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CAUCHY THEORY AND EXPONENTIAL STABILITY FOR INHOMOGENEOUS BOLTZMANN EQUATION FOR HARD POTENTIALS WITHOUT CUT-OFF

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Abstract. In this paper, we investigate both the problems of Cauchy theory and exponential stability for the inhomogeneous Boltzmann equation without angular cut-off. We only deal with the physical case of hard potentials type interactions (with a moderate angular singularity). We prove a result of existence and uniqueness of solutions in a close-to-equilibrium regime for this equation in weighted Sobolev spaces with a polynomial weight, contrary to previous works on the subject, all developed with a weight prescribed by the equilibrium. It is the first result in this more physically relevant framework for this equation. Moreover, we prove an exponential stability for such a solution, with a rate as close as we want to the optimal rate given by the semigroup decay of the linearized equation.

Mathematics Subject Classification (2010): 76P05 Rarefied gas flows, Boltzmann equation; 47H20 Semigroups of nonlinear operators; 35B40 Asymptotic behavior of solutions.

Keywords: Boltzmann equation without cut-off; hard potentials; Cauchy theory; spectral gap; dissipativity; exponential rate of convergence; long-time asymptotic.

Contents

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1. Introduction

1.1. **The model.** In the present paper, we investigate the Cauchy theory and the asymptotic behavior of solutions to the spatially inhomogeneous Boltzmann equation without angular cut-off, that is, for long-range interactions. Previous works have shown that there exist solutions in a close-to-equilibrium regime but in spaces of type $H^q(e^{|v|^2/2})$ which are very restrictive. Here, we are interested in improving this result in the following sense: we enlarge the space in which we develop a Cauchy theory in several ways, we do not require any assumption on the derivatives in velocity and more importantly, our weight is polynomial. We thus only require a condition of finite moments on our data, which is more physically relevant. Moreover, we jointly obtain a convergence to equilibrium for the solutions that we construct with an exponential and explicit rate.

We consider particles described by their space inhomogeneous distribution density $f = f(t,x,v)$ with $t \in \mathbb{R}^+$ the time, $x \in \mathbb{T}^3$ the position and $v \in \mathbb{R}^3$ the velocity. We hence study the so-called spatially inhomogeneous Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f,f). \tag{1.1}$$

The Boltzmann collision operator is defined as

$$Q(g,f) := \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\left[g'_* f' - g_* f\right] d\sigma\, dv_*.$$

Here and below, we are using the shorthand notations $f = f(v)$, $g_* = g(v_*)$, $f' = f(v')$ and $g'_* = g(v'_*)$. In this expression, v, v_* and v', v'_* are the velocities of a pair of particles after and before collision. We make a choice of parametrization of the set of solutions to the conservation of momentum and energy (physical law of elastic collisions):

$$v + v_* = v' + v'_*,$$
$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

so that the pre-collisional velocities are given by:

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$

The Boltzmann collision kernel $B(v-v_*,\sigma)$ only depends on the relative velocity $|v-v_*|$ and on the deviation angle θ through $\cos\theta = \langle \kappa, \sigma\rangle$ where $\kappa = (v-v_*)/|v-v_*|$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^3$. By a symmetry argument, one can always reduce to the case where $B(v-v_*,\sigma)$ is supported on $\langle \kappa,\sigma\rangle \geq 0$ i.e. $0 \leq \theta \leq \pi/2$. So, without loss of generality, we make this assumption.

In this paper, we shall be concerned with the case when the kernel B satisfies the following conditions:

- it takes product form in its arguments as

$$B(v-v_*,\sigma)=\Phi(|v-v_*|)\,b(\cos\theta); \tag{1.2}$$

- the angular function b is locally smooth, and has a nonintegrable singularity for $\theta\to 0$: it satisfies for some $c_b>0$ and $s\in(0,1/2)$ (moderate angular singularity)

$$\forall\,\theta\in(0,\pi/2],\quad \frac{c_b}{\theta^{1+2s}}\le \sin\theta\, b(\cos\theta)\le \frac{1}{c_b\,\theta^{1+2s}}; \tag{1.3}$$

- the kinetic factor Φ satisfies

$$\Phi(|v-v_*|)=|v-v_*|^\gamma \quad\text{with}\quad \gamma\in(0,1), \tag{1.4}$$

this assumption could be relaxed to assuming only that Φ satisfies $\Phi(\cdot)=C_\Phi\,|\cdot|^\gamma$ for some $C_\Phi>0$.

Our main physical motivation comes from particles interacting according to a repulsive potential of the form

$$\phi(r)=r^{-(p-1)},\quad p\in(2,+\infty). \tag{1.5}$$

The assumptions made on B throughout the paper include the case of potentials of the form (1.5) with $p>5$. Indeed, for repulsive potentials of the form (1.5), the collision kernel cannot be computed explicitly but Maxwell [24] has shown that the collision kernel can be computed in terms of the interaction potential ϕ. More precisely, it satisfies the previous conditions (1.2), (1.3) and (1.4) in dimension 3 (see [13, 14, 33]) with $s:=\frac{1}{p-1}\in(0,1)$ and $\gamma:=\frac{p-5}{p-1}\in(-3,1)$.

One traditionally calls *hard potentials* the case $p>5$ (for which $0<\gamma<1$), *Maxwell molecules* the case $p=5$ (for which $\gamma=0$) and *soft potentials* the case $2<p<5$ (for which $-3<\gamma<0$). We can hence deduce that our assumptions made on B include the case of hard potentials.

Let us give a weak formulation of the collision operator Q. For any suitable test function $\varphi=\varphi(v)$, we have:

$$\int_{\mathbb{R}^3} Q(f,f)(v)\,\varphi(v)\,dv=\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,[f'_*f'-f_*f]\,(\varphi+\varphi_*-\varphi'-\varphi'_*)\,d\sigma\,dv_*\,dv. \tag{1.6}$$

From this formula, we can deduce some features of equation (1.1): it preserves mass, momentum and energy. Indeed, at least formally, we have:

$$\int_{\mathbb{R}^3} Q(f,f)(v)\,\varphi(v)\,dv=0\quad\text{for}\quad \varphi(v)=1,v,|v|^2;$$

from which we deduce that a solution f_t to equation (1.1) is conservative, meaning that

$$\forall\,t\ge 0,\quad \int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)\,\varphi(v)\,dv\,dx=\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)\,\varphi(v)\,dv\,dx\quad\text{for}\quad\varphi(v)=1,v,|v|^2. \tag{1.7}$$

We introduce the entropy $H(f)=\int_{\mathbb{T}^3\times\mathbb{R}^3} f\,\log(f)\,dv\,dx$ and the entropy production $D(f)$ defined through:

$$\begin{aligned} D(f)&:=-\frac{d}{dt}H(f)\\ &=\frac{1}{4}\int_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}\,d\sigma\,dv_*\,dv\,dx. \end{aligned} \tag{1.8}$$

Boltzmann's H theorem asserts that

$$\frac{d}{dt}H(f) = -D(f) \leq 0 \tag{1.9}$$

and states that any equilibrium (i.e. any distribution which maximizes the entropy) is a Maxwellian distribution. Moreover, it is known that global equilibria of (1.1) are global Maxwellian distributions that are independent of time t and position x. In this paper, we shall only consider the case of an initial datum satisfying

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0 \, dv \, dx = 1, \quad \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0 \, v \, dv \, dx = 0, \quad \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0 \, |v|^2 dv \, dx = 3, \tag{1.10}$$

and therefore consider μ the Maxwellian with same mass, momentum and energy as f_0:

$$\mu(v) := (2\pi)^{-3/2} e^{-|v|^2/2}. \tag{1.11}$$

1.2. **Notations.** Let X, Y be Banach spaces and consider a linear operator $\Lambda : X \to X$. When defined, we shall denote by $\mathcal{S}_\Lambda(t) = e^{t\Lambda}$ the semigroup generated by Λ. Moreover we denote by $\mathscr{B}(X,Y)$ the space of bounded linear operators from X to Y and by $\| \cdot \|_{\mathscr{B}(X,Y)}$ its norm operator, with the usual simplification $\mathscr{B}(X) = \mathscr{B}(X,X)$.

For simplicity of notations, hereafter, we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$; $a \approx b$ means that there exist constants $c_1, c_2 > 0$ depending only on fixed numbers such that $c_1 b \leq a \leq c_2 b$; we shall use the same notation C for positive constants that may change from line to line or abbreviate " $\leq C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed number.

In what follows, we denote $m(v) := \langle v \rangle^k$ with $k > 0$, the range of admissible k will be specified throughout the paper. We also introduce $\chi \in \mathcal{D}(\mathbb{R})$ a truncation function which satisfies $\mathbb{1}_{[-1,1]} \leq \chi \leq \mathbb{1}_{[-2,2]}$ and we denote $\chi_a(\cdot) := \chi(\cdot/a)$ for $a > 0$.

1.3. **Function spaces.** Through all the paper, we shall consider functions of two variables $f = f(x,v)$ with $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$. Let $\nu = \nu(v)$ be a positive Borel weight function and $1 \leq p \leq \infty$. We define the space $L^p_{x,v}(\nu)$ as the Lebesgue space associated to the norm, for $f = f(x,v)$,

$$\|f\|_{L^p_{x,v}(\nu)} := \big\| \|f\|_{L^p_v(\nu)} \big\|_{L^p_x} := \big\| \|\nu \, f\|_{L^p_v} \big\|_{L^p_x}$$

which writes if $p < \infty$:

$$\begin{aligned} \|f\|_{L^p_{x,v}(\nu)} &= \left(\int_{\mathbb{T}^3_x} \|f(x,\cdot)\|^p_{L^p_v(\nu)} \, dx \right)^{1/p} \\ &= \left(\int_{\mathbb{T}^3_x} \int_{\mathbb{R}^3_v} |f(x,v)|^p \, \nu(v)^p \, dv \, dx \right)^{1/p}. \end{aligned}$$

We define the high-order Sobolev spaces $H^n_x H^\ell_v(\nu)$, for $n, \ell \in \mathbb{N}$:

$$\|f\|^2_{H^n_x H^\ell_v(\nu)} := \sum_{|\alpha| \leq \ell, \, |\beta| \leq n, \, |\alpha| + |\beta| \leq \max(\ell, n)} \|\partial^\alpha_v \partial^\beta_x (f\nu)\|^2_{L^2_{x,v}}. \tag{1.12}$$

This definition reduces to the usual weighted Sobolev space $H^n_{x,v}(\nu)$ when $\ell = n$. We use Fourier transform to define the general space $H^r_{x,v}(\nu)$ for $r \in \mathbb{R}^+$:

$$\|f\|^2_{H^r_{x,v}(\nu)} := \|f\nu\|^2_{H^r_{x,v}} = \sum_{\xi\in\mathbb{Z}^3} \int_{\eta\in\mathbb{R}^3} (1+|\xi|^2+|\eta|^2)^r \, |\widehat{f\nu}(\xi,\eta)|^2 \, d\eta \tag{1.13}$$

where the hat corresponds to the Fourier transform in both x (with corresponding variable $\xi \in \mathbb{Z}^3$) and v (with corresponding variable $\eta \in \mathbb{R}^3$). In this case, the norms given by (1.12) and (1.13) are equivalent. We won't make any difference in the notation and will use one norm or the other at our convenience. It won't have any impact on our estimates since it will only add multiplicative universal constants.

We also introduce the fractional Sobolev space $H^{r,\varsigma}_{x,v}(\nu)$ for $r, \varsigma \in \mathbb{R}^+$ associated to the norm:

$$\|f\|^2_{H^{r,\varsigma}_{x,v}(\nu)} := \|f\nu\|^2_{H^{r,\varsigma}_{x,v}} = \sum_{\xi\in\mathbb{Z}^3} \int_{\mathbb{R}^3} (1+|\xi|^2)^r \, (1+|\eta|^2)^\varsigma \, |\widehat{f\nu}(\xi,\eta)|^2 \, d\eta. \tag{1.14}$$

When $r \in \mathbb{N}$, we can also define the space $H^{r,\varsigma}_{x,v}(\nu)$ through the norm:

$$\|f\|^2_{H^{r,\varsigma}_{x,v}(\nu)} := \sum_{0\le j\le r} \int_{\mathbb{T}^3_x} \|\nabla^j_x f\|^2_{H^\varsigma_v(\nu)} = \sum_{0\le j\le r} \|\nabla^j_x f\|^2_{L^2_x H^\varsigma_v(\nu)}. \tag{1.15}$$

As previously, when $r \in \mathbb{N}$, the norms given by (1.14) and (1.15) are equivalent and we will use one norm or the other at our convenience. Finally, denoting for $\varsigma \in \mathbb{R}^+$,

$$\|f\|^2_{\dot H^\varsigma_v(\nu)} := \|f\nu\|^2_{\dot H^\varsigma_v} = \int_{\mathbb{R}^3} |\eta|^{2\varsigma} \, |\widehat{f\nu}(\eta)|^2 \, d\eta,$$

we introduce the space $\dot H^{n,\varsigma}_{x,v}(\nu)$ for $(n,\varsigma) \in \mathbb{N}\times\mathbb{R}^+$ defined through the norm:

$$\|f\|^2_{\dot H^{n,\varsigma}_{x,v}(\nu)} := \sum_{0\le j\le n} \int_{\mathbb{T}^3_x} \|\nabla^j_x f\|^2_{\dot H^\varsigma_v(\nu)} \, dx = \sum_{0\le j\le n} \|\nabla^j_x f\|^2_{L^2_x \dot H^\varsigma_v(\nu)}. \tag{1.16}$$

Notice also that in the case $\varsigma = 0$, the spaces $H^n_x L^2_v(\nu)$ and $H^{n,0}_{x,v}(\nu)$ associated respectively to the norms given by (1.12) and (1.15) are the same.

We now introduce some "twisted" Sobolev spaces (useful for the development of our Cauchy theory in Section 4), we denote them $\mathcal{H}^{n,\varsigma}_{x,v}(\nu)$ for $(n,\varsigma) \in \mathbb{N}\times\mathbb{R}^+$ and they are associated to the norm:

$$\|f\|^2_{\mathcal{H}^{n,\varsigma}_{x,v}(\nu)} := \sum_{0\le j\le n} \int_{\mathbb{T}^3_x} \|\nabla^j_x f\|^2_{H^\varsigma_v(\langle v\rangle^{-2js}\nu)} = \sum_{0\le j\le n} \|\nabla^j_x f\|^2_{L^2_x H^\varsigma_v(\langle v\rangle^{-2js}\nu)} \tag{1.17}$$

where s is the angular singularity of the Boltzmann kernel introduced in (1.3) and $\langle v\rangle = (1+|v|^2)^{1/2}$. For the case $\varsigma = 0$, since the notation is consistent, we will use the notation $\mathcal{H}^n_x L^2_v(\nu)$ or $\mathcal{H}^{n,0}_{x,v}(\nu)$ indifferently.

Finally, following works from Alexandre et al. (see [6]), we introduce an anisotropic norm that we denote $\|\cdot\|_{\dot H^{s,*}_v}$ (the notation will be explained by Lemma 2.1) and which is defined through

$$\|f\|^2_{\dot H^{s,*}_v} := \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\mu_*\langle v_*\rangle^{-\gamma}(f'\langle v'\rangle^{\gamma/2} - f\langle v\rangle^{\gamma/2})^2 \, d\sigma \, dv_* \, dv.$$

In this definition, γ is the power of the kinetic factor in (1.4) and μ is given by (1.11). Moreover, we recall that b is the angular function of the Boltzmann kernel which satisfies (1.3) and we define b_δ as the following truncation of b: $b_\delta(\cos\theta) := \chi_\delta(\theta) b(\cos\theta)$ with

δ fixed so that the conclusion of Lemma 3.2 holds. Since the constant δ is fixed, we do not mention the dependency of the norm defined above with respect to δ. Let us also introduce the space $H_v^{s,*}(\nu)$ associated with the norm

$$\|f\|^2_{H_v^{s,*}(\nu)} = \|f\|^2_{L^2_v(\langle v\rangle^{\gamma/2}\nu)} + \|f\nu\|^2_{\dot{H}_v^{s,*}}. \tag{1.18}$$

For $n \in \mathbb{N}$, we also define the space $\mathcal{H}_{x,v}^{n,s,*}(\nu)$ associated with the norm

$$\|f\|^2_{\mathcal{H}_{x,v}^{n,s,*}(\nu)} := \sum_{0\le j\le n} \int_{\mathbb{T}^3} \|\nabla_x^j f\|^2_{H_v^{s,*}(\langle v\rangle^{-2js}\nu)}\, dx \tag{1.19}$$

where s is still the angular singularity in (1.3).

1.4. **Main results and known results.**

1.4.1. *Cauchy theory and convergence to equilibrium.* We state now the main result on the fully nonlinear problem (1.1). Let $m(v) = \langle v\rangle^k$ with

$$k > \frac{21}{2} + \gamma + 22s.$$

We then denote $X := \mathcal{H}^3_x L^2_v(m)$ and we introduce $Y^* := \mathcal{H}^{3,s,*}_{x,v}(m)$ (see (1.17) and (1.19) for the definition of the spaces).

Theorem 1.1. *We assume that f_0 has same mass, momentum and energy as μ (i.e. satisfies* (1.10)*). There is a constant $\varepsilon_0 > 0$ such that if $\|f_0 - \mu\|_X \le \varepsilon_0$, then there exists a unique global weak solution f to the Boltzmann equation* (1.1)*, which satisfies, for some constant $C > 0$,*

$$\|f-\mu\|_{L^\infty([0,\infty);X)} + \|f-\mu\|_{L^2([0,\infty);Y^*)} \le C\varepsilon_0.$$

Moreover, this solution satisfies the following estimate: for any $0 < \lambda_2 < \lambda_1$ there exists $C > 0$ such that

$$\forall\, t \ge 0, \quad \|f(t) - \mu\|_X \le C\, e^{-\lambda_2 t}\, \|f_0 - \mu\|_X,$$

where $\lambda_1 > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator in Theorem 3.1.

We refer to Remark 4.1 in which the imposed condition on the power k of our weight is explained. Let us now comment our result and give an overview on the previous works on the Cauchy theory for the inhomogeneous Boltzmann equation. For general large data, we refer to the paper of DiPerna-Lions [17] for global existence of the so-called renormalized solutions in the case of the Boltzmann equation with cut-off. This notion of solution has been extended to the case of long-range interactions by Alexandre-Villani [8] where they construct global renormalized solutions with a defect measure. We also mention the work of Desvillettes-Villani [16] that proves the convergence to equilibrium of a priori smooth solutions for both Boltzmann and Landau equations for general initial data.

