A X^B, \quad A, B \in \mathcal{M}(\mathcal{A}),$$

and is therefore the multivariate analog of (1).

For $n \geq 1$ and for $B, B_1, \dots, B_n \in \mathcal{M}(\mathcal{A})$

$$\binom{B}{B_1 \dots B_n} = \mathbb{1}_{(B_1 \cdot B_2 \dots B_n = B)} \prod_{a \in \mathcal{A}} \frac{B(a)!}{B_1(a)! \dots B_n(a)!}$$

where $B_1 \cdot B_2 \dots B_n$ is the product of B_1, \dots, B_n in $\mathcal{M}(\mathcal{A})$ using the multiplication law defined in (13). Note that for a given $B \in \mathcal{M}(\mathcal{A})$, there exist only finitely many $B_1, \dots, B_n \in \mathcal{M}(\mathcal{A})$ such that $B_1 \cdot B_2 \dots B_n = B$.

Definition 3.1. For every $B \in \mathcal{M}(\mathcal{A})$, we define the cumulant $\mathbf{E}_c(X_B)$ inductively over $|B|$ by $\mathbf{E}_c(X_\emptyset) = 0$ and else

$$\mathbb{E}(X^B) = \sum_{n=1}^{|B|} \frac{1}{n!} \sum_{B_1, \dots, B_n \in \mathcal{M}(\mathcal{A}) \setminus \{\emptyset\}} \binom{B}{B_1 \dots B_n} \prod_{i=1}^n \mathbf{E}_c(X_{B_i}). \quad (14)$$

Remark 3.2. If $B \in \mathcal{M}(\mathcal{A}) \cap \{0, 1\}^{\mathcal{A}}$, then (14) reduces to the first Leonov-Shiryaev relation (10), since on the right-hand side of (14) $B_1, \dots, B_n \in \mathcal{M}(\mathcal{A})$ are also in $\{0, 1\}^{\mathcal{A}}$ and in particular the binomial coefficient (when non-zero) is equal to 1.

As we will show in (16) below, expression (14) is equivalent to the usual formal power series definition of cumulants (whose exponential generating series is the logarithm of the exponential generating series of moments). As for (10), expression (14) does indeed determine the cumulants uniquely by induction over $|B|$. This is because the right hand side only involves $\mathbf{E}_c(X_B)$, which is what we want to define, as well as $\mathbf{E}_c(X_{\bar{B}})$ for some \bar{B} with $|\bar{B}| < |B|$, which is already defined by the inductive hypothesis.

Define two linear functionals on H :

$$\begin{aligned} \mu : H &\rightarrow \mathbb{R} & \kappa : H &\rightarrow \mathbb{R} \\ A &\mapsto \mu(A) := \mathbb{E}(X^A) & A &\mapsto \kappa(A) := \mathbf{E}_c(X_A), \end{aligned} \quad (15)$$

with $\mu(\emptyset) := 1$ and $\kappa(\emptyset) := 0$.

3.2. Exponential generating functions. Let us fix a finite subset $\mathcal{S} = \{a_1, \dots, a_p\} \subset \mathcal{A}$. For $B \in \mathcal{M}(\mathcal{S})$ we set

$$t^B := \prod_{i=1}^p (t_i)^{B(a_i)},$$

where the t_i are commuting variables. Then we define for $B \in \mathcal{M}(\mathcal{A})$ the factorial

$$B! := \prod_{a \in \mathcal{A}} (B(a))!,$$

and the exponential generating function of $\lambda \in H^*$ (seen as a formal power series in the variables t_i)

$$\hat{\lambda}(t, \mathcal{S}) := \sum_{B \in \mathcal{M}(\mathcal{S})} \frac{t^B}{B!} \lambda(B).$$

Then from Definition 3.1 we get the usual exponential relation between the exponential moment and cumulant generating functions of μ and κ , analogous to (5) and (9):

$$\begin{aligned} \hat{\mu}(t, \mathcal{S}) &= \sum_{B \in \mathcal{M}(\mathcal{S})} \frac{t^B}{B!} \mu(B) \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{B \in \mathcal{M}(\mathcal{S})} \sum_{B_1, \dots, B_n \in \mathcal{M}(\mathcal{S})} \frac{1}{B!} \binom{B}{B_1 \dots B_n} \prod_{i=1}^n (t^{B_i} \kappa(B_i)) \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{B_1, \dots, B_n \in \mathcal{M}(\mathcal{S})} \prod_{i=1}^n \left(\frac{t^{B_i}}{B_i!} \kappa(B_i) \right) \\ &= \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{B \in \mathcal{M}(\mathcal{S})} \frac{t^B}{B!} \kappa(B) \right)^n = \exp(\hat{\kappa}(t, \mathcal{S})). \end{aligned} \tag{16}$$

