Classification of the Bounds on the Probability of Ruin for Lévy Processes with Light-tailed Jumps

Jérôme Spielmann

CLASSIFICATION OF THE BOUNDS ON THE PROBABILITY OF RUIN FOR LÉVY PROCESSES WITH LIGHT-TAILED JUMPS

JÉRÔME SPIELMANN*

LAREMA, Université d'Angers, 2 Bd. Lavoisier, 49045 Angers

Abstract. In this note, we study the ultimate ruin probabilities of a real-valued Lévy process X with light-tailed negative jumps. It is well-known that, for such Lévy processes, the probability of ruin decreases as an exponential function with a rate given by the root of the Laplace exponent, when the initial value goes to infinity. Under the additional assumption that X has integrable positive jumps, we show how a finer analysis of the Laplace exponent gives in fact a complete description of the bounds on the probability of ruin for this class of Lévy processes. This leads to the identification of a case that was not considered before. We apply the result to the Cramér-Lundberg model perturbed by Brownian motion.

Keywords: Laplace exponent; Lévy processes; Lundberg equation; Perturbed model; Ruin probabilities.

MSC 2010 subject classifications: 60G51, 91B30
JEL Classification : G220

1. Introduction and Main Result

Ruin theory studies in particular the time of passage below 0 of stochastic processes that represent the capital of an insurance company or a pension fund. In particular, it studies the probability that the process becomes negative on

E-mail address: jerome.spielmann@univ-angers.fr.

*: I would like to thank Lioudmila Vostrikova for helpful discussions and for taking the time to read through the first draft of the manuscript. I would also like to acknowledge financial support from the DéfiMaths project of the "Fédération de Recherche Mathématiques des Pays de Loire", form the PANORisk project of the Région Pays de la Loire and from the French government's "Investissements d'Avenir" program ANR-11-LABX-0020-01.

an infinite time horizon in function of the initial value of the process. The key result of Cramér [4] is that, in the case of the compound Poisson process with drift, this probability decreases as an exponential function with a rate given as a solution to the *Lundberg equation*. It is well-know that, when the initial value goes to infinity, the result of Cramér holds for more general (light-tailed) Lévy processes where the rate is given by the root of the Laplace exponent of the process, see Theorem XI.2.6 in [1], and also [3], [7] and Section 7.2 in [8].

In this note, we show that a finer analysis of the Laplace exponent can lead to a complete description of the bounds on the ultimate probability of ruin. Our main contribution is to give a systematic description of all possible cases (Theorem 1), where the case when it has a root (Theorem 1, Case B) corresponds to the well-known Lundberg bound. This also leads to the identification of a case that is not considered in the literature (Theorem 1, Case D). We show that in this case the ruin probability also decreases at least as an exponential function and identify the rate of decay. Thus, Theorem 1 gives a method for obtaining exponential bounds and conditions for ruin with probability one for a large class of risk models. We illustrate this by applying the method to the Cramér-Lundberg model perturbed by Brownian motion (Proposition 1).

When the Lévy process has jumps only on one side (i.e., it is *spectrally one-sided*), the results contained in Chapter 8 of [8] and the references therein give a precise description of the ultimate ruin probability in terms of the so-called *scale functions*. However, these scale functions are in general not very explicit. In comparison, the method presented here is more elementary and less precise but works also in the case where there are two-sided jumps and is, in some cases, more explicit.

1.1. **Lévy Processes and Laplace Exponents.** In this section, we state some basic facts about Lévy processes and present the main assumptions for the rest of this paper.

Let $X = (X_t)_{t\geq 0}$ be a real-valued Lévy process on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ (in the sense of [6], Definition II.4.1, p.101) where the filtration $\mathbb{F}$ is assumed to satisfy the usual conditions. It is well-known that the characteristic function

of X_t for each $t \geq 0$ is given by the Lévy-Khintchine formula:

$$\mathbf{E}\left(e^{i\lambda X_t}\right) = e^{t\Phi(\lambda)}, \text{ for all } t \geq 0 \text{ and } \lambda \in \mathbb{R},$$

where

$$\Phi(\lambda) = ia\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|<1\}}\right) \Pi(dx), \text{ for all } \lambda \in \mathbb{R},$$

for $a \in \mathbb{R}$, $\sigma \geq 0$ and Π a Lévy measure on $\mathbb{R}$ satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \left(x^2 \wedge 1\right) \Pi(dx) < +\infty.$$

The function Φ and the triplet (a, σ^2, Π) are unique and are called the Lévy exponent and the characteristics (or Lévy triplet) of X respectively.

