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A Fourier Continuation-based solver for 3D elastic wave propagation

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Abstract

We present a spectral numerical algorithm for the fast solution of elastodynamics problems in general 3D domains based on a FFT-speed Fourier Continuation (FC) approximation for accurate Fourier expansion of non-periodic functions. The high-order methodology yields physically correct solutions including those with traction conditions on curved boundaries. The approach is essentially without dispersion errors; entails mild CFL constraints; runs at a cost scaling linearly with discretization size; and can be efficiently parallelized for computing clusters.

Keywords: Fourier Continuation, elastic wave equation, high-performance numerical methods

1 Introduction

This work considers the propagation of elastic waves in a linear, isotropic, possibly heterogeneous medium contained in a general domain $\Omega \in \mathbb{R}^3$ and governed by the Navier equation

$$\rho(\mathbf{x})\mathbf{u}_{tt}(\mathbf{x}, t) = \nabla \cdot \left[\mu(\mathbf{x}) \left(\nabla \mathbf{u}(\mathbf{x}, t) + \nabla \mathbf{u}^T(\mathbf{x}, t) \right) + \lambda(\mathbf{x}) (\nabla \cdot \mathbf{u}(\mathbf{x}, t)) \mathbf{I} \right] + \mathbf{f}(\mathbf{x}, t) \quad (1)$$

for position and displacement vectors \mathbf{x} and \mathbf{u} ; Lamé parameters $\mu(\mathbf{x})$, $\lambda(\mathbf{x})$; density $\rho(\mathbf{x})$; and a given vector of body forces $\mathbf{f}(\mathbf{x}, t)$. Initial conditions $\mathbf{u}(\mathbf{x}, t_0)$, $\mathbf{u}_t(\mathbf{x}, t_0)$ are prescribed at initial time $t = t_0$, and the boundary $\partial\Omega$ is partitioned as a union $\partial\Omega = \Gamma_D \cup \Gamma_T$ of two surfaces Γ_D and Γ_T upon which boundary displacements and boundary tractions,

$$\begin{aligned} \mathbf{u} &= \mathbf{c}(\mathbf{x}, t) \quad \text{on } \Gamma_D \quad \text{and} \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{d}(\mathbf{x}, t) \quad \text{on } \Gamma_T, \end{aligned}$$

are prescribed, where \mathbf{n} is the inward unit normal and $\boldsymbol{\sigma}$ is the (symmetric) stress tensor for an isotropic medium.

2 Methodology

Accelerated FC(Gram). The FC method [1–3] enables high-order convergence of Fourier series

approximations of non-periodic functions by resolving the Gibbs “ringing” effect, extending the applicability of classical Fourier-based PDE solvers (together with their inherent qualities, e.g. limited dispersion, high-order accuracy and mild CFL conditions) to problems with general domains and boundary conditions. Given point values $f(x_i)$ of a function $f : [0, 1] \rightarrow \mathbb{R}$ on a uniform grid $x_i = i/(N - 1)$, $i = 0, \dots, N - 1$, the FC method produces a rapidly-convergent interpolating Fourier series representation $f^c : [0, b] \rightarrow \mathbb{R}$ on a region $[0, b]$ larger than the physical domain $[0, 1]$ as

$$f^c(x) = \sum_{k=-M}^M a_k e^{\frac{2\pi i k x}{b}} \quad \text{s.t. } f^c(x_i) = f(x_i), \quad (2)$$

for suitably chosen FC-parameters M (bandwidth) and $b > 1$ (interval length). The b -periodic continuation function f^c is an approximate *periodic extension* of f that closely approximates f in $[0, 1]$. Derivatives for a PDE solver can be then produced with high-order accuracy by term-wise differentiation. We base the construction of (2) on a “biased-order” technique [1] that uses numbers d_ℓ and d_r of function values near the left and right endpoints 0 and 1, together with projections of the corresponding vectors of function values onto a Gram polynomial basis—whose continuations are precomputed by means of high-precision linear algebra methods. This is then extended to a form suitable for use in traction conditions [2].