From (16) we obtain another recursive relation between moments and cumulants. Let us set $\mathbb{1}_{(a)}(b) := \mathbb{1}_{(a=b)}$ for $a, b \in \mathcal{A}$. Then we have

$$\mu(A \cdot \mathbb{1}_{(a)}) = \sum_{B_1, B_2 \in \mathcal{M}(\mathcal{A})} \binom{A}{B_1 B_2} \kappa(B_1 \cdot \mathbb{1}_{(a)}) \mu(B_2). \tag{17}$$

This recursion is the multivariate analog of the one in (6).

3.3. Moment–cumulant relations and Hopf algebras. We endow now the space H with the commutative and associative product $\cdot : H \otimes H \rightarrow H$ induced by the semigroup structure of $\mathcal{M}(\mathcal{A})$ defined in (13). The unit element is the null function \emptyset (we will also write abusively \emptyset for the unit map – the embedding of \mathbb{R} into H : $\lambda \mapsto \lambda \cdot \emptyset$). We also define a coproduct

$\Delta : H \rightarrow H \otimes H$ on H by

$$\Delta A := \sum_{B_1, B_2 \in \mathcal{M}(\mathcal{A})} \binom{A}{B_1 B_2} [B_1 \otimes B_2], \quad (18)$$

recall (2). The counit $\varepsilon : H \rightarrow \mathbb{R}$ is defined by $\varepsilon(A) = \mathbb{1}_{(A=\emptyset)}$ and turns H into a coassociative counital coalgebra. Coassociativity $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ follows from the associativity of the semigroup $\mathcal{M}(\mathcal{A})$:

$$\begin{aligned} (\Delta \otimes \text{id})\Delta A &= \sum_{(B_1 \cdot B_2) \cdot B_3 = A} \binom{A}{B_1 B_2 B_3} [B_1 \otimes B_2 \otimes B_3] \\ &= \sum_{B_1 \cdot (B_2 \cdot B_3) = A} \binom{A}{B_1 B_2 B_3} [B_1 \otimes B_2 \otimes B_3] \\ &= (\text{id} \otimes \Delta)\Delta A. \end{aligned}$$

Proposition 3.3. *H is a commutative and cocommutative bialgebra and, since H is graded by $|\mathcal{A}|$ as well as connected, it is a Hopf algebra.*

Proof. Indeed, we already noticed that H is isomorphic as a vector space to the polynomial algebra, denoted P , generated by commuting variables x_a , $a \in \mathcal{A}$. The latter is uniquely equipped with a bialgebra and Hopf algebra structure by requiring the x_a to be primitive elements, that is, by defining a coproduct map $\Delta_P : P \rightarrow P \otimes P$ such that it is an algebra map and $\Delta_P(x_a) = x_a \otimes 1 + 1 \otimes x_a$. Recall that since P is a polynomial algebra, these two conditions define Δ_P uniquely. The antipode is the algebra endomorphism of P induced by $S(x_a) := -x_a$. We let the reader check that the natural isomorphism between H and P is an isomorphism of algebras and maps Δ to Δ_P . The proposition follows. \square

Recall that the dual of a coalgebra is an algebra, which is associative (resp. unital, commutative) if the coalgebra is coassociative (resp. counital, cocommutative). In particular, the dual H^* of H is equipped with an associative and commutative unital product written \star , defined for all $f, g \in H^*$ and $A \in H$ by:

$$(f \star g)(A) = (f \otimes g)\Delta A.$$

The unit of this product is the augmentation map ε . For later use, we also mention that the associative product \star extends to linear endomorphisms f, g of H as well as to the product of a linear form on H with a linear endomorphism of H , by the same defining formula.

