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► **To cite this version:**

| Idriss Mazari, Grégoire Nadin, Yannick Privat. Optimal location of resources maximizing the total population size in logistic models. 2017. hal-01607046v1

**HAL Id: hal-01607046**

**<https://hal.science/hal-01607046v1>**

Preprint submitted on 2 Oct 2017 (v1), last revised 28 Jul 2019 (v4)

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# Optimal location of resources maximizing the total population size in logistic models\*

Idriss Mazari<sup>†</sup>      Grégoire Nadin<sup>‡</sup>      Yannick Privat<sup>§</sup>

## Abstract

In this article, we consider a species whose population density solves the steady diffusive logistic equation in a heterogeneous environment modeled with the help of a spatially non constant coefficient standing for a resources distribution. We address the issue of maximizing the total population size with respect to the resources distribution, considering some uniform pointwise bounds as well as prescribing the total amount of resources. By assuming the diffusion rate of the species large enough, we prove that any optimal configuration is bang-bang (in other words an extremal of the admissible set) meaning that this problem can be recast as a shape optimization problem, the unknown domain standing for the resources location. In the one-dimensional case, this problem is deeply analyzed, and for large diffusion rates, all optimal configurations are exhibited. This study is completed by several numerical simulations in the one dimensional case.

**Keywords:** diffusive logistic equation, rearrangement inequalities, symmetrization, optimal control, shape optimization, optimality conditions.

**AMS classification:** 49K20, 35Q92, 49J30, 34B15.

## 1 Introduction

### 1.1 Motivations and state of the art

In this article, we investigate an optimal control problem arising in population dynamics. Let us consider the population density  $\theta_{m,\mu}$  of a given species evolving in a bounded and connected domain  $\Omega$  in  $\mathbb{R}^d$  with a  $C^2$  boundary. In what follows, we will assume that  $\theta_{m,\mu}$  is the positive solution of the so-called steady logistic-diffusive equation (LDEE) writing

$$\begin{cases} \mu \Delta \theta_{m,\mu}(x) + \theta_{m,\mu}(x)(m(x) - \theta_{m,\mu}(x)) = 0 & x \in \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 & x \in \partial\Omega, \end{cases} \quad (\text{LDE})$$

where  $m \in L^\infty(\Omega)$  stands for the resources distribution and  $\mu > 0$  stands for the species velocity also called *diffusion rate*. From a biological point of view, the real number  $m(x)$  is the local intrinsic

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\*The authors were partially supported by the Project “Analysis and simulation of optimal shapes - application to lifesciences” of the Paris City Hall.

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growth rate of species at location  $x$  of the habitat  $\Omega$  and can be seen as a measure of the resources available at  $x$ .

The optimal control problem we will investigate consists of maximizing the functional

$$F_\mu : m \mapsto \int_\Omega \theta_{m,\mu},$$

standing for the total population size.

In the framework of population dynamics, the density  $\theta_{m,\mu}$  solving Equation (LDE) can be interpreted as a steady state associated to the following evolution equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \mu \Delta u(t, x) + u(t, x)(m(x) - u(t, x)) & t > 0, x \in \Omega \\ \frac{\partial u}{\partial \nu}(t, x) = 0 & t > 0, x \in \partial\Omega \\ u(0, x) = u^0(x) & x \in \Omega \end{cases} \quad (\text{LDEE})$$

modeling the spatiotemporal behavior of a population density  $u$  in a domain  $\Omega$  with the spatially heterogeneous resource term  $m$ .

The pioneering works by Fisher [10], Kolmogorov-Petrovski-Piskounov [22] and Skellam [32] sparked a new interest to the study of the heterogeneity influence of the environment on the growth of a population densities. Investigating the existence, uniqueness of solutions for the two previous equations as well as their regularity properties boils down to the study of spectral properties for the linearized operator

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \ni f \mapsto \mu \Delta f + mf,$$

where  $\mathcal{D}(\mathcal{L}) = \{f \in L^2(\Omega) \mid \Delta f \in L^2(\Omega)\}$  and of its first eigenvalue  $\lambda_1(m, \mu)$ , characterized by the Courant-Fischer formula

$$\lambda_1(m, \mu) := \sup_{f \in W^{1,2}(\Omega), \int_\Omega f^2 = 1} \left\{ -\mu \int_\Omega |\nabla f|^2 + \int_\Omega mf^2 \right\}. \quad (1)$$

Indeed, the positiveness of  $\lambda(m, \mu)$  is a sufficient condition ensuring the well-posedness of Equations (LDEE) and (LDE) ([2]). More precisely, assuming that  $m$  and  $\mu$  are such that  $\lambda_1(m, \mu) > 0$ , then, Equation (LDE) has unique positive solution  $\theta_{m,\mu} \in W^{1,2}(\Omega)$ . Furthermore, for any  $p \geq 1$ ,  $\theta_{m,\mu} \in W^{2,p}(\Omega)$ , and there holds

$$0 < \inf_{\Omega} \theta_{m,\mu} \leq \theta_{m,\mu} \leq \|m\|_{L^\infty(\Omega)}. \quad (2)$$

Moreover, the steady state  $\theta_{m,\mu}$  is globally asymptotically stable. Namely, for any  $u_0 \in W^{1,2}(\Omega)$  such that  $u_0 \geq 0$ ,  $u_0 \neq 0$  a.e. in  $\Omega$ , one has

$$\|u(t, \cdot) - \theta_{m,\mu}\|_{L^\infty(\Omega)} \xrightarrow{t \rightarrow +\infty} 0.$$

where  $u$  denotes the unique solution of (LDEE) with initial state  $u_0$  (belonging to  $L^2(0, T; W^{1,2}(\Omega))$  for every  $T > 0$ ).

The importance of  $\lambda_1(m, \mu)$  for stability issues related to population dynamics models was first noted in simple cases by Ludwig, Aronson and Weinberger [28]. Let us mention [8] where the case of diffusive Lotka-Volterra equations is investigated.

To guarantee that  $\lambda_1(m, \mu) > 0$ , it is enough to work with distributions of resources  $m$  satisfying the assumption

$$m \in L_+^\infty(\Omega) \quad \text{where} \quad L_+^\infty(\Omega) = \left\{ m \in L^\infty(\Omega), \int_\Omega m > 0 \right\}. \quad (\text{H1})$$

Under such a condition on the weight function  $m(\cdot)$ , many relevant quantities from the biological point of view are well defined, including the eigenvalue  $\lambda_1(m, \mu)$ . Note that the issue of maximizing this principal eigenvalue was addressed for instance in [17, 18, 23, 27, 31].

In the survey article [26], Lou suggests the following problem: the parameter  $\mu > 0$  being fixed, which weight  $m$  maximizes the total population size among all uniformly bounded elements of  $L^\infty(\Omega)$ ?

In this article, we aim at providing partial answers to this issue, and more generally new results about the understanding of the influence of the weight  $m(\cdot)$  on the total population size.

For that purpose, let us introduce the total population size functional, defined for a given  $\mu > 0$  by

$$F_\mu : L_+^\infty(\Omega) \ni m \longmapsto \int_\Omega \theta_{m,\mu}, \quad (3)$$

where  $\theta_{m,\mu}$  denotes the solution of equation (LDE).

Let us mention several previous works dealing with the maximization of the total population size functional. It is shown in [25] that, among all weights  $m$  such that  $\int_\Omega m = m_0$ , there holds  $F_\mu(m) \geq F_\mu(m_0) = m_0$ ; the inequality is strict whenever  $m$  is nonconstant<sup>1</sup>. Moreover, it is also shown that the problem of maximizing  $F_\mu$  over  $L_+^\infty(\Omega)$  has no solution.

In the recent article [1], it is shown that, when  $\Omega = (0, \ell)$ , one has

$$\forall \mu > 0, \forall m \in L_+^\infty(\Omega) \mid m \geq 0 \text{ a.e.}, \quad \int_\Omega \theta_{m,\mu} \leq 3 \int_\Omega m.$$

Moreover, this inequality is sharp, although the right-hand side is never reached, and the authors exhibit a sequence  $(m_k, \mu_k)_{k \in \mathbb{N}}$  such that  $\int_\Omega \theta_{m_k, \mu_k} / \int_\Omega m_k \rightarrow 3$  as  $k \rightarrow +\infty$ . Moreover, there holds  $\|m_k\|_{L^\infty(\Omega)} \rightarrow +\infty$  and  $\mu_k \rightarrow 0$  as  $k \rightarrow +\infty$ .

It is notable that in [7], the more general functional  $J_B$  defined by

$$J_B(m) = \int_\Omega (u - Bm^2) \text{ for } B \geq 0$$

is introduced. In the case  $B = 0$ , the authors apply the so-called Pontryagin principle, show the Gâteaux-differentiability of  $J_B$  and carry out numerical simulations backing up the conjecture that maximizers are of bang-bang type.

In the present work, we aim at dealing with a more realistic modeling, by taking into consideration uniform pointwise constraints on the weight  $m(\cdot)$ . This way, the analysis of optimality conditions is made rather intricate. Indeed, the sensitivity of the total population size functional with respect to the variations of  $m(\cdot)$  is directly related to the solution of an adjoint state, solving a linearized version of (LDE). Deriving and exploiting properties of optimal configurations needs hence a deep understanding of the behavior  $\theta_{m,\mu}$  as well as the adjoint state.

The main issue that will be addressed in what follows is the *bang-bang* character of optimal weights  $m^*(\cdot)$ , in other words, we wonder whether  $m^*$  is an extremal point of the set  $\mathcal{M}_{\kappa, m_0}(\Omega)$  (roughly speaking, an admissible function equal a.e. to its extremal  $L^\infty$ -bounds in  $\Omega$ ). It is notable that the asymptotic analysis performed in [7] cannot be reproduced anymore in our case since the

<sup>1</sup>This result relies on the following observation: multiplying (LDE) by  $\frac{1}{\theta_{m,\mu}}$  and integrating by parts yields

$$\mu \int_\Omega \frac{|\nabla \theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + \int_\Omega (m - \theta_{m,\mu}) = 0. \quad (4)$$

and therefore,  $F_\mu(m) = m_0 + \mu \int_\Omega \frac{|\nabla \theta_{m,\mu}|^2}{\theta_{m,\mu}^2} \geq m_0 = F_\mu(m_0)$  for all  $m \in \mathcal{M}_{\kappa, m_0}(\Omega)$ . It follows that the constant function equal to  $m_0$  is a global minimizer of  $F_\mu$  over  $\mathcal{M}_{\kappa, m_0}(\Omega)$ .

set of admissible weights will be strengthened into a bounded set of  $L^\infty$ , whereas the  $L^\infty$  norm of maximizing sequences for the problem investigated in [7] diverges at the limit.

Our approach allows to deal with a population dynamics model with a large diffusion rate, and rests upon a well-adapted expansion of the solution  $\theta_{m,\mu}$  of (LDE), as a series involving the solutions of a sequence of cascade systems.

