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DISPERSION RELATIONS IN HOT MAGNETIZED PLASMAS

CHRISTOPHE CHEVERRY ⁽¹⁾, ADRIEN FONTAINE⁽²⁾

ABSTRACT. In the framework of hot magnetized collisionless plasmas, dispersion relations have been extensively studied in the past [2, 11, 12, 22, 31, 32, 35]. This subject is still topical in plasma physics [17, 25, 30, 34, 39]. The aim of this article is to provide a rigorous derivation of the characteristic variety, based on some asymptotic analysis of the relativistic Vlasov-Maxwell system. Special emphasis is made on the modeling of Tokamaks, with spatial variations of the magnetic field and of the equilibrium distribution function. In order to take into account the inhomogeneities, the problem is formulated in terms of geometrical optics [27, 29]. This allows to unify, justify and extend the preceding results. New aspects are indeed included. For instance, the dielectric tensor is defined for real frequencies through singular integrals involving the Hilbert transform.

Keywords. Hot magnetized plasmas ; relativistic Vlasov-Maxwell equations ; tokamaks ; dispersion relations ; characteristic variety ; wave particle interaction ; kinetic resonances ; dielectric tensor ; Hilbert transform.

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1. INTRODUCTION

The dispersion relations have been extensively studied in plasma physics. It is because they are involved in a wide range of astrophysical contexts and laboratory experiments through wave-particle interaction [21, 36], transfer of power between waves and particles, heating of plasmas, reflectometry techniques [19], and so on. The preparatory works from the 1960s, 1970s and 1980s [2, 11, 12, 22, 31, 32, 35] are the template for recent numerical studies [34, 39], for contemporary investigations in more complex situations [17, 25, 28, 30] or, like in the present text which is about tokamaks, for developments up to the case of *non-uniform* magnetized plasmas.

In real fusion machines, the dominant distribution function and the external magnetic field are inhomogeneous. They undergo significant fluctuations in position. These variations have a major effect on the geometry of wave propagation. Their impact is important when performing ray tracing, with many practical consequences. It becomes decisive when looking at the transport equations (to measure power transfers between waves and particles) or in the perspective of long time studies [5, 6]. However, the presence of inhomogeneities is complicated to simulate. This is probably why, despite some attempts [33], this subject has not been completely studied. Another reason is, without a doubt, a general principle of physics according to which a dispersion relation can be obtained by analyzing a plane monochromatic wave in a homogeneous medium, and then letting the medium's properties (in the dielectric tensor) vary slowly in position. After verification, this principle holds true, but it is not so easy to determine what should vary, why and how. There are questions that remain unanswered. The aim of this article is precisely to check what the situation really is. It is to rigorously define the characteristic variety by extracting the corresponding dielectric tensor through a comprehensive study. To this end, it is not enough to extend existing procedures, which give formal results, provide partial information or rely on specific hypotheses. A new approach is needed.

In a plasma, the presence of a strong magnetic field makes the charged particles oscillate at the electron cyclotron frequency ε^{-1} with $\varepsilon \ll 1$. Away from thermal equilibrium, the repartition of the charged particles is therefore described by oscillating kinetic distribution functions whose structures are exhibited in [6]. This produces oscillating currents. Then, by a mesoscopic caustic effect [5], self-consistent oscillating electromagnetic waves are emitted. They act like coherent sources [7]. Roughly speaking, it is as if the rays emanate from a *smooth nonlinear phase* $\phi(\mathbf{t}, \mathbf{x})$. The same applies to waves launching by antennas, in view of the radio frequency heating of tokamak plasmas.

It turns out that the propagation of electromagnetic oscillations in a hot quasi-neutral background of ions and electrons can be described in the framework of some asymptotic analysis. To some extent, we can consider WKB expansions involving a single phase $\phi(\mathbf{t}, \mathbf{x})$, as in (3.3). From there, the matter is to construct for the relativistic Vlasov-Maxwell system an adequate geometrical optics. In comparison with usual theories in hyperbolic equations [27, 29], new difficulties come from the kinetic resonances which are hidden in the self-consistent picture.

As a matter of fact, the propagation of waves is still governed by a dielectric tensor $\sigma(\cdot)$. But now the dielectric property becomes a reactive aspect of the wave-particle interaction. The aim of this article is to derive $\sigma(\cdot)$ from basic principles. Then, it is to rigorously define the content of $\sigma(\cdot)$ in the domain of *real* frequencies. When doing this, complications arise for instance from the singular integrals that play a part in the construction of $\sigma(\cdot)$.

Theorem 1 (eikonal equation in axisymmetric configurations). *There exists a well-defined skew-symmetric matrix $\sigma(\cdot)$ playing the part of a conductivity tensor, such that the eikonal equation governing wave propagation in tokamaks can be determined through the following Hamilton-Jacobi equation:*

$$(1.1) \quad \det(\nabla_{\mathbf{x}}\phi^t \nabla_{\mathbf{x}}\phi + (\partial_t \phi)^2 \text{Id} - |\nabla_{\mathbf{x}}\phi|^2 \text{Id} + i \partial_t \phi \sigma(\mathbf{x}, \partial_t \phi, \nabla_{\mathbf{x}}\phi)) = 0.$$

More precisely, the matrix $\sigma(\cdot)$ is defined by:

$$(1.2) \quad \sigma(\mathbf{x}, \tau, \xi) := -4\pi i G^{\text{d}}(\Psi(\rho, z)) \sum_{n \in \mathbb{Z}} \int_0^{+\infty} \int_0^\pi \frac{r^4}{\langle r \rangle (\tau + \tau_n)} \frac{\partial_r F^{\text{d}}(r^2)}{T_n} dr d\varpi$$

where $G^{\text{d}}(\cdot)$ and $F^{\text{d}}(\cdot)$ are constitutive elements of realistic tokamak distribution functions (see [10, 23] and Definition 2.2), whereas $\Psi(\cdot)$ is a general poloidal flux function. At the level of (1.2), the " T_n " symbol stands for the skew-symmetric matrix:

$$T_n := \begin{pmatrix} \frac{n^2 J_n^2(\zeta)}{\zeta^2} \sin^3 \varpi & \frac{i n J_n(\zeta) J'_n(\zeta)}{\zeta} \sin^3 \varpi & \frac{n J_n^2(\zeta)}{\zeta} \cos \varpi \sin^2 \varpi \\ -\frac{i n J_n(\zeta) J'_n(\zeta)}{\zeta} \sin^3 \varpi & (J'_n(\zeta))^2 \sin^3 \varpi & -i J_n(\zeta) J'_n(\zeta) \cos \varpi \sin^2 \varpi \\ \frac{n J_n^2(\zeta)}{\zeta} \cos \varpi \sin^2 \varpi & i J_n(\zeta) J'_n(\zeta) \cos \varpi \sin^2 \varpi & J_n^2(\zeta) \cos^2 \varpi \sin \varpi \end{pmatrix},$$

where $J_n(\cdot)$ denotes the n -th Bessel function of the first kind. Introduce $\mathbf{b}_e(\cdot)$ as in (2.20). The function $\mathbf{b}_e(\cdot)$ takes into account the variations of the external magnetic field. The scalar functions $\tau_n(\cdot)$ in (1.2) and $\zeta(\cdot)$ in T_n are given by:

$$\tau_n(\mathbf{x}, r, \varpi, \xi) := \langle r \rangle^{-1} (r \xi_{\parallel} \cos \varpi + n \mathbf{b}_e), \quad \zeta(\mathbf{x}, r, \varpi, \xi_{\perp}) := r \xi_{\perp} \sin \varpi \mathbf{b}_e(\mathbf{x})^{-1}.$$

This text is divided into two main chapters, Section 2 and Section 3.

The discussion begins in Section 2 with the modeling of hot magnetized collisionless plasmas in axisymmetric configurations, through the textbook case of tokamaks. The starting point is the relativistic Vlasov-Maxwell (RVM) system. A first step (Part 2.3) is to describe the content of toroidal equilibria. This means (Paragraph 2.3.1) to use practical external magnetic fields $\tilde{\mathbf{B}}_e(\cdot)$ and (Paragraphs 2.3.2 and 2.3.3) to exhibit realistic distribution functions $\tilde{f}_\alpha^{\text{d}}(\cdot)$ satisfying the stationary RVM system (2.9)-(2.10). A second stage (Part 2.4) is to perform some dimensionless analysis of the RVM system. The purpose (Part 2.5) is to interpret the hot regime in terms of some asymptotic analysis, where the size of all physical quantities is expressed in function of the small parameter ε . By this way, we are led to a version of the RVM equations which is much more singular than in the article [8], with a number of new aspects which must be taken into account.

Section 3 contains the core of the analysis. Part 3.2 is devoted to a precise description of the *characteristic variety*. The framework of geometrical optics allows in the preliminary Paragraph 3.2.1 to extract a simplified system of equations. Then, in Paragraph 3.2.2, we perform a Fourier analysis through the Jacobi-Anger identity. In Paragraph 3.2.3, this leads to some interesting kinetic interpretation of the electron cyclotron resonances. By this way, in Paragraph 3.2.4, we can get a formal definition of the dielectric tensor $\sigma(\cdot)$. Now, the aim of Part 3.3 is to clarify the meaning of $\sigma(\cdot)$. This is achieved in several stages. First, in Paragraph 3.3.1, we perform a change of variables. Secondly, in Paragraph 3.3.2, we introduce the *Hilbert transform* and we study its action through Lipschitz estimates (Paragraph 3.3.3) and through L^1 -estimates (Paragraph 3.3.4). Finally, the particular cases of perpendicular and parallel propagation are adressed in Part 3.3.2.

The formula (1.2) is new. It shows through the variations of the functions $G^d(\cdot)$, $F^d(\cdot)$, $\Psi(\cdot)$ and $\mathbf{b}_e(\cdot)$ what can be the concrete influence of the inhomogeneities. On the other hand, the mathematical difficulties which are solved in Part 3.3 in order to rigorously define the dielectric tensor $\sigma(\cdot)$ are original. They are issued from an interpretation of wave-particle interactions, where the gyroballistic dispersion functions $\tau_n(\cdot)$ with $n \in \mathbb{Z}$ appear as constitutive elements of $\sigma(\cdot)$. As an extension of the present work, the mechanisms of power transfer between particles and waves could be further investigated [7].

2. HOT MAGNETIZED PLASMAS IN AXISYMMETRIC CONFIGURATIONS

This section is dedicated to the modelling of hot magnetized plasmas in axisymmetric configurations. We will consider the case of **Tokamaks**. Keeping in mind the physical observations, the discussion will be systematically tested in this Tokamak context. The basic equations are the Relativistic Vlasov-Maxwell equations (RVM equations) recalled in Paragraph 2.1. Paragraph 2.2 introduces the notations and assumptions that will be used throughout this article. Section 2.3 exhibits exact and realistic stationary solutions. Section 2.4 highlights dimensionless equations which are issued from the RVM equations. This leads in Section 2.5 to the notion of what is a *hot asymptotic regime*.

2.1. Relativistic Vlasov-Maxwell equations. The topic of RVM equations has been widely discussed [3, 5, 13, 14, 24]. The corresponding framework is recalled hereafter. The speed of light is $c_0 \simeq 2,99 \times 10^8 \text{ m s}^{-1}$. Let $L \in \mathbb{R}_+^*$ be a characteristic spatial length. The original spatial variable is $\tilde{\mathbf{x}} \in \tilde{\Omega}$, where $\tilde{\Omega}$ is some non-empty open set of \mathbb{R}^3 . We fix the observation time $T \in \mathbb{R}_+^*$ as $T := L/c_0$. The original time variable is $\tilde{t} \in [0, T]$. There are corresponding rescaled versions:

$$(2.1) \quad \mathbf{t} := \frac{\tilde{t}}{T} \in [0, 1], \quad \mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3) := \frac{\tilde{\mathbf{x}}}{L} \in \Omega := \left\{ \frac{\tilde{\mathbf{x}}}{L}; \tilde{\mathbf{x}} \in \tilde{\Omega} \right\}.$$

The original space and momentum variables are $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ with:

$$\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2, \tilde{\mathbf{x}}^3) \in \tilde{\Omega} \subset \mathbb{R}^3, \quad \tilde{\mathbf{p}} = (\tilde{\mathbf{p}}^1, \tilde{\mathbf{p}}^2, \tilde{\mathbf{p}}^3) \in \mathbb{R}^3.$$

We consider a plasma which is confined inside $\tilde{\Omega}$, and which consists of N distinct species labelled by $\alpha \in \{1, \dots, N\}$. The particles of the α^{th} species have charge e_α and rest mass m_α . The number $\alpha = 1$ will be associated with electrons.

