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CLASSIFICATION OF THE BOUNDS ON THE PROBABILITY OF RUIN FOR LÉVY PROCESSES WITH LIGHT-TAILED JUMPS

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ABSTRACT. In this note, we study the ultimate ruin probabilities of a real-valued Lévy process X with light-tailed negative jumps. It is well-known that, for such Lévy processes, the probability of ruin decreases as an exponential function with a rate given by the root of the Laplace exponent, when the initial value goes to infinity. Under the additional assumption that X has integrable positive jumps, we show how a finer analysis of the Laplace exponent gives in fact a complete description of the bounds on the probability of ruin for this class of Lévy processes. This leads to the identification of a case that is not considered in the literature and for which we give an example. We then apply the result to various risk models and in particular the Cramér-Lundberg model perturbed by Brownian motion.

Keywords: Bounds, Laplace exponent, Lévy processes, Lundberg equation, Perturbed model, Ruin probabilities.

MSC 2010 subject classifications: 60G51, 91B30

1. INTRODUCTION

Ruin theory studies the time of passage below 0 of stochastic processes that represent the capital of an insurance company or a pension fund. In particular, it studies the probability that the process becomes negative on an infinite time horizon in function of the initial value of the process. The key result of Cramér [5] is that, in the case of the compound Poisson process with drift, this probability decreases as an exponential function with a rate given as a solution to the *Lundberg equation*. It is well-known that, when the initial value goes to infinity, the result of Cramér holds

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for more general (light-tailed) Lévy processes where the rate is given by the root of the Laplace exponent of the process, see Theorem XI.2.6 in [1], and also [3], [8] and Section 7.2 in [9].

In this note, we show that a finer analysis of the Laplace exponent can lead to a complete description of the bounds on the ultimate probability of ruin. The fact that the proof of the asymptotic result also gives exponential bounds (and sometimes exact probabilities) is well-known. Our contribution is to give a systematic description of all possible cases (Theorem 1), when the Laplace exponent does not necessarily have a root. This leads to the identification of a case that is not treated in the literature (Theorem 1, Case D). We show that in this case the ruin probability also decreases at least as an exponential function and show that this case is indeed possible (Example 3). Thus, Theorem 1 gives a method for obtaining exponential bounds and conditions for ruin with probability one for a large class of risk models. We illustrate this by applying the method to the Cramér-Lundberg model (Example 2), the model with positive risk sums (Example 1) and the Cramér-Lundberg model perturbed by Brownian motion (Proposition 5).

We now give some classical models for the capital of an insurance company or a pension fund. Let $u \geq 0$ be the initial capital and Y^u the process representing the capital or risk process. Classically, it is assumed that the income of the company is growing at a constant rate and that the random expenses happen at random times (see e.g. Section 4 in [1]). The model for the capital is then given by

$$(1) \quad Y_t^u = u + pt - \sum_{n=1}^{N_t} U_n, \text{ for all } t \geq 0,$$

where $p > 0$, $N = (N_t)_{t \geq 0}$ is a standard Poisson process with rate β and $(U_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. non-negative random variables, which are independent from N and with common distribution F . When F is a light-tailed distribution (i.e., there exists $\gamma_0 > 0$ such that $\int_{\mathbb{R}_+} e^{\gamma_0 x} F(dx) < +\infty$), then Theorem 1 allows to classify the ruin probabilities (see Example 2, for the Cramér-Lundberg case).

Another model, sometimes called *perturbed risk process* and studied first in [6] is the following,

$$(2) \quad Y_t^u = u + pt + \sigma W_t - \sum_{n=1}^{N_t} U_n, \text{ for all } t \geq 0,$$

where $p, \sigma > 0$, $N = (N_t)_{t \geq 0}$ and $(U_n)_{n \in \mathbb{N}}$ are defined as above and $(W_t)_{t \geq 0}$ is a Brownian motion that is independent of N and $(U_n)_{n \in \mathbb{N}}$. Again, if F is a light-tailed

distribution, Theorem 1 allows to classify the ruin probabilities (see Proposition 4, for the exponential case).

The last model we mention is sometimes called *risk process with positive risk sums* and is used to model the capital of a pension fund. Here it is assumed that the fund is paying a constant annuity p and receives random sums (in the form of property or accumulated premiums) at the time of death of the insured individuals. This model is given by

$$(3) \quad Y_t^u = u - pt + \sum_{n=1}^{N_t} U_n, \text{ for all } t \geq 0,$$

where $p > 0$, $N = (N_t)_{t \geq 0}$ and $(U_n)_{n \in \mathbb{N}}$ are defined as above. In this model, if F is integrable (i.e., $\int_{\mathbb{R}_+} xF(dx) < +\infty$), Theorem 1 allows to classify the ruin probabilities (see Example 1 for the exponential case).

1.1. Lévy Processes and Laplace Exponents. In this section we state some basic facts about Lévy processes and present our assumptions for the rest of this paper.

Definition 1 ([7], Definition II.4.1, p.101).

- (i) A process with independent increments (abbreviated PII) $X = (X_t)_{t \geq 0}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ is a càdlàg adapted real-valued process with $X_0 = 0$ and for all $0 \leq s \leq t < +\infty$, $X_t - X_s$ is independent of \mathcal{F}_s .
- (ii) A Lévy process (also called process with independent and stationary increments) on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a PII X such that the distribution of the increment $X_t - X_s$ depends only on $t - s$, for all $0 \leq s \leq t$.