We denote $\Delta^0 := \text{id} : H \rightarrow H$, $\Delta^1 := \Delta : H \rightarrow H \otimes H$, and for $n \geq 2$:

$$\Delta^n := (\Delta \otimes \text{id})\Delta^{n-1} : H \rightarrow H^{\otimes(n+1)}.$$

Proposition 3.4. *We have*

$$\mu = \exp^\star(\kappa) = \varepsilon + \sum_{n \geq 1} \frac{1}{n!} \kappa^{\star n}. \quad (19)$$

Proof. By the definitions of Δ and Δ^n we find that

$$\kappa^{\otimes n} \Delta^{n-1} A = \sum_{B_1, \dots, B_n \in \mathcal{M}(A)} \binom{A}{B_1 \cdots B_n} \prod_{i=1}^n \kappa(B_i),$$

and, by (14), this yields the result. \square

Remark 3.5. *Formula (19) is the Hopf-algebraic analog of the first Leonov–Shiryayev relation (10).*

Since $\kappa(\emptyset) = 0$, $\kappa^{\otimes n} \Delta^{n-1}(A)$ vanishes whenever $|A| > n$. Similarly, under the same assumption, $(\mu - \varepsilon)^{\star n}(A) = 0$. It follows that one can handle formal series identities such as $\log^*(\exp^*(\kappa)) = \kappa$ or $\exp^*(\log^*(\mu)) = \mu$ without facing convergence issues. In particular

Proposition 3.6. *We have*

$$\kappa = \log^*(\mu) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\mu - \varepsilon)^{\star n}. \quad (20)$$

From Proposition 3.6 we obtain the formula

$$\mathbf{E}_c(X_B) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{B_1, \dots, B_n \in \mathcal{M}(A) \setminus \{\emptyset\}} \binom{B}{B_1 \cdots B_n} \prod_{i=1}^n \mathbb{E}(X^{B_i}) \quad (21)$$

which may be considered the inverse to (14).

Remark 3.7. *The formula (20) is the Hopf-algebraic analog of the second Leonov–Shiryayev relation (11). Moreover, for $B \in \mathcal{M}(A) \cap \{0, 1\}^A$, then (21) also reduces to the second Leonov–Shiryayev relation (11), since on the right-hand side of (21) $B_1, \dots, B_n \in \mathcal{M}(A)$ are also in $\{0, 1\}^A$.*

3.4. A sub-coalgebra. If one prefers to work in the combinatorial framework of the Leonov–Shiryayev formulae (10)-(11) rather than with (14)-(21), then one may consider the linear space J of $\mathcal{M}(A) \cap \{0, 1\}^A$ (namely of all finite subsets of A , or of their indicator functions).

Then J is a linear subspace of H , which is *not* a sub-algebra of H for the product \cdot defined in (13). The coproduct Δ defined in (18) coacts however nicely on J since for all finite subsets A of A

$$\Delta A = \sum_{B_1 \cdot B_2 = A} B_1 \otimes B_2 \in J \otimes J.$$

Moreover the restriction of ε to J defines a counit for (J, Δ) . Therefore J is a sub-coalgebra of H . With a slight abuse of notation we still write \star for the dual product on J^*

$$(f \star g)(A) := (f \otimes g)\Delta A,$$

for $A \in J$ and $f, g \in J^*$. If we denote as before

$$\begin{aligned} \mu : J &\rightarrow \mathbb{R} & \kappa : J &\rightarrow \mathbb{R} \\ A &\mapsto \mu(A) := \mathbb{E}(X^A) & A &\mapsto \kappa(A) := \mathbf{E}_c(X_A), \end{aligned}$$

with $\mu(\emptyset) := 1$ and $\kappa(\emptyset) := 0$, then the Leonov–Shiryaev relations (10)–(11) can be rewritten in J^* as, respectively,

$$\mu = \exp^*(\kappa) = \varepsilon + \sum_{n \geq 1} \frac{1}{n!} \kappa^{*n}$$

and

$$\kappa = \log^*(\mu) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\mu - \varepsilon)^{*n}.$$

4. WICK PRODUCTS

The theory of Wick products, as well as the related notion of chaos decomposition, play an important role in various fields of applied probability. Both have deep structural features in relation to the fine structure of the algebra of square integrable functions associated to one or several random variables. The aim of this section and the following ones is to revisit the theory on Hopf algebraic grounds. The basic observation is that the formula for the Wick product is closely related to the recursive definition of antipode in a connected graded Hopf algebra. This approach seems to be new, also from the point of view of concurring approaches such as umbral calculus [7, 12] or set partition combinatorics à la Rota–Wallstrom [13].

4.1. Wick polynomials. We are going to use extensively the notion of Wick polynomials for a collection of (not necessarily Gaussian) random variables which is defined as follows.