In a close-to-equilibrium framework, Gressman and Strain [18] in parallel with Alexandre et al. [6] have developed a Cauchy theory in spaces of type $H^n_x H^\ell_v(\mu^{-1/2})$. One of the famous difficulty of the Boltzmann equation without cut-off is to well understand coercivity estimates. In both papers [6] and [18], the gain induced is seen and understood through a non-isotropic norm. Our strategy uses this type of approach but we also exploit the fact that the linearized Boltzmann operator can be seen as a pseudo-differential operator in order to understand the gain induced by the linearized operator. It allows us to obtain regularization estimates (quantified in time) on the semigroup associated to the linearized operator. We refer to the paper of the same authors [22] for more details on the subject.

To end this brief review, we also refer to a series of papers by Alexandre et al. [3, 4, 5, 6, 7] in which the Boltzmann equation without cut-off is studied in various aspects (different type of collision kernels, Cauchy theory in exponentially weighted spaces, regularity of the solutions etc...).

Let us underline the fact that Theorem 1.1 largely improves previous results on the Cauchy theory associated to the Boltzmann equation without cut-off for hard potentials in a perturbative setting. Indeed, we have enlarged the space in which the Cauchy theory has been developed in the sense that the weight of our space is much less restrictive (it is polynomial instead of the inverse Maxwellian equilibrium) and we also require few assumptions on the derivatives, in particular no derivatives in the velocity variable. However, we need three derivatives in the space variable (Gressman and Strain only require two derivatives in x in [18]): this is the counterpart of the gain in weight we have obtained. Indeed, our framework is less favorable and needs more attention due to the lack of symmetry of the operator in our spaces to obtain nonlinear estimates on the Boltzmann collision operator. And thus, to close our estimates, we require regularity on three derivatives in x.

Our strategy is based on the study of the linearized equation. And then, we go back to the fully nonlinear problem. This is a standard strategy to develop a Cauchy theory in a close-to-equilibrium regime. However, we point out that our study of the nonlinear problem is very tricky. Indeed, usually (for example in the case of the non-homogeneous Boltzmann equation for hard spheres in [19]), the gain induced by the linear part of the equation is enough to directly control the loss due to the nonlinear part of the equation so that the linear part is dominant and thus dictates the dynamics of the equation. In our case, it is more difficult because the gain induced by the linear part is not strong enough and it is not possible to conclude using only natural estimates on the Boltzmann collision operator (this fact was for example pointed out by Mouhot and Neumann in [29]). As a consequence, we establish some new very accurate estimates on the Boltzmann collision operator (see Lemma 2.4). We also have to study very carefully the regularization properties of the semigroup associated to the linearized operator: to this end, we use results by the same authors [22] in which the linearized Boltzmann operator is seen as a pseudo-differential operator, following the framework introduced in [2] by Alexandre, Li and the first author. Also, in the spirit of what was done in [12] by Carrapatoso, Wu and the third author, we work in Sobolev spaces in which the weights depend on the order of the derivative in the space variable. Those key elements allow us to close our estimates and thus, to develop our Cauchy theory in our "twisted" Sobolev spaces.

1.4.2. *The linearized equation.* The linearized operator around equilibrium is defined at first order through

$$\Lambda h := Q(\mu, h) + Q(h, \mu) - v \cdot \nabla_x h.$$

We study spectral properties of the linearized operator Λ in various weighted Sobolev spaces of type $H^n_x H^\ell_v(\langle v\rangle^k)$ up to $L^2_{x,v}(\langle v\rangle^k)$ for k large enough. It is important to highlight the fact that, in order to take advantage of symmetry properties, most of the previous studies have been made in Sobolev weighted spaces of type $H^q_{x,v}(\mu^{-1/2})$. We largely improve theses previous results in the sense that we are able to get similar spectral estimates in larger Sobolev spaces, with a polynomial weight and with less assumptions on the derivatives. Here is a rough version of the main result (Theorem 3.1) that we obtain on the linearized operator Λ:

Theorem 1.2. *Let $\mathcal{E}$ be one of the admissible spaces defined in* (3.4)*. Then, there exist explicit constants $\lambda_1 > 0$ and $C \geq 1$ such that*

$$\forall\, t \geq 0, \quad \forall\, h \in \mathcal{E}, \quad \|\mathcal{S}_\Lambda(t)h - \Pi_0 h\|_{\mathcal{E}} \leq C\, e^{-\lambda_1 t}\, \|h - \Pi_0 h\|_{\mathcal{E}},$$

where $\mathcal{S}_\Lambda(t)$ is the semigroup associated to Λ and Π_0 the projector onto the null space of Λ defined by (1.22).

As mentioned above, the operator Λ (and its homogeneous version $\mathcal{L}h := Q(\mu, h) + Q(h, \mu)$) has already been widely studied. Let us first briefly review the existing results concerning spectral gap estimates for the homogeneous case. Pao [31] studied spectral properties of the linearized operator $\mathcal{L}$ for hard potentials by non-constructive and very technical means. This article was reviewed by Klaus [23]. Then, Baranger and Mouhot gave the first explicit estimate on this spectral gap in [9] for hard potentials ($\gamma > 0$). If we denote $\mathcal{D}$ the Dirichlet form associated to $-\mathcal{L}$:

$$\mathcal{D}(h) := \int_{\mathbb{R}^3} (-\mathcal{L}h)\, h\, \mu^{-1},$$

and $\mathcal{N}(\mathcal{L})^\perp$ the orthogonal of the null space of $\mathcal{L}$, $\mathcal{N}(\mathcal{L})$ which is given by

$$\mathcal{N}(\mathcal{L}) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\},$$

the Dirichlet form $\mathcal{D}$ satisfies

$$\forall\, h \in \mathcal{N}(\mathcal{L})^\perp, \quad \mathcal{D}(h) \geq \lambda_0\, \|h\|^2_{L^2(\mu^{-1/2})}, \tag{1.20}$$

for some constructive constant $\lambda_0 > 0$. This result was then improved by Mouhot [27] and later by Mouhot and Strain [30]. In the last paper, it was conjectured that a spectral gap exists if and only if $\gamma + 2s \geq 0$. This conjecture was finally proven by Gressman and Strain in [18]. Finally, let us point out that the analysis that we carry on can be seen as the sequel of the one handled in [32] by the third author which focuses on the homogeneous linearized operator $\mathcal{L}$. We also improve it in several aspects: we are able to deal with the spatial dependency and we are able to do computations in L^2 (only the L^1-case was treated in the latter).

Concerning the non-homogeneous case, we state here a result coming from Mouhot and Neumann [29] (which takes advantage of the results proven in [9] by Baranger and Mouhot), it gives us a spectral gap estimate in $H^q_{x,v}(\mu^{-1/2})$, $q \in \mathbb{N}^*$, thanks to hypocoercivity methods. Let us underline the fact that it provides us the existence of spectral gap and an estimate on the semigroup decay associated to Λ in the "small" space $E = H^q_{x,v}(\mu^{-1/2})$, which is a crucial point in view of applying the enlargement theorem of [19]. It is also important to precise that Mouhot and Neumann [29] only obtain a result on the linearized operator, they are not able to go back to the nonlinear problem.

Theorem 1.3 ([29])**.** *Consider $E := H^q_{x,v}(\mu^{-1/2})$ with $q \in \mathbb{N}^*$. Then, there exists a constructive constant $\lambda_0 > 0$ (spectral gap) such that Λ satisfies on E:*

(i) *the spectrum $\Sigma(\Lambda) \subset \{z \in \mathbb{C} : \Re e\, z \leq -\lambda_0\} \cup \{0\}$;*
(ii) *the null space $N(\Lambda)$ is given by*

$$N(\Lambda) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}, \tag{1.21}$$

and the projection Π_0 *onto* $N(\Lambda)$ *by*

$$\Pi_0 h = \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} h\,dv\,dx\right)\mu + \sum_{i=1}^{3}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} v_i h\,dv\,dx\right) v_i\mu \\ \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v|^2-3}{6}\,h\,dv\,dx\right)\frac{(|v|^2-3)}{6}\,\mu; \tag{1.22}$$

(iii) Λ *is the generator of a strongly continuous semigroup* $\mathcal{S}_\Lambda(t)$ *that satisfies*

$$\forall\, t\geq 0,\ \forall\, h\in E,\quad \|\mathcal{S}_\Lambda(t)h-\Pi_0 h\|_E \leq e^{-\lambda_0 t}\|h-\Pi_0 h\|_E. \tag{1.23}$$

To prove Theorem 1.2, our strategy follows the one initiated by Mouhot in [28] for the homogeneous Boltzmann equation for hard potentials with cut-off. This argument has then been developed and extended in an abstract setting by Gualdani, Mischler and Mouhot [19], and Mischler and Mouhot [26]. Let us describe in more details this strategy. We want to apply the abstract theorem of enlargement of the space of semigroup decay from [19, 26] to our linearized operator Λ. We shall deduce the spectral/semigroup estimates of Theorem 1.2 on "large spaces" $\mathcal{E}$ using the already known spectral gap estimates for Λ on $H^\ell_{x,v}(\mu^{-1/2})$, for $\ell\geq 1$, described in Theorem 1.3. Roughly speaking, to do that, we have to find a splitting of Λ into two operators $\Lambda=\mathcal{A}+\mathcal{B}$ which satisfy some properties. The first part $\mathcal{A}$ has to be bounded, the second one $\mathcal{B}$ has to have some dissipativity properties, and also the operator $(\mathcal{A}\mathcal{S}_\mathcal{B}(t))$ is required to have some regularization properties.

1.5. **Outline of the paper.** We end this introduction by describing the organization of the paper. In Section 2, we prove nonlinear estimates on the Boltzmann collision operator. In Section 3 we consider the linearized equation and prove a precise version of Theorem 1.2. In Section 4, we come back to the nonlinear equation and prove our main result Theorem 1.1.

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2. Preliminaries on the Boltzmann collision operator

In this part, we give estimates on the trilinear form $\langle Q(g,h),f\rangle$ in our physical framework (meaning that the collision kernel B satisfies conditions (1.2), (1.3), (1.4)). We start by recalling some homogeneous estimates and then establish some new estimates in weighted Sobolev (or Lebesgue) non homogeneous spaces. These new estimates will be used both in the linear (Section 3) and nonlinear (Section 4) studies.

For sake of clarity, we recall that $m(v)=\langle v\rangle^k$ with $k>0$ and that we will specify the range of admissible k in each result.

2.1. **Bound on the anisotropic norm.** In this subsection, we compare the anisotropic norm defined in (1.18) with usual Sobolev norms.

Lemma 2.1. *Let $k \geq 0$. We have the following estimate: for $g \in H^s_v(\langle v\rangle^{\gamma/2+s}m)$,*

$$\delta^{2-2s}\|g\|_{H^s_v(\langle v\rangle^{\gamma/2}m)} \lesssim \|g\|_{H^{s,*}_v(m)} \lesssim \|g\|_{H^s_v(\langle v\rangle^{\gamma/2+s}m)}.$$

Proof. Adapting the proof of [21, Theorem 3.1], we know that there exist c_0 and c_1 such that

$$\|gm\|^2_{\dot H^{s,*}_v} \geq c_0\,\delta^{2-2s}\|g\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)} - c_1\,\delta^{2-2s}\|g\|^2_{L^2(\langle v\rangle^{\gamma/2}m)}.$$

As a consequence, we have for $\lambda \in (0,1)$,

$$\begin{aligned}\|g\|^2_{H^{s,*}_v(m)} &= \|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + \|gm\|^2_{\dot H^{s,*}_v}\\ &\geq \|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + \lambda\|gm\|^2_{\dot H^{s,*}_v}\\ &\geq \|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}(1-\lambda\,c_1\,\delta^{2-2s}) + \lambda\,c_0\,\delta^{2-2s}\|g\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)}.\end{aligned}$$

Taking $\lambda > 0$ small enough, we obtain the bound $\delta^{2-2s}\|g\|_{H^s_v(\langle v\rangle^{\gamma/2}m)} \lesssim \|g\|_{H^{s,*}_v(m)}$. The reverse bound is directly given by [6, Lemma 2.4] since

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\mu_*\langle v_*\rangle^{-\gamma}(g'm'\langle v'\rangle^{\gamma/2} - gm\langle v\rangle^{\gamma/2})^2\,d\sigma\,dv_*\,dv\\ &\quad\leq \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\mu_*\langle v_*\rangle^{-\gamma}(g'm'\langle v'\rangle^{\gamma/2} - gm\langle v\rangle^{\gamma/2})^2\,d\sigma\,dv_*\,dv.\end{aligned}$$

□

We will use the fact that our lower bound in the previous lemma depends on δ in the proof of Lemmas 3.2 and 3.3. However, in the next subsection, δ is fixed so that the conclusion of Lemma 3.2 is satisfied, we thus do not mention anymore the dependency of constants with respect to δ.

2.2. **Homogeneous estimates.**

Lemma 2.2 ([20])**.** *For smooth functions f, g, h, one has:*

$$|\langle Q(f,g),h\rangle_{L^2_v}| \lesssim \|f\|_{L^1_v(\langle v\rangle^{\gamma+2s})}\,\|g\|_{H^{\varsigma_1}_v(\langle v\rangle^{N_1})}\,\|h\|_{H^{\varsigma_2}_v(\langle v\rangle^{N_2})}$$

with ς_1, $\varsigma_2 \in [0,2s]$ satisfying $\varsigma_1+\varsigma_2 = 2s$ and N_1, $N_2 \geq 0$ such that $N_1+N_2 = \gamma+2s$.

The goal of what follows is to extend this type of estimates to weighted Lebesgue spaces. Lemma 2.3 is a "weighted version" of Lemma 2.2.

Lemma 2.3. *Assume $k > \gamma/2+3+2s$.*

(i) *For any $\ell > \gamma+1+3/2$, there holds*

$$\begin{aligned}\langle Q(f,g),h\rangle_{L^2_v(m)} &\lesssim \|f\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|_{H^{\varsigma_1}_v(\langle v\rangle^{N_1}m)}\,\|h\|_{H^{\varsigma_2}_v(\langle v\rangle^{N_2}m)}\\ &\quad + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|h\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\end{aligned} \tag{2.1}$$

with ς_1, $\varsigma_2 \in [0,2s]$ satisfying $\varsigma_1+\varsigma_2 = 2s$ and N_1, $N_2 \geq 0$ such that $N_1+N_2 = \gamma+2s$.

(ii) *For any $\ell > 4-\gamma+3/2$, there holds*

$$\langle Q(f,g),g\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|^2_{H^{s,*}_v(m)} + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}. \tag{2.2}$$

Proof of (i). We write

$$\begin{aligned}\langle Q(f,g),h\rangle_{L^2_v(m)} &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,(f'_*g'-f_*g)\,h\,m^2\,d\sigma\,dv_*\,dv\\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,(f'_*g'm'-f_*g\,m)\,h\,m\,d\sigma\,dv_*\,dv\\ &\quad+\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,f'_*g'h\,m\,(m-m')\,d\sigma\,dv_*\,dv\\ &=: I_1+I_2.\end{aligned}$$

We deal with the first term I_1 using Lemma 2.2:

$$\begin{aligned}I_1=\langle Q(f,gm),hm\rangle_{L^2_v} &\lesssim \|f\|_{L^1_v(\langle v\rangle^{\gamma+2s})}\,\|g\|_{H^{\varsigma_1}_v(\langle v\rangle^{N_1}m)}\,\|h\|_{H^{\varsigma_2}_v(\langle v\rangle^{N_2}m)}\\ &\lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})}\,\|g\|_{H^{\varsigma_1}_v(\langle v\rangle^{N_1}m)}\,\|h\|_{H^{\varsigma_2}_v(\langle v\rangle^{N_2}m)}\end{aligned}$$

because $\ell>\gamma+2s+3/2$, with ς_1, $\varsigma_2\in[0,2s]$ satisfying $\varsigma_1+\varsigma_2=2s$ and with N_1, $N_2\geq 0$ such that $N_1+N_2=\gamma+2s$. To deal with I_2, we use the following estimate on $|m'-m|$ (see the proof in [3, Lemma 2.3]):

$$|m'-m|\lesssim \sin(\theta/2)\left(m'+\langle v'_*\rangle\,\langle v'\rangle^{k-1}+\sin^{k-1}(\theta/2)\,m'_*\right). \tag{2.3}$$

Notice that $|v-v_*|=|v'-v'_*|\lesssim|v-v'_*|$ which implies

$$|v-v_*|^\gamma\lesssim|v'-v'_*|^{\gamma/2}\,|v-v'_*|^{\gamma/2}\lesssim\langle v\rangle^{\gamma/2}\,\langle v'\rangle^{\gamma/2}\,\langle v'_*\rangle^{\gamma}. \tag{2.4}$$

Also, we have,

$$|v-v_*|^\gamma\lesssim|v'-v|^{\gamma/2}\,\sin^{-\gamma/2}(\theta/2)\,|v'-v'_*|^{\gamma/2}\lesssim\sin^{-\gamma/2}(\theta/2)\,\langle v'\rangle^{\gamma}\,\langle v\rangle^{\gamma/2}\,\langle v'_*\rangle^{\gamma/2}. \tag{2.5}$$

This bound induces the appearance of a singularity in θ. However, we notice that in the third term of the estimate (2.3) we have a gain in the power of $\sin(\theta/2)$ depending on the value of k, the power of our polynomial weight. As a consequence, if k is large enough, we can keep a power of $\sin(\theta/2)$ that is large enough to remove the singularity of $b(\cos\theta)$ at $\theta=0$. Consequently, we have:

$$\begin{aligned}I_2 &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,|v-v_*|^\gamma\,|f'_*||g'||h|\,m\\ &\qquad\left(m'+\langle v'_*\rangle\,\langle v'\rangle^{k-1}+\sin^{k-1}(\theta/2)\,m'_*\right)\,d\sigma\,dv_*\,dv\\ &=: I_{21}+I_{22}+I_{23}.\end{aligned}$$

The two first terms I_{21} and I_{22} are treated in the same way using the estimate (2.4), we obtain:

$$\begin{aligned}I_{21}+I_{22} &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,|f'_*|\langle v'_*\rangle^{\gamma+1}\,|g'|m'\langle v'\rangle^{\gamma/2}\,|h|\,m\langle v\rangle^{\gamma/2}\,d\sigma\,dv_*\,dv\\ &\lesssim \left(\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,|f'|\langle v'\rangle^{\gamma+1}\,(g'_*)^2(m'_*)^2\langle v'_*\rangle^{\gamma}\,d\sigma\,dv_*\,dv\right)^{1/2}\\ &\quad\times\left(\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,|f'|\langle v'\rangle^{\gamma+1}\,h_*^2\,m_*^2\langle v_*\rangle^{\gamma}\,d\sigma\,dv_*\,dv\right)^{1/2}\\ &=: J_1\times J_2.\end{aligned}$$