The elementary charge of electrons is $e \equiv -e_1 \simeq 1,6 \times 10^{-19} C$ and the electron rest mass is $m_e \equiv m_1 \simeq 9,1 \times 10^{-31} kg$. Recall that the proton-to-electron mass ratio $\beta \simeq 1836$ is a dimensionless quantity, so that:

$$(2.2) \quad \iota_1 := \frac{m_1}{m_e} = 1, \quad \iota_\alpha := \frac{m_1}{m_\alpha} \lesssim \beta^{-1} \simeq 10^{-3}, \quad \forall \alpha \in \{2, \dots, N\}.$$

On the other hand, the charge e_α is an integer multiple of e . More precisely:

$$(2.3) \quad \forall \alpha \in \{2, \dots, N\}, \quad \exists k_\alpha \in \mathbb{N}^*; \quad k_\alpha \simeq 1, \quad e_\alpha = k_\alpha e.$$

The velocity $\tilde{\mathbf{v}}_\alpha$ of a particle of type α is limited by $|\tilde{\mathbf{v}}_\alpha| \leq c_0$, and it is linked to the momentum $\tilde{\mathbf{p}} \in \mathbb{R}^3$ through:

$$(2.4) \quad \frac{\tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}})}{c_0} = \frac{\tilde{\mathbf{p}}}{m_\alpha c_0} \left(1 + \frac{|\tilde{\mathbf{p}}|^2}{m_\alpha^2 c_0^2} \right)^{-1/2}, \quad \frac{\tilde{\mathbf{p}}(\tilde{\mathbf{v}}_\alpha)}{m_\alpha c_0} = \frac{\tilde{\mathbf{v}}_\alpha}{c_0} \left(1 - \frac{|\tilde{\mathbf{v}}_\alpha|^2}{c_0^2} \right)^{-1/2}.$$

The kinetic distribution function (KDF) of the α^{th} species is denoted by $\tilde{f}_\alpha^k(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}})$. It is composed of a dominant stationary part $\tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ and a smaller moving part $\tilde{f}_\alpha^s(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}})$. The density ratio $\nu \in \mathbb{R}_+^*$ between these two populations is assumed to be small and independant of α :

$$(2.5) \quad \tilde{f}_\alpha^k(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) = \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) + \nu \tilde{f}_\alpha^s(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}), \quad (\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) \in \mathbb{R}_+ \times \tilde{\Omega} \times \mathbb{R}^3, \quad \nu \ll 1.$$

The charge density $\tilde{\rho}$ and the current density $\tilde{\mathbf{j}}$ are given by:

$$(2.6a) \quad \tilde{\rho} \equiv \tilde{\rho}(\tilde{f}_1^k, \dots, \tilde{f}_N^k)(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}) \equiv \tilde{\rho}(\tilde{f}_\alpha^k)(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}) := \sum_{\alpha=1}^N \int_{\mathbb{R}^3} e_\alpha \tilde{f}_\alpha^k(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}},$$

$$(2.6b) \quad \tilde{\mathbf{j}} \equiv \tilde{\mathbf{j}}(\tilde{f}_1^k, \dots, \tilde{f}_N^k)(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}) \equiv \tilde{\mathbf{j}}(\tilde{f}_\alpha^k)(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}) := \sum_{\alpha=1}^N \int_{\mathbb{R}^3} e_\alpha \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \tilde{f}_\alpha^k(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}}.$$

We impose a (stationary) external magnetic field $\tilde{\mathbf{B}}_e : \tilde{\Omega} \rightarrow \mathbb{R}^3$. We also take into account some collective self-consistent electromagnetic field $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})(\tilde{\mathbf{t}}, \tilde{\mathbf{x}})$, which is created by all plasma particles. Then, neglecting the collisional effects, the time evolution of the KDF can be modelled through the Vlasov equation:

$$(2.7) \quad \partial_{\tilde{\mathbf{t}}} \tilde{f}_\alpha^k + \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{f}_\alpha^k + e_\alpha [\tilde{\mathbf{E}}(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}) + \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \times (\tilde{\mathbf{B}}(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}) + \tilde{\mathbf{B}}_e(\tilde{\mathbf{x}}))] \cdot \nabla_{\tilde{\mathbf{p}}} \tilde{f}_\alpha^k = 0.$$

On the other hand, the self-consistent electromagnetic field $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})(\tilde{\mathbf{t}}, \tilde{\mathbf{x}})$ is subjected to the Maxwell equations:

$$(2.8a) \quad \partial_{\tilde{\mathbf{t}}} \tilde{\mathbf{E}} - c_0^2 \nabla_{\tilde{\mathbf{x}}} \times (\tilde{\mathbf{B}} + \tilde{\mathbf{B}}_e) = -\epsilon_0^{-1} \tilde{\mathbf{j}}(\tilde{f}_\alpha^k), \quad \nabla_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{E}} = \epsilon_0^{-1} \tilde{\rho}(\tilde{f}_\alpha^k),$$

$$(2.8b) \quad \partial_{\tilde{\mathbf{t}}} \tilde{\mathbf{B}} + \nabla_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{E}} = 0, \quad \nabla_{\tilde{\mathbf{x}}} \cdot (\tilde{\mathbf{B}} + \tilde{\mathbf{B}}_e) = 0.$$

In (2.8), the physical constant $\epsilon_0 \simeq 8,8 \times 10^{-12} F m^{-1}$ stands for the vacuum permittivity.

The unknowns in (2.7)-(2.8) are the $\tilde{f}_\alpha^k(\cdot)$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})(\cdot)$. The strategy is to seek solutions of (2.7)-(2.8) as perturbations of a stationary equilibrium state given by $\tilde{f}_\alpha^k(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) \equiv \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ for all $\alpha \in \{1, \dots, N\}$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}}) \equiv (0, 0)$.

A first stage in this direction is to find functions $\tilde{f}_\alpha^d(\cdot)$ and $\tilde{\mathbf{B}}_e(\cdot)$ satisfying:

$$(2.9) \quad \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{f}_\alpha^d + e_\alpha [\tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \times \tilde{\mathbf{B}}_e(\tilde{\mathbf{x}})] \cdot \nabla_{\tilde{\mathbf{p}}} \tilde{f}_\alpha^d = 0, \quad \forall \alpha \in \{1, \dots, N\},$$

together with:

$$(2.10) \quad c_0^2 \nabla_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{B}}_e = \epsilon_0^{-1} \tilde{j}(\tilde{f}_\alpha^d), \quad \tilde{\rho}(\tilde{f}_\alpha^d) = 0, \quad \nabla_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{B}}_e = 0.$$

2.2. Some basic assumptions. The purpose of this subsection 2.2 is to introduce the notations of this article, together with a number of other relevant physical constraints.

2.2.1. Cold, warm and hot plasma temperatures. A plasma which is spatially in Local Thermodynamic Equilibrium (LTE) can be characterized at the position $\tilde{\mathbf{x}}$ with a few parameters, as is the case with Maxwell-Boltzmann distributions:

$$(2.11) \quad \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) = \frac{\tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}})}{m_\alpha^3 c_0^3} \mathcal{M}_{\tilde{\theta}_\alpha^d(\tilde{\mathbf{x}})}^b \left(\frac{|\tilde{\mathbf{p}}|}{m_\alpha c_0} \right), \quad \mathcal{M}_\theta^b(r) := \frac{1}{\pi^{3/2}} \frac{1}{\theta^3} \exp\left(-\frac{r^2}{\theta^2}\right).$$

Both $\tilde{\theta}_\alpha^d(\cdot)$ and $\tilde{\mathbf{n}}_\alpha^d(\cdot)$ are building blocks in the construction of $\tilde{f}_\alpha^d(\cdot)$. Retain for instance that the density $\tilde{\mathbf{n}}_\alpha^d(\cdot)$ can be recovered from $\tilde{f}_\alpha^d(\cdot)$ through the integral formula:

$$(2.12) \quad \tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}}) := \int_{\mathbb{R}^3} \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}}, \quad \tilde{\mathbf{x}} \in \tilde{\Omega}, \quad \alpha \in \{1, \dots, N\}.$$

Denote simply by $\theta_\alpha^d \in \mathbb{R}_+^*$ and $n_\alpha^d \in \mathbb{R}_+^*$ typical sizes of $\tilde{\theta}_\alpha^d(\cdot)$ and $\tilde{\mathbf{n}}_\alpha^d(\cdot)$. We require that the two quantities $\tilde{\theta}_\alpha^d(\tilde{\mathbf{x}})$ and $\tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}})$ do not deviate too far from θ_α^d and n_α^d . In other words:

Assumption 2.1. *[possible but slight variations in temperatures and densities] There is a constant $c \in]0, 1[$ such that:*

$$(2.13) \quad 0 < c \theta_\alpha^d \leq \tilde{\theta}_\alpha^d(\tilde{\mathbf{x}}) \leq c^{-1} \theta_\alpha^d, \quad 0 < c n_\alpha^d \leq \tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}}) \leq c^{-1} n_\alpha^d, \quad \forall \tilde{\mathbf{x}} \in \tilde{\Omega}.$$

Recall that $k_B \simeq 1,38 \times 10^{-23} m^2 kg s^{-2} K^{-1}$ stands for the Boltzmann constant, and also retain the relationship $1 eV \simeq 1,16 \times 10^4 k_B K$. The electron temperature ($T_e \equiv T_1$) and the ion temperatures (denoted by T_α for $\alpha > 1$) can be expressed either in kelvin (K) or in electronvolt (eV). Because of the large difference in mass, the electrons will come to thermodynamic equilibrium amongst themselves much faster than they will come into equilibrium with the ions or neutral atoms. For this reason, the ion temperatures may be very different from (usually much lower than) the electron temperature:

$$(2.14) \quad T_\alpha \leq T_e \equiv T_1, \quad \forall \alpha \in \{1, \dots, N\}.$$

Based on the relative temperatures of the electrons, ions and neutrals, plasmas are classified as **thermal** or **non-thermal**. Introduce the thermal speed v_α^{th} and its dimensionless version θ_α^d as indicated below:

$$(2.15) \quad v_\alpha^{th} := \left(\frac{k_B T_\alpha}{m_\alpha} \right)^{1/2} \in \mathbb{R}_+^*, \quad \theta_\alpha^d := \frac{v_\alpha^{th}}{c_0} \in \mathbb{R}_+^*.$$

Both v_α^{th} and θ_α^d can be viewed as measures of temperature, albeit in dimensionless units concerning θ_α^d . Combining (2.2) and (2.14), we get:

$$(2.16) \quad \frac{v_\alpha^{th}}{v_1^{th}} = \frac{\theta_\alpha^d}{\theta_1^d} = \left(\frac{T_\alpha}{T_1}\right)^{1/2} \times \left(\frac{m_1}{m_\alpha}\right)^{1/2} \lesssim \left(\frac{T_\alpha}{T_1}\right)^{1/2} \times \left(\frac{1}{\beta}\right)^{1/2} \ll 1.$$

As a rule of thumb, temperatures T_α well below 100 eV ($\theta_\alpha^d \simeq \varepsilon$) are said *cold*; those which are about 100 eV ($\theta_\alpha^d \simeq 10^{-2}$) are considered *warm*; those with T_α ranging from 100 eV to 10 keV ($10^{-2} \lesssim \theta_\alpha^d < 1$) become progressively *hot*; particles with higher energies ($1 \sim \theta_\alpha^d$) are termed *energetic* or *relativistic*.

2.2.2. Quasi-neutrality. A plasma consists of approximately equal numbers of positively charged ions and negatively charged electrons. This property is expressed by the second equation of (2.10). In view of (2.6a) and (2.12), this amounts to the following condition.

Assumption 2.2. *The plasma is quasi-neutral in the sense that:*

$$(2.17) \quad e \tilde{n}_1^d(\tilde{\mathbf{x}}) = \sum_{\alpha=2}^N e_\alpha \tilde{n}_\alpha^d(\tilde{\mathbf{x}}), \quad \forall \tilde{\mathbf{x}} \in \tilde{\Omega}.$$

The interpretation of (2.17) is the existence of a background neutralizing ion population. In view of (2.3), (2.13) and (2.17), we can infer that:

$$(2.18) \quad e n_1^d \simeq \sum_{\alpha=2}^N e_\alpha n_\alpha^d, \quad n_\alpha^d \simeq n_1^d, \quad \forall \alpha \in \{2, \dots, N\}.$$

2.3. Toroidal equilibrium. The discussion is devoted here to the study of (2.9)-(2.10), that is to the determination of $\tilde{\mathbf{B}}_e(\cdot)$ and $\tilde{f}_\alpha^d(\cdot)$. In Paragraph 2.3.1, we select axisymmetric divergence free external magnetic fields $\tilde{\mathbf{B}}_e(\cdot)$ that are issued from physical considerations. In Paragraph 2.3.2, we explain how (2.9)-(2.10) is usually replaced by the *Grad-Shafranov* equations, giving rise to a notion of a fluid equilibrium. Finally, in Paragraph 2.3.3, we investigate directly (2.9)-(2.10) to find special solutions incorporating kinetic aspects.

2.3.1. Axisymmetric inhomogeneous external magnetic fields. In Tokamaks, the charged particles are confined by a strong external inhomogeneous magnetic field $\tilde{\mathbf{B}}_e(\cdot) \in \mathcal{C}^\infty(\tilde{\Omega}; \mathbb{R}^3)$, with amplitude $\tilde{b}_e(\tilde{\mathbf{x}}) := |\tilde{\mathbf{B}}_e(\tilde{\mathbf{x}})|$. The function $\tilde{b}_e(\cdot)$ is assumed to be of the order $b_e \in \mathbb{R}_+^*$. More precisely, we can find $c \in]0, 1[$ such that:

$$(2.19) \quad 0 < c b_e \leq \tilde{b}_e(\tilde{\mathbf{x}}) \leq c^{-1} b_e, \quad \forall \tilde{\mathbf{x}} \in \tilde{\Omega}.$$

In view of (2.1), we can consider the following rescaled version of $\tilde{\mathbf{B}}_e(\cdot)$:

$$(2.20) \quad \mathbf{B}_e(\mathbf{x}) := \frac{\tilde{\mathbf{B}}_e(L\mathbf{x})}{b_e}, \quad \mathbf{b}_e(\mathbf{x}) := |\mathbf{B}_e(\mathbf{x})|.$$

Then, the condition (2.19) becomes:

Assumption 2.3. *[nowhere-vanishing external magnetic field] There is $c \in]0, 1[$ such that:*

$$(2.21) \quad 0 < c \leq \mathbf{b}_e(\mathbf{x}) \leq c^{-1}, \quad \forall \mathbf{x} \in \Omega.$$

The function $\mathbf{B}_e(\cdot)$ generates a unit vector field:

$$(2.22) \quad e_3(\mathbf{x}) := \mathbf{b}_e(\mathbf{x})^{-1} \mathbf{B}_e(\mathbf{x}) \in \mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| = 1\}.$$

Complete $e_3(\mathbf{x})$ into some right-handed orthonormal basis $(e_1, e_2, e_3)(\mathbf{x})$, with:

$$(2.23) \quad e_j(\cdot) = {}^t(e_j^1, e_j^2, e_j^3)(\cdot) \in \mathcal{C}^\infty(\Omega; \mathbb{S}^2), \quad \forall j \in \{1, 2, 3\}.$$

With the preceding ingredients, we can define some orthogonal matrix $O(\mathbf{x})$ and some constant skew-symmetric matrix Λ according to:

$$(2.24) \quad O := \begin{pmatrix} e_1^1 & e_2^1 & e_3^1 \\ e_1^2 & e_2^2 & e_3^2 \\ e_1^3 & e_2^3 & e_3^3 \end{pmatrix} = {}^tO^{-1}, \quad \Lambda := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -{}^t\Lambda.$$

Note that the matrix $O(\mathbf{x})$ allows to straighten out the field lines, since ${}^tO e_3 = {}^t(0, 0, 1)$. On the other hand, the divergence free condition of (2.10) is verified when $\tilde{\mathbf{B}}_e(\cdot)$ is issued from a magnetic potential.