Definition 4.1. *Given a collection $\mathcal{X} = \{X_a\}_{a \in A}$ of random variables with finite moments of all orders, for any $A \in \mathcal{M}(A)$ the Wick polynomial $:X_A:$ is a random variable defined recursively by setting $:X_\emptyset: = 1$ and postulating that*

$$X^A = \sum_{B_1, B_2 \in \mathcal{M}(A)} \binom{A}{B_1 B_2} \mathbb{E}(X^{B_1}) :X_{B_2}:. \quad (22)$$

As for cumulants, (22) is sufficient to define $:X_A:$ by recursion over $|A|$. Indeed, the term with $B_2 = A$ is precisely the quantity we want to define, and all other terms only involve Wick polynomials $:X_B:$, for $B \in \mathcal{M}(A)$ with $|B| < |A|$.

It is now clear that formula (22) can be lifted to H as

$$A = \sum_{B_1, B_2 \in \mathcal{M}(A)} \binom{A}{B_1 B_2} \mathbb{E}(X^{B_1}) :B_2:, \quad (23)$$

and written in Hopf algebraic terms as follows

$$A = (\mu \star W)(A) = (\mu \otimes W)\Delta A, \quad (24)$$

for $A \in \mathcal{M}(\mathcal{A})$. We have set $W : H \rightarrow H$, $W(A) := :A:$ and call W the Wick product map (see Theorem 5.2 for a justification of the terminology). Notice that it depends on the joint distribution of the X_a s. Formula 24 is the Hopf algebraic analog of the definition of the Wick polynomial $:X_B:$ used in references [6, 9]. Moreover, introducing the algebra map $ev : A \mapsto X^A$ from H to the algebra of random variables generated by $(X_a, a \in \mathcal{A})$, one gets by a recursion over $|A|$ that $ev(:A:) = :X_A:$ (for that reason, from now on we will call slightly abusively both $:A:$ and the random variable $:X_A:$ the Wick polynomial associated to A).

4.2. A Hopf algebraic construction. We want to present now a closed Hopf algebraic formula for the Wick polynomials introduced in Definition 4.1. We define the set $\mathcal{G}(H) := \{\lambda \in H^* : \lambda(\emptyset) = 1\}$. Then it is well known that $\mathcal{G}(H)$ is a group for the \star -product. Indeed, any $\lambda \in \mathcal{G}(H)$ has an inverse λ^{-1} in $\mathcal{G}(H)$ given by

$$\lambda^{-1} = \sum_{n \geq 0} (\varepsilon - \lambda)^{\star n}. \quad (25)$$

As usual, this infinite sum defines an element of H^* since, evaluated on any $A \in H$, it reduces to a finite number of terms.

Theorem 4.2. *Let $\mu \in \mathcal{G}(H)$ be given by $\mu(A) = \mathbb{E}(X^A)$ and $W(A) = :A:$, then for all $A \in \mathcal{M}(\mathcal{A})$*

$$:A: = (\mu^{-1} \star \text{id})(A) = (\mu^{-1} \otimes \text{id})\Delta A. \quad (26)$$

Proof. The identity follows from (24) and from the associativity of the \star product. \square

From (25) and (26) we obtain

Proposition 4.3. *Wick polynomials have the explicit expansion*

$$:A: = A + \sum_{n \geq 1} (-1)^n \sum_{B \in \mathcal{M}(\mathcal{A})} \sum_{\substack{B_1, \dots, B_n \in \mathcal{M}(\mathcal{A}) \\ B_i \neq \{\emptyset\}}} \binom{A}{B_1 \cdots B_n B} \mu(B_1) \cdots \mu(B_n) B.$$

4.3. Another characterisation. Let $a \in \mathcal{A}$. We define now the functional $\zeta_a : H \rightarrow \mathbb{R}$ given by $\zeta_a(A) := \mathbb{1}_{(A=\{a\})}$, for every finite $A \subset \mathcal{A}$. Then we define the operator $\partial_a : H \rightarrow H$ as $\partial_a := \zeta_a \star \text{id}$, namely

$$\partial_a A := (\zeta_a \otimes \text{id})\Delta A.$$

It is simple to see that ∂_a acts as a formal partial derivation with respect to a , namely it satisfies for $A, B \in \mathcal{M}(\mathcal{A})$ and $a, b \in \mathcal{A}$

$$\partial_a \mathbb{1}_{\{b\}} = \mathbb{1}_{(a=b)}, \quad \partial_a(A \cdot B) = \partial_a(A) \cdot B + A \cdot \partial_a(B),$$

since ζ_a satisfies $\zeta_a(\emptyset) = 0$ and $\zeta_a(A \cdot B) = \zeta_a(A)\varepsilon(B) + \varepsilon(A)\zeta_a(B)$. Recall that the product $A \cdot B$ has been defined in (13).