## 1.2 Main results

In the whole article, the notation  $\chi_I$  will be used to denote the characteristic function of a measurable subset  $I$  of  $\mathbb{R}^n$ , in other words, the function equal to 1 in  $I$  and 0 elsewhere.

For the reasons mentioned in Section 1.1, it is biologically relevant to consider the class of admissible weights

$$\mathcal{M}_{\kappa,m_0}(\Omega) = \left\{ m \in L^\infty(\Omega), m \in [0; \kappa] \text{ a.e and } \int_{\Omega} m = m_0 \right\},$$

where  $\kappa > 0$  and  $m_0 > 0$  denote two given parameters such that  $m_0 < \kappa$  (so that this set is nontrivial).

We will henceforth consider the following optimal design problem.

**Optimal design problem.** Fix  $n \in \mathbb{N}^*$ ,  $\mu > 0$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$  and let  $\Omega$  be a bounded connected domain of  $\mathbb{R}^n$  having a  $C^2$  boundary. We consider the optimization problem

$$\sup_{m \in \mathcal{M}_{\kappa,m_0}(\Omega)} F_\mu(m). \quad (\mathcal{P}_\mu^n)$$

As will be highlighted in the sequel, the existence of a maximizer follows from a direct argument. We will thus be interested in investigating qualitative properties of maximizers such as: do they enjoy symmetry properties? Are they bang-bang, i.e does  $m$  satisfy  $m \in \{0; \kappa\}$  a.e. in  $\Omega$ ? Quid of the uniqueness?

For the sake of readability, almost all the proofs are postponed to Section 2.

Let us stress that the bang-bang character of maximizer is of practical interest in view of spreading resources in an optimal way. Indeed, in the case where a maximizer  $m^*$  writes  $m^* = \kappa\chi_E$ , the total size of population is maximized by locating all the resources on  $E$ .

We start with a preliminary result related to the saturation of pointwise constraints for Problem  $(\mathcal{P}_\mu^n)$ , valid for all diffusivities  $\mu$ .

**Proposition 1.** Let  $n \in \mathbb{N}^*$ ,  $\mu > 0$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$ . Let  $m^*$  be a solution of Problem  $(\mathcal{P}_\mu^n)$ . Then, either  $\{m = \kappa\}$  or  $\{m = 0\}$  has a positive measure

For large values of  $\mu$ , we will prove that the variational problem can be recast in terms of a shape optimization problem, as underlined in the next results.

**Theorem 1.** Let  $n \in \mathbb{N}^*$ ,  $\mu > 0$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$ . There exists a positive number  $\mu^* = \mu^*(\Omega, \kappa, m_0)$  such that, for every  $\mu \geq \mu^*$ , the functional  $F_\mu$  is strictly convex. As a consequence, for  $\mu \geq \mu^*$ , there exists a maximizer of  $F_\mu$  over  $\mathcal{M}_{\kappa,m_0}(\Omega)$  which is moreover bang-bang.

This theorem justifies that Problem  $(\mathcal{P}_\mu^n)$  can be recast as a shape optimization problem. Indeed, every maximizer  $m^*$  is of the form  $m^* = \kappa\chi_E$  where  $E$  is a measurable subset such that  $|E| = m_0|\Omega|/\kappa$ . This way, the following corollary reformulates this result in terms of shape optimization, by considering as main unknown the subset  $E$  of  $\Omega$  where resources are located.

**Corollary 1.** *Under the assumptions of Theorem 1, there exists a positive number  $\mu^* = \mu^*(\Omega, \kappa, m_0)$  such that, for every  $\mu \geq \mu^*$ , the shape optimization problem*

$$\sup_{E \subset \Omega, |E|=m_0|\Omega|/\kappa} F_\mu(\kappa\chi_E), \quad (5)$$

where the supremum is taken over all measurable subset  $E \subset \Omega$  such that  $|E| = m_0|\Omega|/\kappa$ , has a unique solution.

In the one-dimensional case, one can refine this result by showing that, for  $\mu$  large enough, the maximizer is a step function.

**Theorem 2.** *Let us assume that  $n = 1$  and  $\Omega = (0, 1)$ . Let  $\mu > 0$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$ . There exists  $\hat{\mu} > 0$  such that, for any  $\mu \geq \hat{\mu}$ , any solution  $m^*$  of Problem  $(\mathcal{P}_\mu^n)$  is equal a.e. to either  $\tilde{m}$  or  $\tilde{m}(1 - \cdot)$ , where  $\tilde{m} = \kappa\chi_{(1-\ell, 1)}$  and  $\ell = m_0/\kappa$ .*

Let us conclude by underlining that this result cannot be true for all  $\mu > 0$ . Indeed, we provide an example in Section 3.1 where a “double-crenel” growth rate gives a larger total population size than the “simple-crenel”  $m^*$  of Theorem 2.

### 1.3 Tools and notations

In this section, we gather some useful tools we will use to prove the main results.

**Rearrangements of functions and principal eigenvalue.** Let us first recall several monotonicity and regularity related to the principal eigenvalue of the operator  $\mathcal{L}$ .

**Proposition 2.** [8] *Let  $m \in L_+^\infty(\Omega)$  and  $\mu > 0$ .*

- (i) *The mapping  $\mathbb{R}_+^* \ni \mu \mapsto \lambda_1(m, \mu)$  is continuous and non-increasing.*
- (ii) *If  $m \leq m_1$ , then  $\lambda_1(m, \mu) \leq \lambda_1(m_1, \mu)$ , and the equality is true if, and only if  $m = m_1$  a.e. in  $\Omega$ .*

In the proof of Theorem 2, we will use rearrangement inequalities at length. Let us briefly recall the notion of decreasing rearrangement.

**Definition 1.** *For a given function  $b \in L^1(0, 1)$ , one defines its monotone decreasing (resp. monotone increasing) rearrangement  $b_{dr}$  (resp.  $b_{br}$ ) on  $(0, 1)$  by  $b_{dr}(x) = \sup\{c \in \mathbb{R} \mid x \in \Omega_c^*\}$ , where  $\Omega_c^* = (1 - |\Omega_c|, 1)$  with  $\Omega_c = \{b > c\}$  (resp.  $b_{br}(\cdot) = b_{dr}(1 - \cdot)$ ).*

The functions  $b_{dr}$  and  $b_{br}$  enjoy nice properties. In particular, the Polyà-Szego and Hardy-Littlewood inequalities allow to compare integral quantities depending on  $b$ ,  $b_{dr}$ ,  $b_{br}$  and their derivative.

**Theorem** ([20, 24]). *Let  $u$  be a non-negative and measurable function.*

- (i) *If  $\psi$  is any measurable function from  $\mathbb{R}_+$  to  $\mathbb{R}$ , then*

$$\int_0^1 \psi(u) = \int_0^1 \psi(u_{dr}) = \int_0^1 \psi(u_{br}) \quad (\text{equimeasurability});$$

- (ii) *If  $u$  belongs to  $W^{1,p}(0, 1)$  with  $1 \leq p$ , then*

$$\int_0^1 u'^2 \geq \int_0^1 u_{br}'^2 = \int_0^1 u_{dr}'^2 \quad (\text{Pòlya inequality});$$

(iii) If  $u, v$  belong to  $L^2(0,1)$ , then

$$\int_0^1 uv \leq \int_0^1 u_{br}v_{br} = \int_0^1 u_{dr}v_{dr} \quad (\text{Hardy-Littlewood inequality});$$

The equality case in the Polyà-Szego inequality is the object of the Brothers-Ziemer theorem (see e.g. [9]).

**Poincaré constants and elliptic regularity results.** we will denote by  $c_{ell}^{(p)}$  the optimal positive constant such that: for every  $p \in [1, +\infty)$ ,  $f \in L^p(O)$  and  $u \in W^{1,p}(\Omega)$  satisfying

$$\Delta u = f,$$

there holds

$$\|u\|_{W^{2,p}(\Omega)} \leq c_{ell}^{(p)} (\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

The optimal constant in the Poincaré-Wirtinger inequality will be denoted by  $C_{PW}^{(p)}(\Omega)$ . This inequality reads: for every  $u \in W^{1,p}(\Omega)$ ,

$$\left\| u - \int_{\Omega} u \right\|_{L^p(\Omega)} \leq C_{PW}^{(p)}(\Omega) \|\nabla u\|_{L^p(\Omega)}.$$

## 2 Proofs of the main results

### 2.1 First order optimality conditions for Problem $(\mathcal{P}_{\mu}^n)$

To prove the main results, we first need to state the first order optimality conditions for Problem  $(\mathcal{P}_{\mu}^n)$ . We will show that, for any  $m \in \mathcal{M}$  and any admissible perturbation  $h \in \mathcal{T}_{m, \mathcal{M}_{\kappa, m_0}(\Omega)}$ , where  $\mathcal{T}_{m, \mathcal{M}_{\kappa, m_0}(\Omega)}$  denotes the tangent cone<sup>2</sup> to the set  $\mathcal{M}_{\kappa, m_0}(\Omega)$  at  $m$ , the functional  $F_{\mu}$  is twice Gâteaux-differentiable at  $m$  in direction  $h$ . To do that, we will show that the solution mapping

$$\mathcal{S} : m \in \mathcal{M}_{\kappa, m_0}(\Omega) \mapsto \theta_{m, \mu} \in L^2(\Omega),$$

where  $\theta_{m, \mu}$  denotes the solution of (LDE), is twice Gâteaux-differentiable. In this view, we provide several  $L^2(\Omega)$  estimates of the solution  $\theta_{m, \mu}$ .

**Lemma 1.** *The mapping  $\mathcal{S}$  is twice Gâteaux-differentiable.*

The proof of this result is postponed to Appendix A.

For the sake of simplicity, we will denote by  $\dot{\theta}_{m, \mu} = d\mathcal{S}(m)[h]$  the Gâteaux-differential of  $\theta_{m, \mu}$  at  $m$  in direction  $h$  and by  $\ddot{\theta}_{m, \mu} = d^2\mathcal{S}(m)[h, h]$  its second order derivative at  $m$  in direction  $h$ .

It follows that, for all  $\mu > 0$ , the application  $F_{\mu}$  is Gâteaux-differentiable with respect to  $m$  in direction  $h$  and its Gâteaux derivative writes

$$dF_{\mu}(m)[h] = \int_{\Omega} \dot{\theta}_{m, \mu}.$$

---

<sup>2</sup>For every  $m \in M$ , the tangent cone to the set  $\mathcal{M}_{\kappa, m_0}(\Omega)$  at  $m$ , denoted by  $\mathcal{T}_{m, \mathcal{M}_{\kappa, m_0}(\Omega)}$  is the set of functions  $h \in L^{\infty}(\Omega)$  such that, for any sequence of positive real numbers  $\varepsilon_n$  decreasing to 0, there exists a sequence of functions  $h_n \in L^{\infty}(\Omega)$  converging to  $h$  as  $n \rightarrow +\infty$ , and  $m + \varepsilon_n h_n \in \mathcal{M}_{\kappa, m_0}(\Omega)$  for every  $n \in \mathbb{N}$  (see for instance [13, chapter 7]).