Assumption (I). *X is integrable.*

The first assumption we use is integrability. We say that X is integrable if $\mathbf{E}(|X_1|) < +\infty$ and it can be shown (see e.g. [10], Theorem 25.3, p.159) that this is equivalent to the condition

$$\int_{|x|\geq 1} |x|\Pi(dx) < +\infty.$$

Under assumption (I), we can rewrite the Lévy exponent of X as

$$\Phi(\lambda) = i\delta\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) \Pi(dx), \text{ for all } \lambda \in \mathbb{R}, \tag{1}$$

where

$$\delta \triangleq \mathbf{E}(X_1) = a + \int_{|x|\geq 1} x\Pi(dx).$$

Also, under assumption (I), the Lévy-Itô decomposition of X is

$$X_t = \delta t + \sigma W_t + \int_0^t \int_{\mathbb{R}} x \left(\mu^X - \nu^X\right)(ds, dx), \text{ for all } t \geq 0, \tag{2}$$

where μ^X is the jump measure of X, $\nu^X(ds, dx) = ds\Pi(dx)$ is the compensator of the jump measure (see [6], Theorem I.1.8, p.66) and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

Assumption (II). *X has light-tailed negative jumps.*

The second assumption we will use is a condition on the tail behaviour of the negative jumps. Similar definitions to the one below can be found on p.338 in [1] and p.164-165 in [10].

Definition 1. *Let $X = (X_t)_{t\geq 0}$ be a real-valued Lévy process with characteristics (a, σ^2, Π). Let*

$$\gamma_c \triangleq \sup\left\{\gamma \geq 0 : \int_{-\infty}^{-1} e^{-\gamma x}\Pi(dx) < +\infty\right\}.$$

We say that X has light-tailed negative jumps if $\gamma_c > 0$. (Note that γ_c can take the value $+\infty$.)

Under Assumptions (I) and (II), it is possible to show that the Lévy exponent exists also for any $\lambda = i\gamma$, with $\gamma \in [0, \gamma_c)$. In fact, when $\gamma \in [0, \gamma_c)$,

$$\Phi(i\gamma) = -\delta\gamma + \frac{\sigma^2}{2}\gamma^2 + \int_{\mathbb{R}} \left(e^{-\gamma x} - 1 + \gamma x\right)\Pi(dx),$$

and letting $I_- \triangleq \int_{\mathbb{R}_-} \left|e^{-\gamma x} - 1 + \gamma x\right|\Pi(dx)$, we obtain using the Taylor formula,

$$\begin{aligned} I_- &\leq \int_{-1}^{0} \left|e^{-\gamma x} - 1 + \gamma x\right|\Pi(dx) + \int_{-\infty}^{-1} e^{-\gamma x}\Pi(dx) \\ &\leq \frac{\gamma^2}{2}\int_{-1}^{0} x^2\Pi(dx) + \int_{-\infty}^{-1} e^{-\gamma x}\Pi(dx) < +\infty. \end{aligned}$$

On the other hand, letting $I_+ \triangleq \int_{\mathbb{R}_+} \left|e^{-\gamma x} - 1 + \gamma x\right|\Pi(dx)$ and using the Taylor formula and the assumption of integrability,

$$\begin{aligned} I_+ &= \int_{0}^{1} \left|e^{-\gamma x} - 1 + \gamma x\right|\Pi(dx) + \int_{1}^{\infty} \left|e^{-\gamma x} - 1 + \gamma x\right|\Pi(dx) \\ &\leq \frac{\gamma^2}{2}\int_{0}^{1} x^2\Pi(dx) + \Pi\left([1, +\infty)\right) + \gamma\int_{1}^{\infty} |x|\Pi(dx) < +\infty. \end{aligned}$$

Therefore, it is possible to define the Laplace exponent of X as the function Ψ given by

$$\Psi(\gamma) \triangleq \Phi(i\gamma) = -\delta\gamma + \frac{\sigma^2}{2}\gamma^2 + \int_{\mathbb{R}} \left(e^{-\gamma x} - 1 + \gamma x\right)\Pi(dx), \text{ for all } \gamma \in [0, \gamma_c).$$

Remark 1. *The Laplace exponent is always defined on $\mathbb{R}_-$ and can, under Assumptions (I) and (II), be defined on $(-\infty, \gamma_c)$.*

From the Lévy-Khintchine formula, we see that the Laplace transform of X_t is then given by

$$\mathbf{E}\left(e^{-\gamma X_t}\right) = e^{t\Psi(\gamma)}, \text{ for all } t \geq 0 \text{ and } \gamma \in (-\infty, \gamma_c).$$

1.2. **Main Result and Application.** Suppose that $X = (X_t)_{t\geq 0}$ is a real-valued Lévy process satisfying assumptions (I) and (II). Let $Y_t^u \triangleq u + X_t$, for $t \geq 0$ and $u \geq 0$. We define the ultimate ruin probability as

$$\mathbf{P}\left(\inf_{0\leq t<+\infty} Y_t^u \leq 0\right) = \mathbf{P}\left(\inf_{0\leq t<+\infty} X_t \leq -u\right) = \mathbf{P}\left(\sup_{0\leq t<+\infty} (-X_t) \geq u\right).$$

This can also be written as $\mathbf{P}\left(\tau(u) < +\infty\right)$ where $\tau(u) \triangleq \inf\{t \geq 0 : X_t \leq -u\}$ and $\tau(u) \triangleq +\infty$, if X never goes below $-u$. We are now ready to give the main result.