Let $X = (X_t)_{t \geq 0}$ be a real-valued Lévy process on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ where the filtration \mathbb{F} is assumed to satisfy the usual conditions. It is well-known that the characteristic function of X_t for each $t \geq 0$ is given by the Lévy-Khintchine formula:

$$\mathbf{E} \left(e^{i\lambda X_t} \right) = e^{t\Phi(\lambda)}, \text{ for all } t \geq 0 \text{ and } \lambda \in \mathbb{R},$$

where

$$\Phi(\lambda) = ia\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x| < 1\}} \right) \Pi(dx), \text{ for all } \lambda \in \mathbb{R},$$

for $a \in \mathbb{R}$, $\sigma \geq 0$ and Π a Lévy measure on \mathbb{R} satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1) \Pi(dx) < +\infty.$$

The function Φ and the triplet (a, σ^2, Π) are unique and are called the Lévy exponent and the characteristics (or Lévy triplet) of X respectively.

Assumption (I). X is integrable.

The first assumption we use is integrability. We say that X is integrable if $\mathbf{E}(|X_1|) < +\infty$ and it can be shown (see e.g. [11], Theorem 25.3, p.159) that this is equivalent to the condition

$$\int_{|x| \geq 1} |x| \Pi(dx) < +\infty.$$

Under assumption (I), we can rewrite the Lévy exponent of X as

$$(4) \quad \Phi(\lambda) = i\delta\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x \right) \Pi(dx), \text{ for all } \lambda \in \mathbb{R},$$

where

$$\delta \triangleq \mathbf{E}(X_1) = a + \int_{|x| \geq 1} x \Pi(dx).$$

Also, under assumption (I), the Lévy-Itô decomposition of X is

$$(5) \quad X_t = \delta t + \sigma W_t + \int_0^t \int_{\mathbb{R}} x (\mu^X - \nu^X) (ds, dx), \text{ for all } t \geq 0,$$

where μ^X is the jump measure of X , $\nu^X(ds, dx) = ds\Pi(dx)$ is the compensator of the jump measure (see [7], Theorem I.1.8, p.66) and $(W_t)_{t \geq 0}$ is a standard Brownian motion. The following well-known proposition and its corollary (see Proposition IV.1.2, p.73 in [1] in the case of the Cramér-Lundberg model, discussion p.75 and Proposition 8 p.84 in [2], Exercice 7.3 in [9], and Section 36 starting at p.245 in [11] in the general case) give a strong law of large numbers and the tail behaviour for integrable Lévy processes. For completeness, we give an alternative proof that is not based on random walk theory.

Proposition 1. *Let $X = (X_t)_{t \geq 0}$ be real-valued Lévy process satisfying Assumption (I). Then, $\frac{X_t}{t} \xrightarrow{a.s.} \delta$, as $t \rightarrow +\infty$.*

Proof. Using the Lévy-Itô decomposition (5), we obtain

$$\frac{X_t}{t} = \delta + \sigma \frac{W_t}{t} + \frac{1}{t} \int_0^t \int_{\mathbb{R}} x (\mu^X - \nu^X) (ds, dx), \text{ for all } t > 0.$$

But, $\frac{W_t}{t} \xrightarrow{a.s.} 0$. Now let $M \triangleq \int_0 \int_{\mathbb{R}} x (\mu^X - \nu^X) (ds, dx)$. We will show that $\frac{M_t}{t} \xrightarrow{a.s.} 0$. Note that

$$\begin{aligned} M_t &= M_t^{(1)} + M_t^{(2)} + M_t^{(3)} \\ &\triangleq \int_0^t \int_{|x| < 1} x (\mu^X - \nu^X) (ds, dx) + \int_0^t \int_{|x| \geq 1} x \mu^X (ds, dx) \\ &\quad - \int_0^t \int_{|x| \geq 1} x ds \Pi(dx). \end{aligned}$$

Let's prove first that $\frac{M_t^{(1)}}{t} \xrightarrow{a.s.} 0$. By Theorem 9, p.142 in [10] it is enough to show that $\tilde{B}_\infty < +\infty$ a.s., where \tilde{B} is the compensator of of the process $(B_t)_{t \geq 0}$ defined by

$$B_t = \sum_{0 \leq s < t} \frac{(\Delta M_s^{(1)} / (1+s))^2}{1 + |\Delta M_s^{(1)} / (1+s)|}, \text{ for all } t \geq 0,$$

where $\Delta M_s^{(1)}$ is the jump of $M^{(1)}$ at $s \geq 0$. But, by Theorem 1, p.176 in [10], and using the fact that $\nu^X(\{s\}, dx) = \lambda(\{s\})\Pi(dx) = 0$, because λ is the Lebesgue measure, we obtain $\Delta M_s^{(1)} = \Delta X_s 1_{\{|\Delta X_s| < 1\}}$. Next, note that

$$B_t = \sum_{0 \leq s < t} \frac{(\Delta X_s)^2 1_{\{|\Delta X_s| < 1\}} / (1+s)}{1+s + |\Delta X_s 1_{\{|\Delta X_s| < 1\}}|} = \int_0^t \int_{\mathbb{R}} \frac{x^2 1_{\{|x| < 1\}} / (1+s)}{1+s + |x 1_{\{|x| < 1\}}|} \mu^X(ds, dx).$$

Therefore, \tilde{B} satisfies

$$\begin{aligned} \tilde{B}_t &= \int_0^t \int_{\mathbb{R}} \frac{x^2 1_{\{|x| < 1\}} / (1+s)}{1+s + |x 1_{\{|x| < 1\}}|} ds \Pi(dx) \leq \left(\int_0^t \frac{1}{(1+s)^2} ds \right) \left(\int_{|x| < 1} x^2 \Pi(dx) \right) \\ &\leq \left(\int_0^\infty \frac{1}{(1+s)^2} ds \right) \left(\int_{|x| < 1} x^2 \Pi(dx) \right) = \int_{|x| < 1} x^2 \Pi(dx) < +\infty \end{aligned}$$

for all $t \geq 0$, where the last integral is finite because Π is a Lévy measure. So, $\tilde{B}_\infty < +\infty$ a.s. and, if $\Pi(|x| \geq 1) = 0$, we are finished. Therefore, without loss of generality, we suppose that $\Pi(|x| \geq 1) > 0$. Note that $\frac{M_t^{(3)}}{t} = - \int_{|x| \geq 1} x \Pi(dx)$, for all $t \geq 0$, so we need to show that $\frac{M_t^{(2)}}{t} \xrightarrow{a.s.} \int_{|x| \geq 1} x \Pi(dx)$.