Assumption 2.4. [magnetic potential] *There is a vector field $\tilde{\mathbf{A}} \in \mathcal{C}^\infty(\tilde{\Omega}; \mathbb{R}^3)$ such that:*

$$(2.25) \quad \tilde{\mathbf{B}}_e(\tilde{\mathbf{x}}) = \nabla_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{A}}(\tilde{\mathbf{x}}).$$

Consider the cylindrical coordinate system $(\tilde{\rho}, \tilde{\varphi}, \tilde{z}) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}$ with corresponding orthonormal basis $(e_{\tilde{\rho}}, e_{\tilde{\varphi}}, e_{\tilde{z}})$. The second direction $e_{\tilde{\varphi}}$ is called the *toroidal* direction. On the other hand, the plane generated by the directions $e_{\tilde{\rho}}$ and $e_{\tilde{z}}$ is called the *poloidal* plane. Select a and R_0 with $0 < a < R_0$. Then, define:

$$\tilde{\Omega} := \{(\tilde{\rho}, \tilde{\varphi}, \tilde{z}) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}; (\tilde{\rho} - R_0)^2 + \tilde{z}^2 \leq a^2\}.$$

The domain $\tilde{\Omega}$ is represented on Figure 1. The positive numbers a and R_0 respectively stand for the minor and the major radius of the tokamak. A vector field like $\tilde{\mathbf{A}}$ can be decomposed in the basis $(e_{\tilde{\rho}}, e_{\tilde{\varphi}}, e_{\tilde{z}})$. This yields three components $\tilde{\mathbf{A}}_{\tilde{\rho}}$, $\tilde{\mathbf{A}}_{\tilde{\varphi}}$ and $\tilde{\mathbf{A}}_{\tilde{z}}$. We consider axisymmetric plasmas.

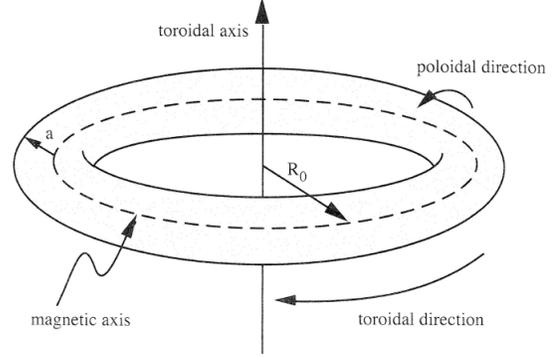


FIGURE 1. Axisymmetric configuration.

Assumption 2.5. [axisymmetric toroidal equilibrium] *The vector field $\tilde{\mathbf{A}}(\cdot)$ and the kinetic distribution functions $\tilde{f}_\alpha^d(\cdot)$ are exhibiting symmetry around the vertical axis $\tilde{\rho} = 0$.*

In particular, all components $\tilde{\mathbf{A}}_{\tilde{\rho}}$, $\tilde{\mathbf{A}}_{\tilde{\varphi}}$ and $\tilde{\mathbf{A}}_{\tilde{z}}$ of $\tilde{\mathbf{A}}$ do not depend on $\tilde{\varphi}$. Then:

$$(2.26) \quad \tilde{\mathbf{B}}_e = \nabla_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{A}} = -\partial_{\tilde{z}} \tilde{\mathbf{A}}_{\tilde{\varphi}} e_{\tilde{\rho}} + (\partial_{\tilde{z}} \tilde{\mathbf{A}}_{\tilde{\rho}} - \partial_{\tilde{\rho}} \tilde{\mathbf{A}}_{\tilde{z}}) e_{\tilde{\varphi}} + \tilde{\rho}^{-1} \partial_{\tilde{\rho}}(\tilde{\rho} \tilde{\mathbf{A}}_{\tilde{\varphi}}) e_{\tilde{z}}.$$

We have $\tilde{\mathbf{B}}_e := \tilde{\mathbf{B}}_e^t + \tilde{\mathbf{B}}_e^p$, where we have put apart the toroidal component $\tilde{\mathbf{B}}_e^t(\cdot)$ of $\tilde{\mathbf{B}}_e(\cdot)$:

$$(2.27) \quad \tilde{\mathbf{B}}_e^t := (\partial_{\tilde{z}} \tilde{\mathbf{A}}_{\tilde{\rho}} - \partial_{\tilde{\rho}} \tilde{\mathbf{A}}_{\tilde{z}}) e_{\tilde{\varphi}},$$

as well as the poloidal component $\tilde{\mathbf{B}}_e^p(\cdot)$ of $\tilde{\mathbf{B}}_e(\cdot)$, namely:

$$(2.28) \quad \tilde{\mathbf{B}}_e^p := -\partial_{\tilde{z}} \tilde{\mathbf{A}}_{\tilde{\varphi}} e_{\tilde{\rho}} + \tilde{\rho}^{-1} \partial_{\tilde{\rho}}(\tilde{\rho} \tilde{\mathbf{A}}_{\tilde{\varphi}}) e_{\tilde{z}}.$$

Use *toroidal* coordinates $(r, \varphi, \theta) \in \mathbb{R}_+ \times \mathbb{T}^2$ as indicated on Figure 2, with associated orthonormal basis $(e_r, e_\varphi, e_\theta)$. The domain of study becomes:

$$\tilde{\Omega} := \{ (r, \varphi, \theta) \in \mathbb{R}_+ \times \mathbb{T}^2; r \leq a \}.$$

Discussion 2.1. [The case of *Tore Supra*] The amplitude of b_e is around $b_e \simeq 4, 5 T$. The major radius is $R_0 \simeq 2, 25 m$, whereas the minor radius is $a \simeq 0, 7 m$.

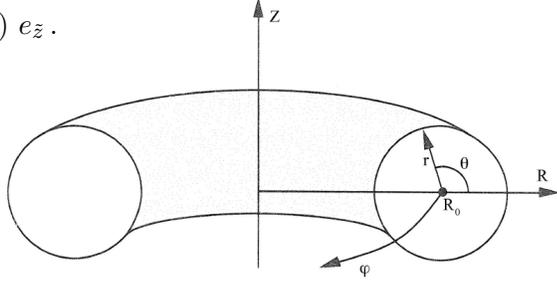


FIGURE 2. Toroidal coordinates.

For axisymmetric systems, the field lines lie in nested magnetic flux surfaces. The cuts of these flux surfaces with the poloidal planes (which are the planes containing the \tilde{z} -axis) give rise to closed curves which can be viewed as the level sets \mathcal{C}_ψ of a *poloidal flux function* $\tilde{\Psi}(\cdot)$. The family of poloidal cross sections $\{\mathcal{C}_\psi\}_\psi$ with $\psi \in \mathbb{R}_+$ is diffeomorphic to concentric circles. The function $\tilde{\Psi}(\cdot)$ can be viewed as depending on the variables $(\tilde{\rho}, \tilde{z})$ or (r, θ) . In the cylindrical coordinate system, the curve \mathcal{C}_ψ takes the following form:

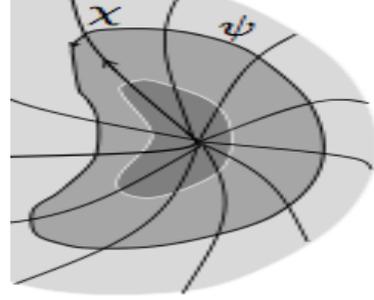


FIGURE 3. Cross sections.

$$(2.29) \quad \mathcal{C}_\psi = \{ (\tilde{\rho}, \tilde{z}) \in \mathbb{R}_+ \times \mathbb{R}; \tilde{\Psi}(\tilde{\rho}, \tilde{z}) = \psi \}.$$

In the toroidal coordinate system, it is:

$$(2.30) \quad \mathcal{C}_\psi = \{ (r, \theta) \in \mathbb{R}_+ \times \mathbb{T}; \tilde{\Psi}(r, \theta) = \psi \}.$$

By definition, we must have $\tilde{\mathbf{B}}_e^p \cdot \nabla \tilde{\Psi} \equiv 0$. It follows directly from (2.28) that $\tilde{\Psi}(\cdot)$ is a function of $\tilde{\rho} \tilde{\mathbf{A}}_{\tilde{\varphi}}$. A possible choice for $\tilde{\Psi}(\cdot)$ is simply (see for instance [18], Section 1.3):

$$(2.31) \quad \tilde{\Psi}(\tilde{\rho}, \tilde{z}) := \tilde{\rho} \tilde{\mathbf{A}}_{\tilde{\varphi}}(\tilde{\rho}, \tilde{z}).$$

The two components $\tilde{\mathbf{B}}_e^t(\cdot)$ and $\tilde{\mathbf{B}}_e^p(\cdot)$ can be written:

$$(2.32) \quad \tilde{\mathbf{B}}_e^t = \tilde{g} \nabla \tilde{\varphi}, \quad \tilde{g}(\tilde{\rho}, \tilde{z}) := \tilde{\rho} (\partial_{\tilde{z}} \tilde{\mathbf{A}}_{\tilde{\rho}} - \partial_{\tilde{\rho}} \tilde{\mathbf{A}}_{\tilde{z}}), \quad \tilde{\mathbf{B}}_e^p = \nabla \tilde{\Psi} \times \nabla \tilde{\varphi}.$$

In [18], the function $\tilde{g}(\cdot)$ is called the *poloidal current function*. It can be freely adjusted since it suffices to integrate the second equation of (2.32) to recover the components $\tilde{\mathbf{A}}_{\tilde{\rho}}$ and $\tilde{\mathbf{A}}_{\tilde{z}}$ of $\tilde{\mathbf{A}}$. Now, a common assumption [6] is to consider that the function $\tilde{g}(\cdot)$ depends only on $\tilde{\Psi}$, say $\tilde{g} = \tilde{g}(\tilde{\Psi})$ for some $\tilde{g} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})$. Let $(\tilde{\rho}_1, \tilde{z})$ and $(\tilde{\rho}_2, \tilde{z})$ be such that $\tilde{\Psi}(\tilde{\rho}_1, \tilde{z}) = \psi_1$ and $\tilde{\Psi}(\tilde{\rho}_2, \tilde{z}) = \psi_2$. The poloidal magnetic flux $\tilde{\Psi}_{pol}$ between the two magnetic surfaces \mathcal{C}_{ψ_1} and \mathcal{C}_{ψ_2} is the difference of $\tilde{\Psi}$ between the two positions $(\tilde{\rho}_1, \tilde{z})$ and $(\tilde{\rho}_2, \tilde{z})$.

In other words [18]:

$$(2.33) \quad \tilde{\Psi}_{pol} := \int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \tilde{\mathbf{B}}_e \cdot e_{\tilde{z}} 2\pi \tilde{\rho} d\tilde{\rho} = 2\pi \int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \partial_{\tilde{\rho}} \tilde{\Psi} d\tilde{\rho} = 2\pi (\psi_2 - \psi_1).$$

There is a similarity between the differential equations contained in (2.25)-(2.31) on the one hand and in (2.10)-(2.32) on the other hand. It follows that the poloidal current \tilde{I}_{pol} enclosed by the two magnetic surfaces \mathcal{C}_{ψ_1} and \mathcal{C}_{ψ_2} is given by:

$$(2.34) \quad \tilde{I}_{pol} = 2\pi \varepsilon_0 c_0^2 [\tilde{g}(\psi_2) - \tilde{g}(\psi_1)].$$

Sometimes the function $g(\cdot)$ is viewed as constant, see for example [38] (Section 3.1). This constant case is highlighted below.

Assumption 2.6. [constant poloidal current density] *The poloidal current function $\tilde{g}(\cdot)$ is constant. More precisely, the toroidal field $\tilde{\mathbf{B}}_e^t(\cdot)$ takes the form:*

$$(2.35) \quad \tilde{\mathbf{B}}_e^t(\tilde{\rho}, \tilde{z}) = b_e R_0 (R_0 + r \cos \theta)^{-1} e_{\tilde{\varphi}} = b_e R_0 \tilde{\rho}^{-1} e_{\tilde{\varphi}}, \quad \tilde{g} \equiv b_e R_0.$$

The electric current that circulates in the primary coil of the tokamak is supposed to produce the poloidal magnetic field $\tilde{\mathbf{B}}_e^p(\cdot)$. In view of (2.32), the function $\tilde{\mathbf{B}}_e^p(\cdot)$ determines the choice of $\tilde{\Psi}(\cdot)$. When $\tilde{\Psi}(\cdot)$ does not depend on the angle θ , the poloidal cross sections form concentric circles. This special situation is described below.

Assumption 2.7. [the cross sections \mathcal{C}_{ψ} are concentric circles] *The external magnetic field $\tilde{\mathbf{B}}_e^p(\cdot)$ takes the form:*

$$(2.36) \quad \tilde{\mathbf{B}}_e^p(r, \theta) = b_e r q(r)^{-1} (R_0 + r \cos \theta)^{-1} e_{\theta}, \quad \tilde{\Psi}(r) = b_e \int_0^r s \iota(s) ds,$$

where the function $\iota : [0, a] \rightarrow \mathbb{R}_+^*$ is called the *rotational transform*, whereas the function $q : [0, a] \rightarrow \mathbb{R}_+^*$ with $q(r) = \iota(r)^{-1}$ is called the *safety factor*.

The value $q(r)$ can be defined as the number of rotations a magnetic field line (located at a distance r from the magnetic axis) makes in the toroidal direction when it makes one loop on the poloidal plane. The term *safety factor* refers to the role it plays in determining the stability of a plasma. Values of $q(\cdot)$ larger than one lead to greater stability. In general (see [1]), the function $q(\cdot)$ is assumed to be increasing, and such that $q(0) \geq 1$.