Since the expression of  $dF_\mu(m)[h]$  above is not workable, we need to introduce an adjoint state  $p_{m,\mu}$  as the solution of the equation

$$\begin{cases} \mu\Delta p_{m,\mu} + p_{m,\mu}(m - 2\theta_{m,\mu}) = 1 & \text{in } \Omega, \\ \frac{\partial p_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (6)$$

Note that  $p_{m,\mu}$  belongs to  $W^{1,2}(\Omega)$  and is unique, according to the Fredholm alternative.

Now, multiplying the main equation of (6) by  $\theta_{m,\mu}$  and integrating two times by parts leads to the expression

$$dF_\mu(m)[h] = - \int_{\Omega} h \theta_{m,\mu} p_{m,\mu}.$$

Now consider a maximizer  $m$ . For every perturbation  $h$  in the cone  $\mathcal{T}_{m,\mathcal{M}_{\kappa,m_0}(\Omega)}$ , there holds  $\langle dF_\mu(m)[h] \rangle \geq 0$ . The analysis of such optimality condition is standard in optimal control theory (see for example [34]) and leads to the following result.

**Proposition 3.** *Let us define  $\varphi_{m,\mu} = \theta_{m,\mu} p_{m,\mu}$ , where  $\theta_{m,\mu}$  and  $p_{m,\mu}$  solve respectively equations (LDE) and (6). There exists  $c \in \mathbb{R}$  such that*

$$\{\varphi_{m,\mu} < c\} = \{m = \kappa\}, \quad \{\varphi_{m,\mu} = c\} = \{0 < m < \kappa\}, \quad \{\varphi_{m,\mu} > c\} = \{m = 0\}.$$

## 2.2 Proof of Proposition 1

An easy but tedious computation shows that the function  $\varphi_{m,\mu}$  introduced in Proposition 3 satisfies the PDE

$$\begin{cases} \mu\Delta\varphi_{m,\mu} + 2\mu \left\langle \nabla\varphi_{m,\mu}, \frac{\nabla\theta_{m,\mu}}{\theta_{m,\mu}} \right\rangle + \varphi_{m,\mu} \left( 2\mu \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + 2m - 3\theta_{m,\mu} \right) = \theta_{m,\mu} & \text{in } \Omega, \\ \frac{\partial\varphi_{m,\mu}}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (7)$$

To prove that  $|\{m = 0\}| + |\{m = \kappa\}| > 0$ , we argue by contradiction, by assuming that  $|\{m = \kappa\}| = |\{m = 0\}| = 0$ . Therefore,  $\varphi_{m,\mu} = c$  a.e. in  $\Omega$  and, according to (7), there holds

$$c \left( 2\mu \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + 2m - 3\theta_{m,\mu} \right) = \theta_{m,\mu}$$

Integrating this identity and using that  $\theta_{m,\mu} > 0$  in  $\Omega$  and  $c \neq 0$ , we get

$$2c \left( \mu \int_{\Omega} \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + \int_{\Omega} (m - \theta_{m,\mu}) \right) = (c + 1) \int_{\Omega} \theta_{m,\mu}.$$

Combining this identity with (4), It follows that one has necessarily  $c = -1$ . Coming back to the equation satisfied by  $\varphi_{m,\mu}$  leads to

$$m = \theta_{m,\mu} - \mu \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2}.$$

The logistic diffusive equation (LDE) is then transformed into

$$\mu\theta_{m,\mu}\Delta\theta_{m,\mu} - \mu|\nabla\theta_{m,\mu}|^2 = 0.$$

Integrating this equation by part yields  $\int_{\Omega} |\nabla\theta_{m,\mu}|^2 = 0$ . Thus,  $\theta_{m,\mu}$  is constant, and so is  $m$ . In other words,  $m = m_0$ , which, according to Remark 4 is impossible. The expected result follows.

### 2.3 Proof of Theorem 1

The proof of Theorem 1 is split into several steps, based on a careful asymptotic analysis with respect to the diffusivity variable  $\mu$ . Let us first explain the outlines of the proof.

According to Proposition 1,  $F_\mu$  is twice Gâteaux-differentiable and its second order Gâteaux-derivative is given by

$$d^2F_\mu(m)[h, h] = \int_{\Omega} \ddot{\theta}_{m,\mu},$$

where  $\ddot{\theta}_{m,\mu}$  is the second Gâteaux derivative of  $\mathcal{S}$ . An elementary computation shows that it satisfies

$$\begin{cases} \mu \Delta \ddot{\theta}_{m,\mu} + \ddot{\theta}_{m,\mu}(m - 2\theta_{m,\mu}) = -2 \left( h \dot{\theta}_{m,\mu} - \dot{\theta}_{m,\mu}^2 \right) & \text{in } \Omega, \\ \frac{\partial \ddot{\theta}_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (8)$$

Let  $m_1$  and  $m_2$  be two elements of  $\mathcal{M}_{\kappa, m_0}(\Omega)$  and define

$$\phi_\mu : [0; 1] \ni t \mapsto F_\mu \left( tm_2 + (1-t)m_1 \right) - tF_\mu(m_1) - (1-t)F_\mu(m_2).$$

One has

$$\frac{d^2\phi_\mu}{dt^2}(t) = \int_{\Omega} \ddot{\theta}_{(1-t)m_1+tm_2,\mu}, \quad \text{and} \quad \phi_\mu(0) = \phi_\mu(1) = 0,$$

where  $\ddot{\theta}_{(1-t)m_1+tm_2,\mu}$  must be interpreted as a bilinear form from  $L^\infty(\Omega)$  to  $W^{1,2}(\Omega)$ , evaluated two times at the same direction  $m_2 - m_1$ . Hence, to get the strict convexity of  $F_\mu$ , it suffices to show that, whenever  $\mu$  is large enough,

$$\int_{\Omega} \ddot{\theta}_{tm_2+(1-t)m_1,\mu} > 0$$

as soon as  $m_1 \neq m_2$  (in  $L^\infty(\Omega)$ ) and  $t \in (0, 1)$ , or equivalently that  $d^2F_\mu(m)[h, h] > 0$  as soon as  $m \in \mathcal{M}_{\kappa, m_0}(\Omega)$  and  $h \in L^\infty(\Omega)$ . Note that since  $h = m_2 - m_1$ , it is possible to assume without loss of generality that  $\|h\|_{L^\infty(\Omega)} \leq 2\kappa$ .

Let us fix  $m \in \mathcal{M}_{\kappa, m_0}(\Omega)$  and  $h \in L^\infty(\Omega)$ . In the sequel, the dot or double dot notation  $\dot{f}$  or  $\ddot{f}$  will respectively denote first and second order Gâteaux-differential of  $f$  at  $m$  in direction  $h$ .

The proof is based on the following asymptotic expansions of  $\theta_{m,\mu}$  in powers of  $1/\mu^n$ :

$$\theta_{m,\mu} = m_0 + \frac{\eta_{1,m}}{\mu} + \frac{\mathcal{R}_{m,\mu}}{\mu^2}, \quad (9)$$

with  $\eta_{1,m} = \widehat{\eta}_{1,m} + \beta_{1,m}$ , where  $\widehat{\eta}_{1,m}$  is the unique solution of the equation

$$\begin{cases} \Delta \widehat{\eta}_{1,m} + m_0(m - m_0) = 0 & \text{in } \Omega \\ \frac{\partial \widehat{\eta}_{1,m}}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases}, \quad \text{with} \quad \int_{\Omega} \widehat{\eta}_{1,m} = 0 \quad (10)$$

and

$$\beta_{1,m} = \frac{1}{m_0} \int_{\Omega} \widehat{\eta}_{1,m} m, \quad (11)$$

and  $\mathcal{R}_{m,\mu}$  is a reminder term.

To avoid technicalities, justifications of the validity of this expansion are postponed to Appendix B.

**Remark 1.** Note that the function  $\eta_{1,m}$  solve in particular the PDE

$$\begin{cases} \Delta \eta_{1,m} + m_0(m - m_0) = 0 & \text{in } \Omega \\ \frac{\partial \eta_{1,m}}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \quad (12)$$

As will be highlighted in the sequel, the expansion (9) has been introduced in such a way that

$$\lim_{\mu \rightarrow +\infty} \left( \mu(\theta_{m,\mu} - m_0) - \eta_{1,m} \right) = 0 \quad \text{a.e. in } \Omega.$$

Indeed, this is a consequence from the fact that  $\lim_{\mu \rightarrow +\infty} \theta_{m,\mu} = m_0$  a.e. in  $\Omega$  [25].

**Remark 2.** One could also notice that the quantity  $\beta_{1,m}$  arose in the recent paper [12], where the authors determine the large time behavior of a diffusive Lotka-Volterra competitive system between two populations with growth rates  $m_1$  and  $m_2$ . If  $\beta_{1,m_1} > \beta_{1,m_2}$ , then when  $\mu$  is large enough, the solution converges as  $t \rightarrow +\infty$  to the steady state solution of a scalar equation associated with the growth rate  $m_1$ . In other words, the species with growth rate  $m_1$  chases the other one.

In the present article, as a byproduct of our results, we maximize the function  $m \mapsto \beta_{1,m}$ . This remark implies that this intermediate result might find other applications of its own.

Now, according to (9), one has for all  $m \in \mathcal{M}_{\kappa, m_0}(\Omega)$ ,

$$d^2 F_\mu(m)[h, h] = \frac{1}{\mu} \int_{\Omega} \ddot{\eta}_{1,m} + \frac{1}{\mu^2} \int_{\Omega} \ddot{\mathcal{R}}_{m,\mu}$$

We will show that there holds

$$d^2 F_\mu(m)[h, h] \geq \frac{C(h)}{\mu} \left( 1 - \frac{\Lambda}{\mu} \right) \quad (13)$$

for all  $\mu > 0$ , where  $C(h)$  and  $\Lambda$  denote some positive constants.

The strict convexity of  $F_\mu$  will then follow. Concerning the bang-bang character of maximizers, notice that the admissible set  $\mathcal{M}_{\kappa, m_0}(\Omega)$  is convex, and that its extremal points are exactly the bang-bang functions of  $\mathcal{M}_{\kappa, m_0}(\Omega)$ . Once the strict convexity of  $F_\mu$  showed, we will then easily infer that  $F_\mu$  reaches its maxima at extremal points, in other words that any maximizer is bang-bang.