The term J_1 is easily handled just using the pre-post collisional change of variable:

$$J_1^2 \lesssim \|f\|_{L^1_v(\langle v\rangle^{\gamma+1})} \|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2} m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})} \|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2} m)}$$

since $\ell > \gamma + 1 + 3/2$. To deal with J_2, we use the regular change of variable $v \to v'$ meaning that for each σ, with v_* still fixed, we perform the change of variables $v \to v'$. This change of variables is well-defined on the set $\{\cos\theta > 0\}$. Its Jacobian determinant is

$$\left|\frac{dv'}{dv}\right| = \frac{1}{8}(1+\kappa\cdot\sigma) = \frac{(\kappa'\cdot\sigma)^2}{4},$$

where $\kappa := (v-v_*)/|v-v_*|$ and $\kappa' := (v'-v_*)/|v'-v_*|$. We have $\kappa'\cdot\sigma = \cos(\theta/2) \geq 1/\sqrt{2}$. The inverse transformation $v' \to \psi_\sigma(v') = v$ is then defined accordingly. Using the fact that

$$\cos\theta = \kappa\cdot\sigma = 2(\kappa'\cdot\sigma)^2 - 1 \quad \text{and} \quad \sin(\theta/2) = \sqrt{1-\cos^2(\theta/2)} = \sqrt{1-(\kappa'\cdot\sigma)^2},$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,|f'|\langle v'\rangle^{\gamma+1}\,d\sigma\,dv \\
&\quad= \int_{\mathbb{R}^3\times\mathbb{S}^2} b(2(\kappa'\cdot\sigma)^2-1)\,\sqrt{1-(\kappa'\cdot\sigma)^2}\,|f'|\langle v'\rangle^{\gamma+1}\,d\sigma\,dv \\
&\quad= \int_{\kappa'\cdot\sigma\geq 1/\sqrt{2}} b(2(\kappa'\cdot\sigma)^2-1)\,\sqrt{1-(\kappa'\cdot\sigma)^2}\,|f'|\langle v'\rangle^{\gamma+1}\,d\sigma\,\frac{4\,dv'}{(\kappa'\cdot\sigma)^2} \\
&\quad\lesssim \int_{\mathbb{S}^2} b(\cos 2\theta)\,\sin\theta\,d\sigma \int_{\mathbb{R}^3} |f|\langle v\rangle^{\gamma+1}\,dv.
\end{aligned}$$

We deduce:

$$J_2^2 \lesssim \|f\|_{L^1_v(\langle v\rangle^{\gamma+1})} \|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2} m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})} \|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2} m)}.$$

In summary, gathering the three previous estimates, we have

$$I_{21} + I_{22} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})} \|g\|_{L^2_v(\langle v\rangle^{\gamma/2} m)} \|h\|_{L^2_v(\langle v\rangle^{\gamma/2} m)}.$$

Concerning I_{23}, we take advantage of the bound given by (2.5):

$$\begin{aligned}
I_{23} &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin^{k-\gamma/2}(\theta/2)\,|f'_*|m'_*\langle v'_*\rangle^{\gamma/2}\,|g'|\langle v'\rangle^{\gamma}\,|h|\,m\,\langle v\rangle^{\gamma/2}\,d\sigma\,dv_*\,dv \\
&\lesssim \left(\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin^{k-\gamma/2}(\theta/2)\,|g'|\langle v'\rangle^{\gamma}\,|f'_*|^2 {m'_*}^2\langle v'_*\rangle^{\gamma}\,d\sigma\,dv_*\,dv\right)^{1/2} \\
&\quad\times \left(\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin^{k-\gamma/2}(\theta/2)\,|g'|\langle v'\rangle^{\gamma}\,h^2\,m^2\,\langle v\rangle^{\gamma}\,d\sigma\,dv_*\,dv\right)^{1/2} \\
&=: T_1 \times T_2.
\end{aligned}$$

As far as T_1 is concerned, a simple pre-post collisional change of variable allows us to get

$$T_1^2 \lesssim \|g\|_{L^1_v(\langle v\rangle^{\gamma})} \|f\|^2_{L^2_v(\langle v\rangle^{\gamma/2} m)} \lesssim \|g\|_{L^2_v(\langle v\rangle^{\ell})} \|f\|^2_{L^2_v(\langle v\rangle^{\gamma/2} m)}$$

since $\ell > \gamma + 3/2$. The second term requires more attention since we have to perform a singular change of variable $v_* \to v'$ showed for example in the proof of Lemma 2.4 in [3]. Recall that the Jacobian of this transform is

$$\left|\frac{dv_*}{dv'}\right| = \frac{4}{\sin^2(\theta/2)} \leq 16\,\theta^{-2},\ \theta\in(0,\pi/2],$$

therefore, this change of variable gives rise to an additional singularity in θ around 0. However, we can take advantage of the fact that we have a power k in $\sin(\theta/2)$, indeed taking k large enough allows us to control this singularity. Notice that θ is no longer the good polar angle to consider, we set $\psi = (\pi-\theta)/2$ for $\psi \in [\pi/4, \pi/2]$ so that

$$\cos\psi = \frac{v'-v}{|v'-v|}\cdot\sigma \quad \text{and} \quad d\sigma = \sin\psi\, d\psi\, d\phi.$$

This measure does not cancel any of the singularity of $b(\cos\theta)$ unlike in the case of the usual polar coordinates but it will be counterbalanced taking k large enough. We then have:

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin^{k-\gamma/2}(\theta/2)\,|g'|\langle v'\rangle^\gamma\,d\sigma\,dv_* \lesssim \int_{\mathbb{R}^3\times\mathbb{S}^2}(\pi-2\psi)^{k-\gamma/2-4-2s}\,|g'|\langle v'\rangle^\gamma\,d\sigma\,dv'$$
$$\lesssim \int_{\pi/4}^{\pi/2}(\pi-2\psi)^{k-\gamma/2-4-2s}\,\sin\psi\,d\psi\int_{\mathbb{R}^3}|g|\,\langle v\rangle^\gamma\,dv \lesssim \int_{\mathbb{R}^3}|g|\,\langle v\rangle^\gamma$$

since $k > \gamma/2+3+2s$. We deduce that

$$T_2^2 \lesssim \|g\|_{L^1_v(\langle v\rangle^\gamma)}\,\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} \lesssim \|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}$$

and thus

$$I_{23} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|h\|_{L^2_v(\langle v\rangle^{\gamma/2}m)},$$

which concludes the proof of estimate (2.1).

Proof of (ii). We have:

$$\begin{aligned}&\langle Q(f,g),g\rangle_{L^2_v(m)}\\ &\quad= \langle Q(f,gm),gm)\rangle_{L^2_v} + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)f'_*g'\,g\,m\,(m-m')\,d\sigma\,dv_*\,dv\\ &\quad=: I+J.\end{aligned}$$

The term J is done in the first step of the proof, it corresponds to the term I_2 replacing h by g, we thus have

$$J \lesssim \|f\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

In order to deal with the term I, we denote $G := gm$. We also recall that

$$b_\delta(\cos\theta) = \chi_\delta(\theta)\,b(\cos\theta)$$

and we introduce the notations

$$b^c_\delta(\cos\theta) := (1-\chi_\delta(\theta))\,b(\cos\theta),$$
$$B_\delta(v-v_*,\sigma) := b_\delta(\cos\theta)\,|v-v_*|^\gamma \quad \text{and} \quad B^c_\delta(v-v_*,\sigma) := b^c_\delta(\cos\theta)\,|v-v_*|^\gamma.$$

The two previous kernels correspond respectively to grazing collisions and non grazing collisions (which encodes the cut-off part of the operator). We also denote Q_δ (resp. Q^c_δ) the operator associated with the kernel B_δ (resp. B^c_δ). We have for $G = gm$:

$$I = \langle Q_\delta(f,G),G\rangle_{L^2_v} + \langle Q^c_\delta(f,G),G\rangle_{L^2_v} =: I^\delta + I^{\delta,c}.$$

We start by dealing with the cut-off part:

$$\begin{aligned}I^{\delta,c} &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B^c_\delta(v-v_*,\sigma)f_*\,G(G'-G)\,d\sigma\,dv_*\,dv\\ &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|v-v_*|^\gamma\,b^c_\delta(\cos\theta)\,|f_*|\,(G^2+(G')^2)\,d\sigma\,dv_*\,dv.\end{aligned}$$

Using that $b_\delta^c(\cos\theta) \le C_\delta$ on $\mathbb{S}^2$ and $|v-v_*|^\gamma \lesssim |v'-v_*|^\gamma$, we get

$$I^{\delta,c} \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |f_*|\langle v_*\rangle^\gamma\, G^2\,\langle v\rangle^\gamma\, d\sigma\, dv_*\, dv + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |f_*|\langle v_*\rangle^\gamma\, G'^2\,\langle v'\rangle^\gamma\, d\sigma\, dv_*\, dv.$$

The first term is directly bounded from above by $\|f\|_{L^1_v(\langle v\rangle^\gamma)}\|G\|^2_{L^2_v(\langle v\rangle^{\gamma/2})}$ and for the second one, we use the regular change of variable $v \to v'$ explained in the proof of (i). We thus get

$$I^{\delta,c} \lesssim \|f\|_{L^1_v(\langle v\rangle^\gamma)}\|G\|^2_{L^2_v(\langle v\rangle^{\gamma/2})} \lesssim \|f\|_{L^2_v(\langle v\rangle^\ell)}\|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

Concerning the grazing collisions part, we write

$$\begin{aligned} I^\delta &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma) f_*\, G\,(G'-G)\, d\sigma\, dv_*\, dv \\ &= -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma) f_*\,(G'-G)^2\, d\sigma\, dv_*\, dv \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma) f_*\,((G')^2-G^2)\, d\sigma\, dv_*\, dv =: I_1^\delta + I_2^\delta. \end{aligned}$$

The second term I_2^δ is treated thanks to the cancellation lemma [1, Lemma 1]:

$$I_2^\delta = \int_{\mathbb{R}^3} (S_\delta * G^2)\, f\, dv,$$

where (for details, see [32, proof of Lemma 2.2])

$$S_\delta(z) := 2\pi \int_0^{\pi/2} \sin\theta\, b_\delta(\cos\theta)\left(\frac{|z|^\gamma}{\cos^{\gamma+3}(\theta/2)} - |z|^\gamma\right) d\theta \lesssim \delta^{2-2s}\,|z|^\gamma. \tag{2.6}$$

We deduce that

$$I_2^\delta \lesssim \|f\|_{L^1_v(\langle v\rangle^\gamma)}\|G\|^2_{L^2_v(\langle v\rangle^{\gamma/2})} \lesssim \|f\|_{L^2_v(\langle v\rangle^\ell)}\|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

It now remains to handle I_1^δ. First, using that $|v-v_*| \lesssim |v'-v_*|$, we have

$$\begin{aligned} I_1^\delta &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)|v-v_*|^\gamma|f_*|\,(G'-G)^2\, d\sigma\, dv_*\, dv \\ &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)|v'-v_*|^\gamma|f_*|\,(G'-G)^2\, d\sigma\, dv_*\, dv \\ &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)|f_*|\langle v_*\rangle^\gamma\,(G'\langle v'\rangle^{\gamma/2} - G\langle v\rangle^{\gamma/2})^2\, d\sigma\, dv_*\, dv \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)|f_*|\langle v_*\rangle^\gamma\, G^2\,(\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2})^2\, d\sigma\, dv_*\, dv =: I_{11}^\delta + I_{12}^\delta. \end{aligned}$$

To deal with I_{12}^δ, we first note that

$$|\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2}| \lesssim |v'-v| \int_0^1 \langle v'+\tau(v-v')\rangle^{\gamma/2-1}\, d\tau \lesssim |v-v_*|\sin(\theta/2)\int_0^1 \langle v_\tau\rangle^{\gamma/2-1}\, d\tau$$

where $v_\tau := v' + \tau(v-v')$. Moreover, for any $\tau\in[0,1]$, we have

$$\langle v\rangle \le \langle v-v_*\rangle + \langle v_*\rangle \le \sqrt{2}\langle v_\tau - v_*\rangle + \langle v_*\rangle \lesssim \langle v_\tau\rangle\langle v_*\rangle$$

which implies (since $\gamma/2-1\le 0$)

$$\langle v_\tau\rangle^{\gamma/2-1} \lesssim \langle v\rangle^{\gamma/2-1}\langle v_*\rangle^{1-\gamma/2}.$$

Consequently, we deduce

$$(\langle v\rangle^{\gamma/2}-\langle v'\rangle^{\gamma/2})^2 \lesssim |v-v_*|^2\sin^2(\theta/2)\langle v\rangle^{\gamma-2}\langle v_*\rangle^{2-\gamma} \lesssim \sin^2(\theta/2)\langle v\rangle^{\gamma}\langle v_*\rangle^{4-\gamma} \tag{2.7}$$

so that

$$I_{12}^{\delta} \lesssim \|f\|_{L^1_v(\langle v\rangle^{4-\gamma})}\|G\|^2_{L^2_v(\langle v\rangle^{\gamma/2})} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})}\|g\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

For the analysis of I_{11}^{δ}, we introduce the following notations: $\tilde f := f\langle\cdot\rangle^{\gamma}$, $\tilde\mu := \mu\langle v\rangle^{-\gamma}$ and $\mathcal{G} := G\langle v\rangle^{\gamma/2}$ so that

$$I_{11}^{\delta} = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)|\tilde f|(\mathcal{G}'-\mathcal{G})^2\,d\sigma\,dv_*\,dv.$$

We then use Bobylev formula [11] (see also [1, Proposition 2]), denoting $\xi^{\pm} = (\xi \pm |\xi|\sigma)/2$, we have:

$$I_{11}^{\delta} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\widehat{|\tilde f|}(0)|\widehat{\mathcal{G}}(\xi)-\widehat{\mathcal{G}}(\xi^+)|^2 + 2\,\Re e\left(\widehat{|\tilde f|}(0)-\widehat{|\tilde f|}(\xi^-)\right)\widehat{\mathcal{G}}(\xi^+)\overline{\widehat{\mathcal{G}}}(\xi)\right)d\sigma\,d\xi.$$

Similarly, we have

$$\|G\|^2_{\dot H^{s,*}_v} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\widehat{\tilde\mu}(0)|\widehat{\mathcal{G}}(\xi)-\widehat{\mathcal{G}}(\xi^+)|^2 + 2\,\Re e\left(\widehat{\tilde\mu}(0)-\widehat{\tilde\mu}(\xi^-)\right)\widehat{\mathcal{G}}(\xi^+)\overline{\widehat{\mathcal{G}}}(\xi)\right)d\sigma\,d\xi.$$

Since $\widehat{|\tilde f|}(0) = \|\tilde f\|_{L^1_v}$ and $\widehat{\tilde\mu}(v) = \|\tilde\mu\|_{L^1_v}$, we deduce that

$$\begin{aligned} I_{11}^{\delta} &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta\left(\frac{\xi}{|\xi|\cdot\sigma}\right)2\,\Re e\left(\widehat{|\tilde f|}(0)-\widehat{|\tilde f|}(\xi^-)\right)\widehat{\mathcal{G}}(\xi^+)\overline{\widehat{\mathcal{G}}}(\xi)\,d\sigma\,d\xi \\ &\quad - \frac{1}{(2\pi)^3}\frac{\|\tilde f\|_{L^1_v}}{\|\tilde\mu\|_{L^1_v}}\int_{\mathbb{R}^3\times\mathbb{S}^2} b_\delta\left(\frac{\xi}{|\xi|\cdot\sigma}\right)2\,\Re e\left(\widehat{\tilde\mu}(0)-\widehat{\tilde\mu}(\xi^-)\right)\widehat{\mathcal{G}}(\xi^+)\overline{\widehat{\mathcal{G}}}(\xi)\,d\sigma\,d\xi \\ &\quad + \frac{\|\tilde f\|_{L^1_v}}{\|\tilde\mu\|_{L^1_v}}\|G\|^2_{\dot H^{s,*}_v} =: I_{111}^{\delta} + I_{112}^{\delta} + I_{113}^{\delta}. \end{aligned}$$

Using then results from the proof of [6, Lemma 2.8] combined with Lemma 2.1, we get that

$$I_{111}^{\delta} \lesssim \|\tilde f\|_{L^1_v(\langle v\rangle^{2s})}\|\mathcal{G}\|^2_{H^s_v} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})}\|g\|^2_{H^{s,*}_v(m)}$$

and

$$I_{112}^{\delta} \lesssim \|\tilde f\|_{L^1_v}\|\mathcal{G}\|^2_{H^s_v} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})}\|g\|^2_{H^{s,*}_v(m)}.$$

We also clearly have

$$I_{113}^{\delta} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\ell})}\|g\|^2_{H^{s,*}_v(m)}.$$

Gathering all the previous estimates, we are able to deduce that (2.2) holds. □

2.3. **Non homogeneous estimates.** We now prove non homogeneous estimates on the trilinear form $\langle Q(f,g),h\rangle$ in order to get some accurate estimates on the terms coming from the nonlinear part of the equation. Basically, we give a non homogeneous version of Lemma 2.3. We introduce the spaces

$$\begin{cases} X := \mathcal{H}^3_x L^2_v(m) \\ Y := \mathcal{H}^{3,s}_{x,v}(\langle v\rangle^{\gamma/2} m) \\ Y^* := \mathcal{H}^{3,s,*}_{x,v}(m) \\ \bar{Y} := \mathcal{H}^{3,s}_{x,v}(\langle v\rangle^{\gamma/2+2s} m) \end{cases} \tag{2.8}$$

that are defined through their norms by (1.17) and (1.19). We also introduce Y' the dual space of Y with respect to the pivot space X, meaning that the Y'-norm is defined through:

$$\|f\|_{Y'} := \sup_{\|\phi\|_Y\leq 1} \langle f,\phi\rangle_X = \sup_{\|\phi\|_Y\leq 1} \sum_{0\leq j\leq 3} \langle \nabla^j_x f, \nabla^j_x \phi\rangle_{L^2_{x,v}(\langle v\rangle^{-2js} m)}. \tag{2.9}$$

Lemma 2.4. *The following estimates hold:*

(i) *For $k > \gamma/2 + 3 + 8s$,*

$$\langle Q(f,g),h\rangle_X \lesssim \|f\|_X \|g\|_{\bar{Y}} \|h\|_Y + \|f\|_Y \|g\|_X \|h\|_Y;$$

therefore,

$$\|Q(f,g)\|_{Y'} \lesssim \|f\|_X \|g\|_{\bar{Y}} + \|f\|_Y \|g\|_X.$$

(ii) *For $k > 4 - \gamma + 3/2 + 6s$,*

$$\langle Q(f,g),g\rangle_X \lesssim \|f\|_X \|g\|^2_{Y^*} + \|f\|_Y \|g\|_X \|g\|_Y.$$

(iii) *For $k > 4 - \gamma + 3/2 + 6s$,*

$$\langle Q(f,f),f\rangle_X \lesssim \|f\|_X \|f\|^2_{Y^*}.$$

Proof. In this proof, we use Lemma 2.3-(i) and (ii) together with the following inequalities when integrating in $x \in \mathbb{T}^3$,

$$\|u\|_{L^\infty(\mathbb{T}^3_x)} \lesssim \|u\|_{H^2(\mathbb{T}^3_x)}, \quad \|u\|_{L^6(\mathbb{T}^3_x)} \lesssim \|u\|_{H^1(\mathbb{T}^3_x)}, \quad \|u\|_{L^3(\mathbb{T}^3_x)} \lesssim \|u\|_{H^1(\mathbb{T}^3_x)}. \tag{2.10}$$

Proof of (i). We write

$$\langle Q(f,g),h\rangle_{\mathcal{H}^3_x L^2_v(m)} = \langle Q(f,g),h\rangle_{L^2_{x,v}(m)} + \sum_{1\leq|\beta|\leq 3} \langle \partial^\beta_x Q(f,g), \partial^\beta_x h\rangle_{L^2_{x,v}(m\langle v\rangle^{-2|\beta|s})},$$

and

$$\partial^\beta_x Q(f,g) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\, Q(\partial^{\beta_1}_x f, \partial^{\beta_2}_x g).$$

In the following steps we will always consider $\ell \in (\gamma + 1 + 3/2, k - 6s]$ which is possible since $k > \gamma/2 + 3 + 8s$, $\gamma \leq 1$ and $s \geq 0$.