Theorem 1. *Let $X = (X_t)_{t\geq 0}$ be a (non-zero) real-valued Lévy process satisfying Assumptions (I) and (II) and $\Psi : [0, \gamma_c) \to \mathbb{R}$ be the Laplace exponent of X. Then, there are only four possible cases.*

(A) If $\Psi(\gamma) > 0$, for all $\gamma \in (0, \gamma_c)$, then $\mathbf{P}(\tau(u) < +\infty) = 1$, for all $u \geq 0$.
(B) If there exists $\gamma_0 \in (0, \gamma_c)$ such that $\Psi(\gamma_0) = 0$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_0 u}$, for all $u \geq 0$.
(C) If $\gamma_c = +\infty$ and $\Psi(\gamma) < 0$, for all $\gamma \in (0, +\infty)$, then $\sigma^2 = 0$, $\Pi(\mathbb{R}_-) = 0$, $\delta > 0$ and which means that X is a subordinator. Therefore, $\mathbf{P}(\tau(u) < +\infty) = 0$, for all $u \geq 0$.
(D) If $\gamma_c < +\infty$ and $\Psi(\gamma) < 0$, for all $\gamma \in (0, \gamma_c)$, then $\mathbf{P}\left(\tau(u) < +\infty\right) \leq e^{-\gamma_c u}$, for all $u \geq 0$.

Thus, Theorem 1 exhausts all possible cases and allows one to classify the behaviour of the ruin probability in function of the behaviour of the Laplace exponent for a large class of risk models. To illustrate how to use Theorem 1, we apply it to the Cramér-Lundberg model perturbed by Brownian motion. This model, which is sometimes also called *perturbed risk process* and was studied first in [5], is given by

$$Y_t^u = u + pt + \sigma W_t - \sum_{n=1}^{N_t} U_n, \text{ for all } t \geq 0, \tag{3}$$

where $p > 0$, $\sigma > 0$, $N = (N_t)_{t\geq 0}$ is a standard Poisson process with rate β, $(W_t)_{t\geq 0}$ is a standard Brownian motion and $U = (U_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. exponential random variables with rate α. Additionally, it is assumed that the processes N, W and the sequence U are independent from each other.

Then, the following proposition gives the description of the ruin probabilities for this model. Note that in contrast to the case when $\sigma^2 = 0$, there are two possible regimes when the *safety loading condition* $p > \frac{\beta}{\alpha}$ is satisfied. This shows how the uncertainty in premium payments affects the ruin probability. Also note that this result is very explicit as the behaviour of the ruin probability only depends on the value of the parameters and that it gives the complete description of the possible cases.

Proposition 1. *Let $X = (X_t)_{t\geq 0}$ be a real-valued Lévy process with Lévy triplet $\Pi(dx) = \beta\alpha e^{\alpha x} 1_{\{x\leq 0\}} dx$, $\sigma^2 > 0$ and $a = p + \int_{|x|<1} x\Pi(dx)$ for some $p, \alpha, \beta > 0$. Then, $Y_t^u = u + X_t$, with $u \geq 0$, corresponds to the perturbed risk process given by (3). Let $\Delta \triangleq (\sigma^2\alpha - 2p)^2 + 8\sigma^2\beta$ and $\gamma_- \triangleq \frac{\sigma^2\alpha + 2p - \sqrt{\Delta}}{2\sigma^2}$.*

- *If $p \leq \frac{\beta}{\alpha}$, then $\mathbf{P}(\tau(u) < +\infty) = 1$, for all $u \geq 0$.*
- *If $p > \frac{\beta}{\alpha}$ and $\gamma_- < \alpha$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_- u}$.*
- *If $p > \frac{\beta}{\alpha}$ and $\gamma_- \geq \alpha$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\alpha u}$.*

Proof. We have $\gamma_c = \alpha$ and $\delta = p - \frac{\beta}{\alpha}$. So, by Theorem 1 (A), we have ruin with probability one when $p \leq \frac{\beta}{\alpha}$ and we assume in the following that $p > \frac{\beta}{\alpha}$. For $\gamma \in (0, \alpha)$, we obtain

$$\begin{aligned}\Psi(\gamma) &= -\delta\gamma + \frac{\sigma^2}{2}\gamma^2 + \beta\alpha \int_{-\infty}^{0} (e^{-\gamma x} - 1 + \gamma x) e^{\alpha x} dx \\ &= -p\gamma + \frac{\sigma^2}{2}\gamma^2 - \frac{\beta\alpha}{\gamma - \alpha} - \beta \\ &= \frac{\gamma\left(\sigma^2\gamma^2 - (\sigma^2\alpha + 2p)\gamma + 2(p\alpha - \beta)\right)}{2(\gamma - \alpha)} = -\frac{1}{2}A(\gamma)B(\gamma),\end{aligned}$$