The rest of the proof is devoted to the proof of the inequality (13). It is divided into the following steps:

**Step 1.** Uniform estimate of  $\int_{\Omega} \ddot{\eta}_{1,m}$  with respect to  $\mu$ .

**Step 2.** Definition and expansion of the reminder term  $\mathcal{R}_{m,\mu}$ .

**Step 3.** Uniform estimate of  $\mathcal{R}_{m,\mu}$  with respect to  $\mu$ .

**Step 1: minoration of  $\int_{\Omega} \ddot{\eta}_{1,m}$ .** One computes successively

$$\dot{\beta}_{1,m} = \frac{1}{m_0} \int_{\Omega} \left( \dot{\hat{\eta}}_{1,m} m + \hat{\eta}_{1,m} h \right), \quad \ddot{\beta}_{1,m} = \frac{1}{m_0} \int_{\Omega} \left( 2\dot{\hat{\eta}}_{1,m} h + \ddot{\hat{\eta}}_{1,m} h \right) \quad (14)$$

where  $\dot{\hat{\eta}}_{1,m}$  solves the equation

$$\begin{cases} \Delta \dot{\hat{\eta}}_{1,m} + m_0 h = 0 & \text{in } \Omega \\ \frac{\partial \dot{\hat{\eta}}_{1,m}}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \quad \text{with } \int_{\Omega} \dot{\hat{\eta}}_{1,m} = 0. \quad (15)$$

Notice moreover that  $\ddot{\hat{\eta}}_{1,m} = 0$ , since  $\dot{\hat{\eta}}_{1,m}$  is linear with respect to  $h$ . Moreover, multiplying the equation above by  $\dot{\hat{\eta}}_{1,m}$  and integrating by parts yields

$$\ddot{\beta}_{1,m} = \frac{2}{m_0^2} \int_{\Omega} |\nabla \dot{\hat{\eta}}_{m,1}|^2. \quad (16)$$

Finally, we obtain

$$\int_{\Omega} \dot{\eta}_{1,m} = |\Omega| \ddot{\beta}_{1,m} + \int_{\Omega} \ddot{\hat{\eta}}_{1,m} = |\Omega| \ddot{\beta}_{1,m} = \frac{2}{m_0^2} \int_{\Omega} |\nabla \dot{\hat{\eta}}_{m,1}|^2.$$

It is then notable that  $\int_{\Omega} \dot{\eta}_{1,m} \geq 0$  and that this quantity does not depend on  $\mu$ .

**Step 2: expansion of the reminder term  $\mathcal{R}_{m,\mu}$ .** Instead of studying directly the equation (8), our strategy consists in providing a well-chosen expansion of  $\ddot{\theta}_{m,\mu}$  of the form

$$\ddot{\theta}_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\zeta_k}{\mu^k}, \text{ where the } \zeta_k \text{ are such that } \sum_{k=2}^{+\infty} \frac{f_{\Omega} \zeta_k}{\mu^{k-1}} \leq M \int_{\Omega} \dot{\eta}_{1,m}.$$

For that purpose, we will expand formally  $\theta_{m,\mu}$  as

$$\theta_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\eta_{k,m}}{\mu^k}. \quad (17)$$

Note that, as underlined previously, since  $\theta_{m,\mu} \xrightarrow{\mu \rightarrow +\infty} m_0$  in  $L^\infty(\Omega)$ , we already know that  $\eta_{0,m} = m_0$ .

Provided that this expansion makes sense and is (two times) differentiable term by term (what will be checked in the sequel) in the sense of Gâteaux, we will get the following expansions

$$\dot{\theta}_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\dot{\eta}_{k,m}}{\mu^k} \quad \text{and} \quad \ddot{\theta}_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\ddot{\eta}_{k,m}}{\mu^k}.$$

Plugging the expression (17) of  $\theta_{m,\mu}$  into the logistic diffusive equation (LDE), a formal computation yields that for all  $k \in \mathbb{N}$ ,  $\eta_{k,m}$  satisfies the induction relation

$$\Delta \eta_{k+1,m} + (m - 2m_0) \eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m} \eta_{k-\ell,m} = 0 \quad \text{in } \Omega,$$

as well as homogeneous Neumann boundary conditions. These relations do not allow to define  $\eta_{k,m}$  in a unique way (it is determined up to a constant), which leads to introducing  $\hat{\eta}_{k+1,m}$  as the solution of the PDE

$$\begin{cases} \Delta \hat{\eta}_{k+1,m} + (m - 2m_0) \eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m} \eta_{k-\ell,m} = 0 & \text{in } \Omega \\ \frac{\partial \hat{\eta}_{k+1,m}}{\partial \nu} = 0 & \text{on } \partial \Omega \end{cases} \quad \text{with } \int_{\Omega} \hat{\eta}_{k+1,m} = 0, \quad (18)$$

and to define the real number  $\beta_{k,m}$  in such a way that

$$\forall k \in \mathbb{N}^*, \quad \eta_{k,m} = \hat{\eta}_{k,m} + \beta_{k,m}.$$

Integrating the main equation of (LDE) yields

$$\int_{\Omega} \theta_{m,\mu} (m - \theta_{m,\mu}) = 0.$$

Plugging the expansion (17) and identifying the terms of order  $k$  indicates that we must define  $\beta_{k,m}$  by the induction relation

$$\beta_{k+1,m} = \frac{1}{m_0} \int_{\Omega} m \hat{\eta}_{k+1,m} - \frac{1}{m_0} \sum_{\ell=1}^k \int_{\Omega} \eta_{\ell,m} \eta_{k+1-\ell,m}.$$

This leads to the following cascade equation on the coefficients  $\{\eta_{k,m}\}_{k \in \mathbb{N}}$ :

$$\boxed{\begin{cases} \eta_{0,m} = m_0, \text{ and for all } k \in \mathbb{N}, \\ \Delta \eta_{1,m} + m_0(m - m_0) = 0 \text{ in } \Omega, \\ \Delta \eta_{k+1,m} + (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m} \eta_{k-\ell,m} = 0 \text{ in } \Omega, \\ \frac{\partial \eta_{k,m}}{\partial \nu} = 0 \text{ over } \partial\Omega, \\ f_{\Omega} \eta_{k+1,m} := \beta_{k+1,m} = \frac{1}{m_0} \int_{\Omega} m \hat{\eta}_{k+1,m} - \frac{1}{m_0} \sum_{\ell=1}^k \int_{\Omega} \eta_{\ell,m} \eta_{k+1-\ell,m}. \end{cases}} \quad (19)$$

Now, the Gâteaux-differentiability of both  $\hat{\eta}_{k,m}$  and  $\beta_{k,m}$  with respect to  $m$  follows from similar arguments as those used to prove Proposition 1. We thus infer that  $\{\dot{\eta}_{k,m}\}_{k \in \mathbb{N}}$  satisfies

$$\begin{cases} \dot{\eta}_{0,m} = 0, \text{ and for all } k \in \mathbb{N}, \\ \Delta \dot{\eta}_{k+1,m} + (m - 2m_0)\dot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \eta_{k-\ell,m} = -h \eta_{k,m} \text{ in } \Omega, \\ \frac{\partial \dot{\eta}_{k,m}}{\partial \nu} = 0 \text{ over } \partial\Omega, \\ f_{\Omega} \dot{\eta}_{k+1,m} = \dot{\beta}_{k+1,m} = \frac{1}{m_0} \int_{\Omega} (h \hat{\eta}_{k+1,m} + m \dot{\hat{\eta}}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \int_{\Omega} \dot{\eta}_{\ell,m} \eta_{k+1-\ell,m}, \end{cases} \quad (20)$$

while  $\{\ddot{\eta}_{k,m}\}_{k \in \mathbb{N}}$  satisfies

$$\begin{cases} \ddot{\eta}_{0,m} = 0, \text{ and for all } k \in \mathbb{N}, \\ \Delta \ddot{\eta}_{k+1,m} + (m - 2m_0)\ddot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \ddot{\eta}_{\ell,m} \eta_{k-\ell,m} = 2 \left( \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \dot{\eta}_{k-\ell,m} - h \dot{\eta}_{k,m} \right) \text{ in } \Omega, \\ \frac{\partial \ddot{\eta}_{k,m}}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ f_{\Omega} \ddot{\eta}_{k+1,m} = \ddot{\beta}_{k+1,m} = \frac{1}{m_0} \int_{\Omega} (h \dot{\hat{\eta}}_{k+1,m} + m \ddot{\hat{\eta}}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \int_{\Omega} (\dot{\eta}_{\ell,m} \dot{\eta}_{k+1-\ell,m} + \ddot{\eta}_{\ell,m} \eta_{k+1-\ell,m}). \end{cases} \quad (21)$$

**Step 3: uniform estimates of  $\mathcal{R}_{m,\mu}$ .** In this section, we will prove the existence of a sequence  $\{\Lambda(k)\}_{k \in \mathbb{N}}$  such that

$$\begin{cases} \forall k \in \mathbb{N}^*, \left| \ddot{\beta}_{k,m} \right| \leq \Lambda(k) \ddot{\beta}_{1,m}, \\ \text{The power series } \sum_{k=1}^{+\infty} \Lambda(k) x^k \text{ has a positive convergence radius.} \end{cases} \quad (22)$$

From the explicit expression of  $\ddot{\beta}_{k,m}$  in (21), one claims that (22) immediately follows from the five following estimates

$$\begin{cases} \|\eta_{k,m}\|_{L^\infty(\Omega)}, \|\nabla \eta_{k,m}\|_{L^\infty(\Omega)} \leq \alpha(k), & (I_\alpha^k) \\ \|\nabla \dot{\eta}_{k,m}\|_{L^2(\Omega)} \leq \beta(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}, & (I_\beta^k) \\ \|\dot{\eta}_{k,m}\|_{L^2(\Omega)} \leq \gamma(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}, & (I_\gamma^k) \\ \|\nabla \ddot{\eta}_{k,m}\|_{L^2(\Omega)} \leq \delta(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2, & (I_\delta^k) \\ \|\ddot{\eta}_{k,m}\|_{L^2(\Omega)} \leq \varepsilon(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2. & (I_\varepsilon^k) \end{cases}$$

where for all  $k \in \mathbb{N}$ , the numbers  $\alpha(k)$ ,  $\beta(k)$ ,  $\gamma(k)$ ,  $\delta(k)$  and  $\varepsilon(k)$  are positive et such that Property (22) holds true.

In what follows, we will write  $f \lesssim g$  when there exists a constant  $C$  (independent of  $k$ ) such that  $f \leq Cg$ .

The end of the proof is devoted to proving the aforementioned estimates .

ESTIMATE ( $I_\alpha^k$ ) This estimate follows from standard elliptic regularity and an iteration procedure. We will define  $\alpha(k)$  by induction. We first choose  $p \in (1, +\infty)$  such that the Sobolev embedding  $W^{2,p}(\Omega) \hookrightarrow C^{1,\frac{1}{2}}(\Omega)$  holds. Let us fix  $\alpha(0) = m_0$  and assume that, for some  $k \in \mathbb{N}^*$ , the estimate ( $I_\alpha^k$ ) holds true.