It is well known that the jump measure μ^X is a Poisson random measure with intensity $\lambda \times \Pi$, where λ is the Lebesgue measure. Then, by Lemma 2.8, p.46-47 in [9], $M^{(2)}$ can be represented as a compound Poisson process with rate $\Pi(|x| \geq 1)$ and jump distribution $\Pi(|x| \geq 1)^{-1} \Pi(dx)|_{\{|x| \geq 1\}}$ (where $\Pi(dx)|_{\{|x| \geq 1\}}$ is the restriction

of the measure Π to the set $\{|x| \geq 1\}$. More precisely,

$$M_t^{(2)} = \sum_{i=1}^{N_t} Y_i, \text{ for all } t \geq 0,$$

where $(N_t)_{t \geq 0}$ is a Poisson process with rate $\Pi(|x| \geq 1)$ and $(Y_i)_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables, which is independent from N and with distribution $\Pi(|x| \geq 1)^{-1} \Pi(dx)|_{\{|x| \geq 1\}}$. Conditioning on N_t , using the strong law of large numbers and noting that $N_t \xrightarrow{a.s.} +\infty$, we obtain

$$\frac{M_t^{(2)}}{N_t} = \frac{1}{N_t} \sum_{i=1}^{N_t} Y_i \xrightarrow{a.s.} \mathbf{E}(Y_1) = \Pi(|x| \geq 1)^{-1} \int_{|x| \geq 1} x \Pi(dx).$$

Finally, using the fact that $\frac{N_t}{t} \xrightarrow{a.s.} \Pi(|x| \geq 1)$, we obtain

$$\frac{M_t^{(2)}}{t} = \frac{N_t}{t} \frac{M_t^{(2)}}{N_t} \xrightarrow{a.s.} \int_{|x| \geq 1} x \Pi(dx).$$

□

Corollary 1. *Let $X = (X_t)_{t \geq 0}$ be a (non-zero) real-valued Lévy process satisfying Assumption (I).*

- (1) *If $\delta > 0$, then $\lim_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} +\infty$.*
- (2) *If $\delta < 0$, then $\lim_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} -\infty$.*
- (3) *If $\delta = 0$, then $\liminf_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} -\infty$ and $\limsup_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} +\infty$.*

Proof. The assertions 1 and 2 follow directly from Proposition 1. For assertion 3, note that the condition $\delta = \mathbf{E}(X_1) = 0$ implies, by Theorem 36.7, p.248 in [11], that X is recurrent. This means that we have neither $\lim_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} +\infty$, nor $\lim_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} -\infty$. Therefore, by Proposition 37.10, p.255 in [11], $\liminf_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} -\infty$ and $\limsup_{t \rightarrow +\infty} X_t \stackrel{a.s.}{=} +\infty$.

□

Assumption (II). *X has light-tailed negative jumps.*

The second assumption we will use is a condition on the tail behaviour of the negative jumps. Similar definitions to the one below can be found on p.338 in [1] and p.164-165 in [11].

Definition 2. *Let $X = (X_t)_{t \geq 0}$ be a real-valued Lévy process with characteristics (a, σ^2, Π) . Let*

$$\gamma_c \triangleq \sup \left\{ \gamma \geq 0 : \int_{-\infty}^{-1} e^{-\gamma x} \Pi(dx) < +\infty \right\}.$$

We say that X has light-tailed negative jumps if $\gamma_c > 0$. (Note that γ_c can take the value $+\infty$.)

Under Assumptions (I) and (II), it is possible to show that the Lévy exponent exists also for any $\lambda = i\gamma$, with $\gamma \in [0, \gamma_c)$. In fact, when $\gamma \in [0, \gamma_c)$,

$$\Phi(i\gamma) = -\delta\gamma + \frac{\sigma^2}{2}\gamma^2 + \int_{\mathbb{R}} (e^{-\gamma x} - 1 + \gamma x) \Pi(dx),$$

and letting $I_- \triangleq \int_{\mathbb{R}_-} |e^{-\gamma x} - 1 + \gamma x| \Pi(dx)$, we obtain using the Taylor formula,

$$\begin{aligned} I_- &\leq \int_{-1}^0 |e^{-\gamma x} - 1 + \gamma x| \Pi(dx) + \int_{-\infty}^{-1} e^{-\gamma x} \Pi(dx) \\ &\leq \frac{\gamma^2}{2} \int_{-1}^0 x^2 \Pi(dx) + \int_{-\infty}^{-1} e^{-\gamma x} \Pi(dx) < +\infty. \end{aligned}$$

On the other hand, letting $I_+ \triangleq \int_{\mathbb{R}_+} |e^{-\gamma x} - 1 + \gamma x| \Pi(dx)$ and using the Taylor formula and the assumption of integrability,

$$\begin{aligned} I_+ &= \int_0^1 |e^{-\gamma x} - 1 + \gamma x| \Pi(dx) + \int_1^{\infty} |e^{-\gamma x} - 1 + \gamma x| \Pi(dx) \\ &\leq \frac{\gamma^2}{2} \int_0^1 x^2 \Pi(dx) + \Pi([1, +\infty)) + \gamma \int_1^{\infty} |x| \Pi(dx) < +\infty. \end{aligned}$$