Geometry and parallelization. Physically realistic configurations with curved geometries are treated by an overset method [4] that decomposes Ω into a union $\Omega = \bigcup_j \Omega_j$ of a finite number of overlapping, boundary-conforming curvilinear patches—endowed with uniform Cartesian-like discretizations—within each one of which a curvilinear formulation of (1) is evolved. Information is exchanged via interpolation, and sharp corners and edges are approximated by rounded patches. Further decomposition of each patch Ω_j into mutually disjoint sub-patches, to-

gether with the use of certain “line-segmented” FC operators (which produce very efficient calculation of the corresponding optimally-sized discrete Fourier transforms), enables a parallelization for distributed-memory computing environments that achieves excellent scalability [2].

3 Numerical experiments

These examples utilize an explicit fourth-order Adams-Bashforth scheme (AB4) in time and a fixed number of 25 discrete points in the periodic extension to construct continuations for fourth-order Gram polynomials ($d_\ell, d_r = 5$). Further parameters and studies are detailed in [2].

A convergence study. A solid cylinder composed of two patches is prescribed with body forces and traction boundary conditions corresponding to a known solution (see [2]). The table below reports the max errors in displacement over all time and space, where fifth-order accuracy can be appreciated.

N (patch Ω_1)	N (patch Ω_2)	L_{err}^∞	$O(L_{\text{err}}^\infty)$
27,000	72,000	$2.82e-03$	—
216,000	576,000	$8.29e-05$	5.09
729,000	1,944,000	$9.37e-06$	5.38
1,728,000	4,608,000	$2.00e-06$	5.37

A parallel performance study. The table below reports errors and CPU-seconds per million unknowns (denoted S) for propagation in a 3D aluminum plate with a circular hole modeled by six different curvilinear patches. Nearly perfect scalability is achieved as the number of cores is increased for a fixed number of grid points.

# grid pts	# cores	L_{err}^∞	S
3,033,360	240	$7.89e-3$	1.51 sec
—	360	$7.98e-3$	1.55 sec
—	480	$8.32e-3$	1.45 sec

A dispersion study. Plane-waves of various numbers W of wavelengths are advanced through a 3D aluminum plate with traction boundary conditions. Figure 1 shows the max errors over all space and over one full temporal cycle (defined as the time required for any one crest to travel the length of the plate) of the solution for increasing W . For each fixed number of points-per-wavelength (PPW), the accuracy of the FC solver remains essentially constant, suggesting that the use of large numbers of sub-domains and sub-patches does not give rise to significant dispersion (up to 512 cores are employed for the highest values of W).

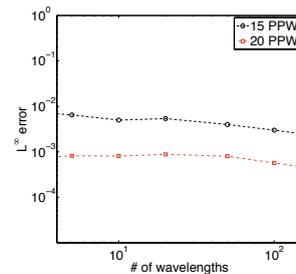


Figure 1: Errors over a cycle for varying W .

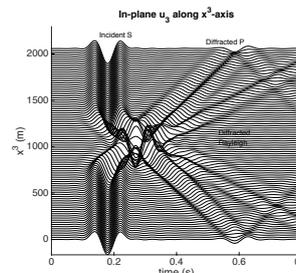


Figure 2: Simulated response of an earthquake.

A classical seismology problem. Wave amplification by a 180m hill in a 3D region impacted by an incident shear (S-)wave is depicted in the seismogram of Figure 2 using the FC-based elasticity solver and a traction-free boundary condition at the surface. The high-definition solutions are constructed in 58 seconds on 96 cores of a high-performance cluster using just shy of 150,000 volumetric discretization points—significantly coarser than a high-order spectral element method (4,935,953 points) and a stable difference method (109,808,412 points) [2].

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