By  $W^{2,p}(\Omega)$  elliptic regularity theorem, there holds

$$\|\eta_{k+1,m}\|_{L^p(\Omega)} \lesssim \|\eta_{k,m}\|_{L^p(\Omega)} + \left\| (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} \right\|_{L^p(\Omega)}.$$

Using the Hölder inequality, one gets

$$\left\| (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} \right\|_{L^p(\Omega)} \lesssim \kappa\alpha(k) + \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell).$$

Moreover, using that  $\|\eta_{k+1,m}\|_{L^p(\Omega)} \leq \|\hat{\eta}_{k+1,m}\|_{L^p(\Omega)} + |\beta_{k+1,m}|$  and the  $L^p$ -Poincaré-Wirtinger inequality (see Section 1.3), we get

$$\|\hat{\eta}_{k+1,m}\|_{L^p(\Omega)} \lesssim \|\nabla \hat{\eta}_{k+1,m}\|_{L^p(\Omega)}.$$

Now, using standard elliptic estimates<sup>3</sup>, we obtain

$$\|\nabla \eta_{k+1,m}\|_{L^p(\Omega)} \lesssim \|\eta_{k,m}\|_{L^p(\Omega)} + \left\| \sum_{\ell=0}^k \eta_{\ell,m}\eta_{k-\ell,m} \right\|_{L^p(\Omega)} \lesssim \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell).$$

The term  $\beta_{k+1,m}$  is controlled similarly, so that

$$|\beta_{k+1,m}| \lesssim \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell) + \sum_{\ell=1}^k \alpha(\ell)\alpha(k+1-\ell).$$

Since it is clear that the sequence  $\{\alpha(k)\}_{k \in \mathbb{N}}$  can be assumed to be increasing, we write

$$\begin{aligned} \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell) + \sum_{\ell=1}^k \alpha(\ell)\alpha(k+1-\ell) &= \sum_{\ell=0}^{k-1} \alpha(k-\ell)(\alpha(\ell) + \alpha(\ell+1)) + \alpha(0)\alpha(k) \\ &\lesssim \sum_{\ell=0}^{k-1} \alpha(\ell+1)\alpha(k-\ell). \end{aligned}$$

Under this assumption, one has

$$\|\eta_{k+1,m}\|_{L^p(\Omega)} \leq |\beta_{k+1,m}| + \|\hat{\eta}_{k+1,m}\|_{L^p(\Omega)} \lesssim \sum_{\ell=0}^k \alpha(\ell+1)\alpha(k-\ell).$$

Standard Sobolev embeddings results enable us to conclude the proof.

This reasoning guarantees the existence of a constant  $C_1$ , depending only on  $\Omega$ ,  $\kappa$  and  $m_0$ , such that the sequence defined by recursively by  $\alpha(0) = m_0$  and

$$\alpha(k+1) = C_1 \sum_{\ell=0}^{k-1} \alpha(\ell+1)\alpha(k-\ell)$$

satisfies the estimate ( $I_\alpha^k$ ).

Setting  $a_k = \alpha(k)/C_1^k$  for all  $k \in \mathbb{N}$ , we know that  $\{a_k\}_{k \in \mathbb{N}}$  is a shifted Catalan sequence (see [30]), and therefore, the power series  $\sum \alpha(k)x^k$  has a positive convergence radius.

<sup>3</sup>More precisely: if  $\Delta f = g$ , then  $\|\nabla f\|_{L^p(\Omega)} \lesssim \|g\|_{L^p(\Omega)}$ , see Agmon-Douglis-Nirenberg

ESTIMATES  $(I_\beta^k)$  AND  $(I_\gamma^k)$ . Obviously, one can assume that  $\beta(0) = \gamma(0) = 0$ . One again, we work by induction, by assuming these two estimates known at a given  $k \in \mathbb{N}$ . Since  $(I_\beta^k)$  is an estimate on the  $L^2(\Omega)$ -norm of the gradient of  $\dot{\eta}_{k,m}$ , it suffices to deal with  $\dot{\eta}_{k+1,m}$ . According to the Poincaré-Wirtinger inequality, one has  $\int_\Omega \dot{\eta}_{k+1,m}^2 \lesssim \int_\Omega |\nabla \dot{\eta}_{k+1,m}|^2$ . Now, using the weak formulation of the equations on  $\dot{\eta}_{k+1,m}$  and  $\eta_{1,m}$ , as well as the uniform boundedness of  $\|h\|_{L^\infty(\Omega)}$ , we get

$$\begin{aligned}
\int_\Omega |\nabla \dot{\eta}_{k+1,m}|^2 &= \int_\Omega (m - 2m_0) \dot{\eta}_{k,m} \dot{\eta}_{k+1,m} - 2 \sum_{\ell=1}^{k-1} \int_\Omega \eta_{k-\ell,m} \dot{\eta}_{\ell,m} \dot{\eta}_{k+1,m} + \int_\Omega h \eta_{k,m} \dot{\eta}_{k+1,m} \\
&\lesssim \|\dot{\eta}_{k,m}\|_{L^2(\Omega)} \|\dot{\eta}_{k+1,m}\|_{L^2(\Omega)} + \sum_{\ell=1}^k \alpha(k-\ell) \|\dot{\eta}_{k+1,m}\|_{L^2(\Omega)} \|\dot{\eta}_{\ell,m}\|_{L^2(\Omega)} + \\
&\quad \int_\Omega \eta_{k,m} \langle \nabla \dot{\eta}_{1,m}, \nabla \dot{\eta}_{k+1,m} \rangle + \int_\Omega \dot{\eta}_{k+1,m} \langle \nabla \dot{\eta}_{1,m}, \nabla \eta_{k,m} \rangle \\
&\lesssim \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)} \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \left( \gamma(k) + \sum_{\ell=1}^k \alpha(k-\ell) \gamma(\ell) + \alpha(k) + \alpha(k) \right) \\
&\lesssim \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)} \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \left( \gamma(k) + \sum_{\ell=0}^k \alpha(k-\ell) \alpha(\ell) \right),
\end{aligned}$$

where the constants appearing in these inequalities only depend on  $\Omega$ ,  $\kappa$  and  $m_0$ . It follows that there exists a constant  $C_2$  such that, by setting for all  $k \in \mathbb{N}$ ,

$$\beta(k+1) = C_2 \left( \gamma(k) + \sum_{\ell=0}^k \alpha(k-\ell) \alpha(\ell) \right),$$

the inequality  $(I_\beta^k)$  is satisfied at rank  $k+1$ .

Let us now state the estimate  $(I_\gamma^k)$ . By using the Poincaré-Wirtinger inequality, there holds

$$\begin{aligned}
|\dot{\beta}_{k+1,n}| &= \left| \frac{1}{m_0} \int_\Omega (h \dot{\eta}_{k+1,m} + m \dot{\eta}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \int_\Omega \dot{\eta}_{\ell,m} \eta_{k+1-\ell,m} \right| \\
&\lesssim \int_\Omega \langle \nabla \dot{\eta}_{1,m}, \nabla \dot{\eta}_{k+1,m} \rangle + \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)} + \|\nabla \eta_{1,m}\|_{L^2(\Omega)} \sum_{\ell=1}^k \gamma(\ell) \alpha(k+1-\ell) \\
&\lesssim \alpha(k+1) \|\nabla \eta_{1,m}\|_{L^2(\Omega)} + \beta(k+1) \|\nabla \eta_{1,m}\|_{L^2(\Omega)} + \|\nabla \eta_{1,m}\|_{L^2(\Omega)} \sum_{\ell=1}^k \gamma(\ell) \alpha(k+1-\ell).
\end{aligned}$$

Once again, since all the constants appearing in the inequalities depend only on  $\Omega$ ,  $\kappa$  and  $m_0$ , we infer that one can choose  $C_3$  such that, by setting

$$\gamma(k+1) = C_3 \left( \beta(k+1) + \alpha(k+1) + \sum_{\ell=1}^k \gamma(\ell) \alpha(k+1-\ell) \right),$$

the estimate  $(I_\gamma^k)$  is satisfied. Notice that, by bounding each term  $\alpha(\ell)$ ,  $\ell \leq k$  by  $\alpha(k)$  and by using the explicit formula for  $\beta(k+1)$ , there exists a constant  $C_4$  depending only on  $\Omega$ ,  $\kappa$  and  $m_0$  such that

$$\gamma(k+1) \leq C_4 \sum_{\ell=1}^k \alpha(k+1-\ell) (\gamma(\ell) + \alpha(\ell)).$$

Under this form, the same arguments as previously guarantee that the associated power series has a positive convergence radius.

ESTIMATES  $(I_\delta^k)$  AND  $(I_\varepsilon^k)$ . As previously, we first set  $\delta(0) = \varepsilon(0) = 0$  and argue by induction. The previous arguments can be mimicked except for the term

$$\sum_{\ell} \dot{\eta}_{\ell,m} \eta_{k-\ell,m}.$$

Using the weak equation satisfied by  $\eta_{k+1,m}$ , it is enough to show an estimate of the form

$$\left\| \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \dot{\eta}_{k-\ell,m} \right\|_{L^2(\Omega)} \leq \omega(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2. \quad (23)$$

for some  $\omega(k)$  such that the associated power series has a positive radius of convergence.

To prove such an estimate, we work by induction, defining the sequence  $\{\zeta_{k,m}\}_{k \in \mathbb{N}}$  by

$$\zeta_{k,m} = \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \dot{\eta}_{k-\ell,m}.$$

Noting that

$$\zeta_{k+1,m} - \zeta_{k,m} = \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} (\dot{\eta}_{k+1-\ell,m} - \dot{\eta}_{k-\ell,m}) + \dot{\eta}_{1,m} \dot{\eta}_{k,m},$$

and using the previous estimates, we are led to show an estimate of the form

$$\|f_k\|_{L^2(\Omega)} \leq \tilde{\omega}(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \quad \text{where} \quad f_k = \dot{\eta}_{k+1-\ell,m} - \dot{\eta}_{k-\ell,m}.$$

Since  $f_k$  solves the equation

$$\Delta f_k + (m - 2m_0)f_k - 2 \sum_{\ell=1}^{k-1} \eta_{k+1-\ell,m} (\dot{\eta}_{\ell,m} - \dot{\eta}_{k,m}) - \eta_{k,m} \dot{\eta}_{1,m} = h(\eta_{k+1,m} - \eta_{k,m}),$$

it is possible to mimic the previous strategies (using the Poincaré-Wirtinger inequality) and to conclude similarly as before.

## 2.4 Proof of Theorem 2

We will prove that, if  $\mu$  large enough, any maximizer of  $F_\mu$  is equal to either  $\tilde{m} = \kappa \chi_{(1-\ell,1]}$  or  $\tilde{m}(1 - \cdot) = \kappa \chi_{(0,\ell]}$  with  $\ell = m_0/\kappa$ .