Therefore, it is possible to define the Laplace exponent of X as the function Ψ given by

$$\Psi(\gamma) \triangleq \Phi(i\gamma) = -\delta\gamma + \frac{\sigma^2}{2}\gamma^2 + \int_{\mathbb{R}} (e^{-\gamma x} - 1 + \gamma x) \Pi(dx), \text{ for all } \gamma \in [0, \gamma_c).$$

Remark 1. *The Laplace exponent is always defined on \mathbb{R}_- and can, under Assumptions (I) and (II), be defined on $(-\infty, \gamma_c)$.*

From the Lévy-Khintchine formula, we see that the Laplace transform of X_t is then given by

$$\mathbf{E}(e^{-\gamma X_t}) = e^{t\Psi(\gamma)}, \text{ for all } t \geq 0 \text{ and } \gamma \in (-\infty, \gamma_c).$$

The following well-known proposition gives the basic properties of the Laplace exponent (see also Lemma 26.4, p.169 in [11]).

Proposition 2. *Let $X = (X_t)_{t \geq 0}$ be a (non-zero) real-valued Lévy process satisfying Assumptions (I) and (II) and $\Psi : [0, \gamma_c) \rightarrow \mathbb{R}$ the Laplace exponent of X . Then,*

- (1) Ψ is convex and starting from 0 and

(2) Ψ is of class C^∞ on $(0, \gamma_c)$ and its derivative Ψ' is non-decreasing and given by

$$(6) \quad \Psi'(\gamma) = -\delta + \sigma^2\gamma + \int_{\mathbb{R}} x(1 - e^{-\gamma x}) \Pi(dx), \text{ for all } \gamma \in (0, \gamma_c).$$

2. RUIN PROBABILITIES

We suppose that $X = (X_t)_{t \geq 0}$ is a real-valued Lévy process satisfying assumptions (I) and (II). Let $Y_t^u \triangleq u + X_t$, for $t \geq 0$ and $u \geq 0$. We define the ultimate ruin probabilities of Y^u as

$$\mathbf{P} \left(\inf_{0 \leq t < +\infty} Y_t^u \leq 0 \right) = \mathbf{P} \left(\inf_{0 \leq t < +\infty} X_t \leq -u \right) = \mathbf{P} \left(\sup_{0 \leq t < +\infty} (-X_t) \geq u \right).$$

This can also be written as $\mathbf{P}(\tau(u) < +\infty)$ where $\tau(u) \triangleq \inf\{t \geq 0 : X_t \leq -u\}$ and $\tau(u) \triangleq +\infty$, if X never goes below $-u$.

2.1. The Martingale Method for Exponential Bounds. The following proposition is a well-known result in ruin theory (see e.g. Proposition II.3.1, p.29 in [1] for the proof).

Proposition 3. *Let $X = (X_t)_{t \geq 0}$ be a real-valued Lévy process. Suppose that*

- (i) *there exists $\gamma_0 > 0$, such that $(e^{-\gamma_0 X_t})_{t \geq 0}$ is a martingale,*
- (ii) *$X_t \xrightarrow{a.s.} +\infty$ as $t \rightarrow +\infty$ on the set $\{\tau(u) < +\infty\}$.*

Then, for all $u \geq 0$, $\mathbf{P}(\tau(u) < +\infty) = C(u)e^{-\gamma_0 u} \leq e^{-\gamma_0 u}$, where

$$C(u) \triangleq \frac{1}{\mathbb{E}(e^{\gamma_0 \xi(u)} \mid \tau(u) < +\infty)},$$

and $\xi(u) \triangleq -u - X_{\tau(u)}$.

Remark 2. *As noted in [1], p.339, it is hard to obtain an explicit expression for $C(u)$. However, in some cases, it is possible to compute $C(u)$. For example, if X has no negative jumps then $C(u) = 1$, and if the jumps are bounded or exponential, it is possible to compute the constant explicitly, see e.g. Section 6c in [1]. There are also asymptotic expressions for $C(u)$ as $u \rightarrow +\infty$, see e.g. Corollary XI.2.7 p.339 in [1] and Section 7.2. in [9]. As we concentrate on the rate of decay of the probability of ruin in the general case, we will set $C(u) = 1$ and keep in mind that more precise results can be obtained for specific models or asymptotics.*

Proposition 4. *Let $X = (X_t)_{t \geq 0}$ be a real-valued Lévy process satisfying Assumptions (I) and (II) and $\Psi : [0, \gamma_c) \rightarrow \mathbb{R}$ be the Laplace exponent of X . Suppose there exists $\gamma_0 \in (0, \gamma_c)$ such that $\Psi(\gamma_0) = 0$. Then, $(e^{-\gamma_0 X_t})_{t \geq 0}$ is a martingale.*

Proof. From the definition of γ_c , we have that $\mathbf{E}(e^{-\gamma X_t}) < +\infty$ for all $t \geq 0$ and $\gamma \in [0, \gamma_c)$. Imitating the proof of Theorem II.1.2, p.23 in [1], we find that the process $(e^{-\gamma X_t} - e^{t\Psi(\gamma)})_{t \geq 0}$ is a martingale for each $\gamma \in [0, \gamma_c)$. In particular, if there exists $\gamma_0 > 0$ such that $\Psi(\gamma_0) = 0$, then $(e^{-\gamma_0 X_t})_{t \geq 0}$ is a martingale. \square

2.2. Bounds on the Ruin Probabilities. In this section, we classify the bounds on the ruin probabilities with help of the properties of the Laplace exponent.