In accordance with (2.6a) and (2.6b), at equilibrium, the *total charge density* $\tilde{\rho}^d$ and the *total current density* \tilde{j}^d can be defined as the sum of what the α^{th} species bring:

$$(2.37a) \quad \tilde{\rho}^d := \tilde{\rho}(\tilde{f}_{\alpha}^d) = \sum_{\alpha=1}^N \tilde{\rho}_{\alpha}^d, \quad \tilde{\rho}_{\alpha}^d := \int_{\mathbb{R}^3} e_{\alpha} \tilde{f}_{\alpha}^d(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}},$$

$$(2.37b) \quad \tilde{j}^d := \tilde{j}(\tilde{f}_{\alpha}^d) = \sum_{\alpha=1}^N \tilde{j}_{\alpha}^d, \quad \tilde{j}_{\alpha}^d := \int_{\mathbb{R}^3} e_{\alpha} \tilde{\mathbf{v}}_{\alpha}(\tilde{\mathbf{p}}) \tilde{f}_{\alpha}^d(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}}.$$

The first condition in (2.10) implies a link between $\tilde{j}^d := \tilde{j}(\tilde{f}_\alpha^d)$ and $\tilde{\mathbf{B}}_e$. Taking into account the relations inside (2.32), this furnishes:

$$(2.38) \quad \tilde{j}^d = \tilde{j}^{dt} + \tilde{j}^{dp}, \quad \tilde{j}^{dt} := -\epsilon_0 c_0^2 \tilde{\rho}^{-1} \Delta^* \tilde{\Psi} e_{\tilde{\varphi}}, \quad \tilde{j}^{dp} := \epsilon_0 c_0^2 \nabla \tilde{g} \times \nabla \tilde{\varphi},$$

where the vector field \tilde{j}^d is decomposed into a toroidal component \tilde{j}^{dt} and a poloidal component \tilde{j}^{dp} , and where Δ^* is the differential operator:

$$(2.39) \quad \Delta^* := \tilde{\rho} \frac{\partial}{\partial \tilde{\rho}} \left(\frac{1}{\tilde{\rho}} \frac{\partial}{\partial \tilde{\rho}} \right) + \frac{\partial^2}{\partial \tilde{z}^2}.$$

At this stage, the axisymmetric magnetic field $\tilde{\mathbf{B}}_e(\cdot)$ is entirely determined by the two functions $\tilde{g}(\cdot)$ and $\tilde{\Psi}(\cdot)$. Additionnal requirements on $\tilde{g}(\cdot)$ and $\tilde{\Psi}(\cdot)$ are coming from (2.9) and (2.38) when specifying the kinetic distribution function $\tilde{f}_\alpha^d(\cdot)$. The determination of $\tilde{f}_\alpha^d(\cdot)$ is achieved in the two next Paragraphs 2.3.2 and 2.3.3.

2.3.2. Fluid equilibria in magnetized plasmas. Magnetohydrodynamics (MHD) or (in the presence of strong external electromagnetic fields) **neoclassical models** are what is meant by fluid theories. By virtue of their relative simplicity, these approaches constitute the most frequently used frameworks to deal with plasma equilibria. They are valid when the plasma is in a quiescent state. They require a minimum level of collisionality, that is an assumption of maxwellianity.

In what follows, we will assume that the plasma is in a non-relativistic regime, that is in the case where $\tilde{\mathbf{p}}(\tilde{\mathbf{v}}_\alpha) = m_\alpha \tilde{\mathbf{v}}_\alpha$. Then, for $\alpha \in \{1, \dots, N\}$, we can define the *flow velocity* $\tilde{\mathbf{u}}_\alpha^d$ and the *pressure tensor* $\tilde{\mathbf{P}}_\alpha^d$ of the α^{th} species as indicated below:

$$(2.40a) \quad \tilde{\mathbf{u}}_\alpha^d \equiv \tilde{\mathbf{u}}_\alpha^d(\tilde{f}_\alpha^d)(\tilde{\mathbf{x}}) := \frac{1}{\tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}})} \int_{\mathbb{R}^3} \tilde{\mathbf{v}}_\alpha \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}},$$

$$(2.40b) \quad \tilde{\mathbf{P}}_\alpha^d \equiv \tilde{\mathbf{P}}_\alpha^d(\tilde{f}_\alpha^d)(\tilde{\mathbf{x}}) := m_\alpha \int_{\mathbb{R}^3} \left(\tilde{\mathbf{v}}_\alpha - \tilde{\mathbf{u}}_\alpha^d \right) \otimes \left(\tilde{\mathbf{v}}_\alpha - \tilde{\mathbf{u}}_\alpha^d \right) \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}}.$$

Introduce also the *charge density* $\tilde{\rho}_\alpha^d := e_\alpha \tilde{\mathbf{n}}_\alpha^d$ and the *current density* $\tilde{j}_\alpha^d := e_\alpha \tilde{\mathbf{n}}_\alpha^d \tilde{\mathbf{u}}_\alpha^d$ of the α^{th} species.

Lemma 2.1. *Assume that the distribution function $\tilde{f}_\alpha^d(\cdot)$ is of the form:*

$$(2.41) \quad \forall (\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) = \tilde{\mathcal{F}}_\alpha^d(\tilde{\mathbf{x}}, |\tilde{\mathbf{p}} - m_\alpha \tilde{\mathbf{u}}_\alpha^d(\tilde{\mathbf{x}})|),$$

where $\tilde{\mathcal{F}}_\alpha^d(\cdot) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}_+; \mathbb{R})$. Then, for some scalar function $\tilde{p}_\alpha^d(\cdot)$, the pressure tensor $\tilde{\mathbf{P}}_\alpha^d$ takes the form:

$$(2.42) \quad \tilde{\mathbf{P}}_\alpha^d(\tilde{\mathbf{x}}) = \tilde{p}_\alpha^d(\tilde{\mathbf{x}}) Id.$$

Proof. The integral over a symmetric domain of an odd function is equal to zero. Therefore, for $i \neq j$, with the change of variable $\tilde{P}_\alpha := \tilde{\mathbf{p}} - m_\alpha \tilde{\mathbf{u}}_\alpha^d$, we have:

$$\int_{\mathbb{R}^3} \left(\left(\tilde{\mathbf{v}}_\alpha - \tilde{\mathbf{u}}_\alpha^d \right) \otimes \left(\tilde{\mathbf{v}}_\alpha - \tilde{\mathbf{u}}_\alpha^d \right) \right)_{ij} \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}} = \frac{1}{m_\alpha^2} \int_{\mathbb{R}^3} \tilde{P}_\alpha^i \tilde{P}_\alpha^j \tilde{\mathcal{F}}_\alpha^d(\tilde{\mathbf{x}}, |\tilde{P}_\alpha^d|) d\tilde{P}_\alpha = 0.$$

On the other hand, for $i \neq 1$, changing \tilde{P}_α^i into \tilde{P}_α^1 , we get:

$$\tilde{p}_\alpha^d(\tilde{\mathbf{x}}) := \frac{1}{m_\alpha^2} \int_{\mathbb{R}^3} (\tilde{P}_\alpha^1)^2 \tilde{\mathcal{F}}_\alpha^d(\tilde{\mathbf{x}}, |\tilde{P}_\alpha|) d\tilde{P}_\alpha = \int_{\mathbb{R}^3} (\tilde{P}_\alpha^i)^2 \tilde{\mathcal{F}}_\alpha^d(\tilde{\mathbf{x}}, |\tilde{P}_\alpha|) d\tilde{P}_\alpha.$$

In view of these elements, we have (2.42). The function $\tilde{p}_\alpha^d(\cdot)$ thus defined is called the *scalar pressure* of the α^{th} species. \square

The condition of quasi-neutrality $\tilde{\rho}^d \equiv 0$ gives rise to the relation (2.17) on $\tilde{\mathbf{n}}_\alpha^d$. On the other hand, multiplying (2.9) by $\tilde{\mathbf{v}}_\alpha$ and integrating with respect to $\tilde{\mathbf{p}}$, we can extract:

$$(2.43) \quad m_\alpha \tilde{\mathbf{n}}_\alpha^d (\tilde{\mathbf{u}}_\alpha^d \cdot \nabla_{\tilde{\mathbf{x}}}) \tilde{\mathbf{u}}_\alpha^d + \nabla_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{P}}_\alpha^d = \tilde{j}_\alpha^d \times \tilde{\mathbf{B}}_e.$$

The introduction of $\tilde{\mathbf{u}}_\alpha^d(\cdot)$ is strongly motivated by the role achieved in tokamaks by the toroidal plasma current \tilde{j}_α^d . A common assumption [10, 23] which turns to be consistent with Assumption 2.5 is to take $\tilde{\mathbf{u}}_\alpha^d(\cdot)$ of the form:

$$(2.44) \quad \tilde{\mathbf{u}}_\alpha^d(\tilde{\mathbf{x}}) := \tilde{\rho} \Omega_\alpha e_{\tilde{\varphi}}, \quad \Omega_\alpha \in \mathbb{R}_+.$$

In (2.44), the number Ω_α is the angular rotation frequency. It is easy to see that:

$$(2.45) \quad (\tilde{\mathbf{u}}_\alpha^d \cdot \nabla_{\tilde{\mathbf{x}}}) \tilde{\mathbf{u}}_\alpha^d = -\tilde{\rho} \Omega_\alpha^2 e_{\tilde{\rho}}.$$

This accounts for the centrifugal force density. Moreover, as pointed out by Lemma 2.1, in an isotropic medium, the matrix $\tilde{\mathbf{P}}_\alpha^d$ is diagonal, of the form $\tilde{\mathbf{P}}_\alpha^d(\tilde{\mathbf{x}}) = \tilde{p}_\alpha^d(\tilde{\mathbf{x}}) Id$ where \tilde{p}_α^d is the scalar pressure of the α^{th} species. With (2.43) and (2.45), we obtain that \tilde{p}_α^d is linked to Ω_α and to the current density \tilde{j}_α^d of the α^{th} species through:

$$(2.46) \quad \nabla_{\tilde{\mathbf{x}}} \tilde{p}_\alpha^d = \tilde{j}_\alpha^d \times \tilde{\mathbf{B}}_e + m_\alpha \tilde{\mathbf{n}}_\alpha^d \tilde{\rho} \Omega_\alpha^2 e_{\tilde{\rho}}, \quad \tilde{j}_\alpha^d = e_\alpha \tilde{\mathbf{n}}_\alpha^d \tilde{\mathbf{u}}_\alpha^d.$$

Now, consider the total scalar pressure $\tilde{p}^d := \tilde{p}_1^d + \dots + \tilde{p}_N^d$. Due to the axisymmetric hypothesis, the function $\tilde{p}^d(\cdot)$ depends only on $(\tilde{\rho}, \tilde{z})$. Then, summing the equation (2.46) over α , we can infer that:

$$(2.47) \quad \nabla_{\tilde{\mathbf{x}}} \tilde{p}^d = \partial_{\tilde{\rho}} \tilde{p}^d e_{\tilde{\rho}} + \partial_{\tilde{z}} \tilde{p}^d e_{\tilde{z}} = \tilde{j}^d \times \tilde{\mathbf{B}}_e + \tilde{\rho} \left(\sum_{\alpha=1}^N m_\alpha \tilde{\mathbf{n}}_\alpha^d \Omega_\alpha^2 \right) e_{\tilde{\rho}}.$$

Using the decomposition $\tilde{j}^d = \tilde{j}^{dt} + \tilde{j}^{dp}$ of (2.38), it follows directly from (2.47) that:

$$(2.48a) \quad \tilde{j}^{dp} \times \tilde{\mathbf{B}}_e^p = 0,$$

$$(2.48b) \quad \partial_{\tilde{\rho}} \tilde{p}^d e_{\tilde{\rho}} + \partial_{\tilde{z}} \tilde{p}^d e_{\tilde{z}} = \tilde{j}^{dp} \times \tilde{\mathbf{B}}_e^t + \tilde{j}^{dt} \times \tilde{\mathbf{B}}_e^p + \tilde{\rho} \left(\sum_{\alpha=1}^N m_\alpha \tilde{\mathbf{n}}_\alpha^d \Omega_\alpha^2 \right) e_{\tilde{\rho}}.$$

From the MHD point of view expressed in (2.48), the centrifugal force is balanced by the magnetic and pressure forces at all points. In particular, inserting the expressions (2.32) and (2.38) of $\tilde{\mathbf{B}}_e^p$ and \tilde{j}^{dp} , the relation (2.48a) implies that $\nabla \tilde{\Psi} \cdot \nabla \tilde{g} = 0$. In particular, $\tilde{g}(\cdot)$ is constant along the curves \mathcal{C}_ψ , and therefore $\tilde{g}(\cdot)$ can be expressed in terms of $\tilde{\Psi}$.

In other words, $\tilde{g} = \tilde{g}(\tilde{\Psi})$ for some $\tilde{g} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})$. Then, (2.48b) together with (2.32) and (2.38) gives rise to the following vectorial equilibrium equation [26]:

$$(2.49) \quad \nabla \tilde{p}^d = -\frac{\epsilon_0 c_0^2}{\tilde{\rho}^2} \Delta^* \tilde{\Psi} \nabla \tilde{\Psi} - \frac{\epsilon_0 c_0^2}{\tilde{\rho}^2} \tilde{g}(\tilde{\Psi}) \tilde{g}'(\tilde{\Psi}) \nabla \tilde{\Psi} + \tilde{\rho} \left(\sum_{\alpha=1}^N m_\alpha \tilde{\mathbf{n}}_\alpha^d \Omega_\alpha^2 \right) e_{\tilde{\rho}}.$$

Moreover, the function $\tilde{p}^d(\cdot)$ can always be written in the form $\tilde{p}^d = \tilde{P}(\tilde{\rho}, \tilde{\Psi})$ for some function $\tilde{P} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$. A first projection of equation (2.49) in the $e_{\tilde{\rho}}$ direction yields:

$$(2.50) \quad \left. \frac{\partial \tilde{P}}{\partial \tilde{\rho}} \right|_{\tilde{\Psi}=cste} = \tilde{\rho} \left(\sum_{\alpha=1}^N m_\alpha \tilde{\mathbf{n}}_\alpha^d \Omega_\alpha^2 \right).$$

Then, a second projection of equation (2.49) in the direction of $\nabla \tilde{\Psi}$ gives the *extended Grad-Shafranov* equation:

$$(2.51) \quad \Delta^* \tilde{\Psi} = -\frac{\tilde{\rho}^2}{\epsilon_0 c_0^2} \left. \frac{\partial \tilde{P}}{\partial \tilde{\Psi}} \right|_{\tilde{\rho}=cste} - \tilde{g}(\tilde{\Psi}) \tilde{g}'(\tilde{\Psi}).$$

The equation (2.51) has been much studied [15, 17, 18, 26] because it gives access to the geometry of the magnetic surfaces. It is scale invariant through:

$$(2.52) \quad \tilde{\Psi} / \alpha \tilde{\Psi}, \quad \tilde{P} / \alpha^2 \tilde{P}, \quad \tilde{g} / \alpha \tilde{g}, \quad \alpha \in \mathbb{R}.$$

Note that the change (2.52) does not affect the shape of magnetic surfaces. Observe also that the number Ω_α plays only an indirect role at the level of (2.51), through (2.50).