As a preliminary remark, we claim that the function  $\theta_{\tilde{m},\mu}$  solving (LDE) with  $m = \tilde{m}$  is positive increasing. Indeed, notice that standard variational analysis arguments yield that  $\theta_{\tilde{m},\mu}$  is the unique minimizer of the energy functional

$$\mathcal{E} : H^1(\Omega, \mathbb{R}_+) \ni u \mapsto \frac{1}{2} \int_0^1 u'^2 - \frac{1}{2} \int_0^1 m^* u^2 + \frac{1}{3} \int_0^1 u^3.$$

By using the rearrangement inequalities recalled in Section 1.3, using the relation  $(m^*)_{br} = m^*$ , one easily shows that

$$\mathcal{E}(\theta_{\tilde{m},\mu}) \geq \mathcal{E}((\theta_{\tilde{m},\mu})_{br}),$$

and therefore, one has necessarily  $\theta_{\tilde{m},\mu} = (\theta_{\tilde{m},\mu})_{br}$  by uniqueness of the steady-state (see Section 1.1). Hence,  $\theta_{\tilde{m},\mu}$  is non-decreasing. Moreover, according to (LDE),  $\theta_{\tilde{m},\mu}$  is convex on  $(0, 1 - \ell)$  and concave on  $(1 - \ell, 1)$  which, combined with the boundary conditions on  $\theta_{\tilde{m},\mu}$ , justifies the positiveness of its derivative. The expected result follows.

**Step 1: convergence of sequences of maximizers.** Let  $\mu > 0$  and  $m_\mu$  be a solutions of Problem  $(\mathcal{P}_\mu^1)$ . This step is devoted to proving the following result.

**Lemma 2.** *The functions  $\tilde{m} = \kappa\chi_{(1-\ell,1)}$  or  $\tilde{m}(1 - \cdot) = \kappa\chi_{(0,\ell)}$  are the only closure points of the family  $(m_\mu)_{\mu>0}$  for the  $L^1(0,1)$  topology.*

Let us prove this result. According to Theorem 1, there exists  $\mu^* > 0$  such that  $m_\mu = \kappa\chi_{E_\mu}$  for all  $\mu \geq \mu^*$ , where  $E_\mu \subset \Omega$  is such that  $|E_\mu| = m_0|\Omega|/\mu$ . Since the family  $(m_\mu)_{\mu>0}$  is uniformly bounded in  $L^\infty(0,1)$ , it converges weakly star in  $L^\infty(0,1)$  up to a subsequence to some element  $m_\infty$  in  $\mathcal{M}_{\kappa,m_0}(0,1)$ .

Now, using standard variational formulation for equation (LDE) as well as elliptic estimates, we infer that the family  $\{\theta_{m_\mu,\mu}\}_{\mu>\mu^*}$  converges up to a subsequence in  $W^{1,2}((0,1))$  to  $\theta_\infty$ , the solution of (LDE) with  $m = m_\infty$ . Therefore,  $(F_\mu(m_\mu)_{\mu>0})$  converges up to a subsequence to  $\int_\Omega \theta_\infty$ .

Observe that the maximizers of  $F_\mu$  over  $\mathcal{M}_{\kappa,m_0}(0,1)$  are the same as the maximizers of  $\mu(F_\mu(m) - m_0)$ . Recall that, according to the proof of Theorem 1, there holds  $\mu(F_\mu(m) - m_0) = \int_0^1 \eta_{1,m} + O(1/\mu)$  for a fixed  $m$  in  $\mathcal{M}_{\kappa,m_0}(0,1)$ , where the notation  $O(1/\mu)$  stands for a function uniformly bounded in  $L^\infty(0,1)$ , where  $\eta_{1,m}$  is defined by (9)-(10)-(11). Let  $m \in \mathcal{M}_{\kappa,m_0}(0,1)$  be arbitrary. According to the previous arguments and the estimates proved in the proof, one can pass to the limit in the inequality  $\mu(F_\mu(m_\mu) - m_0) \geq \mu(F_\mu(m) - m_0)$  yielding that  $m_\infty$  is necessarily a maximizer of the functional

$$f_1 : \mathcal{M}_{\kappa,m_0}(0,1) \ni m \mapsto \int_0^1 \eta_{1,m}.$$

We have then shown that any closure point of  $(m_\mu)_{\mu>0}$  is a maximizer of  $f_1$ .

Let us show that  $\tilde{m}$  and  $\tilde{m}(1 - \cdot)$  are the only maximizers of  $f_1$ . We have already seen in the proof of Theorem 1 that  $\eta_{1,m} = \hat{\eta}_{1,m} + \frac{1}{m_0} \int_0^1 m \hat{\eta}_{1,m}$ , where  $\hat{\eta}_{1,m}$  solves the ODE

$$\begin{cases} \hat{\eta}_{1,m}'' + m_0(m - m_0) = 0 & \text{in } (0,1) \\ \hat{\eta}_{1,m}'(0) = \hat{\eta}_{1,m}'(1) = 0, \end{cases} \quad \text{with } \int_\Omega \hat{\eta}_{1,m} = 0.$$

Therefore, by multiplying the main equation of this ODE by  $\hat{\eta}_{1,m}$  and integrating by parts, we get that for every  $m \in \mathcal{M}_{\kappa,m_0}(0,1)$ , there holds

$$f_1(m) = \frac{1}{m_0} \int_0^1 m \hat{\eta}_{1,m} = \frac{1}{m_0^2} \int_0^1 (\hat{\eta}_{1,m}')^2 = \frac{1}{m_0^2} \int_0^1 (\eta_{1,m}')^2.$$

Introduce the set  $H_0 = \left\{ u \in H^1(0,1), \int_0^1 u = 0 \right\}$ . A standard variational analysis yields that

$$f_1(m) = -\frac{2}{m_0^2} \min_{u \in H_0} T_1^m(u), \quad \text{where } T_1^m : H_0 \ni u \mapsto \frac{1}{2} \int_0^1 (u')^2 - \int_0^1 u m_0(m - m_0).$$

Noting that the function  $\delta = \hat{\eta}_{1,m} - \min_{(0,1)} \hat{\eta}_{1,m}$  is non-negative and using the Polyà-Szego and the Hardy-Littlewood inequalities (see Section 1.3) yields successively

$$\int_0^1 \delta'^2 \geq \int_0^1 \delta_{dr}'^2, \quad \int_0^1 m \delta \leq \int_0^1 m_{dr} \delta_{dr}, \quad \int_0^1 m_{dr} = \int_0^1 m \quad \text{and} \quad \int_0^1 \delta = \int_0^1 \delta_{dr}$$

and therefore, we infer that

$$T_1^m(\hat{\eta}_{1,m}) \geq T_1^{m_{dr}}((\hat{\eta}_{1,m})_{dr}) \geq T_1^{m_{dr}}(\hat{\eta}_{1,m_{dr}}) = \min_{u \in H_0} T_1^{m_{dr}}(u).$$

It follows that  $m_{dr}$  and  $\tilde{m} = m_{dr}(1 - \cdot)$  are the only maximizers of  $f_1$  over  $\mathcal{M}_{\kappa,m_0}(0,1)$  (the necessary character comes from the investigation of the equality case in the Polyà-Szego inequality).

To sum-up, we have proved that the only closure points of  $\{m_\mu\}_{\mu>0}$  for the  $L^\infty$  weak star topology are  $\tilde{m}$  or  $\tilde{m}(1 - \cdot)$ . Finally, since  $\tilde{m}$  is a characteristic function, the convergence is in fact strong in  $L^1(0,1)$  [13, Proposition 2.2.2].

**Step 2. Conclusion:**  $m_\mu = \tilde{m}$  or  $\tilde{m}(1 - \cdot)$  whenever  $\mu$  is large enough. According to Theorem 1 and Proposition 3, we know at this step that for  $\mu$  large enough, there exists  $c_\mu \in \mathbb{R}$  such that

$$\{\varphi_{m_\mu, \mu} > c_\mu\} = \{m_\mu = 0\}, \quad \{\varphi_{m_\mu, \mu} < c_\mu\} = \{m_\mu = \kappa\}.$$

Introduce the function  $z_\mu = \mu(\varphi_{m_\mu, \mu} + 1)$ . Using the convergence results established in the previous steps, in particular that  $(m_\mu)_{\mu > 0}$  converges to  $\tilde{m}$  in  $L^1(\Omega)$  and that  $\varphi_{m_\mu, \mu} = -1 + O(1/\mu)$  uniformly in  $C^{1,\alpha}(\Omega)$  as  $\mu \rightarrow +\infty$ , one infers that  $(z_\mu)_{\mu > 0}$  is uniformly bounded in  $C^{1,\alpha}(\Omega)$  and converges, up to a subsequence to  $z_\infty$  in  $C^1([0, 1])$ , where  $z_\infty$  satisfies in particular

$$z_\infty'' + 2(m_0 - \tilde{m}) = 0,$$

with Neumann Boundary conditions.

We will show that, provided that  $\mu$  be large enough, one has necessarily  $m_\mu = \tilde{m}$  or  $m_\mu = \tilde{m}(1 - \cdot)$ . Since  $\tilde{m} = \kappa$  in  $(0, \ell)$ , it follows that  $z_\infty$  is strictly convex on this interval and since  $z_\infty'(0) = 0$ , one has necessarily  $z_\infty' > 0$  in  $(0, \ell)$ . Similarly, by concavity of  $z_\infty$  in  $(\ell, 1)$ , one has  $z_\infty' > 0$  in this interval.

Furthermore, let us introduce  $d_\mu = \mu(c_\mu + 1)$ . Since  $(z_\mu)_{\mu > 0}$  is bounded in  $C^0([0, 1])$ ,  $(d_\mu)_{\mu > 0}$  converges up to a subsequence to some  $d_\infty$ . By monotonicity of  $z_\infty$  and a compactness argument, there exists a unique  $x_\infty \in [0, \ell]$  such that  $z_\infty(x_\infty) = d_\infty$ . The dominated convergence theorem hence yields

$$|\{z_\infty \leq d_\infty\}| = \kappa\ell, \quad |\{z_\infty \geq d_\infty\}| = \kappa(1 - \ell),$$

and the the aforementioned local convergence results yield

$$\{z_\infty > d_\infty\} \subset \{\tilde{m} = 0\}, \quad \{z_\infty < d_\infty\} \subset \{\tilde{m} = \kappa\}.$$

Hence, the inclusions are equalities (the equality of sets must be understood up to a zero Lebesgue measure set) by using that  $z_\infty$  is increasing.