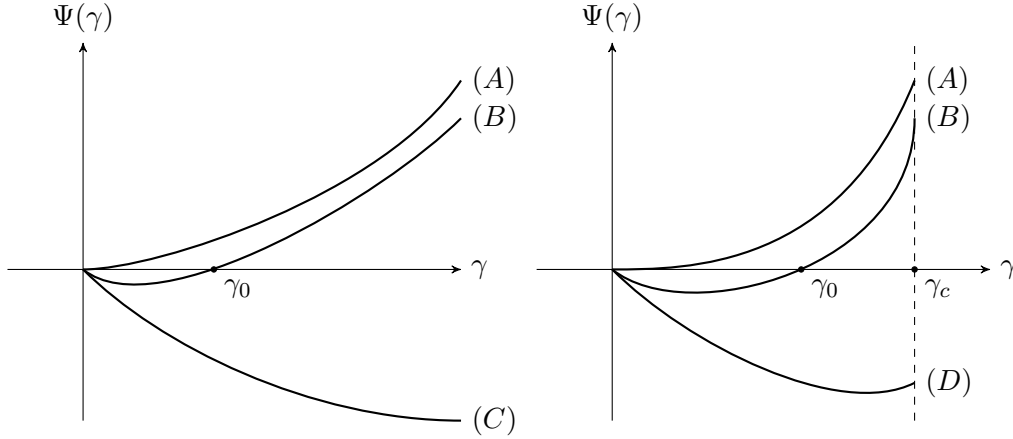


FIGURE 1. Possible behaviours of the Laplace exponent Ψ when $\gamma_c = +\infty$ (left) and $\gamma_c < +\infty$ (right).

Proposition 2 implies that there are only four possible cases which are summarized in the following theorem (see Figure 1). Note that case (B) is contained in Proposition XI.2.3 and Theorem XI.2.6 p.337-338 in [1] and that case (A) is generally implicitly excluded by the *safety loading requirement* $\delta > 0$.

Theorem 1. Let $X = (X_t)_{t \geq 0}$ be a (non-zero) real-valued Lévy process satisfying Assumptions (I) and (II) and $\Psi : [0, \gamma_c) \rightarrow \mathbb{R}$ be the Laplace exponent of X .

- (A) If $\Psi(\gamma) > 0$, for all $\gamma \in (0, \gamma_c)$, then $\mathbf{P}(\tau(u) < +\infty) = 1$, for all $u \geq 0$.
- (B) If there exists $\gamma_0 \in (0, \gamma_c)$ such that $\Psi(\gamma_0) = 0$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_0 u}$, for all $u \geq 0$.
- (C) If $\gamma_c = +\infty$ and $\Psi(\gamma) < 0$, for all $\gamma \in (0, +\infty)$, then $\sigma^2 = 0$, $\Pi(\mathbb{R}_-) = 0$, $\delta > 0$ and which means that X is a subordinator. Therefore, $\mathbf{P}(\tau(u) < +\infty) = 0$, for all $u \geq 0$.
- (D) If $\gamma_c < +\infty$ and $\Psi(\gamma) < 0$, for all $\gamma \in (0, \gamma_c)$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_c u}$, for all $u \geq 0$.

Proof. Note that from (6) we obtain

$$\lim_{\gamma \rightarrow 0^+} \Psi'(\gamma) = \lim_{\gamma \rightarrow 0^+} \left(-\delta + \sigma^2 \gamma + \int_{\mathbb{R}} x (1 - e^{-\gamma x}) \Pi(dx) \right) = -\delta.$$

Therefore, from the study of the function Ψ , we see that $\delta \leq 0$ in case (A), and $\delta > 0$ in cases (B), (C) and (D).

Case (A). Let $u \geq 0$. In case (A), we have $\delta \leq 0$. Suppose first that $\delta < 0$, then, by Corollary 1, $X_t \xrightarrow{a.s.} -\infty$ as $t \rightarrow +\infty$. This immediately implies that

$$\mathbf{P} \left(\inf_{t \geq 0} X_t \leq -u \right) \geq \mathbf{P} \left(\inf_{t \geq 0} X_t = -\infty \right) = 1.$$

If $\delta = 0$, then by Corollary 1, $\mathbf{P}(\liminf_{t \rightarrow +\infty} X_t \leq -u) = 1$. As $(\{\inf_{t \geq n} X_t \leq -u\})_{n \in \mathbb{N}}$ is a decreasing sequence of events, $\mathbf{P}(\inf_{t \geq m} X_t \leq -u) \leq \mathbf{P}(\inf_{t \geq 0} X_t \leq -u)$, for each $m \in \mathbb{N}$ and

$$\begin{aligned} \mathbf{P} \left(\inf_{t \geq 0} X_t \leq -u \right) &\geq \lim_{m \rightarrow \infty} \mathbf{P} \left(\inf_{t \geq m} X_t \leq -u \right) = \lim_{m \rightarrow \infty} \mathbf{P} \left(\bigcap_{n=0}^m \left\{ \inf_{t \geq n} X_t \leq -u \right\} \right) \\ &= \mathbf{P} \left(\bigcap_{n \in \mathbb{N}} \left\{ \inf_{t \geq n} X_t \leq -u \right\} \right) = \mathbf{P} \left(\liminf_{t \rightarrow \infty} X_t \leq -u \right) = 1. \end{aligned}$$

Case (B). We will show that (i) and (ii) of Proposition 3 hold. Because (B) holds, by Proposition 4, (i) is satisfied. Now note that in case (B) we have $\delta > 0$ and, by Proposition 1, that $X_t \xrightarrow{a.s.} +\infty$, as $t \rightarrow +\infty$. So (ii) is also satisfied.