2.3.3. Kinetic equilibria in magnetized plasmas. The fluid theory that has been outlined in Paragraph 2.3.2 is the most common way to study tokamak equilibria. However, it faces significant challenges due to the well-known **closure problem**. In (2.43), the pressure tensor $\tilde{\mathbf{P}}_\alpha^d(\cdot)$ is an unknown. Except under special restrictions (see Paragraph 2.3.2), it cannot be expressed in terms of $\tilde{\mathbf{u}}_\alpha^d$. The equation (2.43) is not self-contained. From that perspective, the kinetic framework offers a more consistent, thorough and precise approach. As a consequence, the study of tokamak equilibria through a kinetic approach has been the subject of intensive research over the last few years in both physics [10, 23] and mathematics [24]. In [10, 23], the purpose is to construct exact (or approximate) solutions to the stationary equation:

$$(2.53) \quad \tilde{\mathbf{p}} \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{f}_\alpha^d + e_\alpha [\tilde{\mathbf{p}} \times \tilde{\mathbf{B}}_e(\tilde{\mathbf{x}})] \cdot \nabla_{\tilde{\mathbf{p}}} \tilde{f}_\alpha^d = 0.$$

To this end, the existence of constants of motion (or of adiabatic invariants) is useful. Obviously, in the case $\tilde{\mathbf{E}} \equiv 0$, the *kinetic energy*:

$$(2.54) \quad E_\alpha \equiv E_\alpha(\tilde{\mathbf{p}}) := \frac{1}{2m_\alpha} |\tilde{\mathbf{p}}|^2,$$

is preserved by the flow associated to the Vlasov equation. The same applies to any function of $|\tilde{\mathbf{p}}|$. It follows that the function $E_\alpha(\cdot)$ is a solution to (2.53). There are others.

Definition 2.1. [*angular momentum*] The angular momentum is the quantity defined by:

$$(2.55) \quad C_\alpha \equiv C_\alpha(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) := m_\alpha (\tilde{\rho} \tilde{\mathbf{p}} \cdot e_{\tilde{\rho}} + e_\alpha \tilde{\Psi}).$$

Under Assumption 2.5 (of axisymmetry), another invariant is available.

Lemma 2.2. *The angular momentum is a constant of motion. The function $C_\alpha(\cdot)$ is a solution to the equation (2.53).*

Proof. Decompose $\tilde{\mathbf{p}}$ into $\tilde{\mathbf{p}} = \tilde{\rho}_\beta e_\rho + \tilde{\rho}_\varphi e_\varphi + \tilde{\rho}_z e_z$. Recall that $\tilde{\mathbf{B}}_e = \tilde{\mathbf{B}}_e^t + \tilde{\mathbf{B}}_e^p$ with $\tilde{\mathbf{B}}_e^t$ and $\tilde{\mathbf{B}}_e^p$ as in (2.32). The condition (2.53) tested with $\tilde{f}_\alpha^d \equiv C_\alpha$ amounts to the same thing as:

$$(2.56) \quad \left[(\tilde{\rho}_\beta \partial_\beta + \tilde{\rho}^{-1} \tilde{\rho}_\varphi \partial_\varphi) (\tilde{\rho} \tilde{\mathbf{p}}_\varphi) \right] + e_\alpha \left[\tilde{\mathbf{p}} \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{\Psi} + (\tilde{\mathbf{p}} \times (\nabla_{\tilde{\mathbf{x}}} \tilde{\Psi} \times e_\varphi)) \cdot e_\varphi \right] = 0.$$

In (2.56), the two parts inside brackets are zero. \square

More generally, any function of E_α and C_α is a solution to (2.53). Such prototypes of (non-maxwellian) axisymmetric equilibria are introduced and studied in [10, 23] for their practical relevance. In particular, they allow to incorporate some anisotropy (in $\tilde{\mathbf{p}}$), to take into account the existence of a non trivial toroidal current, and to work with spatially confined data. To avoid technicalities, we will only consider functions of separate variables.

Definition 2.2. *[tokamak transient distributions] Any function $\tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ having the form:*

$$(2.57) \quad \begin{aligned} \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) &:= \frac{n_\alpha^d}{(m_\alpha c_0 \theta_\alpha^d)^3} F_\alpha^d \left(\frac{|\tilde{\mathbf{p}}|^2}{(m_\alpha c_0 \theta_\alpha^d)^2} \right) G_\alpha^d \left(\frac{1}{e_\alpha L^2 b_e} \cdot (\tilde{\rho} \tilde{\mathbf{p}} \cdot e_\varphi + e_\alpha \tilde{\Psi}) \right) \\ &= \frac{n_\alpha^d}{(m_\alpha c_0 \theta_\alpha^d)^3} F_\alpha^d \left(\frac{2 E_\alpha}{m_\alpha (c_0 \theta_\alpha^d)^2} \right) G_\alpha^d \left(\frac{1}{e_\alpha L^2 b_e} \cdot \frac{C_\alpha}{m_\alpha} \right) \end{aligned}$$

with $F_\alpha^d(\cdot) \in \mathcal{S}(\mathbb{R}_+; \mathbb{R})$ and $G_\alpha^d(\cdot) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})$ is called a tokamak transient distribution.

As long as the temperature θ_α^d (see Paragraph 2.2.1) is small enough, such that $\theta_\alpha^d \leq 10^{-3}$, one can consider that the momentum $\tilde{\mathbf{p}}$ satisfies some usual statistical repartition around the toroidal flow velocity $\tilde{\mathbf{u}}_\alpha^d(\cdot)$. Then, the choice of $\tilde{f}_\alpha^d(\cdot)$ can be further specified.

Definition 2.3. *[shifted Maxwell-Boltzmann distribution] The notion of a shifted Maxwell-Boltzmann distribution refers to the special choice:*

$$(2.58) \quad \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) = \frac{\tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}})}{m_\alpha^3 c_0^3} \mathcal{M}_{\theta_\alpha^d}^b \left(\frac{|\tilde{\mathbf{p}} - m_\alpha \tilde{\mathbf{u}}_\alpha^d(\tilde{\mathbf{x}})|}{m_\alpha c_0} \right), \quad \mathcal{M}_\theta^b(r) := \frac{1}{\pi^{3/2}} \frac{1}{\theta^3} \exp \left(-\frac{r^2}{\theta^2} \right).$$

In (2.58), the vector field $\tilde{\mathbf{u}}_\alpha^d(\cdot)$ represents the flow velocity of the α^{th} species, as it can be given by (2.44). On the other hand, the function $\tilde{\mathbf{n}}_\alpha^d(\cdot)$ is determined by:

$$(2.59) \quad \tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}}) := n_\alpha^d \exp \left(\frac{1}{(c_0 \theta_\alpha^d)^2} \left[|\tilde{\mathbf{u}}_\alpha^d(\tilde{\mathbf{x}})|^2 + \frac{2 e_\alpha}{m_\alpha} \Omega_\alpha \tilde{\Psi} \right] \right).$$

Remark that (2.58) is indeed some particular case of (2.57), with:

$$(2.60) \quad F_\alpha^d(r) := \frac{1}{\pi^{3/2}} \exp(-r), \quad G_\alpha^d(s) := \exp \left(\frac{2 \Omega_\alpha e_\alpha L^2 b_e}{m_\alpha (c_0 \theta_\alpha^d)^2} s \right).$$

Recall (2.5) which says that $\tilde{f}_\alpha^k = \tilde{f}_\alpha^d + \nu \tilde{f}_\alpha^s$ with $\nu \ll 1$. The relativistic features come from the possible presence of energetic particles. As long as the fraction of such particles remains small, the relativistic effects are restricted to the perturbation $\tilde{f}_\alpha^s(\cdot)$. On the other hand, the part \tilde{f}_α^d can take into account cold, warm or hot aspects (see Paragraph 2.2.1). When adjusting $\tilde{f}_\alpha^d(\cdot)$, it is important to look at the physical data.

Discussion 2.2. *[about the dominant stationary part \tilde{f}_α^d] Tokamaks involve special values of temperatures. Two regions can be distinguished. The edge region implies a diluted plasma with $T_e \simeq 50$ eV, so that $\theta_1^d \simeq 10^{-2}$ whereas $\theta_\alpha^d \ll \theta_1^d$ for $\alpha \neq 1$. It separates the cooler vessel wall from the plasma core. The core region contains a hotter plasma, but not that much. We find $T_e \simeq 5$ keV, and we still have $\theta_\alpha^d \ll \theta_1^d \simeq 10^{-1}$ for $\alpha \neq 1$. \circ*

Taking into account these observations, we can point out the following.

Assumption 2.8. *[ions in a state of local cold thermodynamic equilibrium] For all $\alpha \neq 1$, we impose (2.58) with $\theta_\alpha^d \leq 10^{-3}$.*

The shifted Maxwell-Boltzmann distribution $\tilde{f}_\alpha^d(\cdot)$ of (2.58) is clearly subjected to (2.41). Therefore, for $\alpha \neq 1$, the pressure tensor $\tilde{\mathbf{P}}_\alpha^d$ is diagonal. It is of the form $\tilde{\mathbf{P}}_\alpha^d = \tilde{p}_\alpha^d Id$, where \tilde{p}_α^d is some scalar pressure. For $\alpha \neq 1$, the equation (2.46) is satisfied and the fluid approach of Paragraph 2.3.2 does make sense. Now, the model (2.58) is not sure to apply when dealing with electrons. As a matter of fact, as revealed by (2.2), electrons are much lighter than ions and neutral atoms. This has two main consequences [10, 23]:

- First, the electrons can faster reach high energies. Thus, a certain level of flexibility must be achieved concerning $\tilde{f}_1^d(\cdot)$. It is preferable to use (2.57) instead of (2.58). This allows to better localize the support or the properties of the function $\tilde{f}_1^d(\cdot)$;
- Secondly, the electrons are more sensitive to perturbations. Such disturbances are induced for instance by the injection of neutral beams or radio waves. They can induce anisotropic effects with respect to the variable $|\tilde{\mathbf{p}} - m_\alpha \tilde{\mathbf{u}}_\alpha^d(\tilde{\mathbf{x}})|$. The corresponding aspects are not detected by (2.58) which involves only $|\tilde{\mathbf{p}} - m_\alpha \tilde{\mathbf{u}}_\alpha^d(\tilde{\mathbf{x}})|$. They can be taken into account at the level of $\tilde{f}_\alpha^s(\cdot)$. They can also be incorporated (albeit partially) through (2.57).

Assumption 2.9. *[electrons in a state of transient kinetic equilibrium] The distribution function $\tilde{f}_1^d(\cdot)$ is chosen as indicated in (2.57), with expressions $F_1^d(\cdot)$ and $G_1^d(\cdot)$ adjusted in order to achieve the quasi-neutrality condition:*

$$(2.61) \quad e \int_{\mathbb{R}^3} \tilde{f}_1^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) d\tilde{\mathbf{p}} = \sum_{\alpha=2}^N e_\alpha \tilde{\mathbf{n}}_\alpha^d(\tilde{\mathbf{x}}), \quad \forall \tilde{\mathbf{x}} \in \tilde{\Omega}.$$

For general choices of $\tilde{f}_1^d(\cdot)$, the pressure tensor $\tilde{\mathbf{P}}_1^d$ is not at all diagonal. It follows that the equation (2.46) does not hold. On the other hand, the relation (2.38) can still be exploited. In particular, the second equation of (2.38) yields:

$$(2.62) \quad \Delta^* \tilde{\Psi} = \frac{\tilde{\rho}}{\epsilon_0 c_0^2} \int_{\mathbb{R}^3} (-\sin \tilde{\varphi} \tilde{\mathbf{v}}^1 + \cos \tilde{\varphi} \tilde{\mathbf{v}}^2) \left(\sum_{\alpha=1}^N e_\alpha \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) \right) d\tilde{\mathbf{p}}.$$

In view of (2.57) and (2.58)-(2.59), the expressions $\tilde{f}_\alpha^d(\cdot)$ are non linear functions of $\tilde{\Psi}$. Thus, the condition (2.62) can be interpreted as a non linear elliptic equation. The dependence of (2.62) on Ω_α can still appear through (2.60). However, the equation (2.62) cannot be scaled as in (2.52). There are clear differences between (2.51) and (2.62).

Under reasonable assumptions, that is with adequate (possibly free) boundary conditions, the equation (2.62) can be solved. We refer for instance to the appendix of [24] or to the sections III and IV of [23]. A realistic adjustment through $\tilde{\Psi}(\cdot)$ of the magnetic flux surfaces is an important piece of information before looking at more refined aspects. In what follows, the function $\tilde{\Psi}(\cdot)$ is assumed to satisfy the condition (2.62), which is equivalent to (2.51) when the fluid approach is affordable. From now on, the external magnetic field $\tilde{\mathbf{B}}_e(\cdot)$ is fixed as indicated in (2.32), and the kinetic distribution functions $\tilde{f}_\alpha^d(\cdot)$ are adjusted as in Assumptions 2.8 and 2.9.

2.4. Dimensionless equations. The aim here is to obtain a dimensionless form of the system satisfied by the perturbation $\tilde{f}_\alpha^s(\cdot)$. This means first to perform a stability analysis in the vicinity of $\tilde{f}_\alpha^d(\cdot)$ (Paragraph 2.4.1). In Paragraph 2.4.2, we define rescaled unknowns and variables. Then, it is useful to straighten the field lines (Paragraph 2.4.3). Moreover, in connection with the application to the tokamak's context, it is important to give (Paragraph 2.4.4) a precise description of (the relative sizes of) the various physical parameters.