Step 1. Using Lemma 2.3-(i) applied with $\varsigma_1 = \varsigma_2 = s$, $N_1 = \gamma/2+2s$, $N_2 = \gamma/2$ and (2.10) we have

$$\begin{aligned}
&\langle Q(f,g),h\rangle_{L^2_{x,v}(m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|f\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|_{H^s_v(\langle v\rangle^{\gamma/2+2s}m)}\,\|h\|_{H^s_v(\langle v\rangle^{\gamma/2}m)}\\
&\quad+\|f\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|h\|_{L^2_v(\langle v\rangle^{\gamma/2}m)}\Big)\\
&\lesssim \|f\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|g\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2+2s}m)}\,\|h\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2}m)}\\
&\quad+\|f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m)}\,\|g\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y+\|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

Step 2. Case $|\beta|=1$. Arguing as in the previous step,

$$\begin{aligned}
&\langle Q(f,\partial_x^\beta g),\partial_x^\beta h\rangle_{L^2_{x,v}(\langle v\rangle^{-2s}m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|f\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x g\|_{H^s_v(\langle v\rangle^{\gamma/2}m)}\,\|\nabla_x h\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)}\\
&\quad+\|f\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x h\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)}\Big)\\
&\lesssim \|f\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|\nabla_x g\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2}m)}\,\|\nabla_x h\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2-2s}m)}\\
&\quad+\|f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x g\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|\nabla_x h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y+\|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\langle Q(\partial_x^\beta f,g),\partial_x^\beta h\rangle_{L^2_{x,v}(\langle v\rangle^{-2s}m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|\nabla_x f\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|_{H^s_v(\langle v\rangle^{\gamma/2}m)}\,\|\nabla_x h\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)}\\
&\quad+\|\nabla_x f\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x h\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)}\Big)\\
&\lesssim \|\nabla_x f\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|g\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2}m)}\,\|\nabla_x h\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2-2s}m)}\\
&\quad+\|\nabla_x f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)}\,\|g\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|\nabla_x h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y+\|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

Step 3. Case $|\beta|=2$. When $\beta_2=\beta$, we have

$$\begin{aligned}
&\langle Q(f,\partial_x^\beta g),\partial_x^\beta h\rangle_{L^2_{x,v}(\langle v\rangle^{-4s}m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|f\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 g\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x^2 h\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\\
&\quad+\|f\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 h\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)}\Big)\\
&\lesssim \|f\|_{H^2_xL^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 g\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x^2 h\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2-4s}m)}\\
&\quad+\|f\|_{H^2_xL^2_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^\ell)}\,\|\nabla_x^2 h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y+\|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

When $\beta_1 = \beta$, we have

$$\begin{aligned}
&\langle Q(\partial_x^\beta f, g), \partial_x^\beta h\rangle_{L^2_{x,v}(\langle v\rangle^{-4s}m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^\ell)}\,\|g\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x^2 h\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\\
&\quad + \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 h\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)}\Big)\\
&\lesssim \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^\ell)}\,\|g\|_{H^{2,s}_{x,v}(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x^2 h\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\\
&\quad + \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)}\,\|g\|_{H^2_x L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y + \|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

Finally, when $|\beta_1| = |\beta_2| = 1$, we obtain

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^\beta h\rangle_{L^2_x L^2_v(\langle v\rangle^{-4s}m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|\nabla_x f\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x g\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x^2 h\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\\
&\quad + \|\nabla_x f\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 h\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)}\Big)\\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-2s}m)}\,\|\nabla_x^2 h\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\\
&\quad + \|\nabla_x f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x g\|_{H^2_x L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y + \|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

Step 4. Case $|\beta| = 3$. When $\beta_2 = \beta$ we obtain

$$\begin{aligned}
&\langle Q(f, \partial_x^\beta g), \partial_x^\beta h\rangle_{L^2_{x,v}(\langle v\rangle^{-6s}m)}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|f\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^3 g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x^3 h\|_{H^s_v(\langle v\rangle^{\gamma/2-6s}m)}\\
&\quad + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)}\,\|\nabla_x^3 g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^3 h\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)}\Big)\\
&\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^3 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x^3 h\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s}m)}\\
&\quad + \|f\|_{H^2_x L^2_v(\langle v\rangle^{\gamma/2-6s}m)}\,\|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^\ell)}\,\|\nabla_x^3 h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y + \|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

If $|\beta_1| = 1$ and $|\beta_2| = 2$ then

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^\beta h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})}\\
&\lesssim \int_{\mathbb{T}^3}\Big(\|\nabla_x f\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x^3 h\|_{H^s_v(\langle v\rangle^{\gamma/2-6s}m)}\\
&\quad + \|\nabla_x f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)}\,\|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^3 h\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)}\Big)\\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\ell)}\,\|\nabla_x^2 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)}\,\|\nabla_x^3 h\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s}m)}\\
&\quad + \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^{\gamma/2-6s}m)}\,\|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^\ell)}\,\|\nabla_x^3 h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)}\\
&\lesssim \|f\|_X\,\|g\|_{\bar Y}\,\|h\|_Y + \|f\|_Y\,\|g\|_X\,\|h\|_Y.
\end{aligned}$$

When $|\beta_1| = 2$ and $|\beta_2| = 1$ then we get

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^{\beta} h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x^3 h\|_{H^s_v(\langle v\rangle^{\gamma/2-6s} m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \|\nabla_x g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 h\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \Big) \\
&\lesssim \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\ell})} \|\nabla_x g\|_{H^{2,s}_{x,v}(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x^3 h\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s} m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s} m)} \|\nabla_x g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s} m)} \\
&\lesssim \|f\|_X \|g\|_{\bar{Y}} \|h\|_Y + \|f\|_Y \|g\|_X \|h\|_Y .
\end{aligned}$$

Finally, when $\beta_1 = \beta$, it follows

$$\begin{aligned}
&\langle Q(\partial_x^{\beta} f, g), \partial_x^{\beta} h\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x^3 f\|_{L^2_v(\langle v\rangle^{\ell})} \|g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x^3 h\|_{H^s_v(\langle v\rangle^{\gamma/2-6s} m)} \\
&\quad + \|\nabla_x^3 f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \|g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 h\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \Big) \\
&\lesssim \|\nabla_x^3 f\|_{L^2_{x,v}(\langle v\rangle^{\ell})} \|g\|_{H^{2,s}_{x,v}(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x^3 h\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s} m)} \\
&\quad + \|\nabla_x^3 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s} m)} \|g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 h\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s} m)} \\
&\lesssim \|f\|_X \|g\|_{\bar{Y}} \|h\|_Y + \|f\|_Y \|g\|_X \|h\|_Y .
\end{aligned}$$

Proof of (ii). As in the proof of (i), we write

$$\langle Q(f,g), g\rangle_{\mathcal{H}^3_x L^2_v(m)} = \langle Q(f,g), g\rangle_{L^2_{x,v}(m)} + \sum_{1\le|\beta|\le 3} \langle \partial_x^{\beta} Q(f,g), \partial_x^{\beta} g\rangle_{L^2_{x,v}(m\langle v\rangle^{-2|\beta|s})},$$

and

$$\partial_x^{\beta} Q(f,g) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\, Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g).$$

In the following steps, we will always consider $\ell \in (4-\gamma+3/2, k-6s]$. Notice that since $\gamma \le 1$ and $s \le 1/2$, the condition $k > 4-\gamma+3/2+6s$ implies $k > \gamma/2+3+8s$ so that we can apply results from Lemma 2.3.

Step 1. Using Lemma 2.3-(ii) and (2.10), we have

$$\begin{aligned}
&\langle Q(f,g), g\rangle_{L^2_{x,v}(m)} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|f\|_{L^2_v(\langle v\rangle^{\ell})} \|g\|^2_{H^{s,*}_v(m)} + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2} m)} \|g\|_{L^2_v(\langle v\rangle^{\ell})} \|g\|_{L^2_v(\langle v\rangle^{\gamma/2} m)} \Big) \\
&\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|g\|^2_{L^2_x H^{s,*}_v(m)} + \|f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2} m)} \|g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2} m)} \\
&\lesssim \|f\|_X \|g\|^2_{Y^*} + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

Step 2. Case $|\beta| = 1$. Arguing as in the previous step,

$$\begin{aligned}
\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_{x,v}(\langle v\rangle^{-2s}m)} &\lesssim \int_{\mathbb{T}^3} \Big(\|f\|_{L^2_v(\langle v\rangle^\ell)} \|\nabla_x g\|^2_{H^{s,*}_v(\langle v\rangle^{-2s}m)} \\
&\quad + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)} \|\nabla_x g\|_{L^2_v(\langle v\rangle^\ell)} \|\nabla_x g\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)} \Big) \\
&\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|\nabla_x g\|^2_{L^2_x H^{s,*}_v(\langle v\rangle^{-2s}m)} \\
&\quad + \|f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)} \|\nabla_x g\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|\nabla_x g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)} \\
&\lesssim \|f\|_X \|g\|^2_{Y^*} + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

Moreover, we also have using Lemma 2.3-(i),

$$\begin{aligned}
&\langle Q(\partial_x^\beta f, g), \partial_x^\beta g\rangle_{L^2_{x,v}(\langle v\rangle^{-2s}m)} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x f\|_{L^2_v(\langle v\rangle^\ell)} \|g\|_{H^s_v(\langle v\rangle^{\gamma/2}m)} \|\nabla_x g\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)} \\
&\quad + \|\nabla_x f\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)} \|g\|_{L^2_v(\langle v\rangle^\ell)} \|\nabla_x g\|_{L^2_v(\langle v\rangle^{\gamma/2-2s}m)} \Big) \\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2}m)} \|\nabla_x g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-2s}m)} \\
&\quad + \|\nabla_x f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)} \|g\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|\nabla_x g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-2s}m)} \\
&\lesssim \|f\|_X \|g\|^2_Y + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

Step 3. Case $|\beta| = 2$. When $\beta_2 = \beta$, we have

$$\begin{aligned}
\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_{x,v}(\langle v\rangle^{-4s}m)} &\lesssim \int_{\mathbb{T}^3} \Big(\|f\|_{L^2_v(\langle v\rangle^\ell)} \|\nabla_x^2 g\|^2_{H^{s,*}_v(\langle v\rangle^{-4s}m)} \\
&\quad + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)} \|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^\ell)} \|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)} \Big) \\
&\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|\nabla_x^2 g\|^2_{L^2_x H^{s,*}_v(\langle v\rangle^{-4s}m)} \\
&\quad + \|f\|_{H^2_x L^2_v(\langle v\rangle^{\gamma/2-4s}m)} \|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^\ell)} \|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)} \\
&\lesssim \|f\|_X \|g\|^2_{Y^*} + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

When $\beta_1 = \beta$, we have

$$\begin{aligned}
&\langle Q(\partial_x^\beta f, g), \partial_x^\beta g\rangle_{L^2_{x,v}(\langle v\rangle^{-4s}m)} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^\ell)} \|g\|_{H^s_v(\langle v\rangle^{\gamma/2-2s}m)} \|\nabla_x^2 g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)} \|g\|_{L^2_v(\langle v\rangle^\ell)} \|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^{\gamma/2-4s}m)} \Big) \\
&\lesssim \|\nabla_x^2 f\|_{L^6_x L^2_v(\langle v\rangle^\ell)} \|g\|_{L^3_x H^s_v(\langle v\rangle^{\gamma/2-2s}m)} \|\nabla_x^2 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)} \|g\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)} \\
&\lesssim \|\nabla_x^2 f\|_{H^1_x L^2_v(\langle v\rangle^\ell)} \|g\|_{H^{1,s}_{x,v}(\langle v\rangle^{\gamma/2-2s}m)} \|\nabla_x^2 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)} \|g\|_{H^2_x L^2_v(\langle v\rangle^\ell)} \|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s}m)} \\
&\lesssim \|f\|_X \|g\|^2_Y + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

Finally, when $|\beta_1| = |\beta_2| = 1$, we obtain

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^{\beta} g\rangle_{L^2_x L^2_v(\langle v\rangle^{-4s} m)} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x f\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x g\|_{H^s_v(\langle v\rangle^{\gamma/2-2s} m)} \|\nabla_x^2 g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s} m)} \\
&\quad + \|\nabla_x f\|_{L^2_v(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^{\gamma/2-4s} m)} \Big) \\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-2s} m)} \|\nabla_x^2 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s} m)} \\
&\quad + \|\nabla_x f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-4s} m)} \\
&\lesssim \|f\|_X \|g\|_Y^2 + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

Step 4. Case $|\beta| = 3$. When $\beta_2 = \beta$ we obtain

$$\begin{aligned}
\langle Q(f, \partial_x^{\beta} g), \partial_x^{\beta} g\rangle_{L^2_{x,v}(\langle v\rangle^{-6s} m)} &\lesssim \int_{\mathbb{T}^3} \Big(\|f\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|^2_{H^{s,*}_v(\langle v\rangle^{-6s} m)} \\
&\quad + \|f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \|\nabla_x^3 g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \Big) \\
&\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|^2_{L^2_x H^{s,*}_v(\langle v\rangle^{-6s} m)} \\
&\quad + \|f\|_{H^2_x L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s} m)} \\
&\lesssim \|f\|_X \|g\|^2_{Y^*} + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

If $|\beta_1| = 1$ and $|\beta_2| = 2$ then

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^{\beta} g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x f\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^2 g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x^3 g\|_{H^s_v(\langle v\rangle^{\gamma/2-6s} m)} \\
&\quad + \|\nabla_x f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \Big) \\
&\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^2 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-4s} m)} \|\nabla_x^3 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s} m)} \\
&\quad + \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^{\gamma/2-6s} m)} \|\nabla_x^2 g\|_{L^2_{x,v}(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s} m)} \\
&\lesssim \|f\|_X \|g\|_Y^2 + \|f\|_Y \|g\|_X \|g\|_Y .
\end{aligned}$$

When $|\beta_1| = 2$ and $|\beta_2| = 1$, we get

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^{\beta} g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)} \|\nabla_x^3 g\|_{H^s_v(\langle v\rangle^{\gamma/2-6s}m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)} \|\nabla_x g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)} \Big) \\
&\lesssim \|\nabla_x^2 f\|_{L^6_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x g\|_{L^3_x H^s_v(\langle v\rangle^{\gamma/2-4s}m)} \|\nabla_x^3 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s}m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)} \|\nabla_x g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)} \\
&\lesssim \|\nabla_x^2 f\|_{H^1_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x g\|_{H^{1,s}_{x,v}(\langle v\rangle^{\gamma/2-4s}m)} \|\nabla_x^3 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s}m)} \\
&\quad + \|\nabla_x^2 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)} \|\nabla_x g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)} \\
&\lesssim \|f\|_X \|g\|_Y^2 + \|f\|_Y \|g\|_X \|g\|_Y.
\end{aligned}$$

Finally, when $\beta_1 = \beta$, it follows

$$\begin{aligned}
&\langle Q(\partial_x^{\beta} f, g), \partial_x^{\beta} g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-6s})} \\
&\lesssim \int_{\mathbb{T}^3} \Big(\|\nabla_x^3 f\|_{L^2_v(\langle v\rangle^{\ell})} \|g\|_{H^s_v(\langle v\rangle^{\gamma/2-4s}m)} \|\nabla_x^3 g\|_{H^s_v(\langle v\rangle^{\gamma/2-6s}m)} \\
&\quad + \|\nabla_x^3 f\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)} \|g\|_{L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^2 g\|_{L^2_v(\langle v\rangle^{\gamma/2-6s}m)} \Big) \\
&\lesssim \|\nabla_x^3 f\|_{L^2_{x,v}(\langle v\rangle^{\ell})} \|g\|_{H^{2,s}_{x,v}(\langle v\rangle^{\gamma/2-4s}m))} \|\nabla_x^3 g\|_{L^2_x H^s_v(\langle v\rangle^{\gamma/2-6s}m)} \\
&\quad + \|\nabla_x^3 f\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)} \|g\|_{H^2_x L^2_v(\langle v\rangle^{\ell})} \|\nabla_x^3 g\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2-6s}m)} \\
&\lesssim \|f\|_X \|g\|_Y^2 + \|f\|_Y \|g\|_X \|g\|_Y.
\end{aligned}$$

We conclude noticing that $\|g\|_Y^2 \lesssim \|g\|_{Y^*}^2$ from Lemma 2.1.
Proof of (iii). The result is immediate from *(ii)* and the fact that $\|f\|_Y^2 \lesssim \|f\|_{Y^*}^2$. □

3. The linearized equation

We linearize the equation around the equilibrium μ. If we set $f = \mu + h$, h satisfies the equation

$$\begin{cases} \partial_t h = Q(\mu, h) + Q(h, \mu) - v \cdot \nabla_x h + Q(h, h) \\ h_{|t=0} = h_0 = f_0 - \mu. \end{cases}$$

We recall the notations

$$\mathcal{L}h = Q(h, \mu) + Q(\mu, h) \quad \text{and} \quad \Lambda h = \mathcal{L}h - v \cdot \nabla_x h. \tag{3.1}$$

The aim of the present section is to prove that the semigroup associated to Λ enjoys exponential decay properties in various Sobolev spaces.