Case (C). Because (C) holds, we have $\Psi(\gamma) < 0$, $\lim_{\gamma \rightarrow 0^+} \Psi'(\gamma) = -\delta < 0$. We also have $\Psi'(\gamma) < 0$, for all $\gamma > 0$. But, from (6), we see that $\Psi'(\gamma) \leq 0$, for all $\gamma > 0$, if, and only if,

$$\sigma^2 \gamma + \int_{\mathbb{R}} x (1 - e^{-\gamma x}) \Pi(dx) \leq \delta, \text{ for all } \gamma > 0.$$

If $\sigma^2 > 0$, the limit of the left-hand side when $\gamma \rightarrow +\infty$ goes to $+\infty$, so this immediately implies that $\sigma^2 = 0$. Now let $I \triangleq \int_{\mathbb{R}} x (1 - e^{-\gamma x}) \Pi(dx)$, and note that

$$I = \int_{\mathbb{R}_-} x (1 - e^{-\gamma x}) \Pi(dx) + \int_0^1 x (1 - e^{-\gamma x}) \Pi(dx) + \int_1^{+\infty} x (1 - e^{-\gamma x}) \Pi(dx).$$

Note that $x(1 - e^{-\gamma x}) \leq x$, for all $x \geq 1$ and $\gamma > 0$. So, taking the limit as $\gamma \rightarrow +\infty$ and using the dominated convergence theorem on the integral over $(1, +\infty)$ with Assumption (I), we obtain

$$\lim_{\gamma \rightarrow +\infty} \int_{-\infty}^1 x (1 - e^{-\gamma x}) \Pi(dx) \leq \delta - \int_1^{+\infty} x \Pi(dx).$$

But, $x(1 - e^{-\gamma x}) \geq \gamma x^2$, for all $x < 0$ and $\gamma > 0$. The above inequality, therefore implies

$$\lim_{\gamma \rightarrow +\infty} \left(\gamma \int_{\mathbb{R}_-} x^2 \Pi(dx) + \int_0^1 x(1 - e^{-\gamma x}) \Pi(dx) \right) < +\infty,$$

which implies that $\int_{\mathbb{R}_-} x^2 \Pi(dx) = 0$. Now note that the function $x \mapsto x^2$ is strictly positive on \mathbb{R}_- except in 0. But, by definition of the Lévy measure $\Pi(\{0\}) = 0$, so $x \mapsto x^2$ is strictly positive Π -a.e. So, $\int_{\mathbb{R}_-} x^2 \Pi(dx) = 0$ if, and only if, $\Pi(\mathbb{R}_-) = 0$.

Case (D). Let $u \geq 0$. Fix $\epsilon \in (0, \gamma_c)$ and define

$$Z_t^\epsilon = \frac{\Psi(\gamma_c - \epsilon)}{\gamma_c - \epsilon} t + X_t, \text{ for all } t \geq 0.$$

Then, because (D) holds $\Psi(\gamma_c - \epsilon) < 0$, so that $Z_t^\epsilon \leq X_t$, for all $t \geq 0$, and

$$\mathbf{P} \left(\inf_{0 \leq t < +\infty} X_t \leq -u \right) \leq \mathbf{P} \left(\inf_{0 \leq t < +\infty} Z_t^\epsilon \leq -u \right).$$

Note that the Laplace exponent Ψ^ϵ of Z^ϵ is defined for $\gamma \in [0, \gamma_c)$ and given by

$$\begin{aligned} \Psi^\epsilon(\gamma) &= - \left(\frac{\Psi(\gamma_c - \epsilon)}{\gamma_c - \epsilon} + \delta \right) \gamma + \frac{\sigma^2}{2} \gamma^2 + \int_{\mathbb{R}} (e^{-\gamma x} - 1 + \gamma x) \Pi(x) \\ (7) \quad &= - \frac{\Psi(\gamma_c - \epsilon)}{\gamma_c - \epsilon} \gamma + \Psi(\gamma). \end{aligned}$$

Now, we will show that Z^ϵ satisfies (i) and (ii) of Proposition 3. Condition (i) is satisfied for $\gamma_0 = \gamma_c - \epsilon$, because $\Psi^\epsilon(\gamma_c - \epsilon) = 0$. For condition (ii), note that because Ψ^ϵ has a root and is convex, we have $\lim_{\gamma \rightarrow 0+} (\Psi^\epsilon)'(\gamma) < 0$. Thus, by Proposition 1, we obtain that $Z_t^\epsilon \xrightarrow{a.s.} +\infty$, so that (ii) is also satisfied. Therefore, we obtain

$$\mathbf{P}(\tau(u) < +\infty) \leq e^{-(\gamma_c - \epsilon)u}.$$

As this is true for each $\epsilon \in (0, \gamma_c)$, we can let $\epsilon \rightarrow 0+$ to finish the proof. \square

2.3. Application to Risk Models. In this section, we apply our result to the risk models given in the introduction.

Example 1. Consider a real-valued Lévy process $X = (X_t)_{t \geq 0}$ with Lévy triplet $\Pi(dx) = \beta \alpha e^{-\alpha x} 1_{\{x \geq 0\}} dx$, $\sigma^2 = 0$ and $a = -p + \int_{|x| < 1} x \Pi(dx)$ for some $p, \alpha, \beta > 0$. Then, $Y_t^u = u + X_t$, with $u \geq 0$, corresponds to the model with positive risk sums given by (3) with exponential jumps with intensity $\alpha > 0$. We have $\gamma_c = +\infty$ because $\Pi(\mathbb{R}_-) = 0$. We obtain $\delta = \mathbf{E}(X_1) = -p + \frac{\beta}{\alpha}$. Then, Theorem 1 (A), tells us that we have ruin with probability one when $p \geq \frac{\beta}{\alpha}$. So, we assume that $p < \frac{\beta}{\alpha}$