Then the following result is a reformulation in our setting of [9, Proposition 3.4].

Theorem 4.4. *The family of polynomials $(:A: , A \in \mathcal{M}(\mathcal{A}))$ is the only collection such that $: \emptyset: = \emptyset$ and for all non-null $A \in \mathcal{M}(\mathcal{A})$ and $a \in \mathcal{A}$*

$$\partial_a :A: = : \partial_a A: \quad \text{and} \quad \mu(:A:) = 0. \quad (27)$$

Proof. Since $\mu \in \mathcal{G}(H)$ equation (26) implies

$$\mu(:A:) = (\mu^{-1} \star \mu)(A) = \varepsilon(A) = \mathbb{1}_{(A=\emptyset)}.$$

Now let $\lambda : H \rightarrow \mathbb{R}$ be linear and set $\phi_\lambda : H \rightarrow H$,

$$\phi_\lambda(A) := (\lambda \otimes \text{id})\Delta A \quad (28)$$

for $A \in H$. We claim that

$$\Delta\phi_\lambda = (\text{id} \otimes \phi_\lambda)\Delta. \quad (29)$$

Indeed, we have

$$\begin{aligned} \Delta\phi_\lambda &= (\lambda \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta)\Delta \\ &= (\lambda \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})\Delta \\ &= (\text{id} \otimes \lambda \otimes \text{id})(\Delta \otimes \text{id})\Delta \\ &= (\text{id} \otimes \lambda \otimes \text{id})(\text{id} \otimes \Delta)\Delta \\ &= (\text{id} \otimes \phi_\lambda)\Delta, \end{aligned}$$

where we have used, in this order, coassociativity, cocommutativity and then coassociativity again. In particular for $\lambda = \zeta_a$ we obtain

$$\Delta\partial_a = (\text{id} \otimes \partial_a)\Delta.$$

We conclude from (26) that $:A:$ satisfies (27). The converse follows from the fact that (27) defines by recurrence a unique family. \square

4.4. Back to simple subsets. As in Subsection 3.4, we can restrict the whole discussion to Wick polynomials associated to finite sets $B \in \mathcal{M}(\mathcal{A}) \cap \{0, 1\}^A$ and their linear span J . Indeed, if $A \in J$ then $:A: = W(A)$ also belongs to J and is defined by the recursion

$$A = \sum_{B_1 \cdot B_2 = A} \mathbb{E}(X^{B_1}) :B_2: .$$

As in Theorem 4.2, we have $W = \mu^{-1} \star \text{id}$ and $\text{id} = \mu \star W$, and, as in Proposition 4.3,

$$\begin{aligned} :A: &= A + \\ &+ \sum_{n \geq 1} (-1)^n \sum_{B \in \mathcal{M}(\mathcal{A})} \sum_{B_1, \dots, B_n \in \mathcal{M}(\mathcal{A}) \setminus \{\emptyset\}} \mathbb{1}_{(B \cdot B_1 \dots B_n = A)} \mu(B_1) \cdots \mu(B_n) B \end{aligned}$$

for all $A \in \mathcal{M}(\mathcal{A}) \cap \{0, 1\}^A$.

However, as we shall see in Section 5 below, it is more interesting to work on the bialgebra H than on the coalgebra J , see in particular Theorem 5.2.

5. WICK PRODUCTS AS HOPF ALGEBRA DEFORMATIONS

The group $\mathcal{G}(H) = \{\lambda \in H^* : \lambda(\emptyset) = 1\}$ equipped with the \star product acts canonically on H by means of the map $\phi_\lambda : H \rightarrow H$

$$\phi_\lambda(A) := (\lambda \otimes \text{id})\Delta A,$$

for $\lambda \in \mathcal{G}(H)$ and $A \in H$. In other words, $\phi_\lambda = \lambda \star \text{id} = \text{id} \star \lambda$, since Δ is cocommutative. This is a group action since one checks easily using the coassociativity of Δ that

$$\phi_{\lambda_1 \star \lambda_2} = \phi_{\lambda_1} \circ \phi_{\lambda_2},$$

so that in particular

$$(\phi_\lambda)^{-1} = \phi_{\lambda^{-1}}.$$

These maps ϕ_λ , being invertible, allow to define *deformations* of the product \cdot defined in (13), as well as of the coproduct Δ defined in (18) and of the counit. Namely we define $\cdot_\lambda : H \otimes H \rightarrow H$, $\Delta_\lambda : H \rightarrow H \otimes H$ and ε_λ by

$$A \cdot_\lambda B := \phi_\lambda^{-1}(\phi_\lambda(A) \cdot \phi_\lambda(B)),$$

$$\Delta_\lambda A := (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1})\Delta\phi_\lambda A,$$

respectively,

$$\varepsilon_\lambda(A) := \varepsilon \circ \phi_\lambda(A) = \lambda(A).$$

Although $\varepsilon_\lambda = \lambda$, we find useful to introduce the notation ε_λ to feature the new role of λ as a counit.