3.1. **Functional spaces.** We recall that m is a polynomial weight $m(v) = \langle v\rangle^k$. We introduce the spaces $H^n_x\mathcal{H}^\ell_v(m)$ and $\mathcal{H}^n_x\mathcal{H}^\ell_v(m)$, $(n,\ell)\in\mathbb{N}^2$ which are respectively associated to the following norms:

$$\|h\|^2_{H^n_x\mathcal{H}^\ell_v(m)} := \sum_{|\alpha|\le\ell,\ |\beta|\le n,\ |\alpha|+|\beta|\le\max(\ell,n)} \|\partial^\alpha_v\partial^\beta_x h\|^2_{L^2_{x,v}(m\langle v\rangle^{-2|\alpha|s})}, \tag{3.2}$$

and

$$\|h\|^2_{\mathcal{H}^n_x\mathcal{H}^\ell_v(m)} := \sum_{|\alpha|\le\ell,\ |\beta|\le n,\ |\alpha|+|\beta|\le\max(\ell,n)} \|\partial^\alpha_v\partial^\beta_x h\|^2_{L^2_{x,v}(m\langle v\rangle^{-2|\alpha|s-2|\beta|s})}. \tag{3.3}$$

We want to establish exponential decay of the semigroup $\mathcal{S}_\Lambda(t)$ in various Lebesgue and Sobolev spaces that we will denote $\mathcal{E}$:

$$\begin{aligned}\mathcal{E} &:= \begin{cases} H^n_x\mathcal{H}^\ell_v(m),\ (n,\ell)\in\mathbb{N}^2,\ n\ge\ell \\ \mathcal{H}^n_x\mathcal{H}^\ell_v(m),\ (n,\ell)\in\mathbb{N}^2,\ n\ge\ell \end{cases} \\ &\text{with}\quad k > \frac{\gamma}{2} + 3 + 2(\max(1,n)+1)s.\end{aligned} \tag{3.4}$$

Notice that those definitions include the case $L^2_{x,v}(m)$ obtained taking $n=\ell=0$ in one or the other type of space.

3.2. **Main results on the linearized operator.** The main result of this section is a precise version of Theorem 1.2 and reads

Theorem 3.1. *Let us consider $\mathcal{E}$ be one of the admissible spaces defined in* (3.4) *and introduce $E = H^{\max(1,n)}_{x,v}(\mu^{-1/2})$ where $n\in\mathbb{N}$ is the order of x-derivatives in the definition of $\mathcal{E}$. Then, for any $\lambda<\lambda_0$, where we recall that $\lambda_0>0$ is the spectral gap of Λ on E (see* (1.23)*), there is a constructive constant $C\ge 1$ such that the operator Λ satisfies on $\mathcal{E}$:*

(i) $\Sigma(\Lambda)\subset\{z\in\mathbb{C}\mid \Re e\, z\le-\lambda\}\cup\{0\}$;
(ii) *the null-space $N(\Lambda)$ is given by* (1.21) *and the projection Π_0 onto $N(\Lambda)$ by* (1.22)*;*
(iii) *Λ is the generator of a strongly continuous semigroup $\mathcal{S}_\Lambda(t)$ on $\mathcal{E}$ that verifies*

$$\forall\, t\ge 0,\ \forall\, h\in\mathcal{E},\quad \|\mathcal{S}_\Lambda(t)h-\Pi_0 h\|_{\mathcal{E}} \le C\, e^{-\lambda t}\,\|h-\Pi_0 h\|_{\mathcal{E}}.$$

To prove this theorem, we exhibit a splitting of the linearized operator into two parts, one which is regular and the second one which is dissipative. We shall also study the regularization properties of the semigroup. The latter point is based on the paper [22] in which a precise study of the short time regularization properties of the linearized operator are studied. We can then use the abstract theorem of enlargement of the functional space of the semigroup decay from Gualdani et al. [19] using the result of Mouhot and Neumann [29] (Theorem 1.3) as a starting point.

3.3. **Splitting of the linearized operator.** We recall that $\chi\in\mathcal{D}(\mathbb{R})$ is a truncation function which satisfies $\mathbb{1}_{[-1,1]}\le\chi\le\mathbb{1}_{[-2,2]}$ and we denote $\chi_a(\cdot):=\chi(\cdot/a)$ for $a>0$. We then introduce

$$\mathcal{A}h := M\chi_R h \quad\text{and}\quad \mathcal{B}h := \Lambda h - \mathcal{A}h = -v\cdot\nabla_x h + \mathcal{L}h - \mathcal{A}h$$

for some positive constants M and R to be chosen later. In the next subsection, we are going to prove a coercivity-type inequality of the following form: for δ small enough,

$$\langle\mathcal{L}h,h\rangle_{L^2_v(m)} \le -c_{0,\delta}\|h\|^2_* + c_{1,\delta}\|h\|^2_{L^2_v}$$

where $\|\cdot\|_*$ is a stronger norm than the $L^2_v(m)$-norm and $c_{0,\delta}$, $c_{1,\delta}$ are positive constants depending on δ. Then, choosing suitable constants M and R, we will be able to deduce that our operator $\mathcal{B}$ is indeed dissipative in $L^2_{x,v}(m)$ and that it provides us a gain of regularity coming from the term $-c_{0,\delta}\|h\|_*^2$.

3.4. **Dissipativity properties.**

Lemma 3.2. *Let $k > \gamma/2 + 3 + 2s$. For $\delta > 0$ small enough, we have:*

$$\langle \mathcal{L}h, h\rangle_{L^2_v(m)} \leq -c_0\,\delta^{2-2s}\|h\|^2_{\dot{H}^{s,*}_v(m)} - c_0\,\delta^{-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + C_\delta\|h\|^2_{L^2_v}.$$

where c_0 is a universal positive constant and C_δ is a positive constant depending on δ.

Proof. In what follows, we denote $H := hm$. We start by spliting the scalar product $\langle Q(\mu,h),h\rangle_{L^2_v(m)}$ into two parts:

$$\begin{aligned}\langle Q(\mu,h),h\rangle_{L^2_v(m)} &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\left[\mu'_*\,h' - \mu_*\,h\right]\,h\,m^2\,d\sigma\,dv_*\,dv\\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\left[\mu'_*\,H' - \mu_*\,H\right]\,H\,d\sigma\,dv_*\,dv\\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,\mu'_*\,h'\,h\,m\,(m-m')\,d\sigma\,dv_*\,dv\\ &=: \langle Q(\mu,H),H\rangle_{L^2_v} + R.\end{aligned}$$

We recall that for $\delta > 0$, b_δ and b^c_δ are given by

$$b_\delta(\cos\theta) = \chi_\delta(\theta)\,b(\cos\theta) \quad \text{and} \quad b^c_\delta(\cos\theta) = (1-\chi_\delta(\theta))\,b(\cos\theta)$$

and we denote B_δ, B^c_δ (resp. Q_δ, Q^c_δ) the associated kernels (resp. operators). We then write that

$$\langle Q(\mu,h),h\rangle_{L^2_v(m)} = \langle Q_\delta(\mu,H),H\rangle_{L^2_v} + \langle Q^c_\delta(\mu,H),H\rangle_{L^2_v} + R \tag{3.5}$$

and we are going to estimate each part of this decomposition. First, concerning grazing collisions, using the pre-post change of variables, we have:

$$\begin{aligned}\langle Q_\delta(\mu,H),H\rangle_{L^2_v} &= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu_*\,H\,(H'-H)\,d\sigma\,dv_*\,dv\\ &= -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu_*\left(H'-H\right)^2\,d\sigma\,dv_*\,dv\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2} B_\delta(v-v_*,\sigma)\,\mu_*\left((H')^2-H^2\right)\,d\sigma\,dv_*\,dv =: -I_1 + I_2.\end{aligned}$$

Using the cancellation lemma [1, Lemma 1], we have that

$$I_2 = \frac{1}{2}\int_{\mathbb{R}^3}(S_\delta * H^2)\,\mu\,dv$$

with S_δ defined in (2.6) which satisfies $S_\delta(z) \lesssim \delta^{2-2s}|z|^\gamma$. We deduce that

$$I_2 \lesssim \delta^{2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}. \tag{3.6}$$

We now treat I_1. To do that, we first notice that for $\varepsilon \in (0,1/2)$, we have

$$|v-v_*|^\gamma \geq \varepsilon\langle v-v_*\rangle^\gamma - \varepsilon\,\mathbb{1}_{|v-v_*|^\gamma \leq \varepsilon/(1-\varepsilon)}.$$

Together with the fact that

$$\langle v - v_*\rangle^\gamma \gtrsim \langle v' - v_*\rangle^\gamma \gtrsim \langle v_*\rangle^{-\gamma}\langle v'\rangle^\gamma,$$

we deduce that

$$\begin{aligned}
I_1 &\geq \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\langle v - v_*\rangle^\gamma\,\mu_*(H' - H)^2\,d\sigma\,dv_*\,dv \\
&\quad - \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mathbb{1}_{|v-v_*|^\gamma\leq\varepsilon/(1-\varepsilon)}\,\mu_*(H' - H)^2\,d\sigma\,dv_*\,dv \\
&\geq C\varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mu_*\langle v_*\rangle^{-\gamma}(H'\langle v'\rangle^{\gamma/2} - H\langle v'\rangle^{\gamma/2})^2\,d\sigma\,dv_*\,dv \\
&\quad - \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mathbb{1}_{|v-v_*|^\gamma\leq\varepsilon/(1-\varepsilon)}\,\mu_*(H' - H)^2\,d\sigma\,dv_*\,dv \\
&\geq C\frac{\varepsilon}{2} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mu_*\langle v_*\rangle^{-\gamma}(H'\langle v'\rangle^{\gamma/2} - H\langle v\rangle^{\gamma/2})^2\,d\sigma\,dv_*\,dv \\
&\quad - C\varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mu_*\langle v_*\rangle^{-\gamma}\,H^2\,(\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2})^2\,d\sigma\,dv_*\,dv \\
&\quad - \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mathbb{1}_{|v-v_*|^\gamma\leq\varepsilon/(1-\varepsilon)}\,\mu_*(H' - H)^2\,d\sigma\,dv_*\,dv =: I_{11} - I_{12} - I_{13}.
\end{aligned}$$

First, we clearly have

$$I_{11} \gtrsim \varepsilon\|h\|^2_{\dot{H}^{s,*}_v(m)}.$$

For I_{12}, we can use (2.7) to get

$$I_{12} \lesssim \varepsilon\,\delta^{2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

Concerning I_{13}, we use that for $\varepsilon \leq 1/2$, $\mathbb{1}_{|v-v_*|^\gamma\leq\varepsilon/(1-\varepsilon)} \leq \mathbb{1}_{|v-v_*|\leq 1}$ so that

$$\begin{aligned}
I_{13} &\lesssim \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b_\delta(\cos\theta)\,\mathbb{1}_{|v-v_*|\leq 1}\,\mu_*(H' - H)^2\,d\sigma\,dv_*\,dv \\
&\lesssim \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\mathbb{1}_{|v-v_*|\leq 1}\,\mu_*(H' - H)^2\,d\sigma\,dv_*\,dv.
\end{aligned}$$

From the proof of [15, Theorem 1.2], we get

$$I_{13} \lesssim \varepsilon\,\|h\|^2_{H^s_v(m)}.$$

We thus have obtained

$$\frac{1}{2}I_1 \geq c_1\,\varepsilon\|h\|^2_{\dot{H}^{s,*}_v(m)} - c_2\,\varepsilon\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)}, \quad c_1,\,c_2 > 0.$$

On the other hand, as already mentioned in the proof of Lemma 2.1, adapting the proof of [21, Theorem 3.1], we can get that

$$\frac{1}{2}I_1 \geq c_3\,\delta^{2-2s}\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)} - c_4\,\delta^{2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}, \quad c_3,\,c_4 > 0.$$

Combining the two previous inequalities, we get that there exist positive constants c_i, $i = 1,\dots,4$ such that

$$\begin{aligned}
I_1 &\geq c_1\,\varepsilon\,\|h\|^2_{\dot{H}^{s,*}_v(m)} + (c_3\,\delta^{2-2s} - c_2\,\varepsilon)\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)} \\
&\quad - c_4\,\delta^{2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.
\end{aligned} \tag{3.7}$$

Gathering (3.6) and (3.7), up to changing the value of c_4, we have obtained:

$$\begin{aligned}(3.8)\qquad &\langle Q_\delta(\mu,H),H\rangle_{L^2_v}\\ &\quad\le -c_1\,\varepsilon\,\|h\|^2_{\dot H^{s,*}_v(m)}-(c_3\,\delta^{2-2s}-c_2\,\varepsilon)\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)}+c_4\,\delta^{2-2s}\,\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.\end{aligned}$$

We now deal with the cut-off part $\langle Q^c_\delta(\mu,H),H\rangle_{L^2_v}$. In this term, grazing collisions are removed, we can thus separate gain and loss terms:

$$\begin{aligned}\langle Q^c_\delta(\mu,H),H\rangle_{L^2_v}&\le\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}B^c_\delta(v-v_*,\sigma)\,\mu_*\,|H'|\,|H|\,d\sigma\,dv_*\,dv\\ &\quad-\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}B^c_\delta(v-v_*,\sigma)\,\mu_*\,d\sigma\,dv_*\,H^2\,dv.\end{aligned}$$

The loss term is multiplicative and can be rewritten as

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}B^c_\delta(v-v_*,\sigma)\,\mu_*\,d\sigma\,dv_*\,H^2\,dv=K_\delta\int_{\mathbb{R}^3}(\mu*|\cdot|^\gamma)\,H^2\,dv$$

with

$$(3.9)\qquad K_\delta:=\int_{\mathbb{S}^2}b^c_\delta(\cos\theta)\,d\sigma\approx\int_\delta^{\pi/2}b(\cos\theta)\,\sin\theta\,d\theta\approx\delta^{-2s}-\Big(\frac{\pi}{2}\Big)^{-2s}\xrightarrow[\delta\to0]{}+\infty$$

using the spherical coordinates to get the second equality and (1.3) to get the final one. Since we also have

$$(\mu*|\cdot|^\gamma)(v)\approx\langle v\rangle^\gamma,$$

we can deduce that there exists $\nu_0>0$ such that

$$(3.10)\qquad -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}B^c_\delta(v-v_*,\sigma)\,\mu_*\,d\sigma\,dv_*\,H^2\,dv\le-\nu_0\,\delta^{-2s}\,\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

Concerning the gain term, following ideas from [25], we are going to split it into two parts. To do that, we denote $w:=v+v_*$ and $\widehat w:=w/|w|$. We then have

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}B^c_\delta(v-v_*,\sigma)\,\mu_*\,|H'|\,|H|\,d\sigma\,dv_*\,dv\\ &\quad=\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\mathbb{1}_{|\widehat w\cdot\sigma|\ge1-\delta^3}\,B^c_\delta(v-v_*,\sigma)\,\mu_*\,|H'|\,|H|\,d\sigma\,dv_*\,dv\\ &\qquad+\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\mathbb{1}_{|\widehat w\cdot\sigma|\le1-\delta^3}\,B^c_\delta(v-v_*,\sigma)\,\mu_*\,|H'|\,|H|\,d\sigma\,dv_*\,dv=:J_1+J_2.\end{aligned}$$

We first deal with J_1: using Young inequality, we have

$$\begin{aligned}J_1&\lesssim\delta^{-1/2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\mathbb{1}_{|\widehat w\cdot\sigma|\ge1-\delta^3}\,B^c_\delta(v-v_*,\sigma)\,\mu_*\,H^2\,d\sigma\,dv_*\,dv\\ &\quad+\delta^{1/2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\mathbb{1}_{|\widehat w\cdot\sigma|\ge1-\delta^3}\,B^c_\delta(v-v_*,\sigma)\,\mu'_*\,H^2\,d\sigma\,dv_*\,dv=:J_{11}+J_{12}\end{aligned}$$

where we have used the pre-post collisional change of variables noticing that $\widehat{w'}=\widehat w$ (with obvious notations). Using that $b^c_\delta(\cos\theta)\lesssim\delta^{-2-2s}$ on the sphere and $(\mu*|\cdot|^\gamma)(v)\lesssim\langle v\rangle^\gamma$, we get

$$J_{11}\lesssim\delta^{-5/2-2s}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\mathbb{1}_{|\widehat w\cdot\sigma|\ge1-\delta^3}\,d\sigma\,H^2\,\langle v\rangle^\gamma\,dv.$$

Then, since for any $z\in\mathbb{S}^2$, we have $\int_{\mathbb{S}^2}\mathbb{1}_{|z\cdot\sigma|\ge1-\delta^3}\,d\sigma\lesssim\delta^3$, we obtain

$$(3.11)\qquad J_{11}\lesssim\delta^{1/2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}.$$

As far as J_{12} is concerned, we roughly bound it from above as:

$$J_{12} \lesssim \delta^{1/2} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B^c_\delta(v-v_*,\sigma)\,\mu'_*\,H^2\,d\sigma\,dv_*\,dv.$$

We then perform the regular change of variable $v_* \to v'_*$ as shown in the proof of Lemma 2.3 and notice that $|v-v_*|^\gamma \lesssim |v-v'_*|^\gamma$ to obtain:

$$J_{12} \lesssim \delta^{1/2} \int_{\mathbb{S}^2} b^c_\delta(\cos\theta)\,d\sigma \int_{\mathbb{R}^3\times\mathbb{R}^3} \mu_*|v-v_*|^\gamma\,H^2\,dv_*\,dv \lesssim \delta^{1/2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}. \tag{3.12}$$

The analysis of J_2 starts similarly as the one of J_1 using Young inequality:

$$\begin{aligned} J_2 \lesssim\ & \delta^{1/2} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \mathbb{1}_{|\widehat{w}\cdot\sigma|\le 1-\delta^3}\,B^c_\delta(v-v_*,\sigma)\,\mu_*\,H^2\,d\sigma\,dv_*\,dv \\ & + \delta^{-1/2} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \mathbb{1}_{|\widehat{w}\cdot\sigma|\le 1-\delta^3}\,B^c_\delta(v-v_*,\sigma)\,\mu'_*\,H^2\,d\sigma\,dv_*\,dv =: J_{21}+J_{22}. \end{aligned}$$