2.4.1. Perturbation theory. To grasp **plasma instabilities** or plasma processes which are not in thermal equilibrium, we can perform a stability analysis in the vicinity of $\tilde{f}_\alpha^d(\cdot)$. This amounts to add a small term $\nu \tilde{f}_\alpha^s(\cdot)$ with $\nu \ll 1$ to the exact solution $\tilde{f}_\alpha^d(\cdot)$. In other words, the solution $\tilde{f}_\alpha^k(\cdot)$ to (2.7) is broken into:

$$(2.63) \quad \tilde{f}_\alpha^k(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) = \tilde{f}_\alpha^d(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) + \nu \tilde{f}_\alpha^s(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}), \quad (\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}) \in \mathbb{R}_+ \times \tilde{\Omega} \times \mathbb{R}^3.$$

With a dominant stationary part $\tilde{f}_\alpha^d(\cdot)$ given by the shifted Maxwell-Boltzmann distribution (2.58) or by the tokamak transient distribution (2.57), the extra part $\tilde{f}_\alpha^s(\cdot)$ is governed by:

$$(2.64) \quad \begin{aligned} \partial_{\tilde{\mathbf{t}}} \tilde{f}_\alpha^s + \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{f}_\alpha^s + e_\alpha [\tilde{\mathbf{E}} + \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \times (\tilde{\mathbf{B}} + \tilde{\mathbf{B}}_e)] \cdot \nabla_{\tilde{\mathbf{p}}} \tilde{f}_\alpha^s \\ = - \frac{e_\alpha}{\nu} [\tilde{\mathbf{E}} + \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}) \times \tilde{\mathbf{B}}] \cdot \nabla_{\tilde{\mathbf{p}}} \tilde{f}_\alpha^d. \end{aligned}$$

In view of (2.10), the Maxwell's equations give rise to:

$$(2.65) \quad \begin{cases} \partial_{\tilde{\mathbf{t}}} \tilde{\mathbf{B}} + \nabla_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{E}} = 0, \\ \partial_{\tilde{\mathbf{t}}} \tilde{\mathbf{E}} - c_0^2 \nabla_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{B}} = -\epsilon_0^{-1} \nu \tilde{j}(\tilde{f}_1^s, \dots, \tilde{f}_N^s), \end{cases}$$

together with:

$$(2.66) \quad \nabla_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{E}} = \frac{\nu}{\epsilon_0} \tilde{\rho}(\tilde{f}_1^s, \dots, \tilde{f}_N^s), \quad \nabla_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{B}} = 0.$$

2.4.2. *Rescalings.* The dimensionless version $\mathbf{A}(\cdot)$ of $\tilde{\mathbf{A}}(\cdot)$ is:

$$(2.67) \quad \mathbf{A}(\mathbf{x}) := \frac{1}{L b_e} \tilde{\mathbf{A}}(L \mathbf{x}), \quad \mathbf{x} := \frac{\tilde{\mathbf{x}}}{L} \in \Omega, \quad \rho := \frac{\tilde{\rho}}{L} \in \mathbb{R}_+, \quad z := \frac{\tilde{z}}{L} \in \mathbb{R}.$$

The dimensionless function $\Psi(\cdot)$ associated with the magnetic flux function $\tilde{\Psi}(\cdot)$ is:

$$(2.68) \quad \Psi(\rho, z) := \frac{1}{L^2 b_e} \tilde{\Psi}(L \rho, L z).$$

From the Ampère's law in (2.65), we can infer that $\tilde{\mathbf{B}} \simeq \nu \theta_\alpha^d b_e$. With this in mind, we can further define new unknowns by the relations:

$$(2.69a) \quad \mathbf{v}_\alpha(\mathbf{p}) := (c_0)^{-1} \tilde{\mathbf{v}}_\alpha(\tilde{\mathbf{p}}), \quad \mathbf{p}_\alpha := (m_\alpha c_0 \theta_\alpha^d)^{-1} \tilde{\mathbf{p}}_\alpha, \quad \forall \alpha \in \{1, \dots, N\},$$

$$(2.69b) \quad f_\alpha(\mathbf{t}, \mathbf{x}, \mathbf{p}_\alpha) := (n_\alpha^d)^{-1} m_\alpha^3 c_0^3 (\theta_\alpha^d)^3 \tilde{f}_\alpha^s(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}, \tilde{\mathbf{p}}), \quad \forall \alpha \in \{1, \dots, N\},$$

$$(2.69c) \quad \mathbf{E}(\mathbf{t}, \mathbf{x}) := (\nu \theta_1^d c_0 b_e)^{-1} \tilde{\mathbf{E}}(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}), \quad \mathbf{B}(\mathbf{t}, \mathbf{x}) := (\nu \theta_1^d b_e)^{-1} \tilde{\mathbf{B}}(\tilde{\mathbf{t}}, \tilde{\mathbf{x}}).$$

From now on, the time and spatial positions are \mathbf{t} and \mathbf{x} with $(\mathbf{t}, \mathbf{x}) \in M := [0, T] \times \Omega$ for some $T \in \mathbb{R}_+^*$. Let T^*M be the cotangent bundle associated with M . With (2.69a), the vectors \mathbf{v}_α and \mathbf{p}_α are linked by the relations issued from (2.4), that is:

$$(2.70) \quad \mathbf{p}_\alpha(\mathbf{v}_\alpha) := \frac{\mathbf{v}_\alpha}{\theta_\alpha^d (1 - |\mathbf{v}_\alpha|^2)^{1/2}}, \quad \mathbf{v}_\alpha(\mathbf{p}_\alpha) := \frac{\theta_\alpha^d \mathbf{p}_\alpha}{\langle \theta_\alpha^d | \mathbf{p}_\alpha | \rangle}, \quad \langle r \rangle := \sqrt{1 + r^2}.$$

Among the fundamental **plasma parameters**, we can mention (for $\alpha = 1$) the electron gyrofrequency (or cyclotron frequency) $\omega_{ce} \equiv \omega_{c1}$ and the electron plasma frequency (or plasma oscillation) $\omega_{pe} \equiv \omega_{p1}$. For $\alpha \in \{2, \dots, N\}$, we can cite the ion gyrofrequencies $\omega_{c\alpha}$ and the ion plasma frequencies $\omega_{p\alpha}$. For simplicity of presentation, we define below these quantities with an algebraic sign:

$$(2.71) \quad \omega_{c\alpha} := \frac{e_\alpha b_e}{m_\alpha}, \quad \omega_{p\alpha} := \sqrt{\frac{n_\alpha^d e_\alpha^2}{m_\alpha \epsilon_0}}, \quad \forall \alpha \in \{1, \dots, N\}.$$

There are corresponding dimensionless coefficients ε_α and μ_α , given by:

$$(2.72) \quad \varepsilon_\alpha := (L \omega_{c\alpha})^{-1} c_0, \quad \mu_\alpha := (\omega_{c\alpha})^{-1} \omega_{p\alpha}, \quad \forall \alpha \in \{1, \dots, N\}.$$

Then, the new Vlasov equation is:

$$(2.73) \quad \begin{aligned} & \partial_t f_\alpha + \frac{\theta_\alpha^d}{\langle \theta_\alpha^d | \mathbf{p}_\alpha | \rangle} \mathbf{p}_\alpha \cdot \nabla_{\mathbf{x}} f_\alpha + \frac{\theta_1^d}{\theta_\alpha^d} \frac{\nu}{\varepsilon_\alpha} \left[\mathbf{E} + \frac{\theta_\alpha^d}{\langle \theta_\alpha^d | \mathbf{p}_\alpha | \rangle} \mathbf{p}_\alpha \times \mathbf{B} \right] \cdot \nabla_{\mathbf{p}_\alpha} f_\alpha \\ & + \frac{1}{\varepsilon_\alpha} \frac{1}{\langle \theta_\alpha^d | \mathbf{p}_\alpha | \rangle} (\mathbf{p}_\alpha \times \mathbf{B}_e(\mathbf{x})) \cdot \nabla_{\mathbf{p}_\alpha} f_\alpha \\ & + \frac{\theta_1^d}{\theta_\alpha^d \varepsilon_\alpha} \left[\mathbf{E} + \frac{\theta_\alpha^d}{\langle \theta_\alpha^d | \mathbf{p}_\alpha | \rangle} \mathbf{p}_\alpha \times \mathbf{B} \right] \cdot \nabla_{\mathbf{p}_\alpha} f_\alpha^d(\mathbf{x}, \mathbf{p}_\alpha) = 0, \end{aligned}$$

where:

$$(2.74) \quad \begin{aligned} f_\alpha^d(\mathbf{x}, \mathbf{p}_\alpha) &:= (n_\alpha^d)^{-1} m_\alpha c_0 (\theta_\alpha^d)^3 \tilde{f}_\alpha^d(L \mathbf{x}, m_\alpha c_0 \theta_\alpha^d \mathbf{p}_\alpha) \\ &= F_\alpha^d(|\mathbf{p}_\alpha|^2) G_\alpha^d(\varepsilon_\alpha \theta_\alpha^d \rho \mathbf{p}_\alpha \cdot e_\varphi + \Psi(\rho, z)). \end{aligned}$$

On the other hand, the Maxwell's equations become:

$$(2.75a) \quad \partial_t \mathbf{B} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \quad \partial_t \mathbf{E} - \nabla_{\mathbf{x}} \times \mathbf{B} = -j(f_\alpha),$$

$$(2.75b) \quad \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0, \quad \nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho(f_\alpha),$$

where we have introduced:

$$(2.76a) \quad \rho(f_1, \dots, f_N)(t, \mathbf{x}) \equiv \rho(f_\alpha)(t, \mathbf{x}) := \sum_{\alpha=1}^N \frac{1}{\theta_1^d} \frac{\mu_\alpha^2}{\varepsilon_\alpha} \int_{\mathbb{R}^3} f_\alpha(t, \mathbf{x}, \mathbf{p}_\alpha) d\mathbf{p}_\alpha,$$

$$(2.76b) \quad j(f_1, \dots, f_N)(t, \mathbf{x}) \equiv j(f_\alpha)(t, \mathbf{x}) := \sum_{\alpha=1}^N \frac{\theta_\alpha^d}{\theta_1^d} \frac{\mu_\alpha^2}{\varepsilon_\alpha} \int_{\mathbb{R}^3} \frac{\mathbf{p}_\alpha}{\langle \theta_\alpha^d |\mathbf{p}_\alpha| \rangle} f_\alpha(t, \mathbf{x}, \mathbf{p}_\alpha) d\mathbf{p}_\alpha.$$

2.4.3. *Straightening the field lines.* Equation (2.73) is not yet in a suitable form. Still, we need to straighten out the field lines. Recall (2.22)-(2.24) so that:

$$(2.77) \quad \mathbf{B}_e(\mathbf{x}) = \mathbf{b}_e(\mathbf{x}) e_3(\mathbf{x}) = \mathbf{b}_e(\mathbf{x}) O(\mathbf{x}) {}^t(0, 0, 1), \quad \forall \mathbf{x} \in \Omega.$$

In view of Assumptions 2.6 and 2.7, the directions of the unit vector field $e_3(\cdot)$, and therefore of $\mathbf{B}_e(\cdot)$, can vary with changes in $\mathbf{x} \in \Omega$. To remedy this situation, we replace simultaneously \mathbf{B}_e , \mathbf{B} , \mathbf{E} and \mathbf{p}_α according to:

$$(2.78) \quad \mathbf{b}_e {}^t(0, 0, 1) = {}^tO \mathbf{B}_e, \quad B := {}^tO \mathbf{B}, \quad E := {}^tO \mathbf{E}, \quad p_\alpha := {}^tO \mathbf{p}_\alpha.$$

For the sake of simplicity, the subscript α that identifies the different momentum variables p_α will be omitted. Concerning $p \equiv p_\alpha \in \mathbb{R}^3$, we can pass from cartesian to spherical coordinates, with orthonormal basis $(e_r, e_\varpi, e_\omega)$. This gives rise to:

$$(2.79) \quad p = r {}^t(\cos \omega \sin \varpi, \sin \omega \sin \varpi, \cos \varpi), \quad (\varpi, \omega, r) \in \mathbb{T}^2 \times \mathbb{R}_+, \quad r = |p| = |\mathbf{p}|.$$

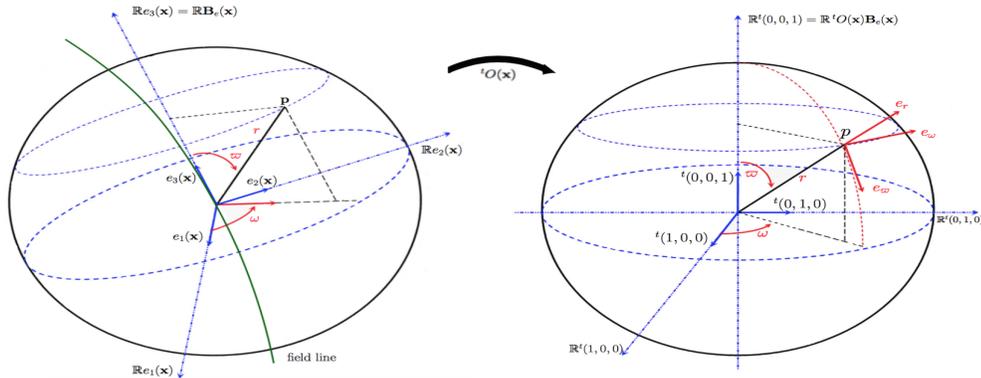


FIGURE 4. Spherical coordinates of $p \in \mathbb{R}^3$ after straightening.