(i.e., the *safety loading requirement* mentioned earlier.) For $\gamma > 0$, we have

$$\begin{aligned}\Psi(\gamma) &= -\delta\gamma + \beta\alpha \int_{\mathbb{R}_+} (e^{-\gamma x} - 1 + \gamma x)e^{-\alpha x} dx \\ &= p\gamma + \frac{\beta\alpha}{\gamma + \alpha} - \beta = \frac{p\gamma^2 + (p\alpha - \beta)\gamma}{\gamma + \alpha}.\end{aligned}$$

We see that Ψ has a root given by $\frac{\beta - p\alpha}{p} > 0$. Therefore, by Theorem 1 (B), we obtain $\mathbf{P}(\tau(u) < +\infty) = \exp\left(-\left(\frac{\beta - p\alpha}{p}\right)u\right)$ (as, in this case, $C(u) = 1$).

Example 2. Consider a real-valued Lévy process $X = (X_t)_{t \geq 0}$ with Lévy triplet $\Pi(dx) = \beta\alpha e^{\alpha x} 1_{\{x \leq 0\}} dx$, $\sigma^2 = 0$ and $a = p + \int_{|x| < 1} x \Pi(dx)$ for some $p, \alpha, \beta > 0$. Then, $Y_t^u = u + X_t$, with $u \geq 0$, corresponds to the classical Cramér-Lundberg model given by (1) with exponential jumps with intensity $\alpha > 0$. In this case, we have $\gamma_c = \alpha$. In fact, for $\gamma \in [0, \alpha)$,

$$\int_{-\infty}^{-1} e^{-\gamma x} \Pi(dx) = \beta\alpha \int_{-\infty}^{-1} e^{(\alpha - \gamma)x} dy = \beta\alpha \frac{e^{\gamma - \alpha}}{\alpha - \gamma} < +\infty,$$

and the integral diverges if $\gamma \geq \alpha$. We obtain $\delta = \mathbf{E}(X_1) = p - \frac{\beta}{\alpha}$. Then, Theorem 1 (A) tells us that we have ruin with probability one when $p \leq \frac{\beta}{\alpha}$. So, we assume that $p > \frac{\beta}{\alpha}$. For $\gamma \in (0, \alpha)$, we have

$$\begin{aligned}\Psi(\gamma) &= -\delta\gamma + \beta\alpha \int_{-\infty}^0 (e^{-\gamma x} - 1 + \gamma x)e^{\alpha x} dx \\ &= -p\gamma - \frac{\beta\alpha}{\gamma - \alpha} - \beta = \frac{-p\gamma^2 + (p\alpha - \beta)\gamma}{\gamma - \alpha}.\end{aligned}$$

We see that Ψ a root in $(0, \alpha)$ given by $\frac{p\alpha - \beta}{p}$. Therefore, by Theorem 1 (B), we obtain $\mathbf{P}(\tau(u) < +\infty) \leq \exp\left(-\left(\frac{p\alpha - \beta}{p}\right)u\right)$.

We now give an example for case (D).

Example 3. Let $\alpha > 0$, consider the Lévy process $X = (X_t)_{t \geq 0}$ with $\sigma^2 = 0$ and $\Pi(dx) = K(\alpha)x^{-2}e^{\alpha x} 1_{\{x \leq -1\}} dx$ where $K(\alpha) > 0$ is a constant such that $\int_{\mathbb{R}} \Pi(dx) = 1$ and let $Y_t^u = u + X_t$, for $u \geq 0$. In fact, Π represents the large negative jumps of a tempered 1-stable process (see e.g. [4], p.119 for the definition of these processes.) It is possible to show that $\gamma_c = \alpha$. In fact, if $\gamma < \alpha$,

$$\int_{-\infty}^{-1} K(\alpha)e^{-\gamma x} x^{-2} e^{\alpha x} dx \leq K(\alpha) \int_{-\infty}^{-1} e^{(\alpha - \gamma)x} dx = K(\alpha) \frac{e^{\alpha - \gamma}}{\gamma - \alpha} < +\infty,$$

and if $\gamma > \alpha$, $\lim_{x \rightarrow -\infty} x^{-2} e^{(\alpha - \gamma)x} = +\infty$ and the integral diverges. We can now choose $\delta > 0$, so that $\Psi(\gamma) < 0$ for all $\gamma \in (0, \alpha)$. In fact, it is enough to find $\delta > 0$

such that $\Psi(\alpha) < 0$. But,

$$(8) \quad \begin{aligned} \Psi(\alpha) &= -\delta\alpha + \int_{-\infty}^{-1} (e^{-\alpha x} - 1 + \alpha x)K(\alpha)x^{-2}e^{\alpha x} dx \\ &= -\delta\alpha + K(\alpha) - 1 - \alpha K(\alpha)\Gamma(0, \alpha), \end{aligned}$$

where $\Gamma(0, \alpha) \triangleq \int_{\alpha}^{\infty} t^{-1}e^{-t}dt$ is the upper incomplete gamma function. Therefore, $\Psi(\alpha) < -\delta\alpha + K(\alpha)$, and if we choose $\delta > \frac{K(\alpha)}{\alpha}$, we have then $\Psi(\alpha) < 0$ and, by Theorem 1 (D), we obtain $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\alpha u}$.