Notice that, as $\lambda(\emptyset) = 1$, we have $\phi_\lambda(\emptyset) = \emptyset$ and $A \cdot_\lambda \emptyset = A$. Dually,

$$(\varepsilon_\lambda \otimes \text{id}) \circ \Delta_\lambda(A) = (\varepsilon \otimes \phi_{\lambda^{-1}})\Delta\phi_\lambda(A) = A.$$

Then we have

Theorem 5.1. *For any $\lambda \in \mathcal{G}(H)$, the quintuple $(H, \cdot_\lambda, \emptyset, \Delta_\lambda, \varepsilon_\lambda)$ defines a Hopf algebra. The map*

$$\phi_\lambda^{-1} : (H, \cdot, \emptyset, \Delta, \varepsilon) \rightarrow (H, \cdot_\lambda, \emptyset, \Delta_\lambda, \varepsilon_\lambda)$$

is an isomorphism of Hopf algebras.

Proof. Although the Theorem follows directly from the properties of coniu-gacy, we detail the proof. Associativity of \cdot_λ and coassociativity of Δ_λ follow directly. First,

$$(A \cdot_\lambda B) \cdot_\lambda C = \phi_\lambda^{-1}(\phi_\lambda(A) \cdot \phi_\lambda(B) \cdot \phi_\lambda(C)) = A \cdot_\lambda (B \cdot_\lambda C),$$

which shows associativity. Coassociativity is simple to see as well

$$\begin{aligned} (\Delta_\lambda \otimes \text{id})\Delta_\lambda A &= (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1} \otimes \phi_\lambda^{-1})(\Delta \otimes \text{id})\Delta\phi_\lambda A \\ &= (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1} \otimes \phi_\lambda^{-1})(\text{id} \otimes \Delta)\Delta\phi_\lambda A \\ &= (\text{id} \otimes \Delta_\lambda)\Delta_\lambda A. \end{aligned}$$

We check now the compatibility relation between \cdot_λ and Δ_λ :

$$\begin{aligned} (\Delta_\lambda A) \cdot_\lambda (\Delta_\lambda B) &= (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1}) \left(((\phi_\lambda \otimes \phi_\lambda) \Delta_\lambda A) \cdot ((\phi_\lambda \otimes \phi_\lambda) \Delta_\lambda B) \right) \\ &= (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1}) ((\Delta \phi_\lambda A) \cdot (\Delta \phi_\lambda B)) \\ &= (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1}) \Delta (\phi_\lambda A \cdot \phi_\lambda B) \\ &= \Delta_\lambda (A \cdot_\lambda B). \end{aligned}$$

Finally, we check that $\phi_\lambda^{-1} : (H, \cdot, \emptyset, \Delta, \varepsilon) \rightarrow (H, \cdot_\lambda, \emptyset, \Delta_\lambda, \varepsilon_\lambda)$ is a bialgebra morphism:

$$\begin{aligned} \phi_\lambda^{-1}(A \cdot B) &= \phi_\lambda^{-1}(A) \cdot_\lambda \phi_\lambda^{-1}(B), \\ (\phi_\lambda^{-1} \otimes \phi_\lambda^{-1}) \Delta A &= \Delta_\lambda \phi_\lambda^{-1} A. \end{aligned}$$

Notice that all these formulae would hold if we were to replace ϕ_λ with any linear invertible map $\phi : H \rightarrow H$ such that $\phi(\emptyset) = \emptyset$. \square

In the particular case of $\lambda = \mu$, where μ is the moment functional defined in (15), we obtain by Theorem 4.2 and Theorem 5.1:

Theorem 5.2. *The Wick product map $W(A) = :A:$ is equal to $\phi_{\mu^{-1}}$. Therefore $W : (H, \cdot, \emptyset, \Delta, \varepsilon) \rightarrow (H, \cdot_\mu, \emptyset, \Delta_\mu, \varepsilon_\mu)$ is a Hopf algebra isomorphism, in particular*

$$:A_1 \cdot A_2: = :A_1: \cdot_\mu :A_2:,$$

for $A_1, A_2 \in H$.