From now on, the spatial-velocity position is marked by $\mathbf{y} := (\mathbf{x}, \varpi, \omega, r) \in \Omega \times \mathbb{T}^2 \times \mathbb{R}_+$. We modify $f_\alpha(\cdot)$ to fit with the preceding adjustments:

$$f_\alpha(\mathbf{t}, \mathbf{y}) \equiv f(\mathbf{t}, \mathbf{x}, \varpi, \omega, r) := f_\alpha(\mathbf{t}, \mathbf{x}, r O(\mathbf{x}) (\cos \omega \sin \varpi, \sin \omega \sin \varpi, \cos \varpi)).$$

As usual, the symbol \mathcal{S} refers to the Schwartz space. We consider functions $f(\cdot)$ satisfying uniformly in $(\mathbf{t}, \mathbf{x}, \varpi, \omega) \in M \times \mathbb{T}^2$ the conditions:

$$(2.80) \quad f \in \mathcal{C}^\infty(M \times \mathbb{T}^2 \times \mathbb{R}_+; \mathbb{R}), \quad f(\mathbf{t}, \mathbf{x}, \varpi, \omega, \cdot) \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}).$$

The gradient $\nabla_{\mathbf{p}}$ is converted into the spherical gradient ∇_p , with:

$$\nabla_p f := \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \varpi} e_\varpi + \frac{1}{r \sin \varpi} \frac{\partial f}{\partial \omega} e_\omega.$$

The change of variables $(\mathbf{x}, \mathbf{p}) \rightarrow (\mathbf{x}, p)$ on the right of (2.78) induces some extra term when transforming $(\mathbf{v} \cdot \nabla_{\mathbf{x}})f$ accordingly. Some application $Q(\cdot)$ does appear. This is a vector valued quadratic form in p , namely:

$$Q(\mathbf{x}, p) := \begin{pmatrix} O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^1} e_1 & O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^2} e_1 & O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^3} e_1 \\ O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^1} e_2 & O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^2} e_2 & O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^3} e_2 \\ O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^1} e_3 & O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^2} e_3 & O(\mathbf{x}) p \cdot \partial_{\mathbf{x}^3} e_3 \end{pmatrix} O(\mathbf{x}) p \in \mathbb{R}^3.$$

Put aside the integral operators:

$$(2.81a) \quad \rho(f) := \int_0^{+\infty} \int_0^\pi \int_{-\pi}^\pi f(\varpi, \omega, r) r^2 \sin \varpi dr d\varpi d\omega,$$

$$(2.81b) \quad \mathcal{J}(\theta; f) := \int_0^{+\infty} \int_0^\pi \int_{-\pi}^\pi \frac{r^3}{\langle \theta r \rangle} \begin{pmatrix} \cos \omega \sin \varpi \\ \sin \omega \sin \varpi \\ \cos \varpi \end{pmatrix} f(\varpi, \omega, r) \sin \varpi dr d\varpi d\omega.$$

After straighttenning, we obtain:

$$(2.82) \quad \begin{aligned} & \partial_{\mathbf{t}} f_\alpha + \frac{\theta_\alpha^{\text{d}}}{\langle \theta_\alpha^{\text{d}} r \rangle} O(\mathbf{x}) p \cdot \nabla_{\mathbf{x}} f_\alpha + \frac{\theta_\alpha^{\text{d}}}{\langle \theta_\alpha^{\text{d}} r \rangle} Q(\mathbf{x}, p) \cdot \nabla_p f_\alpha \\ & + \frac{\theta_1^{\text{d}}}{\theta_\alpha^{\text{d}}} \frac{\nu}{\varepsilon_\alpha} \left[E + \frac{\theta_\alpha^{\text{d}}}{\langle \theta_\alpha^{\text{d}} r \rangle} p \times B \right] \cdot \nabla_p f_\alpha - \frac{1}{\varepsilon_\alpha} \frac{\mathbf{b}_e(\mathbf{x})}{\langle \theta_\alpha^{\text{d}} r \rangle} (-p_2 \partial_{p_1} + p_1 \partial_{p_2}) f_\alpha \\ & + \frac{\theta_1^{\text{d}}}{\theta_\alpha^{\text{d}}} \frac{1}{\varepsilon_\alpha} \left[E + \frac{\theta_\alpha^{\text{d}}}{\langle \theta_\alpha^{\text{d}} r \rangle} p \times B \right] \cdot \nabla_p (f_\alpha^{\text{d}}(\mathbf{x}, O(\mathbf{x}) p)) = 0. \end{aligned}$$

On the other hand, the Maxwell's equations become:

$$(2.83a) \quad O(\mathbf{x}) \partial_{\mathbf{t}} B + \nabla_{\mathbf{x}} \times (O(\mathbf{x}) E) = 0, \quad \nabla_{\mathbf{x}} \cdot (O(\mathbf{x}) B) = 0,$$

$$(2.83b) \quad O(\mathbf{x}) \partial_{\mathbf{t}} E - \nabla_{\mathbf{x}} \times (O(\mathbf{x}) B) = -j^{\text{h}}(f_\alpha), \quad \nabla_{\mathbf{x}} \cdot (O(\mathbf{x}) E) = \rho^{\text{h}}(f_\alpha).$$

In (2.92), the expressions $j^h(f_\alpha)$ and $\rho^h(f_\alpha)$ are given by:

$$(2.84a) \quad \rho^h(f_1, \dots, f_N)(\mathbf{t}, \mathbf{x}) \equiv \rho^h(f_\alpha)(\mathbf{t}, \mathbf{x}) := \sum_{\alpha=1}^N \frac{1}{\theta_1^d} \frac{\mu_\alpha^2}{\varepsilon_\alpha} \rho(f_\alpha),$$

$$(2.84b) \quad j^h(f_1, \dots, f_N)(\mathbf{t}, \mathbf{x}) \equiv j^h(f_\alpha)(\mathbf{t}, \mathbf{x}) := \sum_{\alpha=1}^N \frac{\theta_\alpha^d}{\theta_1^d} \frac{\mu_\alpha^2}{\varepsilon_\alpha} \mathcal{J}(\theta_\alpha^d; f_\alpha).$$

2.4.4. *Comparison of the dimensionless parameters.* For further analysis, it is crucial to produce values for the parameters ε_α , θ_α^d and μ_α which could be meaningful from a physical viewpoint. It is also important to compare these quantities to one another. To this end, the following dimensionless number:

$$(2.85) \quad \varepsilon \equiv |\varepsilon_1| := \frac{c_0}{L |\omega_{c1}|} = \frac{c_0 m_e}{L e b_e} \simeq \frac{10^{-3}}{L b_e},$$

(coming from the inverse of the **electron cyclotron frequency**) will serve as a unit of measure.

Discussion 2.3. [about the size of ε] As indicated in (2.72), the number ε is defined as the ratio between the reference frequency $1/T = c_0/L$ and the gyrofrequency ω_{ce} . This turns out to be a small parameter. For fusion devices like ITER [38], we find $\varepsilon \simeq 10^{-4}$. \circ

From now on, we take $\varepsilon := 10^{-4} \ll 1$ as the small reference parameter to which all other quantities will be compared. For instance, with (2.2), keep in mind that:

$$(2.86) \quad |\varepsilon_\alpha| = \frac{e}{|e_\alpha|} \frac{m_\alpha}{m_e} \varepsilon \simeq \frac{\varepsilon}{\iota_\alpha} \gtrsim \beta \varepsilon \simeq 1, \quad \forall \alpha \in \{2, \dots, N\}.$$

Discussion 2.4. [about the size of the coefficients θ_α^d] In view of (2.15), the numbers θ_α^d increase with the temperatures T_α , see (2.14). They decrease with the masses m_α , see (2.2). The ratio $\theta_\alpha^d/\theta_1^d$, which is given by (2.16), remains relevant. The comparison of θ_1^d with 1 and of all the other coefficients θ_α^d for $\alpha \neq 1$ with ε seems acceptable. \circ

Discussion 2.5. [about the size of the coefficients μ_α] In view of (2.71)-(2.72), the access to μ_1 requires to evaluate b_e , m_e and n_1^d . How to fix b_e and m_e has already been explained. As regards the electron density, we can take the value $n_1^d \equiv n_\alpha^d = 10^{20}$ electrons/ m^3 [28, 38]. Let us recall that:

$$(2.87) \quad |\mu_\alpha| = \left(\frac{n_\alpha^d m_\alpha}{\varepsilon_0} \right)^{1/2} \frac{1}{b_e} = \left(\frac{n_\alpha^d m_\alpha}{n_1^d m_1} \right)^{1/2} |\mu_1| \gtrsim |\mu_1|, \quad \forall \alpha \in \{1, \dots, N\}.$$

The higher value $n_1^d \simeq 10^{20}$ electrons/ m^3 is compensated by the presence of a stronger magnetic field $b_e \simeq 4,5 T$. We can again assert that $|\mu_1| \simeq 1$. We also remind that:

$$(2.88) \quad \frac{\mu_\alpha^2}{\varepsilon_\alpha} = \frac{e_\alpha n_\alpha^d}{e_1 n_1^d} \frac{\mu^2}{\varepsilon} \simeq \frac{\mu^2}{\varepsilon}, \quad \forall \alpha \in \{1, \dots, N\}. \quad \circ$$

In practice, the value of $\mu := |\mu_1|$ is of size 1. To track the influence of μ , this parameter will not be normalized in what follows. At all events, retain that the size of ε is always small, and far below μ .

Discussion 2.6. [about the size of ν] We adjust ν in such a way that $\nu \sim \varepsilon$. By this way, we can stay in a perturbative regime, even if $\theta_1^d \simeq 1$. Indeed:

- *Smallness of $\tilde{\mathbf{B}}$ in comparison with $\tilde{\mathbf{B}}_e$:* In view of (2.69c), the condition $\mathbf{B} \simeq 1$ implies $\tilde{\mathbf{B}} \simeq \nu \theta_1^d b_e \lesssim \varepsilon b_e$. With (2.19), we can be sure that $|\tilde{\mathbf{B}}| \lesssim \varepsilon |\tilde{\mathbf{B}}_e|$.
- *Smallness of $\nu \tilde{f}_\alpha^s$ in comparison with \tilde{f}_α^d :* When computing $\tilde{f}_\alpha^k(\cdot)$ through (2.5), the part $\nu \tilde{f}_\alpha^s(\cdot)$ appears as a small modification of $\tilde{f}_\alpha^d(\cdot)$. This makes sense whatever the parameter θ_α^d is, small or large. Indeed, the amplitude of $\tilde{f}_\alpha^s(\cdot)$ as prescribed by (2.69b) with $f_\alpha \simeq 1$ is equivalent to the amplitude of $\tilde{f}_\alpha^d(\cdot)$ given by (2.57) or (2.58).

2.5. The hot asymptotic regime. The framework specified below is intended to describe what happens in fusion devices. This requires to fix the size of the various parameters in function of ε . Of course, this can be done only under simplifying assumptions that make the model tractable. In the light of Paragraph 2.4.4, we can retain the following choices:

- (Hp1): We have $\varepsilon_1 = -\varepsilon \ll 1$ together with $\varepsilon_\alpha = 1$ for all $\alpha \in \{2, \dots, N\}$;
- (Hp2): We have $\theta_1^d(\varepsilon) = 1$ together with $\theta_\alpha^d(\varepsilon) = \varepsilon$ for all $\alpha \in \{2, \dots, N\}$;
- (Hp3): For all $\alpha \in \{2, \dots, N\}$, the dominant stationary part $\tilde{f}_\alpha^d(\cdot)$ is given by the shifted Maxwell-Boltzmann KDF (2.58);
- (Hp4): For $\alpha = 1$, the dominant stationary part $\tilde{f}_1^d(\cdot)$ is given by (2.57) with (2.61);
- (Hp5): The perturbation in (2.63) is such that $\nu = \varepsilon$;
- (Hp6): We have $\mu := |\mu_1| = 1$ and $\mu_\alpha^2 = \varepsilon^{-1}$ for all $\alpha \in \{2, \dots, N\}$.

Thus, we have $|\varepsilon_\alpha| \theta_\alpha^d = \varepsilon$ for all $\alpha \in \{1, \dots, N\}$. Coming back to (2.74), this yields:

$$(2.89) \quad f_\alpha^d(\mathbf{x}, p) = F_\alpha^d(|p|^2) G_\alpha^d(\Psi(\mathbf{x})) + O(\varepsilon), \quad \forall \alpha \in \{1, \dots, N\}.$$

By combining informations from Section 2.4 with assumptions (Hp1), \dots , (Hp6), we can simplify (2.82)-(2.75). For all $\alpha \in \{2, \dots, N\}$, with $F_\alpha^d(\cdot)$ and $G_\alpha^d(\cdot)$ as in (2.60), we find:

$$(2.90) \quad \begin{aligned} & \partial_t f_\alpha + \frac{\varepsilon}{\langle \varepsilon r \rangle} O(\mathbf{x}) p \cdot \nabla_{\mathbf{x}} f_\alpha + \frac{\varepsilon}{\langle \varepsilon r \rangle} Q(\mathbf{x}, p) \cdot \nabla_p f_\alpha - \frac{\mathbf{b}_e(\mathbf{x})}{\langle \varepsilon r \rangle} \partial_\omega f_\alpha \\ & + \left[E + \frac{\varepsilon}{\langle \varepsilon r \rangle} p \times B \right] \cdot \nabla_p f_\alpha + \frac{1}{\varepsilon} \left[E + \frac{\varepsilon}{\langle \varepsilon r \rangle} p \times B \right] \cdot \nabla_p f_\alpha^d = 0. \end{aligned}$$

On the other hand, for $\alpha = 1$, the equation (2.82) yields:

$$(2.91) \quad \begin{aligned} & \partial_t f_1 + \frac{1}{\langle r \rangle} O(\mathbf{x}) p \cdot \nabla_{\mathbf{x}} f_1 + \frac{1}{\langle r \rangle} Q(\mathbf{x}, p) \cdot \nabla_p f_1 + \frac{1}{\varepsilon} \frac{\mathbf{b}_e(\mathbf{x})}{\langle r \rangle} \partial_\omega f_1 \\ & - \left[E + \frac{1}{\langle r \rangle} p \times B \right] \cdot \nabla_p f_1 - \frac{1}{\varepsilon} \left[E + \frac{1}{\langle r \rangle} p \times B \right] \cdot \nabla_p f_1^d = 0. \end{aligned}$$

With $B(\cdot)$ and $E(\cdot)$ as in (2.78), the equation (2.75) becomes:

$$(2.92a) \quad O(\mathbf{x}) \partial_t B + \nabla_{\mathbf{x}} \times (O(\mathbf{x}) E) = 0, \quad \nabla_{\mathbf{x}} \cdot (O(\mathbf{x}) B) = 0,$$

$$(2.92b) \quad O(\mathbf{x}) \partial_t E - \nabla_{\mathbf{x}} \times (O(\mathbf{x}) B) = -j^h(f_\alpha), \quad \nabla_{\mathbf{x}} \cdot (O(\mathbf{x}) E) = \rho^h(f_\alpha),$$

where $\rho^h(f_\alpha)$ and $j^h(f_\alpha)$ can be specified by using (Hp1) and (2.88). With integral operators $\rho(\cdot)$ and $\mathcal{J}(\cdot)$ as in (2.81), this furnishes:

$$(2.93a) \quad \rho^h(f_\alpha)(\mathbf{t}, \mathbf{x}) := -\frac{1}{\varepsilon} \rho(f_1) + \sum_{\alpha=2}^N \rho(f_\alpha),$$

$$(2.93b) \quad j^h(f_\alpha)(\mathbf{t}, \mathbf{x}) := -\frac{1}{\varepsilon} \mathcal{J}(1; f_1) + \sum_{\alpha=2}^N \mathcal{J}(\varepsilon; f_\alpha).$$

3. HOT PLASMA DISPERSION RELATIONS

The dispersion relations inform about various properties of wave propagation. They say if a wave can propagate. By looking at complex frequencies, they can indicate if a wave is damped or amplified. They furnish the phase velocity, the group velocity and the refractive index. They allow to determine the eikonal equation, and they are crucial in reflectometry [19]. They are a prerequisite to understand turbulence phenomena [7]. And the list of potential applications goes on.