Finally, we obtain classification of the bounds on the ruin probabilities for the Cramér-Lundberg model perturbed by Brownian motion. Note that in contrast to the case when $\sigma^2 = 0$, there are two possible regimes when the safety loading condition is satisfied. This shows how the uncertainty in premium payments affects the ruin probability.

Proposition 5. *Let $X = (X_t)_{t \geq 0}$ be a real-valued Lévy process with Lévy triplet $\Pi(dx) = \beta\alpha e^{\alpha x} 1_{\{x \leq 0\}} dx$, $\sigma^2 > 0$ and $a = p + \int_{|x| < 1} x\Pi(dx)$ for some $p, \alpha, \beta > 0$. Then, $Y_t^u = u + X_t$, with $u \geq 0$, corresponds to the perturbed risk process given by (2) with exponential jumps with intensity $\alpha > 0$. Let $\Delta \triangleq (\sigma^2\alpha - 2p)^2 + 8\sigma^2\beta$ and $\gamma_- \triangleq \frac{\sigma^2\alpha + 2p - \sqrt{\Delta}}{2\sigma^2}$.*

- If $p \leq \frac{\beta}{\alpha}$, then $\mathbf{P}(\tau(u) < +\infty) = 1$, for all $u \geq 0$.
- If $p > \frac{\beta}{\alpha}$ and $\gamma_- < \alpha$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_- u}$.
- If $p > \frac{\beta}{\alpha}$ and $\gamma_- \geq \alpha$, then $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\alpha u}$.

Proof. Similarly to Example 2, we have $\gamma_c = \alpha$ and $\delta = p - \frac{\beta}{\alpha}$. So, by Theorem 1 (A), we have ruin with probability one when $p \leq \frac{\beta}{\alpha}$ and we assume in the following that $p > \frac{\beta}{\alpha}$. For $\gamma \in (0, \alpha)$, we obtain

$$\begin{aligned} \Psi(\gamma) &= -\delta\gamma + \frac{\sigma^2}{2}\gamma^2 + \beta\alpha \int_{-\infty}^0 (e^{-\gamma x} - 1 + \gamma x)e^{\alpha x} dx \\ &= -p\gamma + \frac{\sigma^2}{2}\gamma^2 - \frac{\beta\alpha}{\gamma - \alpha} - \beta \\ &= \frac{\gamma(\sigma^2\gamma^2 - (\sigma^2\alpha + 2p)\gamma + 2(p\alpha - \beta))}{2(\gamma - \alpha)} = -\frac{1}{2}A(\gamma)B(\gamma), \end{aligned}$$

where $A(\gamma) \triangleq \frac{\gamma}{\alpha - \gamma}$ and $B(\gamma) \triangleq \sigma^2\gamma^2 - (\sigma^2\alpha + 2p)\gamma + 2(p\alpha - \beta)$. To see if Ψ has an other root along 0, we need to consider the solutions of $B(\gamma) = 0$. This is an equation of second order with determinant Δ . As $\Delta > 0$, B has two distinct roots γ_+ and γ_- , given by

$$\gamma_{\pm} = \frac{\sigma^2\alpha + 2p \pm \sqrt{\Delta}}{2\sigma^2}.$$

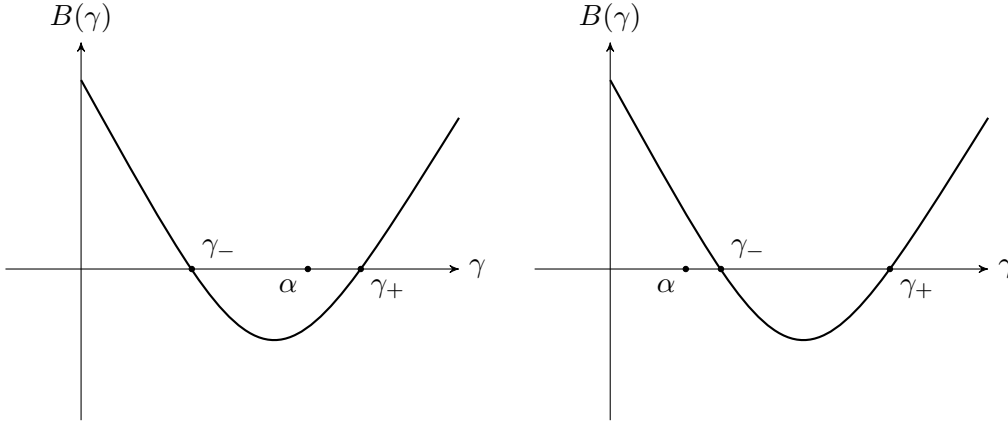


FIGURE 2. Behaviour of B when $\gamma_- < \alpha$ (left) and $\gamma_- \geq \alpha$ (right).

First note that $\gamma_- < \gamma_+$ and that $\gamma_+ \geq \alpha$ and $\gamma_- \geq 0$, because $(\sigma^2\alpha + 2p)^2 \geq \Delta \geq (\sigma^2\alpha - 2p)^2$. Additionally, note that $B''(\gamma) = 2\sigma^2 > 0$, so that B is convex. Therefore, we only have two possible cases (see Figure 2) : either $\gamma_- < \alpha$ and then γ_- is a root of B and of Ψ , or $\gamma_- \geq \alpha$ and then $B(\gamma) > 0$ and $\Psi(\gamma) < 0$, for all $\gamma \in [0, \alpha)$. So, if $\gamma_- < \alpha$, then, by Theorem 1 (B), we obtain $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\gamma_- u}$ and if $\gamma_- \geq \alpha$, then, by Theorem 1 (D), we obtain $\mathbf{P}(\tau(u) < +\infty) \leq e^{-\alpha u}$. \square

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