More generally, we obtain for any $A_1, \dots, A_n \in H$ that

$$:A_1 \cdots A_n: = :A_1: \cdot_\mu \cdots \cdot_\mu :A_n:.$$

We notice at last an interesting additional result expressing abstractly compatibility relations between the two Hopf algebra structures on H (see also Proposition 6.3 below). We recall that a linear space M is a left comodule over the coalgebra (H, Δ, ε) if there is linear map $\rho : M \rightarrow H \otimes M$ such that

$$(\Delta \otimes \text{id}_M) \rho = (\text{id}_H \otimes \rho) \rho, \quad (\varepsilon \otimes \text{id}_M) \rho = \text{id}_M. \quad (30)$$

A left comodule endomorphism of M is then a linear map $f : M \rightarrow M$ such that

$$\rho \circ f = (\text{id}_H \otimes f) \rho.$$

In particular the coalgebra (H, Δ, ε) is a left comodule over itself, with $\rho = \Delta$.

We recall now that we have shown in the proof of Theorem 4.4 above the formula (29)

$$\Delta \phi_\lambda = (\text{id} \otimes \phi_\lambda) \Delta,$$

which holds for all linear $\lambda : H \rightarrow \mathbb{R}$. This yields

Proposition 5.3. *If we consider H as a left comodule over itself, then ϕ_λ is a left comodule morphism for all linear $\lambda : H \rightarrow \mathbb{R}$. In particular the Wick product map W is a left comodule endomorphism of (H, Δ, ε) .*

6. ON THE INVERSE OF UNITAL FUNCTIONALS

As we have seen in Theorem 4.2, the element $\mu^{-1} \in \mathcal{G}(H)$ plays an important role in the Hopf algebraic representation (26) of Wick products. From (25) we obtain a general way to compute μ^{-1} by the formula (25). We discuss now another way to represent μ^{-1} by means of a comodule structure, which is directly inspired by [3, 5].

Let us consider now a linear functional $\lambda : H \rightarrow \mathbb{R}$ which is also an unital algebra morphism (or *character*), namely such that $\lambda(\emptyset) = 1$ and $\lambda(A \cdot B) = \lambda(A)\lambda(B)$ for all $A, B \in H$. Then we have a simpler way to compute its inverse: namely as $\lambda^{-1} = \lambda \circ S$, where $S : H \rightarrow H$ is the *antipode*, i.e. the only linear map such that

$$S \star \text{id} = \text{id} \star S = \emptyset \varepsilon,$$

where \emptyset is the unit and ε the counit of H .

However, this does not seem to help in the computation of μ , since moments are notoriously not multiplicative in general. This problem is circumvented by extending μ to a character $\hat{\mu}$ defined on a larger Hopf algebra \hat{H} which is constructed from H , such that the inverse $\hat{\mu}^{-1}$ will be computed via the antipode of \hat{H} .

Definition 6.1. *Let \hat{H} be the free commutative unital algebra (the algebra of polynomials) generated by $\mathcal{M}(\mathcal{A})$. We denote by \bullet the product in \hat{H} and we define the coproduct $\hat{\Delta} : \hat{H} \rightarrow \hat{H} \otimes \hat{H}$ given by $\hat{\Delta}(\iota A) = (\iota \otimes \iota)\Delta A$ and*

$$\hat{\Delta}(A_1 \bullet A_2 \bullet \cdots \bullet A_n) = (\hat{\Delta}A_1) \bullet (\hat{\Delta}A_2) \bullet \cdots \bullet (\hat{\Delta}A_n),$$

where $\iota : H \rightarrow \hat{H}$ is the canonical injection (which we will omit whenever this does not cause confusion). The unit of \hat{H} is \emptyset and the counit is defined by $\hat{\varepsilon}(A_1 \bullet A_2 \bullet \cdots \bullet A_n) = \varepsilon(A_1) \cdots \varepsilon(A_n)$.