where $A(\gamma) \triangleq \frac{\gamma}{\alpha - \gamma}$ and $B(\gamma) \triangleq \sigma^2\gamma^2 - (\sigma^2\alpha + 2p)\gamma + 2(p\alpha - \beta)$. To see if Ψ has an other root along 0, we need to consider the solutions of $B(\gamma) = 0$. This is an equation of second order with determinant Δ. As $\Delta > 0$, B has two distinct roots γ_+ and γ_-, given by

$$\gamma_\pm = \frac{\sigma^2\alpha + 2p \pm \sqrt{\Delta}}{2\sigma^2}.$$

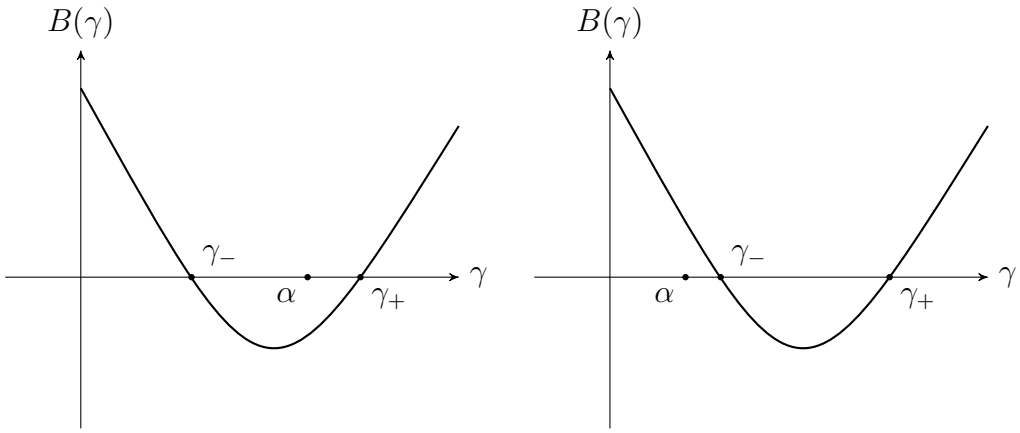

FIGURE 1. Behaviour of B when $\gamma_- < \alpha$ (left) and $\gamma_- \geq \alpha$ (right).

First note that $\gamma_- < \gamma_+$ and that $\gamma_+ \geq \alpha$ and $\gamma_- \geq 0$, because $(\sigma^2\alpha+2p)^2 \geq \Delta \geq (\sigma^2\alpha - 2p)^2$. Additionally, note that $B''(\gamma) = 2\sigma^2 > 0$, so that B is convex. Therefore, we only have two possible cases (see Figure 1) : either $\gamma_- < \alpha$ and then γ_- is a root of B and of Ψ, or $\gamma_- \geq \alpha$ and then $B(\gamma) > 0$ and $\Psi(\gamma) < 0$, for all $\gamma \in [0, \alpha)$. So, if $\gamma_- < \alpha$, then, by Theorem 1 (B), we obtain $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_- u}$ and if $\gamma_- \geq \alpha$, then, by Theorem 1 (D), we obtain $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\alpha u}$. □

2. Proof of Theorem 1

2.1. Law of Large Numbers and Properties of the Laplace Exponent.

We start with the following well-known proposition and corollary (see Proposition IV.1.2, p.73 in [1] in the case of the compound Poisson process with drift, disscussion p.75 and Proposition 8 p.84 in [2], Exercice 7.3 in [8], and Section 36 starting at p.245 in [10] in the general case) that give a strong law of large numbers and the tail behaviour for integrable Lévy processes. For completeness, we give an alternative proof which is not based on the random walk approximation.

Proposition 2. *Let $X = (X_t)_{t\geq 0}$ be real-valued Lévy process satisfying Assumption (I). Then, $\frac{X_t}{t} \overset{a.s.}{\to} \delta$, as $t \to +\infty$.*

Proof. Using the Lévy-Itô decomposition (2), we obtain

$$\frac{X_t}{t} = \delta + \sigma\frac{W_t}{t} + \frac{1}{t}\int_0^t \int_{\mathbb{R}} x\left(\mu^X - \nu^X\right)(ds, dx), \text{ for all } t > 0.$$

But, $\frac{W_t}{t} \overset{a.s.}{\to} 0$. Now let $M \triangleq \int_0^{\cdot} \int_{\mathbb{R}} x \left(\mu^X - \nu^X\right)(ds, dx)$. We will show that $\frac{M_t}{t} \overset{a.s.}{\to} 0$. Note that

$$\begin{aligned} M_t &= M_t^{(1)} + M_t^{(2)} + M_t^{(3)} \\ &\triangleq \int_0^t \int_{|x|<1} x \left(\mu^X - \nu^X\right)(ds, dx) + \int_0^t \int_{|x|\geq 1} x\mu^X(ds, dx) \\ &- \int_0^t \int_{|x|\geq 1} x ds \Pi(dx). \end{aligned}$$