The treatment of J_{21} is simple and similar as the one of J_{12}, we get:

$$J_{21} \lesssim \delta^{1/2-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}. \tag{3.13}$$

For J_{22}, we are going to use the following computation: denoting $u := v - v_*$ the relative velocity, we have

$$|v'_*|^2 = \frac{1}{4}(|w|^2+|u|^2) - \frac{|w||u|}{2}\widehat{w}\cdot\sigma$$

so that if $|\widehat{w}\cdot\sigma| \le 1-\delta^3$, then

$$|v'_*|^2 \ge \frac{1}{4}(|w|^2+|u|^2) - (1-\delta^3)\frac{|w||u|}{2} \ge \frac{\delta^3}{4}(|w|^2+|u|^2) = \frac{\delta^3}{2}(|v|^2+|v_*|^2).$$

From this, we deduce that

$$\mu'_* \le e^{-\delta^3|v|^2/4}e^{-\delta^3|v_*|^2/4}.$$

Consequently,

$$J_{22} \lesssim \delta^{-5/2-2s} \int_{\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^\gamma\,e^{-\delta^3|v_*|^2/4}\,H^2\,e^{-\delta^3|v|^2/4}\,dv_*\,dv \lesssim C_\delta\|h\|^2_{L^2_v}. \tag{3.14}$$

Combining (3.10), (3.11), (3.12), (3.13) and (3.14), we obtain

$$\langle Q^c_\delta(\mu,H),H\rangle_{L^2_v} \le \delta^{-2s}\left(c_5\delta^{1/2}-\nu_0\right)\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + C_\delta\|h\|^2_{L^2_v}, \quad c_5>0. \tag{3.15}$$

Coming back to (3.5), it remains to analyse the rest term:

$$R = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,\mu'_*\,h'\,h\,m\,(m-m')\,d\sigma\,dv_*\,dv.$$

First, let us remark that

$$|m'-m| \le \left(\sup_{z\in\mathrm{B}(v,|v'-v|)} |\nabla m|\,(z)\right)|v'-v|,$$

with

$$|v'-v| \lesssim |v-v_*|\,\sin(\theta/2).$$

Then, we use the fact that

$$\sup_{z\in\mathrm{B}(v,|v'-v|)} |\nabla m|\,(z) \lesssim \langle v\rangle^{k-1} + \langle v'\rangle^{k-1} \lesssim \langle v'\rangle^{k-1}\,\langle v'_*\rangle^{k-1},$$

which implies that

$$|m' - m| \lesssim \sin(\theta/2)\,|v - v_*|\,\langle v'\rangle^{k-1}\,\langle v'_*\rangle^{k-1}.$$

Consequently, we have:

$$\begin{aligned} R &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,\mu'_*\,\langle v'_*\rangle^{k-1}\,|v - v_*|^{\gamma+1}\,|h'|\,\langle v'\rangle^{k-1}\,|h|\,m\,d\sigma\,dv_*\,dv \\ &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,\mu'_*\,\langle v'_*\rangle^{k-1}\,|v - v_*|^{\gamma+2}\,(h')^2\langle v'\rangle^{2k-2}\,d\sigma\,dv_*\,dv \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\,\sin(\theta/2)\,\mu'_*\,\langle v'_*\rangle^{k-1}\,|v - v_*|^{\gamma}\,h^2\,m^2\,d\sigma\,dv_*\,dv. \end{aligned}$$

For the first part, we use the pre-post collisional change of variables and for the second one, we use the regular change of variable $v_* \to v'_*$ explained in the proof of Lemma 2.3. It gives us

$$R \le c_6\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}, \quad c_6 > 0. \tag{3.16}$$

Gathering (3.8), (3.15) and (3.16) yields

$$\begin{aligned} \langle Q(\mu,h),h\rangle_{L^2_v(m)} &\le -(c_3\,\delta^{2-2s} - c_2\,\varepsilon)\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)} - c_1\,\varepsilon\|h\|^2_{\dot H^{s,*}_v(m)} \\ &\quad + \Big(c_6 + \delta^{-2s}\,\Big(c_4\,\delta^2 + c_5\,\delta^{1/2} - \nu_0\Big)\Big)\,\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + C_\delta\|h\|^2_{L^2_v}. \end{aligned}$$

We also have from Lemma 2.3-(i) applied with $\varsigma_1 = 2s$, $\varsigma_2 = 0$, $N_1 = \gamma + 2s$ and $N_2 = 0$:

$$\langle Q(h,\mu),h\rangle_{L^2_v(m)} \le c_7\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)}, \quad c_7 > 0.$$

The two previous inequalities imply

$$\begin{aligned} \langle \mathcal{L}h,h\rangle_{L^2(m)} &\le -(c_3\,\delta^{2-2s} - c_2\,\varepsilon)\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)} - c_1\,\varepsilon\|h\|^2_{\dot H^{s,*}_v(m)} \\ &\quad + \Big(c_6 + c_7 + \delta^{-2s}\,\Big(c_4\,\delta^2 + c_5\,\delta^{1/2} - \nu_0\Big)\Big)\,\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + C_\delta\|h\|^2_{L^2_v}. \end{aligned}$$

Taking δ small enough and then ε small enough of the order of δ^{2-2s}, we obtain the wanted estimate:

$$\begin{aligned} &\langle \mathcal{L}h,h\rangle_{L^2_v(m)} \\ &\quad \le -c_0\,\delta^{2-2s}\|h\|^2_{\dot H^{s,*}_v(m)} - c_0\,\delta^{2-2s}\|h\|^2_{H^s_v(\langle v\rangle^{\gamma/2}m)} - c_0\,\delta^{-2s}\|h\|^2_{L^2_v(\langle v\rangle^{\gamma/2}m)} + C_\delta\|h\|^2_{L^2_v} \end{aligned}$$

for some $c_0 > 0$. □

We can now prove the dissipativity properties of $\mathcal{B} = -v\cdot\nabla_x + \mathcal{L} - M\chi_R$ in $L^2_{x,v}(m)$.

Lemma 3.3. *Let us consider $k > \gamma/2 + 3 + 2s$ and $a < 0$. There exist M and $R > 0$ such that $\mathcal{B} - a$ is dissipative in $L^2_{x,v}(m)$, namely*

$$\forall\, t \ge 0, \quad \|\mathcal{S}_{\mathcal{B}}(t)h\|_{L^2_{x,v}(m)} \le e^{at}\|h\|_{L^2_{x,v}(m)}.$$

We even have the following estimate (which is better that simple dissipativity as stated above), for any $h \in L^2_{x,v}(m)$:

$$\forall\, t \ge 0, \quad \frac{1}{2}\frac{d}{dt}\|\mathcal{S}_{\mathcal{B}}(t)h\|^2_{L^2_{x,v}(m)} \le -c_1\,\|\mathcal{S}_{\mathcal{B}}(t)h\|^2_{L^2_x H^{s,*}_v(m)} + a\,\|\mathcal{S}_{\mathcal{B}}(t)h\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m)}$$

for some constant $c_1 > 0$.

Proof. Consider $a < 0$ and $\delta > 0$ small enough so that the conclusion of Lemma 3.2 holds and such that $c_0\,\delta^{-2s} > -a$. We are going to estimate the integral $\int_{\mathbb{R}^3\times\mathbb{R}^3}(\mathcal{B}h)\,h\,m^2\,dv\,dx$. We first notice that the term coming from the transport operator gives no contribution:

$$\int_{\mathbb{T}^3\times\mathbb{R}^3}(v\cdot\nabla_x h)\,h\,m^2\,dv\,dx = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3}(v\cdot\nabla_x h^2)\,m^2\,dv\,dx = 0.$$

Then, using Lemma 3.2 and integrating in x, we obtain

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3}&(\mathcal{L}h)\,h\,m^2\,dv\,dx\\ &\leq -c_0\,\delta^{2-2s}\|h\|^2_{L^2_x H^{s,*}_v(m)} - c_0\,\delta^{-2s}\|h\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m)} + C_\delta\|h\|^2_{L^2_{x,v}}.\end{aligned}$$

In summary, we have obtained

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3}(\mathcal{B}h)\,h\,m^2\,dv\,dx &\leq -c_0\,\delta^{2-2s}\|h\|^2_{L^2_x H^{s,*}_v(m)}\\ &\quad+\int_{\mathbb{T}^3\times\mathbb{R}^3} h^2\,m^2\langle v\rangle^\gamma\left(-c_0\,\delta^{-2s}+C_\delta\langle v\rangle^{-\gamma}-M\chi_R(v)\right)dv\,dx.\end{aligned}$$

Since $-c_0\,\delta^{-2s}+C_\delta\langle v\rangle^{-\gamma}$ goes to $-c_0\,\delta^{-2s} < a$ as $|v|$ goes to infinity, we can choose M and R large enough so that for any $v\in\mathbb{R}^3$, $-c_0\,\delta^{-2s}+C_\delta\langle v\rangle^{-\gamma}-M\chi_R \leq a$, which concludes the proof. □

The goal of the next lemma is to generalize previous dissipativity results to higher order derivatives spaces of type $H^n_x\mathcal{H}^\ell_v(m)$ and $\mathcal{H}^n_x\mathcal{H}^\ell_v(m)$ defined through their norms in (3.2) and (3.3). Notice that, in order to get our dissipativity result, it is necessary to have less weight on v-derivatives (which is induced by the weight $\langle v\rangle^{-2|\alpha|s}$ in the definitions of the norms of $H^n_x\mathcal{H}^\ell_v(m)$ and $\mathcal{H}^n_x\mathcal{H}^\ell_v(m)$). However, the introduction of the weight $\langle v\rangle^{-2|\beta|s}$ in order to have less weight on the x-derivatives in the space $\mathcal{H}^n_x\mathcal{H}^\ell_v(m)$ is not needed at this point but dissipativity results still hold true doing that and we will make use of it in the nonlinear study in Section 4.

Lemma 3.4. *Let us consider $(n,\ell)\in\mathbb{N}^2$ with $n\geq\ell$. In what follows, $\mathcal{E} = H^n_x\mathcal{H}^\ell_v(m)$ with $k > \gamma/2+3+2(n+1)s$ or $\mathcal{E} = \mathcal{H}^n_x\mathcal{H}^\ell_v(m)$ with $k > \gamma/2+3+2(n+1)s$. Then for any $a<0$, there exist M, $R>0$ such that $\mathcal{B}-a$ is hypodissipative in $\mathcal{E}$ in the sense that*

$$\forall\, t\geq 0,\quad \|\mathcal{S}_{\mathcal{B}}(t)h\|_{\mathcal{E}} \lesssim e^{at}\|h\|_{\mathcal{E}}.$$

Proof. The case $n=\ell=0$ is nothing but Lemma 3.3. Let us notice that the operator ∇_x commutes with the operator $\mathcal{B}$, the treatment of x-derivatives is thus simple and one can always reduce to the case $n=\ell$. Moreover, we only handle the case $\mathcal{E} = H^n_x\mathcal{H}^\ell_v(m)$, the other case being similar. We now deal with the case $n=\ell=1$, the higher-order derivatives being treatable in the same way. To do that, we introduce the following norm on $H^1_x\mathcal{H}^1_v(m)$:

$$|||h|||^2_{H^1_x\mathcal{H}^1_v(m)} := \|h\|^2_{L^2_{x,v}(m)} + \|\nabla_x h\|^2_{L^2_{x,v}(m)} + \zeta\,\|\nabla_v h\|^2_{L^2_{x,v}(m_0)}$$

where $\zeta>0$ is a positive constant to be chosen later and $m_0(v) := \langle v\rangle^{-2s}\,m(v) = \langle v\rangle^{k_0}$ with $k_0 := -2s+k$. This norm is equivalent to the classical norm on $H^1_x\mathcal{H}^1_v(m)$ defined through (3.2).
In the subsequent proof, η is a positive constant that will be fixed later on. Let us introduce $h_t := \mathcal{S}_{\mathcal{B}}(t)h$ with $h\in H^1_x\mathcal{H}^1_v(m)$.

Coming back to the proof of Lemma 3.3, we have that

$$\begin{aligned}\forall\, t\geq 0,\quad \frac{1}{2}\frac{d}{dt}\|h_t\|^2_{L^2_{x,v}(m)} &\leq -c_0\,\delta^{2-2s}\,\|h_t\|^2_{L^2_xH^{s,*}_v(m)}\\ &\quad+\int_{\mathbb{T}^3\times\mathbb{R}^3}\big(-c_0\,\delta^{-2s}+C_\delta\langle v\rangle^{-\gamma}-M\chi_R(v)\big)\,h_t^2\,m^2\langle v\rangle^{\gamma}\,dv\,dx.\end{aligned}\tag{3.17}$$

Moreover, since the x-derivatives commute with $\mathcal{B}$,

$$\begin{aligned}\forall\, t\geq 0,\quad \frac{1}{2}\frac{d}{dt}\|\nabla_x h_t\|^2_{L^2_{x,v}(m)} &\leq -c_0\,\delta^{2-2s}\,\|\nabla_x h_t\|^2_{L^2_xH^{s,*}_v(m)}\\ &\quad+\int_{\mathbb{T}^3\times\mathbb{R}^3}\big(-c_0\,\delta^{-2s}+C_\delta\langle v\rangle^{-\gamma}-M\chi_R(v)\big)\,|\nabla_x h_t|^2\,m^2\langle v\rangle^{\gamma}\,dv\,dx.\end{aligned}\tag{3.18}$$

Therefore, it remains to consider the v-derivatives. In what follows ∂_x and ∂_v stand for ∂_{x_1},∂_{x_2} or ∂_{x_3} and ∂_{v_1},∂_{v_2} or ∂_{v_3}, respectively.
We have

$$\partial_t(\partial_v h_t)=\mathcal{B}(\partial_v h_t)-\partial_x h_t-M\,(\partial_v\chi_R)\,h_t+Q(h_t,\partial_v\mu)+Q(\partial_v\mu,h_t),$$

thus, we can split $\frac{1}{2}\frac{d}{dt}\|\partial_v h_t\|^2_{L^2_{x,v}(m_0)}$ into five terms, according to the previous computation,

$$\frac{1}{2}\frac{d}{dt}\|\partial_v h_t\|^2_{L^2_{x,v}(m_0)}:=I_1+\cdots+I_5.$$

For the first term we can use again Lemma 3.3, obtaining

$$\begin{aligned}\forall\, t\geq 0,\quad I_1 &\leq -c_0\,\delta^{2-2s}\,\|\partial_v h_t\|^2_{L^2_xH^{s,*}_v(m_0)}\\ &\quad+\int_{\mathbb{T}^3\times\mathbb{R}^3}\big(-c_0\,\delta^{-2s}+C_\delta\langle v\rangle^{-\gamma}-M\chi_R(v)\big)\,|\partial_v h_t|^2\,m_0^2\,\langle v\rangle^{\gamma}\,dv\,dx.\end{aligned}\tag{3.19}$$

For the second term, we have

$$I_2=-\int_{\mathbb{T}^3\times\mathbb{R}^3}(\partial_x h_t)\,(\partial_v h_t)\,m_0^2\,dv\,dx\leq\frac{1}{2}\|\partial_v h_t\|^2_{L^2_{x,v}(m_0)}+\frac{1}{2}\|\partial_x h_t\|^2_{L^2_{x,v}(m_0)}.\tag{3.20}$$

The term I_3 is simply handled as follows:

$$\begin{aligned}I_3&\lesssim\frac{M}{R}\int_{\mathbb{T}^3\times\mathbb{R}^3}\mathbb{1}_{R\leq|v|\leq 2R}\,h_t\,(\partial_v h_t)\,m_0^2\,dx\,dv\\ &\lesssim\frac{M}{R}\int_{\mathbb{T}^3\times\mathbb{R}^3}\mathbb{1}_{R\leq|v|\leq 2R}\,h_t^2\,m_0^2\,dx\,dv+\frac{M}{R}\int_{\mathbb{T}^3\times\mathbb{R}^3}\mathbb{1}_{R\leq|v|\leq 2R}\,(\partial_v h_t)^2\,m_0^2\,dx\,dv.\end{aligned}\tag{3.21}$$

Let us now consider I_4. Using Lemma 2.3-(i), we have

$$\begin{aligned}I_4&=\int_{\mathbb{T}^3}\langle Q(h_t,\partial_v\mu),\partial_v h_t\rangle_{L^2_v(m_0)}\,dx\lesssim\|h_t\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m_0)}\|\partial_v h_t\|_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m_0)}\\ &\lesssim\frac{1}{\eta}\|h_t\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m_0)}+\eta\|\partial_v h_t\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2}m_0)}.\end{aligned}\tag{3.22}$$

Concerning I_5, still using Lemma 2.3-(i), we have:

$$\begin{aligned}I_5&=\int_{\mathbb{T}^3}\langle Q(\partial_v\mu,h_t),\partial_v h_t\rangle_{L^2_v(m_0)}\,dx\lesssim\|h_t\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2+2s}m_0)}\|\partial_v h_t\|_{L^2_xH^s_v(\langle v\rangle^{\gamma/2}m_0)}\\ &\lesssim\frac{1}{\eta}\|h_t\|^2_{L^2_xH^s_v(\langle v\rangle^{\gamma/2}m)}+\eta\|\partial_v h_t\|^2_{L^2_xH^s_v(\langle v\rangle^{\gamma/2}m_0)}.\end{aligned}\tag{3.23}$$

Before concluding, let us remark that from Lemma 2.1,

$$\|h\|^2_{L^2_x H^{s,*}_v(m)} \gtrsim \delta^{2-2s} \|h\|^2_{L^2_x H^s_v(\langle v\rangle^{\gamma/2} m)}.$$