Moreover, since  $z_\infty$  is increasing, one has  $z_\infty(0) < d_\infty$  and  $z_\infty(1) > d_\infty$ . Since the family  $(z_\mu)_{\mu > 0}$  is uniformly Lipschitz-continuous, there exists  $\varepsilon > 0$  such that for  $\mu$  large enough, there holds

$$z_\mu < d_\mu \text{ in } (0, \varepsilon), \quad z_\mu > d_\mu \text{ in } (1 - \varepsilon, 1), \quad z_\mu' > 0 \text{ in } (\varepsilon, 1 - \varepsilon).$$

This implies the existence of  $x_\mu \in (0, 1)$  such that

$$\{z_\mu < d_\mu\} = [0, x_\mu) \quad \text{and} \quad \{z_\mu > d_\mu\} = (x_\mu, 1],$$

whence the result.

## 3 Conclusion and further comments

### 3.1 About the 1D case

Let us assume in this section that  $n = 1$  and  $\Omega = (0, 1)$ . We provide hereafter several numerical simulations based on the primal formulation of the optimal design problem  $(\mathcal{P}_\mu^n)$ : on Fig. 1, the objective function is plotted with respect to  $x_0$  for several values of  $\mu$ , where we assumed the control function  $m$  having the particular form  $m = \kappa\chi_I$  with  $I = (x_0 - m_0/2, x_0 + m_0/2)$ . On Fig. 2, we come back to the general problem  $(\mathcal{P}_\mu^n)$  and we plot the optimal  $m$  determined numerically for several values of  $\mu$ .

These simulations were obtained with an interior point method applied to the optimal control problem  $(\mathcal{P}_\mu^n)$ . We used a Runge-Kutta method of order 4 to discretize the underlying differential

equations. The control  $m$  has been also discretized, which has allowed to reduce the optimal control problem to some finite dimensional minimization problem with constraints. We used the code IPOPT (see [33]) combined with AMPL (see [11]) on a standard desktop machine. We considered a regular subdivision of  $(0, 1)$  with 1000 points. The resulting code works out the solution quickly (around 5 to 10 seconds depending on the choice of the parameter  $\mu$ ).

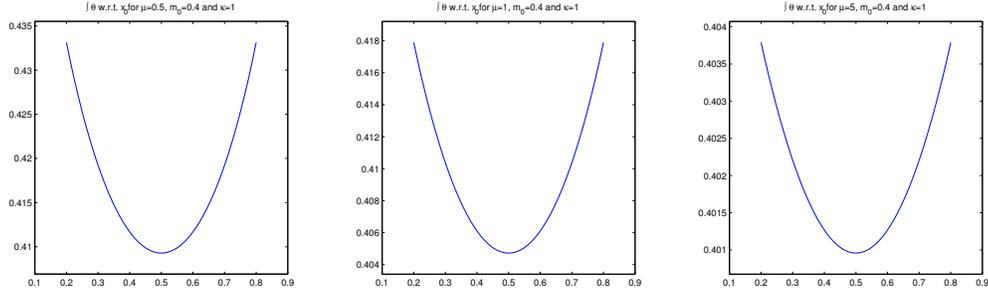


Figure 1:  $m_0 = 0.4$ ,  $\kappa = 1$ . Graph of  $\int_0^1 \theta_{m, \mu}$  with respect to  $x_0$  where  $m = \kappa \chi_{(x_0 - m_0/2, x_0 + m_0/2)}$ . From left to right:  $\mu = 0.5, 1, 5$ .

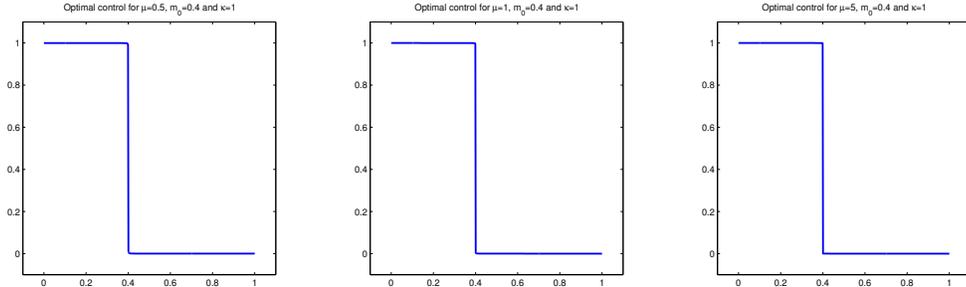


Figure 2:  $m_0 = 0.4$ ,  $\kappa = 1$ . Plot of the optimal solution of Problem  $(\mathcal{P}_\mu^n)$  computed with the help of an interior point method. From left to right:  $\mu = 0.5, 1, 5$ .

These simulations highlighted that, in the 1D case, assuming that  $m = \kappa \chi_I$ , where  $I$  is an interval, the best locations of  $I$  are the extremal one (so that  $m$  is either decreasing or increasing on  $\Omega$ ).

Nevertheless, one encounters a problem when dealing with too small values of  $\mu$  (for instance  $\mu = 0.01$ ). Indeed, in that case, the stiffness of the discretized system seems to become huge as  $\mu$  takes small positive values and makes the numerical computations hard to converge. Improvements of the numerical method should be found for further numerical investigations.

In addition, it is notable that a theoretical argument can be used to prove that  $\tilde{m} = \kappa \chi_{(1-\ell, 1)}$  or  $\tilde{m} = \kappa \chi_{(0, \ell)}$  is not a maximizer anymore for small values of  $\mu$ . Indeed, we claim that

$$\exists \mu > 0 \quad \text{s.t.} \quad F_\mu(\tilde{m}(2 \cdot)) > F_\mu(\tilde{m}),$$

where  $\tilde{m}$  has been extended outside of  $(0, 1)$  by periodicity.

In order to prove this result, as the function  $\mu > 0 \mapsto F_\mu(\tilde{m}(2 \cdot))$  has a finite number of maximizers ([25]), we could define  $\mu_1$  as its first local maximizer. One gets from a simple change of variables that  $\theta_{\tilde{m}, \mu_1}(2x) = \theta_{\tilde{m}(2 \cdot), \mu_1/4}(x)$  for all  $x \in \Omega$  and thus one has

$$F_{\mu_1}(\tilde{m}) = F_{\frac{\mu_1}{4}}(\tilde{m}(2 \cdot))$$

But our choice of  $\mu_1$  yields that  $\mu \mapsto F_\mu(\tilde{m}(2 \cdot))$  is increasing on  $(0, \mu_1)$  and thus:

$$F_{\mu_1}(\tilde{m}) = F_{\frac{\mu_1}{4}}(\tilde{m}(2 \cdot)) < F_{\mu_1}(\tilde{m}(2 \cdot)). \quad (24)$$

### 3.2 Is the maximizer bang-bang for all $\mu$ ?

We proved in this paper that any maximizer  $m^*$  of the total population size is bang-bang when  $\mu$  is large enough. This question remains open for intermediate  $\mu > 0$ . We state here a partial result giving an explicit  $\mu^*$  over which any maximizer is bang-bang in the particular case where  $4\kappa \leq 5m_0$ . Of course this result is not optimal since we know from Theorem 1 that maximizers are always bang-bang for  $\mu$  large (but implicit), without any such condition needed on  $\kappa$  and  $m_0$ .

**Proposition 4.** *If  $N = 1$  and  $4\kappa \leq 5m_0 e^{\sqrt{\kappa/\mu}}$ , then any maximizer is of bang-bang type.*

*Proof.* Consider a maximizer  $m$  and assume that  $\{0 < m < \kappa\}$  has a positive measure. According to (7), as  $\varphi = \varphi_{m, \mu} = c$  a.e. on  $\{0 < m < \kappa\}$ , one has

$$2\mu \frac{|\theta_{m, \mu}'|^2}{\theta_{m, \mu}} + 2m - 3\theta_{m, \mu} = \theta_{m, \mu}/c \quad \text{in } \{0 < m < \kappa\}. \quad (25)$$

Let  $z = \theta_{m, \mu}'/\theta_{m, \mu}$ . By (LDE), one has

$$\mu z' + \mu z^2 = m - \theta_{m, \mu} \leq \kappa - \inf_{\Omega} \theta_{m, \mu} \quad \text{in } \Omega.$$

Considering each point where  $z$  reaches an extremum, one gets

$$\sup_{\Omega} |z| \leq \sqrt{\frac{\kappa - \inf_{\Omega} \theta_{m, \mu}}{\mu}}.$$

This yields

$$\mu \frac{|\theta_{m, \mu}'|^2}{\theta_{m, \mu}} \leq \kappa - \inf_{\Omega} \theta_{m, \mu} \quad \text{in } \Omega.$$

Coming back to (25), as  $c < 0$  since  $\theta_{m, \mu} > 0$  and  $p_{m, \mu} < 0$ , we infer that

$$m \geq -\kappa + \frac{5}{2} \inf_{\Omega} \theta_{m, \mu}.$$

Consider now the unique positive eigenfunction  $\phi$  of

$$\begin{cases} \mu \phi''(x) + m(x)\phi(x) = \lambda_1(m, \mu)\phi(x) & x \in \Omega, \\ \partial_n \phi(x) = 0 & x \in \partial\Omega, \end{cases}$$

normalized by  $\sup_{\Omega} \phi = 1$ . With the same arguments as above, one could show that

$$\sup_{\Omega} |\phi'/\phi| \leq \sqrt{\frac{\lambda_1(m, \mu)}{\mu}}.$$

It follows that

$$1 = \sup_{\Omega} \phi \leq \inf_{\Omega} \phi e^{\sqrt{\frac{\lambda_1(m, \mu)}{\mu}}}.$$

On the other hand, one could notice that  $\lambda_1(m, \mu)\phi$  is a subsolution of (LDE), and thus, as  $\theta_{m, \mu}$  is the unique solution of this equation, one gets  $\theta_{m, \mu} \geq \lambda_1(m, \mu)\phi$  and thus

$$\inf_{\Omega} \theta_{m, \mu} \geq \lambda_1(m, \mu) \inf_{\Omega} \phi \geq \lambda_1(m, \mu) e^{-\sqrt{\frac{\lambda_1(m, \mu)}{\mu}}}.$$

Coming back to our inequality on  $m$ , one gets:

$$\kappa > m \geq -\kappa + \frac{5}{2} \inf_{\Omega} \theta_{m, \mu} \geq -\kappa + \frac{5}{2} \lambda_1(m, \mu) e^{-\sqrt{\frac{\lambda_1(m, \mu)}{\mu}}}.$$

We conclude by noticing that according to (1), there holds

$$\lambda_1(m, \mu) \geq m_0 \quad \text{and} \quad \lambda_1(m, \mu) \leq \kappa.$$

□

### 3.3 Comments on the Dirichlet case

When dealing with Dirichlet boundary conditions instead of Neumann ones, all the results of this article can be directly extended to that case, by noting that the assumption  $\lambda_1(m, \mu) > 0$  needs to be replaced by the condition

$$\lambda_1(m, \mu) > \lambda_1(0, \mu) + m_0,$$

that is, we need to replace (H1) with (H1'):

$$m_0 + \lambda_1(0, \mu) > 0. \tag{H1'}$$

Once this condition is satisfied, all of the previous proofs can be written along the same lines and are even sometimes simplified. Indeed, one of the main difficulties when dealing with Neumann boundary conditions was to control the integral quantities  $\eta_{k, m}$  by using a Poincaré-Wirtinger inequality. Working with Dirichlet boundary conditions enables to work with the standard Poincaré inequality in  $H_0^1$  and to overcome this technical point.