Let's prove first that $\frac{M_t^{(1)}}{t} \overset{a.s.}{\to} 0$. By Theorem 9, p.142 in [9] it is enough to show that $\tilde{B}_\infty < +\infty$ a.s., where $\tilde{B}$ is the compensator of of the process $(B_t)_{t\geq 0}$ defined by

$$B_t = \sum_{0\leq s<t} \frac{(\Delta M_s^{(1)}/(1+s))^2}{1 + |\Delta M_s^{(1)}/(1+s)|}, \text{ for all } t \geq 0,$$

where $\Delta M_s^{(1)}$ is the jump of $M^{(1)}$ at $s \geq 0$. But, by Theorem 1, p.176 in [9], and using the fact that $\nu^X(\{s\}, dx) = \lambda(\{s\})\Pi(dx) = 0$, because λ is the Lebesgue measure, we obtain $\Delta M_s^{(1)} = \Delta X_s 1_{\{|\Delta X_s|<1\}}$. Next, note that

$$B_t = \sum_{0\leq s<t} \frac{(\Delta X_s)^2 1_{\{|\Delta X_s|<1\}}/(1+s)}{1 + s + |\Delta X_s 1_{\{|\Delta X_s|<1\}}|} = \int_0^t \int_{\mathbb{R}} \frac{x^2 1_{\{|x|<1\}}/(1+s)}{1 + s + |x 1_{\{|x|<1\}}|} \mu^X(ds, dx).$$

Therefore, $\tilde{B}$ satisfies

$$\begin{aligned} \tilde{B}_t = \int_0^t \int_{\mathbb{R}} \frac{x^2 1_{\{|x|<1\}}/(1+s)}{1 + s + |x 1_{\{|x|<1\}}|} ds\Pi(dx) &\leq \left(\int_0^t \frac{1}{(1+s)^2} ds\right) \left(\int_{|x|<1} x^2 \Pi(dx)\right) \\ &\leq \left(\int_0^\infty \frac{1}{(1+s)^2} ds\right) \left(\int_{|x|<1} x^2 \Pi(dx)\right) = \int_{|x|<1} x^2 \Pi(dx) < +\infty \end{aligned}$$

for all $t \geq 0$, where the last integral is finite because Π is a Lévy measure. So, $\tilde{B}_\infty < +\infty$ a.s. and, if $\Pi(|x| \geq 1) = 0$, we are finished. Therefore, without loss of generality, we suppose that $\Pi(|x| \geq 1) > 0$. Note that $\frac{M_t^{(3)}}{t} = -\int_{|x|\geq 1} x\Pi(dx)$, for all $t \geq 0$, so we need to show that $\frac{M_t^{(2)}}{t} \overset{a.s.}{\to} \int_{|x|\geq 1} x\Pi(dx)$.

It is well known that the jump measure μ^X is a Poisson random measure with intensity $\lambda \times \Pi$, where λ is the Lebesgue measure. Then, by Lemma 2.8, p.46-47 in [8], $M^{(2)}$ can be represented as a compound Poisson process with rate $\Pi(|x| \geq 1)$ and jump distribution $\Pi(|x| \geq 1)^{-1}\Pi(dx)|_{\{|x|\geq 1\}}$ (where $\Pi(dx)|_{\{|x|\geq 1\}}$ is the restriction of the measure Π to the set $\{|x| \geq 1\}$). More

precisely,

$$M_t^{(2)} = \sum_{i=1}^{N_t} Y_i, \text{ for all } t \geq 0,$$

where $(N_t)_{t\geq 0}$ is a Poisson process with rate $\Pi(|x| \geq 1)$ and $(Y_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables, which is independent from N and with distribution $\Pi(|x| \geq 1)^{-1}\Pi(dx)|_{\{|x|\geq 1\}}$. Conditioning on N_t, using the strong law of large numbers and noting that $N_t \overset{a.s.}{\to} +\infty$, we obtain

$$\frac{M_t^{(2)}}{N_t} = \frac{1}{N_t}\sum_{i=1}^{N_t} Y_i \overset{a.s.}{\to} \mathbf{E}(Y_1) = \Pi(|x| \geq 1)^{-1} \int_{|x|\geq 1} x\Pi(dx).$$

Finally, using the fact that $\frac{N_t}{t} \overset{a.s.}{\to} \Pi(|x| \geq 1)$, we obtain

$$\frac{M_t^{(2)}}{t} = \frac{N_t}{t}\frac{M_t^{(2)}}{N_t} \overset{a.s.}{\to} \int_{|x|\geq 1} x\Pi(dx).$$

□

Corollary 1. *Let $X = (X_t)_{t\geq 0}$ be a (non-zero) real-valued Lévy process satisfying Assumption (I).*

(1) If $\delta > 0$, then $\lim_{t\to+\infty} X_t \overset{a.s}{=} +\infty$.
(2) If $\delta < 0$, then $\lim_{t\to+\infty} X_t \overset{a.s}{=} -\infty$.
(3) If $\delta = 0$, then $\liminf_{t\to+\infty} X_t \overset{a.s.}{=} -\infty$ and $\limsup_{t\to+\infty} X_t \overset{a.s.}{=} +\infty$.