Combining this fact with estimates (3.17), (3.18) and (3.19) to (3.23), we get:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|||h_t|||^2_{H^1_x\mathcal{H}^1_v(m)} &= \frac{1}{2}\frac{d}{dt}\|h_t\|^2_{L^2_{x,v}(m)} + \frac{1}{2}\frac{d}{dt}\|\nabla_x h_t\|^2_{L^2_{x,v}(m)} + \zeta\,\frac{1}{2}\frac{d}{dt}\|\nabla_v h_t\|^2_{L^2_{x,v}(m_0)} \\
&\leq -\frac{c_0}{2}\,\delta^{2-2s}\left(\|h_t\|^2_{L^2_x H^{s,*}_v(m)} + \|\nabla_x h_t\|^2_{L^2_x H^{s,*}_v(m)} + \zeta\|\nabla_v h_t\|^2_{L^2_x H^{s,*}_v(m_0)}\right) \\
&\quad + \left(-\frac{c_0}{2}\,\delta^{4-4s} + \frac{C\zeta}{\eta}\right)\|h_t\|^2_{L^2_x H^s_v(\langle v\rangle^{\gamma/2} m)} \\
&\quad + \zeta\left(-\frac{c_0}{2}\,\delta^{4-4s} + C\eta\right)\|\nabla_v h_t\|^2_{L^2_x H^s_v(\langle v\rangle^{\gamma/2} m_0)} \\
&\quad + \int_{\mathbb{T}^3\times\mathbb{R}^3}\Bigg(-c_0\,\delta^{-2s} + C_\delta\langle v\rangle^{-\gamma} \\
&\qquad\qquad + \frac{C\zeta M}{R}\mathbb{1}_{R\leq|v|\leq 2R}\langle v\rangle^{-\gamma-4s} - M\chi_R(v)\Bigg) h_t^2\, m^2\langle v\rangle^\gamma\,dv\,dx \\
&\quad + \int_{\mathbb{T}^3\times\mathbb{R}^3}\Big(-c_0\,\delta^{-2s} + C_\delta\langle v\rangle^{-\gamma} + C\zeta\langle v\rangle^{-\gamma-4s} - M\chi_R(v)\Big)|\nabla_x h_t|^2\, m^2\langle v\rangle^\gamma\,dv\,dx \\
&\quad + \zeta\int_{\mathbb{T}^3\times\mathbb{R}^3}\Bigg(-c_0\,\delta^{-2s} + C_\delta\langle v\rangle^{-\gamma} + C\langle v\rangle^{-\gamma} \\
&\qquad\qquad + \frac{CM}{R}\mathbb{1}_{R\leq|v|\leq 2R}\langle v\rangle^{-\gamma} - M\chi_R(v)\Bigg)|\nabla_v h_t|^2\, m_0^2\,\langle v\rangle^\gamma\,dv\,dx
\end{aligned}$$

for a constant $C > 0$. Consider now $a < 0$ and δ small enough such that $c_0\,\delta^{-2s} > -a$. We can then choose, in this order, η and ζ small enough and then M and R large enough such that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|||h_t|||^2_{H^1_x\mathcal{H}^1_v(m)} &\leq a\,\|h_t\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2} m)} + a\,\|\nabla_x h_t\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2} m)} + \zeta a\,\|\partial_v h_t\|^2_{L^2_{x,v}(\langle v\rangle^{\gamma/2} m_0)} \\
&\quad - c_1\left(\|h_t\|^2_{L^2_x H^{s,*}_v(m)} + \|\nabla_x h_t\|^2_{L^2_x H^{s,*}_v(m)} + \|\nabla_v h_t\|^2_{L^2_x H^{s,*}_v(m_0)}\right)
\end{aligned}$$

for some $c_1 > 0$, which concludes the proof. □

3.5. **Regularization properties of $\mathcal{A}S_{\mathcal{B}}$.** In this part, we focus on the regularization properties of the semigroup $\mathcal{S}_{\mathcal{B}}$ which are crucial in order to get a result on the linearized equation. To do that, we first introduce some notations and tools.

We define the convolution of two semigroups $\mathcal{S}_1 * \mathcal{S}_2$ by

$$(\mathcal{S}_1 * \mathcal{S}_2)(t) := \int_0^t \mathcal{S}_1(\tau)\,\mathcal{S}_2(t-\tau)\,d\tau,$$

and, for $p \in \mathbb{N}^*$, we define $\mathcal{S}^{(*p)}$ by $\mathcal{S}^{(*p)} = \mathcal{S} * \mathcal{S}^{(*(p-1))}$ with $\mathcal{S}^{(*1)} = \mathcal{S}$. For $\varsigma \in \mathbb{R}^+$ and ν a polynomial weight, we also introduce intermediate spaces

$$X_\varsigma(\nu) := \left[H^{\lfloor\varsigma\rfloor}_x\mathcal{H}^{\lfloor\varsigma\rfloor}_v(\nu), H^{\lfloor\varsigma\rfloor+1}_x\mathcal{H}^{\lfloor\varsigma\rfloor+1}_v(\nu)\right]_{\varsigma-\lfloor\varsigma\rfloor,2}.$$

The notation used below is the classical one of real interpolation (see [10]). For sake of completeness, we briefly recall the meaning of this notation. For C and D two Banach

spaces which are both embedded in the same topological separating vector space, for any $z \in A + B$, we define the K-function by

$$K(t,z) := \inf_{z=c+d} \left(\|c\|_C + t\|d\|_D \right), \quad \forall\, t > 0.$$

We then give the definition of the space $[C,D]_{\theta,p}$ for $\theta \in (0,1)$ and $p \in [1,+\infty]$:

$$[C,D]_{\theta,p} := \left\{ z \in C + D,\ t \mapsto K(t,z)/t^\theta \in L^p\left(dt/t^{1/p}\right) \right\}.$$

Notice that by standard results of interpolation, if $\mathcal{B} - a$ is hypodissipative in both spaces $H_x^{\lfloor \varsigma \rfloor}\mathcal{H}_v^{\lfloor \varsigma \rfloor}(\nu)$ and $H_x^{\lfloor \varsigma \rfloor+1}\mathcal{H}_v^{\lfloor \varsigma \rfloor+1}(\nu)$, it is also in $X_\varsigma(\nu)$. Notice also that we have the following continuous embeddings:

$$X_\varsigma(\nu\langle v\rangle^{2(\lfloor \varsigma \rfloor+1)s}) \hookrightarrow H^\varsigma_{x,v}(\nu) \hookrightarrow X_\varsigma(\nu). \tag{3.24}$$

Let us now state a lemma on the regularization properties of the semigroup $S_{\mathcal{B}}(t)$.

Lemma 3.5. *Let $r \in \mathbb{N}^*$, $k' > (1-\gamma)/2$ and $k > k' + \gamma + 5/2 + 2(\lfloor (r-1)s \rfloor + 2)s$. We consider $a < 0$ and the operator $\mathcal{B}$ is defined such that the conclusion of Lemma 3.4 is satisfied in $H_x^{\lfloor (r-1)s \rfloor+1}\mathcal{H}_v^{\lfloor (r-1)s \rfloor+1}(\langle v\rangle^k)$. Then, we have:*

$$\|S_{\mathcal{B}}(t)h\|_{X_{rs}(\langle v\rangle^{k'})} \lesssim \frac{e^{at}}{1 \wedge t^{1/2+s}} \|h\|_{X_{(r-1)s}(\langle v\rangle^k)}, \quad \forall\, t \geq 0.$$

Proof. Step 1. In the first step, we focus on the short time regularization properties of $S_{\mathcal{B}}(t)$: we are going to prove that

$$\|S_{\mathcal{B}}(t)h\|_{X_{rs}(\langle v\rangle^{k'})} \lesssim \frac{1}{t^{1/2+s}} \|h\|_{X_{(r-1)s}(\langle v\rangle^k)}, \quad \forall\, t \in (0,1].$$

This estimate yields the conclusion of the lemma for short times $t \in (0,1]$. To do that, we start by stating a few estimates coming from [22]. We first split Λ as in Subsection 5.1 from [22]: to do that, we introduce $\tilde{B}_\delta$ and $\tilde{B}^c_\delta$ defined through

$$\tilde{B}_\delta(v - v_*, \sigma) := \chi_\delta(|v' - v|)\, b(\cos\theta)\, |v - v_*|^\gamma$$

and

$$\tilde{B}^c_\delta(v - v_*, \sigma) := (1 - \chi_\delta(|v' - v|))\, b(\cos\theta)\, |v - v_*|^\gamma.$$

Then, we split Λ as

$$\begin{aligned}
\Lambda h = &\left(-K\langle v\rangle^{\gamma+2s} - v \cdot \nabla_x h + \int_{\mathbb{R}^3 \times \mathbb{S}^2} \tilde{B}_\delta(v - v_*, \sigma) \mu'_* (h' - h)\, d\sigma\, dv_* \right) \\
&+ \left(K\langle v\rangle^{\gamma+2s} + \int_{\mathbb{R}^3 \times \mathbb{S}^2} \tilde{B}_\delta(v - v_*, \sigma)(\mu'_* - \mu_*) h\, d\sigma\, dv_* \right. \\
&\qquad \left. + \int_{\mathbb{R}^3 \times \mathbb{S}^2} \tilde{B}^c_\delta(v - v_*, \sigma)(\mu'_* h' - \mu_* h)\, d\sigma\, dv_* + Q(h,\mu) \right) \\
=: &\ \widetilde{\Lambda}_1 h + \widetilde{\Lambda}_2 h.
\end{aligned}$$

We have from [22] Theorem 5.1 that for $q \geq 0$,

$$\|S_{\widetilde{\Lambda}_1}(t)h\|_{H^{rs}_{x,v}(\langle v\rangle^q)} \lesssim \frac{1}{t^{1/2+s}} \|h\|_{H^{(r-1)s}_{x,v}(\langle v\rangle^q)}, \quad \forall\, t \in (0,1] \tag{3.25}$$

and for any $\varsigma \in \mathbb{R}^+$

$$\|\widetilde{\Lambda}_2 h\|_{H^\varsigma_{x,v}(\langle v\rangle^{q'})} \lesssim \|h\|_{H^\varsigma_{x,v}(\langle v\rangle^q)}, \quad q > q' + \gamma + 5/2. \tag{3.26}$$

We now show how to propagate the regularization properties of $S_{\widetilde{\Lambda}_1}(t)$ to $S_{\mathcal{B}}(t)$, using the Duhamel formula. We write:

$$\mathcal{B} = \widetilde{\Lambda}_1 + (\widetilde{\Lambda}_2 - \mathcal{A})$$

so that we have:

$$S_{\mathcal{B}}(t) = S_{\widetilde{\Lambda}_1}(t) + \left(S_{\widetilde{\Lambda}_1} * (\widetilde{\Lambda}_2 - \mathcal{A})S_{\mathcal{B}}\right)(t).$$

For the first term, using (3.24) and (3.25), we have:

$$\begin{aligned}\|S_{\widetilde{\Lambda}_1}(t)h\|_{X_{rs}(\langle v\rangle^{k'})} &\lesssim \|S_{\widetilde{\Lambda}_1}(t)h\|_{H^{rs}_{x,v}(\langle v\rangle^{k'})} \lesssim \frac{1}{t^{1/2+s}}\|h\|_{H^{(r-1)s}_{x,v}(\langle v\rangle^{k'})}\\ &\lesssim \frac{1}{t^{1/2+s}}\|h\|_{X_{(r-1)s}(\langle v\rangle^{k'+2(\lfloor (r-1)s\rfloor+1)s})} \lesssim \frac{1}{t^{1/2+s}}\|h\|_{X_{(r-1)s}(\langle v\rangle^{k})}.\end{aligned}$$

For the second one, we introduce k'' such that

$$k \geq k'' + 2(\lfloor (r-1)s\rfloor + 1)s > k' + \gamma + 5/2 + 2(\lfloor (r-1)s\rfloor + 1)s$$

and we use (3.24), (3.25) and (3.26):

$$\begin{aligned}\left\|\left(S_{\widetilde{\Lambda}_1} * (\widetilde{\Lambda}_2 - \mathcal{A})S_{\mathcal{B}}\right)(t)h\right\|_{X_{rs}(\langle v\rangle^{k'})} &\lesssim \int_0^t \left\|S_{\widetilde{\Lambda}_1}(t-\tau)(\widetilde{\Lambda}_2 - \mathcal{A})S_{\mathcal{B}}(\tau)h\right\|_{H^{rs}_{x,v}(\langle v\rangle^{k'})}\,d\tau\\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2+s}}\left\|(\widetilde{\Lambda}_2 - \mathcal{A})S_{\mathcal{B}}(\tau)h\right\|_{H^{(r-1)s}_{x,v}(\langle v\rangle^{k'})}\,d\tau\\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2+s}}\|S_{\mathcal{B}}(\tau)h\|_{H^{(r-1)s}_{x,v}(\langle v\rangle^{k''})}\,d\tau \lesssim \|h\|_{X_{(r-1)s}(\langle v\rangle^{k})}.\end{aligned}$$

Step 2. In this step, we use Lemma 3.4 and interpolation combined with the previous estimates for short times to prove the final estimate which holds for all times. If $t \geq 1$, we have

$$\begin{aligned}&\|S_{\mathcal{B}}(t)h\|_{X_{rs}(\langle v\rangle^{k'})}\\ &\quad = \|S_{\mathcal{B}}(1)S_{\mathcal{B}}(t-1)h\|_{X_{rs}(\langle v\rangle^{k'})} \lesssim \|S_{\mathcal{B}}(t-1)h\|_{X_{(r-1)s}(\langle v\rangle^{k})} \lesssim e^{at}\|h\|_{X_{(r-1)s}(\langle v\rangle^{k})},\end{aligned}$$

which concludes the proof. □

To apply Theorem 2.13 from [19], we study the regularization properties of $(\mathcal{A}S_{\mathcal{B}})^{(*p)}$ for $p \in \mathbb{N}$ in the following corollary. We recall that the "large" space $\mathcal{E}$ is given by (3.4) and the associated "small" one by $E = H^{\max(1,n)}_{x,v}(\mu^{-1/2})$. Let $a < -\lambda_0$ where $\lambda_0 > 0$ is the spectral gap of Λ on E (see (1.23)). We then consider $\mathcal{B}$ such that the conclusion of Lemma 3.4 is satisfied in $H^{\max(1,n)}_x\mathcal{H}^{\max(1,n)}_v(m)$ (resp. $\mathcal{H}^{\max(1,n)}_x\mathcal{H}^{\max(1,n)}_v(m)$) if $\mathcal{E} = H^n_x\mathcal{H}^\ell_v(m)$ (resp. $\mathcal{E} = \mathcal{H}^n_x\mathcal{H}^\ell_v(m)$). Let us mention that it in particular implies that the conclusion of Lemma 3.4 is also satisfied in $\mathcal{E}$ and the one of Lemma 3.3 is also true in $L^2_{x,v}(m)$.

Corollary 3.6. *There exists $p \in \mathbb{N}$ such that*

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*p)}(t)h\|_E \lesssim e^{at}\|h\|_{\mathcal{E}}, \quad \forall\, t \geq 0.$$

Proof. Let us treat the case $\mathcal{E} = L^2_{x,v}(m)$ and $E = H^1_{x,v}(\mu^{-1/2})$ which is indicative of all the difficulties since we need to regularize both in space and velocity variables. We consider $r_0 \in \mathbb{N}^*$ the smallest positive integer such that $\lfloor r_0 s\rfloor = 1$. Using then the fact that $\mathcal{A}$ is a truncation operator, Lemma 3.4 and Lemma 3.5, we get that for any $1 \leq r \leq r_0$,

$$\|(\mathcal{A}S_{\mathcal{B}})(t)\|_{\mathscr{B}(X_{(r-1)s}(m),X_{rs}(m))} \lesssim \frac{e^{at}}{t^{1/2+s}\wedge 1}.$$

To conclude, we use Lemmas 3.3, 3.4 combined with the last estimate. Indeed, all those results allow us to use the criterion given in [19, Lemma 2.17] and gives us the conclusion. □

3.6. **Proof of Theorem 3.1.** Thanks to the estimates proven in the previous subsections, we now turn to the proof of Theorem 3.1. Let $\mathcal{E}$ be one of the admissible space (3.4) and $E = H^{\max(1,n)}_{x,v}(\mu^{-1/2})$ so that in all the cases, we have $E \subset \mathcal{E}$ and we already have the decay of the semigroup $\mathcal{S}_\Lambda(t)$ in E from Theorem 1.3. We then apply Theorem 2.13 from [19] whose assumptions are fulfilled thanks to Lemmas 3.3, 3.4 and Corollary 3.6. □

4. The nonlinear equation

This section is devoted to the proof of Theorem 1.1: we develop a Cauchy theory in a perturbative framework. Our proof is based on the study of the linearized equation that we made in previous sections. The idea is to prove that, using suitable norms, there exists a neighborhood of the equilibrium in which the linear part of the equation is dominant and thus dictates the dynamic. Consequently, taking an initial datum close enough to the equilibrium, one can construct solutions to the equation and prove exponential stability.

4.1. **Functional spaces.** In what follows, we use notations of Subsection 2.3. More precisely, we define the spaces X, Y, Y^*, $\bar{Y}$ and Y' as in (2.8) and (2.9) with a weight

$$m(v) = \langle v\rangle^k, \quad k > \frac{21}{2} + \gamma + 22s.$$

Similarly, for $i = 0, \dots, 3$, we define the spaces X_i, Y_i, $\bar{Y}_i$ and Y_i' as in (2.8) and (2.9) associated to the weights $m_i(v) = \langle v\rangle^{k_i}$. The exponents k_0 and k_1 satisfy the following conditions:

$$k_0 := k - 2s \quad \text{and} \quad 8 + 14s < k_1 < k_0 - \gamma - \frac{5}{2} - 6s.$$

Concerning k_2 and k_3, we set:

$$k_2 := k_1 - 2s \quad \text{and} \quad 4 - \gamma + \frac{3}{2} + 6s < k_3 < k_2 - \gamma - \frac{5}{2} - 6s.$$

Remark 4.1. *Notice first that*

$$k > k_0 > k_1 > k_2 > k_3.$$

Let us also comment briefly the conditions imposed on the weights and explain the introduction of so many spaces.

- *First, in the proof of Proposition 4.5, we need to be able to apply the result from Proposition 4.2 in X_1, this explains the introduction of the spaces X_2 and X_3.*
- *The last condition*

  $$k_3 > 4 - \gamma + \frac{3}{2} + 6s$$

  comes from the fact that we want to apply Theorem 3.1 and Lemma 2.4 in X_3.

- *In our argument explained in the two next subsections, there are two levels in which we have a loss of weight. The first one comes from the regularization estimate* (4.2) *(m_0 to m_1 and m_2 to m_3), which explains the conditions: $k_1 < k_0 - \gamma - 5/2 - 6s$ and $k_3 < k_2 - \gamma - 5/2 - 6s$. The second one comes from the nonlinear estimates in Lemma 2.4 (m to m_0 and m_1 to m_2), which explains the conditions: $k_0 := k - 2s$ and $k_2 := k_1 - 2s$ (a key element is that we have $\|f\|_{\bar{Y}_0} \lesssim \|f\|_Y$ and $\|f\|_{\bar{Y}_2} \lesssim \|f\|_{Y_1}$).*
- *The two first conditions*
$$k_1 > 8 + 14s$$
and
$$k > \gamma + \frac{21}{2} + 22s$$
are then naturally induced.