It is also interesting to note that, in dimension  $N \geq 2$  with Dirichlet boundary conditions and  $\Omega = \mathbb{B}(0; R)$ , the approach developed within this article can be adapted to prove the existence of  $\mu^*$  such that, for any  $\mu \geq \mu^*$ , any sequence of maximizers of  $F_{\mu}$  converges in  $L^1(\Omega)$  to  $m^*$ , where  $m^* = \kappa \chi_{\mathbb{B}(0, r)}$  and  $r$  is chosen so that  $\int_{\Omega} m^* = m_0 |\Omega|$ . Note that the symmetrizations used in the proof of Theorem 2 have to be replaced by the so-called (radial) Schwarz symmetrization (see e.g. [20, 13]).

### 3.4 Comments and open issues

It is also interesting, from a biological point of view, to investigate a more general version of Problem ( $\mathcal{P}_{\mu}^n$ ) for changing-sign weights. In that case, the admissible class of weights is then transformed (for instance) into

$$\widetilde{\mathcal{M}}_{\kappa, m_0}(\Omega) = \left\{ m \in L^{\infty}(\Omega), m \in [-1; \kappa] \text{ a.e and } \int_{\Omega} m = m_0 \right\},$$

with  $m_0 \in (0, 1)$  (so that  $\lambda_1(m, \mu) > 0$  and Equation (LDE) is well-posed). We claim that the main results of this article can be extended without effort to this new framework and that we will

still obtain the bang-bang character of maximizers provided that  $\mu$  be large enough. Such a class has also been considered in the context of principal eigenvalue minimization (see [16, 23]).

Finally, we end this section by providing some open problems for which we did not manage to bring complete answer and that deserve and remain, to our opinion, to be investigated. They are in order:

- (for general domains  $\Omega$ ) obtain a sharp estimate of  $\mu^*$ ;
- (for general domains  $\Omega$ ) use the main results of the present article to determine numerically the maximizer  $m^*$  with the help of an adapted shape optimization algorithm;
- (if  $\Omega = (0, 1)$ ) obtain a sharp estimate of  $\hat{\mu}$ ;
- (for general domains  $\Omega$ ) investigate the asymptotic behavior of maximizer as the parameter  $\mu$  tends to 0? Such a issue appears intricate since it requires a refine study of singular limits for Problem (LDE).

## A Proof of Lemma 1

Once the first order differentiability property shown, one gets easily the second order differentiability by reproducing the reasoning. For this reason, we will only prove the Gâteaux-differentiability of  $\mathcal{S}$ . A proof was already provided in [7] but we provide our own proof here for the sake of completeness.

Our proof rests upon the following estimate: there exists a positive constant  $\Lambda_0 = \Lambda_0(m_0, \kappa, \mu)$  such that, for every  $m, m' \in \mathcal{M}_{\kappa, m_0}(\Omega)$ , there holds

$$\|\theta_{m, \mu} - \theta_{m', \mu}\|_{L^2(\Omega)} \leq \Lambda_0 \|m - m'\|_{L^2(\Omega)}. \quad (26)$$

Indeed, assuming this inequality proved, let us fix  $m \in \mathcal{M}_{\kappa, m_0}(\Omega)$  and  $h \in \mathcal{T}_{m, \mathcal{M}_{\kappa, m_0}(\Omega)}$ . From (26) we infer that, whenever  $t$  is small enough, there holds

$$\left\| \frac{\theta_{m+th, \mu} - \theta_{m, \mu}}{t} \right\|_{L^2(\Omega)} \leq \Lambda_0 \|h\|_{L^2(\Omega)}.$$

Defining for every  $t > 0$  the function  $z : t \mapsto \frac{\theta_{m+th, \mu} - \theta_{m, \mu}}{t}$ , one shows with an easy computation that  $z$  satisfies

$$\text{for all } t > 0, \quad \mu \Delta z(t) + z(t) \left( m - (\theta_{m+th, \mu} - \theta_{m, \mu}) \right) = -h \theta_{m, \mu} \quad \text{in } \Omega, \quad (27)$$

with homogeneous Neumann boundary conditions on  $\partial\Omega$ .

Hence,  $W^{1,2}(\Omega)$ -regularity estimates yields

$$\begin{aligned} \|z(t)\|_{W^{1,2}(\Omega)} &\lesssim (\|z(t)\|_{L^2(\Omega)} + \|z(t)(m - (\theta_{m+th, \mu} - \theta_{m, \mu}))\|_{L^2(\Omega)}) + \|h \theta_{m, \mu}\|_{L^2(\Omega)} \\ &\lesssim \Lambda_0 \|h\|_{L^2(\Omega)} (1 + 2\kappa + \|\theta_{m+th, \mu}\|_{L^\infty(\Omega)}) + \kappa \|h\|_{L^2(\Omega)}. \end{aligned}$$

As a consequence, the family  $(z(t))_{t>0}$  is uniformly bounded in  $W^{1,2}(\Omega)$ . Sobolev embeddings combined with the Rellich-Kondrachov theorem yields the existence of a closure point  $z_\infty$ . To prove the differentiability property, it remains to show the uniqueness of  $z_\infty$ . Passing to the limit in the weak formulation of (27) shows that every closure point  $z_\infty$  necessarily satisfies (in the sense of distributions)

$$\mu \Delta z_\infty + z_\infty (m - 2\theta_{m, \mu}) = -h \theta_{m, \mu} \quad \text{in } \Omega,$$

with homogeneous Neumann boundary conditions on  $\partial\Omega$ . Since  $\theta_{m,\mu} > 0$  and  $\lambda_1(m - \theta_{m,\mu}, \mu) = 0$  (according to Section 1.3 and the condition satisfied by  $\theta_{m,\mu}$ ), we infer that  $\lambda_1(m - 2\theta_{m,\mu}, \mu) < 0$ . We conclude by applying the Fredholm alternative which provides the uniqueness of the closure point  $z_\infty$ .

It now remains to prove the inequality (26). To this aim, let us introduce  $\hat{z} = \theta_{m,\mu} - \theta_{m',\mu}$ . Following the computations done in [25],  $\hat{z}$  satisfies

$$\mu\Delta\hat{z} + (m - 2\theta_{m,\mu})\hat{z} + \hat{z}^2 + (m - m')\theta_{m',\mu} = 0.$$

Using as a test function  $\psi_+ = (\hat{z})_+$  (the positive part of  $\hat{z}$ ), an integration by part combined with the Cauchy-Schwarz inequality and (2) yields

$$\begin{aligned} -\mu \int_{\Omega} |\nabla(\hat{z})_+|^2 + \int_{\Omega} (m - 2\theta_{m,\mu})(\hat{z})_+^2 &= - \int_{\Omega} (m - m')(\hat{z})_+ \theta_{m,\mu} - \int_{\Omega} (\hat{z})_+^3 \\ &\leq \int_{\Omega} \kappa |m - m'| (\hat{z})_+ \\ &\leq \kappa \|m - m'\|_{L^2(\Omega)} \|(\hat{z})_+\|_{L^2(\Omega)}. \end{aligned}$$

Combining this inequality with the Courant-Fischer formula (1) yields

$$-\lambda_1(m - 2\theta_{m,\mu}, \mu) \int_{\Omega} (\hat{z})_+^2 \leq -\mu \int_{\Omega} |\nabla(\hat{z})_+|^2 + \int_{\Omega} (m - 2\theta_{m,\mu})(\hat{z})_+^2.$$

Replacing the test function  $\psi_+$  by the function  $\psi_- = (\hat{z})_-$  (the negative part of  $\hat{z}$ ) and applying the same reasoning yields

$$-\lambda_1(m - 2\theta_{m,\mu}, \mu) \int_{\Omega} (\hat{z})_-^2 \leq -\mu \int_{\Omega} |\nabla(\hat{z})_-|^2 + \int_{\Omega} (m - 2\theta_{m,\mu})(\hat{z})_-^2.$$

The desired estimate follows by combining the two last inequalities.

## B Convergence of the series

Let  $\frac{1}{\mu_1^*}$  be the minimum of the convergence radii associated to the power series  $\sum \alpha(k)x^k$ ,  $\sum \beta(k)x^k$ ,  $\sum \gamma(k)x^k$ ,  $\sum \delta(k)x^k$  and  $\sum \varepsilon(k)x^k$  introduced in the proof of Theorem 1.

We will show that, whenever  $\mu \geq \mu_1^*$ , the following expansions

$$\sum_{\ell=0}^{+\infty} \frac{\eta_{k,m}}{\mu^\ell} = \theta_{m,\mu}, \quad \sum_{k=1}^{+\infty} \frac{\dot{\eta}_{k,m}}{\mu^k} = \dot{\theta}_{m,\mu}, \quad \sum_{k=1}^{+\infty} \frac{\ddot{\eta}_{k,m}}{\mu^k} = \ddot{\theta}_{m,\mu}$$

make sense in  $L^2(\Omega)$ . Since the proofs for the series defining  $\dot{\theta}_{m,\mu}$  and  $\ddot{\theta}_{m,\mu}$  are exactly similar to the one for  $\theta_{m,\mu}$ , we only concentrate on the expansion of  $\theta_{m,\mu}$ . By construction, the series  $g_{\infty,\mu} := \sum_{\ell=0}^{+\infty} \frac{\eta_{k,m}}{\mu^\ell}$  converges in  $L^2(\Omega)$ . In order to do so, set, for any  $N \in \mathbb{N}^*$ ,

$$g_{N,\mu} := \sum_{k=0}^N \frac{\eta_{k,m}}{\mu^k}.$$

Then, it is readily checked that  $g_{N,\mu}$  solves the following equation:

$$\mu\Delta g_{N,\mu} + g_{N-1,\mu}(m - g_{N-1,\mu}) = 0,$$

along with Neumann boundary conditions. Thus the function  $g_{\infty,\mu}$  solves the steady logistic equation. Finally, we know that  $g_{\infty,\mu} \xrightarrow{\mu \rightarrow +\infty} m_0 > 0$  so that, for  $\mu$  large enough,  $g_{\infty,\mu}$  is positive. The uniqueness of positive solutions of equation (LDE) entails that, for  $\mu$  large enough,  $g_{\infty,\mu} = \theta_{m,\mu}$ .

This concludes the proof of the convergence of the expansions, and furthermore concludes the proof of theorem 1.

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