Proof. The assertions 1 and 2 follow directly from Proposition 2. For assertion 3, note that the condition $\delta = \mathbf{E}(X_1) = 0$ implies, by Theorem 36.7, p.248 in [10], that X is recurrent. This means that we have neither $\lim_{t\to+\infty} X_t \overset{a.s}{=} +\infty$, nor $\lim_{t\to+\infty} X_t \overset{a.s}{=} -\infty$. Therefore, by Proposition 37.10, p.255 in [10], $\liminf_{t\to+\infty} X_t \overset{a.s.}{=} -\infty$ and $\limsup_{t\to+\infty} X_t \overset{a.s.}{=} +\infty$.

□

Next, the following proposition gives the basic properties of the Laplace exponent (see Lemma 26.4, p.169 in [10]).

Proposition 3. *Let $X = (X_t)_{t\geq 0}$ be a (non-zero) real-valued Lévy process satisfying Assumptions (I) and (II) and $\Psi : [0, \gamma_c) \to \mathbb{R}$ the Laplace exponent of X. Then,*

(1) Ψ is convex and starting from 0 and

(2) Ψ *is of class* C^∞ *on* $(0, \gamma_c)$ *and its derivative* Ψ' *is non-decreasing and given by*

$$\Psi'(\gamma) = -\delta + \sigma^2\gamma + \int_{\mathbb{R}} x\left(1 - e^{-\gamma x}\right)\Pi(dx), \text{ for all } \gamma \in (0, \gamma_c). \tag{4}$$

The convexity of the Laplace exponent then implies that there are only four possible cases which are illustrated in Figure 2 and reflect the possible cases for the behaviour of the ruin probability.

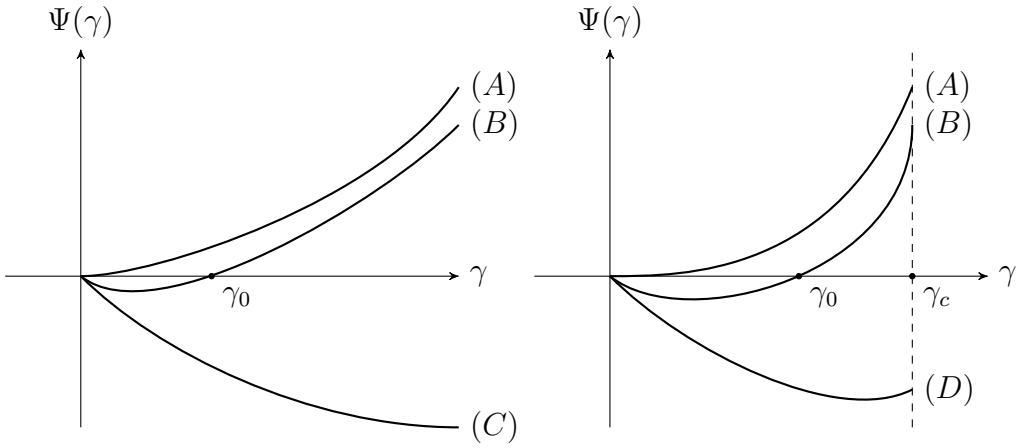

FIGURE 2. Possible behaviours of the Laplace exponent Ψ when $\gamma_c = +\infty$ (left) and $\gamma_c < +\infty$ (right).

2.2. **The Martingale Method in Ruin Theory and the Proof.** In this final section, we recall the martingale method in ruin theory and apply it to prove Theorem 1. For the proof of the following well-known martingale method see e.g. Proposition II.3.1, p.29 in [1].

Proposition 4. *Let* $X = (X_t)_{t\geq 0}$ *be a real-valued Lévy process. Suppose that*

(i) there exists $\gamma_0 > 0$*, such that* $(e^{-\gamma_0 X_t})_{t\geq 0}$ *is a martingale,*

(ii) $X_t \overset{a.s.}{\to} +\infty$ *as* $t \to +\infty$ *on the set* $\{\tau(u) = +\infty\}$.

Then, for all $u \geq 0$, $\mathbf{P}\left(\tau(u) < +\infty\right) = C(u)e^{-\gamma_0 u} \leq e^{-\gamma_0 u}$*, where*

$$C(u) \triangleq \frac{1}{\mathbb{E}(e^{\gamma_0 \xi(u)} \mid \tau(u) < +\infty)},$$

and $\xi(u) \triangleq -u - X_{\tau(u)}$.

Remark 2. *As noted in* [1]*, p.339, it is hard to obtain an explicit expression for* $C(u)$*. However, in some cases, it is possible to compute* $C(u)$*. For example, if* X *has no negative jumps then* $C(u) = 1$*, and if the jumps are bounded or exponential, it is possible to compute the constant explicitly, see e.g. Section 6c in* [1]*. There are also asymptotic expressions for* $C(u)$ *as* $u \to +\infty$*, see e.g. Corollary XI.2.7 p.339 in* [1] *and Section 7.2. in* [8]*. As we concentrate on the rate of decay of the probability of ruin in the general case, we will set* $C(u) = 1$ *and keep in mind that more precise results can be obtained for specific models or asymptotics.*

The following proposition now gives a simple sufficient condition for (i) in Proposition 4 in terms of the Laplace exponent.