Since \hat{H} is a polynomial algebra, $\hat{\Delta}$ is well-defined by specifying its action on the elements of $\mathcal{M}(\mathcal{A})$, and requiring it to be multiplicative. It turns the space \hat{H} into a connected graded Hopf algebra, where the grading is

$$|A_1 \bullet A_2 \bullet \cdots \bullet A_n| := |A_1| + |A_2| + \cdots + |A_n|.$$

The antipode $\hat{S} : \hat{H} \rightarrow \hat{H}$ of \hat{H} can be computed by recurrence with the classical formula

$$\hat{S}A = -A - \sum_{B_1, B_2 \in \mathcal{M}(\mathcal{A}) \setminus \{\emptyset\}} \binom{A}{B_1 B_2} [\hat{S}B_1] \bullet B_2,$$

where we dropped the injection ι for notational convenience. A closed formula for \hat{S} follows

$$\hat{S}A = -A + \sum_{n \geq 2} (-1)^n \sum_{B_1, \dots, B_n \notin \{\emptyset\}} \binom{A}{B_1 \cdots B_n} B_1 \bullet B_2 \bullet \cdots \bullet B_n. \quad (31)$$

We denote by $\mathcal{C}(\hat{H})$ the set of characters on \hat{H} . This is a group for the $\hat{\star}$ convolution, dual to $\hat{\Delta}$.

Proposition 6.2. *The restriction map $R : \mathcal{C}(\hat{H}) \rightarrow \mathcal{G}(H)$, $R\hat{\lambda} := \hat{\lambda}|_H$ defines a group isomorphism.*

Proof. The map is clearly bijective, since a character on \hat{H} is uniquely determined by its values on H , and every $\lambda \in \mathcal{G}(H)$ gives rise in this way to a $\hat{\lambda} \in \mathcal{C}(\hat{H})$ such that $R\hat{\lambda} = \lambda$.

It remains to show that R is a group morphism. This follows from

$$R(\hat{\alpha} \hat{\star} \hat{\beta})(A) = (\hat{\alpha} \otimes \hat{\beta})\hat{\Delta}A = (\hat{\alpha}|_H \otimes \hat{\beta}|_H)\Delta A = (R\hat{\alpha}) \star (R\hat{\beta})(A),$$

where $\hat{\alpha}, \hat{\beta} \in \mathcal{C}(\hat{H})$ and $A \in H$. \square

For all $\lambda \in \mathcal{G}(H)$ we call $\hat{\lambda}$ the only character on \hat{H} which is mapped to λ by the isomorphism R . By the previous proposition we obtain, in particular, that $(\hat{\lambda})^{-1}|_H = \lambda^{-1}$ for all $\lambda \in \mathcal{G}(H)$. Since $\hat{\lambda}$ is a character on \hat{H} , we have $(\hat{\lambda})^{-1} = \hat{\lambda} \circ \hat{S}$. Therefore

$$\lambda^{-1} = (\hat{\lambda} \circ \hat{S})|_H. \quad (32)$$

This formula can be used specifically to compute the inverse μ^{-1} of the functional μ in (26).

6.1. A comodule structure. The above considerations suggest that we can introduce the following additional structure: if we define $\delta : H \rightarrow \hat{H} \otimes H$, $\delta := (\iota \otimes \text{id})\Delta$, where $\iota : H \rightarrow \hat{H}$ is the canonical injection of Definition 6.1, then H is turned into a *left comodule* over \hat{H} , namely we have

$$(\hat{\Delta} \otimes \text{id}_H) \delta = (\text{id}_{\hat{H}} \otimes \Delta) \delta, \quad (33)$$

and $\text{id}_H = (\hat{\varepsilon} \otimes \text{id}_H) \delta$, see (30) above. Note that (33) is in fact just the coassociativity of Δ on H in disguise.

Then we can rewrite the Hopf algebraic representation (26) of Wick polynomials as follows:

$$:A: = (\hat{\mu} \circ \hat{S} \otimes \text{id}) \delta A, \quad (34)$$

for $A \in H$, where $\hat{\mu}$ is the \bullet -multiplicative extension of μ from H to \hat{H} . Expanding this formula by means of the closed formula for \hat{S} , one recovers, by different means, Proposition 4.3.

From Proposition 5.3 above, we obtain

Proposition 6.3. *If we define the action of $\mathcal{C}(\hat{H})$ on H by*

$$\psi_{\hat{\lambda}} : H \rightarrow H, \quad \psi_{\hat{\lambda}}(A) = (\hat{\lambda} \otimes \text{id}) \delta A,$$

for $A \in H$. Then $\psi_{\hat{\lambda}}$ is comodule morphism for all $\hat{\lambda} \in \mathcal{C}(\hat{H})$, namely

$$\delta \circ \psi_{\hat{\lambda}} = (\text{id}_{\hat{H}} \otimes \psi_{\hat{\lambda}}) \delta.$$

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