4.2. **Dissipative norm for the whole linearized operator.** Before going into the proof of an a priori estimate which is going to be the cornerstone of our Cauchy theory, we introduce a norm which is (better than) dissipative for the whole linearized operator Λ.

Proposition 4.2. *Define for any $\eta > 0$ and any $\lambda_1 < \lambda$ (where $\lambda > 0$ is the optimal rate in Theorem 3.1) the equivalent norm on X for $\Pi_0 h = 0$,*

$$|||h|||_X^2 := \eta \|h\|_X^2 + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_1 \tau} h\|_{X_1}^2 \, d\tau. \tag{4.1}$$

Then there is $\eta > 0$ small enough such that the solution $\mathcal{S}_\Lambda(t)h$ to the linearized equation satisfies, for any $t \geq 0$ and some constant $K > 0$,

$$\frac{1}{2}\frac{d}{dt}|||\mathcal{S}_\Lambda(t)h|||_X^2 \leq -\lambda_1 |||\mathcal{S}_\Lambda(t)h|||_X^2 - K\|\mathcal{S}_\Lambda(t)h\|_{Y^*}^2, \quad \forall\, h \in X,\, \Pi_0 h = 0.$$

Proof. First we remark that the norm $|||\cdot|||_{\mathcal{H}^3_x L^2_v(m)}$ is equivalent to the norm $\|\cdot\|_{\mathcal{H}^3_x L^2_v(m)}$ defined in (1.17) for any $\eta > 0$ and any $\lambda_1 < \lambda$. Indeed, using Theorem 3.1, we have

$$\begin{aligned}\eta\|h\|_{\mathcal{H}^3_x L^2_v(m)}^2 \leq |||h|||_{\mathcal{H}^3_x L^2_v(m)}^2 &= \eta\|h\|_{\mathcal{H}^3_x L^2_v(m)}^2 + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_1\tau} h\|_{\mathcal{H}^3_x L^2_v(m_1)}^2 \, d\tau \\ &\leq \eta\|h\|_{\mathcal{H}^3_x L^2_v(m)}^2 + \int_0^\infty C^2 e^{-2(\lambda-\lambda_1)\tau} \|h\|_{\mathcal{H}^3_x L^2_v(m_1)}^2 \, d\tau \leq C\|h\|_{\mathcal{H}^3_x L^2_v(m)}^2.\end{aligned}$$

We now compute, denoting $h_t = \mathcal{S}_\Lambda(t)h$,

$$\frac{1}{2}\frac{d}{dt}|||h_t|||_{\mathcal{H}^3_x L^2_v(m)}^2 = \eta\langle \Lambda h_t \, h_t\rangle_{\mathcal{H}^3_x L^2_v(m)} + \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\|\mathcal{S}_\Lambda(\tau) e^{\lambda_1 t} h_t\|_{\mathcal{H}^3_x L^2_v(m_1)}^2 \, d\tau =: I_1 + I_2.$$

For I_1 we write $\Lambda = \mathcal{A} + \mathcal{B}$. Using the fact that $\mathcal{A}$ is a truncation operator, we first obtain that

$$\langle \mathcal{A} h_t, h_t\rangle_{\mathcal{H}^3_x L^2_v(m)} \leq C\|h_t\|_{\mathcal{H}^3_x L^2_v(m_1)}^2.$$

Moreover, repeating the estimates for the hypodissipativity of $\mathcal{B}$ in Lemmas 3.3 and 3.4 we easily get that for some $K > 0$,

$$\langle \mathcal{B}h, h\rangle_{\mathcal{H}^3_x L^2_v(m)} \leq -\lambda\|h\|_{\mathcal{H}^3_x L^2_v(m)}^2 - K\|h\|_{\mathcal{H}^{3,s,*}_{x,v}(m)}^2,$$

therefore it follows

$$I_1 \leq -\lambda\eta\|h_t\|_{\mathcal{H}^3_x L^2_v(m)}^2 - \eta K\|h_t\|_{\mathcal{H}^{3,s,*}_{x,v}(m)}^2 + \eta C\|h_t\|_{\mathcal{H}^3_x L^2_v(m_1)}^2.$$

The second term is computed exactly

$$\begin{aligned} I_2 &= \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\|\mathcal{S}_\Lambda(\tau+t)e^{\lambda_1\tau}h\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \\ &= \frac{1}{2}\int_0^\infty \frac{\partial}{\partial \tau}\|\mathcal{S}_\Lambda(\tau+t)e^{\lambda_1\tau}h\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau - \lambda_1\int_0^\infty \|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \\ &= \frac{1}{2}\left[\|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\right]_{\tau=0}^{\tau=+\infty} - \lambda_1\int_0^\infty \|\mathcal{S}_\Lambda(\tau+t)e^{\lambda_1\tau}h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \\ &= -\frac{1}{2}\|h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)} - \lambda_1\int_0^\infty \|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \end{aligned}$$

where we have used the semigroup decay from Theorem 3.1.

Gathering previous estimates and using that $\lambda \geq \lambda_1$, we obtain

$$\begin{aligned} I_1 + I_2 \leq -\lambda_1 \left\{ \eta\|h_t\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \right\} \\ - \eta K\|h_t\|^2_{\mathcal{H}^{3,s,*}_{x,v}(m)} + \eta C\|h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)} - \frac{1}{2}\|h_t\|^2_{\mathcal{H}^3_x L^2_v(m_1)}. \end{aligned}$$

We complete the proof choosing $\eta > 0$ small enough. □

4.3. **Regularization properties of S_Λ.** In this subsection, we state a result on the regularization properties of S_Λ which is a key point for having a priori estimates on the nonlinear problem in the next subsection.

Lemma 4.3. *We have the following estimate:*

$$\|\mathcal{S}_\Lambda(t)h\|_{X_1} \lesssim \frac{1}{t^{1/2}}\|h\|_{Y_0'}, \quad \forall\, t \in (0,1]. \tag{4.2}$$

Proof. Let us start this proof noticing an embedding property:

$$\forall\, q_1 \leq q_2,\, \varsigma \in \mathbb{R}^+, \quad H^\varsigma_v(\langle v\rangle^{q_2}) \hookrightarrow H^\varsigma_v(\langle v\rangle^{q_1}). \tag{4.3}$$

This property is clear in the case $\varsigma \in \mathbb{N}$. It is less evident in the case $\varsigma \in \mathbb{R}^+ \setminus \mathbb{N}$. This case can be shown using real interpolation (see Subsection 3.5 for the notations). Indeed, since the weighted space $H^\varsigma_v(\langle v\rangle^{q_i})$ is defined through

$$h \in H^\varsigma_v(\langle v\rangle^{q_i}) \Leftrightarrow h\langle v\rangle^{q_i} \in H^\varsigma_v,$$

we can use that (see Subsection 3.5):

$$H^\varsigma_v = \left[H^{\lfloor\varsigma\rfloor}_v, H^{\lfloor\varsigma\rfloor+1}_v\right]_{\varsigma-\lfloor\varsigma\rfloor,2}$$

to prove that

$$H^\varsigma_v(\langle v\rangle^{q_i}) = \left[H^{\lfloor\varsigma\rfloor}_v(\langle v\rangle^{q_i}), H^{\lfloor\varsigma\rfloor+1}_v(\langle v\rangle^{q_i})\right]_{\varsigma-\lfloor\varsigma\rfloor,2}, \quad i = 1,2.$$

From this, since $H^\ell_v(\langle v\rangle^{q_2}) \hookrightarrow H^\ell_v(\langle v\rangle^{q_1})$ for $\ell \in \mathbb{N}$, we deduce the desired embedding result: $H^\varsigma_v(\langle v\rangle^{q_2}) \hookrightarrow H^\varsigma_v(\langle v\rangle^{q_1})$.

The result that we want to prove is a twisted version of Theorem 1.2 from [22], the only difference being in the weights. First, we notice that

$$\|\mathcal{S}_\Lambda(t)h\|_{X_1} \lesssim \|\mathcal{S}_\Lambda(t)h\|_{H^{3,0}_{x,v}(\langle v\rangle^{k_1})}.$$

The result from [22] gives us that for $k' > k_1 + \gamma + 5/2$, we have:

$$\|\mathcal{S}_\Lambda(t)h\|_{H^{3,0}_{x,v}(\langle v\rangle^{k_1})} \lesssim \frac{1}{t^{1/2}}\|h\|_{(H^{3,s}_{x,v}(\langle v\rangle^{k'}))'}, \quad \forall\, t \in (0,1]$$

where $(H^{3,s}_{x,v}(\langle v\rangle^{k'}))'$ is the dual space of $H^{3,s}_{x,v}(\langle v\rangle^{k'})$ with respect to $H^{3,0}_{x,v}(\langle v\rangle^{k'})$. It remains to show that if $k_0 = k' + 6s > k_1 + \gamma + 5/2 + 6s$, we have

$$\|h\|_{(H^{3,s}_{x,v}(\langle v\rangle^{k'}))'} \lesssim \|h\|_{(\mathcal{H}^{3,s}_{x,v}(\langle v\rangle^{k_0}))'}.$$

Indeed,

$$\begin{aligned}
\|h\|_{(H^{3,s}_{x,v}(\langle v\rangle^{k'}))'} &= \sup_{\sum_{j=0}^3 \|\nabla^j_x(\varphi\langle v\rangle^{k'})\|_{H^{0,s}_{x,v}}\le 1} \sum_{j=0}^3 \big\langle \nabla^j_x h\langle v\rangle^{k_0-2js}, \nabla^j_x\varphi\langle v\rangle^{2k'-(k_0-2js)}\big\rangle_{L^2_{x,v}} \\
&= \sup_{\sum_{j=0}^3 \|\nabla^j_x(\psi\langle v\rangle^{2(k_0-2js)-k'})\|_{H^{0,s}_{x,v}}\le 1} \sum_{j=0}^3 \big\langle \nabla^j_x h\langle v\rangle^{k_0-2js}, \nabla^j_x\psi\langle v\rangle^{k_0-2js}\big\rangle_{L^2_{x,v}} \\
&\le \sup_{\sum_{j=0}^3 \|\nabla^j_x(\psi\langle v\rangle^{k_0-2js})\|_{H^{0,s}_{x,v}}\le 1} \sum_{j=0}^3 \big\langle \nabla^j_x h\langle v\rangle^{k_0-2js}, \nabla^j_x\psi\langle v\rangle^{k_0-2js}\big\rangle_{L^2_{x,v}} \\
&= \sup_{\|\psi\|_{\mathcal{H}^{3,s}_{x,v}(\langle v\rangle^{k_0})}\le 1} \langle h,\psi\rangle_{\mathcal{H}^{3,0}_{x,v}(\langle v\rangle^{k_0})} \\
&\le \|h\|_{(\mathcal{H}^{3,s}_{x,v}(\langle v\rangle^{k_0}))'},
\end{aligned}$$

where we used (4.3) to obtain the third bound and this concludes the proof of (4.2). □

4.4. **Proof of Theorem 1.1.** We consider the Cauchy problem for the perturbation h defined through $h = f - \mu$. The equation satisfied by $h = h(t,x,v)$ is

$$\begin{cases} \partial_t h = \Lambda h + Q(h,h) \\ h_{|t=0} = h_0 = f_0 - \mu. \end{cases} \tag{4.4}$$

From the conservation laws (see (1.7)), for all $t > 0$, $\Pi_0 h_t = 0$ since $\Pi_0 h_0 = 0$, more precisely $\int_{\mathbb{T}^3\times\mathbb{R}^3} h_t(x,v)\,dv\,dx = \int_{\mathbb{T}^3\times\mathbb{R}^3} v_j h_t(x,v)\,dv\,dx = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 h_t(x,v)\,dv\,dx = 0$ for $j = 1,2,3$. Note that we also have $\Pi_0 Q(h_t,h_t) = 0$.

4.4.1. *A priori estimates.*

Proposition 4.4. *Any solution $h = h_t$ to (4.4) satisfies, at least formally, the following differential inequality: for any $\lambda_1 < \lambda$ (where $\lambda > 0$ is one rate given by Theorem 3.1), there holds*

$$\frac{1}{2}\frac{d}{dt}|||h|||^2_X \le -\lambda_1 |||h|||^2_X - \big(K - C|||h|||_X\big)\|h\|^2_{Y^*},$$

for some constants $K, C > 0$ and where we recall that the norm $|||\cdot|||$ is defined in Proposition 4.2.

Proof. We compute the evolution of $|||h|||$ where $h = h_t$ is solution of (4.4):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||h|||_X^2 &= \eta\langle h, \Lambda h\rangle_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h, \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}\Lambda h\rangle_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \\ &\quad + \eta\langle h, Q(h,h)\rangle_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h, \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}Q(h,h)\rangle_{\mathcal{H}^3_x L^2_v(m_1)}\,d\tau \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

For the linear part $I_1 + I_2$, we already have from Proposition 4.2 that, for any $\lambda_1 < \lambda$,

$$I_1 + I_2 \le -\lambda_1 |||h|||_X^2 - K\|h\|_{Y^*}^2.$$

We now deal with the nonlinear part, using first Lemma 2.4:

$$I_3 \lesssim \langle Q(h,h), h\rangle_X \lesssim \|h\|_X \|h\|_{Y^*}^2 \lesssim |||h|||_X \|h\|_{Y^*}^2.$$

For the last term I_4, we use the fact that $\Pi_0 f_t = 0$ and $\Pi_0 Q(f_t, f_t) = 0$ for all $t \ge 0$, together with the estimate (4.2) from Lemma 4.3. More precisely, if $\Pi_0 h = 0$, using Theorem 3.1 in X_1, we have:

$$\forall\, t \ge 0, \quad \|S_\Lambda(t)h\|_{X_1} \lesssim e^{-\lambda t}\|h\|_{X_1}.$$

Combined with the estimate (4.2) from Lemma 4.3, we deduce that for $\Pi_0 h = 0$,

$$\forall\, t > 0, \quad \|S_\Lambda(t)h\|_{X_1} \lesssim \frac{e^{-\lambda t}}{1 \wedge \sqrt{t}}\|h\|_{Y_0'}.$$

It implies

$$\begin{aligned}&\int_0^\infty \langle \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h, \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}Q(h,h)\rangle_{X_1}\,d\tau \\ &\quad \le \int_0^\infty \|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h\|_{X_1}\,\|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}Q(h,h)\|_{X_1}\,d\tau \\ &\quad \lesssim \|h\|_{X_1}\,\|Q(h,h)\|_{Y_0'}\int_0^\infty e^{-(\lambda-\lambda_1)\tau}\,\frac{e^{-(\lambda-\lambda_1)\tau}}{1\wedge\sqrt{\tau}}\,d\tau \\ &\quad \lesssim \|h\|_{X_1}\,\|Q(h,h)\|_{Y_0'}.\end{aligned}$$

To conclude, we use Lemma 2.4:

$$I_4 \lesssim \|h\|_{X_1}\,\|h\|_{X_0}\,\|h\|_{\bar{Y}_0} \lesssim |||h|||_X\,\|h\|_Y^2 \lesssim |||h|||_X\,\|h\|_{Y^*}^2.$$

□

We prove now an a priori estimate on the difference of two solutions to (4.4).

Proposition 4.5. *Consider two solutions g and h to (4.4) associated to initial data g_0 and h_0, respectively. Then, at least formally, the difference $g - h$ satisfies the following differential inequality*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||g-h|||_{X_1}^2 &\le -K\|g-h\|_{Y_1^*}^2 + C\big(\|g\|_{X_1} + \|h\|_{X_1}\big)\,\|g-h\|_{Y_1^*}^2 \\ &\quad + C\big(\|h\|_{Y_1} + \|g\|_Y\big)\,\|g-h\|_{X_1}\,\|g-h\|_{Y_1},\end{aligned}$$

for some constants $K, C > 0$ and where $|||\cdot|||_{X_1}$ is defined as $|||\cdot|||_X$ in (4.1):

$$|||h|||_{X_1}^2 := \eta\|h\|_{X_1}^2 + \int_0^\infty \|\mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}h\|_{X_3}^2\,d\tau.$$

Proof. We write the equation safisfied by $g-h$:

$$\begin{cases} \partial_t(g-h) = \Lambda(g-h) + Q(h,g-h) + Q(g-h,g), \\ (g-h)_{|t=0} = g_0 - h_0. \end{cases}$$

We compute

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|||g_t - h_t|||^2_{X_1} \\ &= \eta\langle (g-h), \Lambda(g-h)\rangle_{X_1} + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}(g-h), \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}\Lambda(g-h)\rangle_{X_3}\, d\tau \\ &\quad + \eta\langle (g-h), Q(h,g-h)\rangle_{X_1} + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}(g-h), \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}Q(h,g-h)\rangle_{X_3}\, d\tau \\ &\quad + \eta\langle (g-h), Q(g-h,g)\rangle_{X_1} + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}(g-h), \mathcal{S}_\Lambda(\tau)e^{\lambda_1\tau}Q(g-h,g)\rangle_{X_3}\, d\tau \\ &=: T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \end{aligned}$$

Since the proof follows closely the one of Proposition 4.4, we do not give too much details here (notice that the spaces indexed by 2 are implicitly used in the following estimates as the spaces indexed by 0 were used in Proposition 4.4). We have:

$$T_1 + T_2 \leq -K\|g-h\|^2_{Y_1^*},$$

and also

$$T_3 + T_4 \lesssim \|h\|_{X_1}\|g-h\|^2_{Y_1^*} + \|h\|_{Y_1}\|g-h\|_{X_1}\|g-h\|_{Y_1}.$$

Moreover, for the last part T_5+T_6, using Lemma 2.4-(i), we get

$$\begin{aligned} T_5 + T_6 &\lesssim \|g-h\|_{X_1}\|g\|_{\bar{Y}_1}\|g-h\|_{Y_1} + \|g\|_{X_1}\|g-h\|^2_{Y_1} \\ &\lesssim \|g-h\|_{X_1}\|g\|_{Y}\|g-h\|_{Y_1} + \|g\|_{X_1}\|g-h\|^2_{Y_1}, \end{aligned}$$

which completes the proof. □

4.4.2. *End of the proof.* The end of the proof of Theorem 1.1 is classical and we do not enter into details here. It follows a standard argument by introducing an iterative scheme whose convergence and stability is shown thanks to Propositions 4.4 and 4.5. The framework being exactly the same, we refer to Subsections 3.4.2. and 3.4.3 from [12] in which a more precise proof is given.

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