Proposition 5. *Let* $X = (X_t)_{t\geq 0}$ *be a real-valued Lévy process satisfying Assumptions (I) and (II) and* $\Psi : [0, \gamma_c) \to \mathbb{R}$ *be the Laplace exponent of* X*. Suppose there exists* $\gamma_0 \in (0, \gamma_c)$ *such that* $\Psi(\gamma_0) = 0$*. Then,* $(e^{-\gamma_0 X_t})_{t\geq 0}$ *is a martingale.*

Proof. From the definition of γ_c, we have that $\mathbf{E}(e^{-\gamma X_t}) < +\infty$ for all $t \geq 0$ and $\gamma \in [0, \gamma_c)$. Imitating the proof of Theorem II.1.2, p.23 in [1], we find that the process $(e^{-\gamma X_t} - e^{t\Psi(\gamma)})_{t\geq 0}$ is a martingale for each $\gamma \in [0, \gamma_c)$. In particular, if there exists $\gamma_0 > 0$ such that $\Psi(\gamma_0) = 0$, then $(e^{-\gamma_0 X_t})_{t\geq 0}$ is a martingale. □

Putting everything together, we can now prove the main theorem. Note that case (B) can also be deduced with some work from Proposition XI.2.3 and Theorem XI.2.6 p.337-338 in [1] and that case (A) is generally implicitly excluded by the *safety loading requirement* $\delta > 0$.

Proof of Theorem 1. Note that from (4) we obtain

$$\lim_{\gamma\to 0+} \Psi'(\gamma) = \lim_{\gamma\to 0+} \left(-\delta + \sigma^2\gamma + \int_{\mathbb{R}} x \left(1 - e^{-\gamma x}\right) \Pi(dx) \right) = -\delta.$$

Therefore, from the study of the function Ψ, we see that $\delta \leq 0$ in case (A), and $\delta > 0$ in cases (B), (C) and (D).

Case (A). Let $u \geq 0$. In case (A), we have $\delta \leq 0$. Suppose first that $\delta < 0$, then, by Corollary 1, $X_t \overset{a.s.}{\to} -\infty$ as $t \to +\infty$. This immediately implies that

$$\mathbf{P}\left(\inf_{t\geq 0} X_t \leq -u\right) \geq \mathbf{P}\left(\inf_{t\geq 0} X_t = -\infty\right) = 1.$$

If $\delta = 0$, then by Corollary 1, $\mathbf{P}\left(\liminf_{t\to+\infty} X_t \leq -u\right) = 1$. As $\left(\{\inf_{t\geq n} X_t \leq -u\}\right)_{n\in\mathbb{N}}$ is a decreasing sequence of events, $\mathbf{P}\left(\inf_{t\geq m} X_t \leq -u\right) \leq \mathbf{P}\left(\inf_{t\geq 0} X_t \leq -u\right)$, for each $m \in \mathbb{N}$ and

$$\begin{aligned}\mathbf{P}\left(\inf_{t\geq 0} X_t \leq -u\right) &\geq \lim_{m\to\infty} \mathbf{P}\left(\inf_{t\geq m} X_t \leq -u\right) = \lim_{m\to\infty} \mathbf{P}\left(\bigcap_{n=0}^{m}\left\{\inf_{t\geq n} X_t \leq -u\right\}\right)\\ &= \mathbf{P}\left(\bigcap_{n\in\mathbb{N}}\left\{\inf_{t\geq n} X_t \leq -u\right\}\right) = \mathbf{P}\left(\liminf_{t\to\infty} X_t \leq -u\right) = 1.\end{aligned}$$

Case (B). We will show that (i) and (ii) of Proposition 4 hold. Because (B) holds, by Proposition 5, (i) is satisfied. Now note that in case (B) we have $\delta > 0$ and, by Corollary 1, that $X_t \overset{a.s.}{\to} +\infty$, as $t \to +\infty$. So (ii) is also satisfied.

Case (C). Because (C) holds, we have $\Psi(\gamma) < 0$, $\lim_{\gamma\to 0+} \Psi'(\gamma) = -\delta < 0$. We also have $\Psi'(\gamma) < 0$, for all $\gamma > 0$. But, from (4), we see that $\Psi'(\gamma) \leq 0$, for all $\gamma > 0$, if, and only if,

$$\sigma^2\gamma + \int_{\mathbb{R}} x\left(1 - e^{-\gamma x}\right)\Pi(dx) \leq \delta, \text{ for all } \gamma > 0.$$

If $\sigma^2 > 0$, the limit of the left-hand side when $\gamma \to +\infty$ goes to $+\infty$, so this immediately implies that $\sigma^2 = 0$. Now let $I \triangleq \int_{\mathbb{R}} x\left(1 - e^{-\gamma x}\right)\Pi(dx)$, and note that

$$I = \int_{\mathbb{R}_-} x\left(1 - e^{-\gamma x}\right)\Pi(dx) + \int_0^1 x\left(1 - e^{-\gamma x}\right)\Pi(dx) + \int_1^{+\infty} x\left(1 - e^{-\gamma x}\right)\Pi(dx).$$

Note that $x(1 - e^{-\gamma x}) \leq x$, for all $x \geq 1$ and $\gamma > 0$. So, taking the limit as $\gamma \to +\infty$ and using the dominated convergence theorem on the integral over $(1, +\infty)$ with Assumption (I), we obtain

$$\lim_{\gamma\to+\infty} \int_{-\infty}^1 x\left(1 - e^{-\gamma x}\right)\Pi(dx) \leq \delta - \int_1^{+\infty} x\Pi(dx).$$

But, $x(1-e^{-\gamma x}) \geq \gamma x^2$, for all $x < 0$ and $\gamma > 0$. The above inequality, therefore implies

$$\lim_{\gamma \to +\infty} \left(\gamma \int_{\mathbb{R}_-} x^2 \Pi(dx) + \int_0^1 x \left(1 - e^{-\gamma x}\right) \Pi(dx) \right) < +\infty,$$

which implies that $\int_{\mathbb{R}_-} x^2 \Pi(dx) = 0$. Now note that the function $x \mapsto x^2$ is strictly positive on $\mathbb{R}_-$ except in 0. But, by definition of the Lévy measure $\Pi(\{0\}) = 0$, so $x \mapsto x^2$ is strictly positive Π-a.e. So, $\int_{\mathbb{R}_-} x^2 \Pi(dx) = 0$ if, and only if, $\Pi(\mathbb{R}_-) = 0$.

Case (D). Let $u \geq 0$. Fix $\epsilon \in (0, \gamma_c)$ and define

$$Z_t^\epsilon = \frac{\Psi(\gamma_c - \epsilon)}{\gamma_c - \epsilon} t + X_t, \text{ for all } t \geq 0.$$

Then, because (D) holds $\Psi(\gamma_c - \epsilon) < 0$, so that $Z_t^\epsilon \leq X_t$, for all $t \geq 0$, and

$$\mathbf{P}\left(\inf_{0 \leq t < +\infty} X_t \leq -u \right) \leq \mathbf{P}\left(\inf_{0 \leq t < +\infty} Z_t^\epsilon \leq -u \right).$$

Note that the Laplace exponent Ψ^ϵ of Z^ϵ is defined for $\gamma \in [0, \gamma_c)$ and given by

$$\begin{aligned} \Psi^\epsilon(\gamma) &= -\left(\frac{\Psi(\gamma_c - \epsilon)}{\gamma_c - \epsilon} + \delta \right) \gamma + \frac{\sigma^2}{2} \gamma^2 + \int_{\mathbb{R}} \left(e^{-\gamma x} - 1 + \gamma x \right) \Pi(x) \\ &= -\frac{\Psi(\gamma_c - \epsilon)}{\gamma_c - \epsilon} \gamma + \Psi(\gamma). \end{aligned} \tag{5}$$

Now, we will show that Z^ϵ satisfies (i) and (ii) of Proposition 4. Condition (i) is satisfied for $\gamma_0 = \gamma_c - \epsilon$, because $\Psi^\epsilon(\gamma_c - \epsilon) = 0$. For condition (ii), note that because Ψ^ϵ has a root and is convex, we have $\lim_{\gamma \to 0+} (\Psi^\epsilon)'(\gamma) < 0$. Thus, by Corollary 1, we obtain that $Z_t^\epsilon \overset{a.s.}{\to} +\infty$, so that (ii) is also satisfied. Therefore, we obtain

$$\mathbf{P}\left(\tau(u) < +\infty\right) \leq e^{-(\gamma_c - \epsilon)u}.$$

As this is true for each $\epsilon \in (0, \gamma_c)$, we can let $\epsilon \to 0+$ to finish the proof. □

References

[1] S. Asmussen and H. Albrecher. *Ruin Probabilities*. World Scientific, 2010.

[2] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[3] J. Bertoin and R.A. Doney. Cramér's estimate for Lévy processes. *Statistics & Probability Letters*, 21(5):363 – 365, 1994.

[4] H. Cramér. Sur un nouveau théorème-limite de la théorie des probabilités. *Actualits Scientifiques et Industrielles*, 736:5 23, 1938.

[5] H.U. Gerber. An extension of the renewal equation and its application in the collective theory of risk. *Scandinavian Actuarial Journal*, 1970(3-4):205 – 210, 1970.

[6] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer-Verlag Berlin Heidelberg, second edition, 2003.

[7] C. Klüppelberg, A.E. Kyprianou, and R.A. Maller. Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Probab.*, 14(4):1766–1801, 2004.

[8] A. Kyprianou. *Fluctuations of Lévy Processes with Applications*. Springer-Verlag Berlin Heidelberg, 2014.

[9] R.Sh. Liptser and A.N. Shiryayev. *Theory of martingales*, volume 49 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by K. Dzjaparidze [Kacha Dzhaparidze].

[10] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.