Based on the Vlasov theory of hot collisionless and magnetized plasmas, several approaches have been proposed in order to obtain the dispersion relations. Derivations can be found in Trubnikov [35] (1959), Bekefi [2] (1966), Krall and Trivelpiece [22] (1973), Davidson [11] (1983), Swanson [32] (1989), etc. They give access to a preliminary treatment of wave propagation. There are more recent works dealing with the relativistic features [16, 25] or with the numerical aspects [34, 39]. Most of these contributions [2, 11, 22, 35] are restricted to the case of a *constant* external magnetic field and also to the case of a *homogeneous* velocity distribution function. The improvements concerning the choice of more realistic functions $\tilde{f}_\alpha^d(\cdot)$ have been principally related to the dependence on $\tilde{\mathbf{p}}$ of $\tilde{f}_\alpha^d(\cdot)$. As a matter of fact, the behaviour of $\tilde{f}_\alpha^d(\cdot)$ in $\tilde{\mathbf{p}}$ can be of quasi-Maxwellian type [12] or of gyrotropic type [30]. One of the advantages of Section 2 is to incorporate through (2.57) realistic variations in \mathbf{x} of $\tilde{f}_\alpha^d(\cdot)$

Many practical situations in space and laboratory plasmas [28] involve variations in position $\tilde{\mathbf{x}}$ of the distribution $\tilde{f}_\alpha^d(\cdot)$ and of the magnetic field $\tilde{\mathbf{B}}_e(\cdot)$. These variations have an effect on the dispersion relations, and by this way they can modify the geometry of the propagation. They have an impact on ray tracing, and beyond they can induce caustics [4, 20]. They have first been taken into account through the *Kinetic Theory of Drift Waves* (KTDW), see for instance Paragraph 6.6.3 in [32]. This approach implies very specific assumptions (electrostatic approximation, modelling of the curvature effects through some gravitational potential, ...). In fact, the idea behind KTDW is to come back to the case of a constant external magnetic field $\tilde{\mathbf{B}}_e(\cdot)$ and to handle the variations in $\tilde{\mathbf{x}}$ as perturbations. This allows to expand the particle orbits around their trajectories, to integrate the unperturbed trajectories through explicit formulas, to perform a Fourier analysis of the linearized Vlasov equation, and to employ a fixed decomposition of the velocity $\tilde{\mathbf{p}}$ into two components $\tilde{\mathbf{p}}_{\parallel}$ and $\tilde{\mathbf{p}}_{\perp}$ which are respectively parallel and perpendicular to the magnetic field.

For many technical reasons, the preceding procedures [2, 11, 22, 32, 33, 35] do not apply appropriately in the presence of realistic inhomogeneities. On the one hand, they rely on hypotheses that could be questionable. On the other hand, they often use non local arguments in space or in time (especially when integrating the Vlasov equation), while the dispersion relations should emanate from a *local* space-time analysis. For all these reasons, the approaches [2, 11, 22, 32, 33, 35] bring answers that need to be completed. Indeed, they are not able to fully capture the underlying *geometry*, which is essential to really understand wave propagation. Now, of course, a dielectric tensor is a macroscopic notion. In some ways, it summarizes the average macroscopic outcome of the underlying kinetic effects. Thus, it should depend on \mathbf{t} and \mathbf{x} , but not on \mathbf{p} . In the end, the momentum variable \mathbf{p} should disappear. Some *global* analysis is needed, but only in \mathbf{p} .

In contrast with [2, 11, 12, 22, 32], and as required by tokamak configurations, the modelling and the dimensional analysis of Section 2 takes into account the concrete dependence on $\tilde{\mathbf{x}}$ of both $\tilde{\mathbf{B}}_e(\cdot)$ and $\tilde{f}_\alpha^d(\cdot)$. They combine together the various physical data in order to evaluate their relative importance and to provide a coherent description of the phenomena. They allow to formulate the problem in terms of geometrical optics. This is a prerequisite which gives rise in this Section 3 to a complete understanding of the dispersion relations, valid in the presence of inhomogeneities.

3.1. In the framework of geometric optics. From now on, we are interested in the asymptotic analysis (when the parameter ε goes to zero) of the oscillating solutions to the system (2.90)-...-(2.93). To this end, the tools of geometric optics [27, 29] are particularly well-suited. Being interested in the propagation of *electromagnetic waves* means to focus on oscillations of the self-consistent field ${}^t(E, B)(\cdot)$, rather than on kinetic oscillations of the density distribution $f(\cdot)$ as is the case in [5, 6]. Since the function ${}^t(E, B)(\cdot)$ depends only on (\mathbf{t}, \mathbf{x}) , a key point is that only *time-space oscillations* can be involved at this level. With this in mind, we can introduce some smooth phase function:

$$\phi \in \mathcal{C}^\infty(M; \mathbb{R}), \quad M := [0, T] \times \Omega, \quad T \in \mathbb{R}_+^*$$

depending on the *macroscopic* variable $(\mathbf{t}, \mathbf{x}) \in M$ but not on the *kinetic* variable $p \in \mathbb{R}^3$.

Assumption 3.1. [*non-stationary phase*] The function $\phi(\cdot)$ is such that:

$$(3.1) \quad \forall (\mathbf{t}, \mathbf{x}) \in M, \quad (\partial_{\mathbf{t}}\phi, \nabla_{\mathbf{x}}\phi)(\mathbf{t}, \mathbf{x}) \neq 0.$$

Usually, the time evolution of ${}^t(E, B)(\cdot)$ is studied in the framework of MHD descriptions, through *fluid* models based on Maxwell's equations, involving only the variables (\mathbf{t}, \mathbf{x}) . This has the advantage of simplicity. But this also means various simplifying assumptions which are irrelevant when dealing with hot plasma phenomena out of equilibrium. As we will see, the dependence of $f(\cdot)$ on the variable $p \in \mathbb{R}^3$ has a real impact. To take this aspect into account, it is necessary to come back to the original RVM system. To this end, given some $M \in \mathbb{N}^*$, select profiles $\check{\mathcal{U}}_j = {}^t(\check{\mathcal{F}}_{j,1}, \dots, \check{\mathcal{F}}_{j,N}, \check{\mathcal{B}}_j, \check{\mathcal{E}}_j)$ such that:

$$(3.2) \quad \check{\mathcal{U}}_j \in \mathcal{C}^\infty([0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{T}; \mathbb{R}^{N+6}), \quad j \in \{0, \dots, M\}, \quad \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}).$$

In (3.2), the profiles $\check{\mathcal{U}}_j(\mathbf{t}, \mathbf{x}, \mathbf{p}, \theta)$ are periodic in the *fast* variable $\theta \in \mathbb{T}$. On the other hand, the coordinates inside $(\mathbf{t}, \mathbf{x}, \mathbf{p})$ are considered as *slow* variables. When dealing with capital letters like U , the different font styles \mathcal{U} , \mathcal{U} and U will be used for expressions depending respectively on the variables $(\mathbf{t}, \mathbf{x}, \mathbf{p}, \theta)$, $(\mathbf{t}, \mathbf{x}, \mathbf{p})$ and $(\mathbf{t}, \mathbf{x}, \varpi, \omega)$. We look for approximate solutions u_a^ε to the system (2.90)–(2.93) in the form of *monophase* representations. More precisely, we consider expansions of the form:

$$(3.3) \quad \begin{pmatrix} f_{a,1}^\varepsilon(\mathbf{t}, \mathbf{x}, \mathbf{p}) \\ \vdots \\ f_{a,N}^\varepsilon(\mathbf{t}, \mathbf{x}, \mathbf{p}) \\ B_a^\varepsilon(\mathbf{t}, \mathbf{x}) \\ E_a^\varepsilon(\mathbf{t}, \mathbf{x}) \end{pmatrix} = \sum_{j=0}^M \varepsilon^j \check{\mathcal{U}}_j\left(\mathbf{t}, \mathbf{x}, \mathbf{p}, \frac{\phi(\mathbf{t}, \mathbf{x})}{\varepsilon}\right).$$

In (3.3), with $p = {}^tO(\mathbf{x})\mathbf{p}$ as in (2.78) and with $\check{\mathcal{U}}_j(\mathbf{t}, \mathbf{x}, p, \theta) := \check{\mathcal{U}}_j(\mathbf{t}, \mathbf{x}, O(\mathbf{x})p, \theta)$, the profiles $\check{\mathcal{U}}_j(\cdot)$ can be decomposed into:

$$\check{\mathcal{U}}_j(\mathbf{t}, \mathbf{x}, \mathbf{p}, \theta) = \check{\mathcal{U}}_j(\mathbf{t}, \mathbf{x}, {}^tO(\mathbf{x})\mathbf{p}, \theta) = \begin{pmatrix} \tilde{\mathcal{F}}_{j,1}(\mathbf{t}, \mathbf{x}, {}^tO(\mathbf{x})\mathbf{p}, \theta) \\ \vdots \\ \tilde{\mathcal{F}}_{j,N}(\mathbf{t}, \mathbf{x}, {}^tO(\mathbf{x})\mathbf{p}, \theta) \\ \mathcal{B}_j(\mathbf{t}, \mathbf{x}, \theta) \\ \mathcal{E}_j(\mathbf{t}, \mathbf{x}, \theta) \end{pmatrix}, \quad \tilde{\mathcal{F}}_{j,k}(\mathbf{t}, \mathbf{x}, p, \theta).$$

With p represented in spherical coordinates as in (2.79), the functions $\check{\mathcal{U}}_j(\mathbf{t}, \mathbf{x}, \cdot, \theta)$ and the functions $\tilde{\mathcal{F}}_{j,k}(\mathbf{t}, \mathbf{x}, \cdot, \theta)$ can be viewed as functions $\mathcal{U}_j(\mathbf{t}, \mathbf{x}, \cdot, \theta)$ and $\tilde{\mathcal{F}}_{j,k}(\mathbf{t}, \mathbf{x}, \cdot, \theta)$ of the variables $(\varpi, \omega, r) \in \mathbb{T}^2 \times \mathbb{R}_+$. Mark by $\mathbf{y} := (\mathbf{x}, \varpi, \omega, r) \in \Omega \times \mathbb{T} \times \mathbb{T} \times \mathbb{R}_+$ the spatial-velocity position. We can introduce the Fourier series expansion with respect to $\theta \in \mathbb{T}$ of the profile $\mathcal{U}_j(\cdot)$ to obtain:

$$(3.4) \quad \mathcal{U}_j(\mathbf{t}, \mathbf{y}, \theta) = \sum_{l \in \mathbb{Z}} \mathcal{U}_j^l(\mathbf{t}, \mathbf{y}) e^{il\theta}, \quad \mathcal{U}_j^l = ({}^t\mathcal{F}_{j,1}^l, \dots, \mathcal{F}_{j,N}^l, B_j^l, E_j^l) \equiv \bar{\mathcal{U}}_j^{-l}.$$

It is understood that the function $\mathcal{F}_{j,\alpha}^l(\cdot)$ and its derivatives at all orders satisfy (2.80). Plugg the real valued function u_a^ε into (2.90)–(2.93). Collect the contributions having the same power of ε in factor, sorted in increasing order. By this way, we get:

$$(3.5) \quad \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{G}_j\left(\mathbf{t}, \mathbf{y}, \frac{\phi(\mathbf{t}, \mathbf{x})}{\varepsilon}\right) = 0, \quad \mathcal{G}_j(\mathbf{t}, \mathbf{y}, \theta) = \sum_{l \in \mathbb{Z}} \mathcal{G}_j^l(\mathbf{t}, \mathbf{y}) e^{il\theta}, \quad \mathcal{G}_j^l \equiv \bar{\mathcal{G}}_j^{-l}.$$

It turns out that the expressions $\mathcal{G}_j(\cdot)$ depend only on terms \mathcal{U}_i with $i \leq j+1$. In particular, for $j = -1$, we get the preliminary constraint:

$$(3.6) \quad \mathcal{G}_{-1}(\mathbf{t}, \mathbf{y}, \theta, \mathcal{U}_0) = 0.$$

Then, an approximate solution u_a^ε can be derived by solving successively the conditions $\mathcal{G}_j \equiv 0$ for $j = 0$, $j = 1$ and so on, up to $j = N - 1$. In this text, we focus on the initialization procedure, based on (3.6), which already requires a substantial amount of work. The condition (3.6) is interesting and difficult to solve. It includes especially the so-called *eikonal equation* which allows to determine ϕ , and which therefore governs the geometry of the propagation.

In Part 3.2, starting from (3.6), we give a precise definition of the characteristic variety sustaining wave propagation. The rigorous analysis of the dielectric tensor is performed in Part 3.3. Finally, Part 3.4 is devoted to the study of interesting special cases.

3.2. Description of the characteristic variety. The condition (3.6) is expressed in an abstract form. In Paragraph 3.2.1, we extract from (3.6) a simplified system of equations that is amenable to the Fourier analysis performed in Paragraph 3.2.2. As explained in Paragraph 3.2.3 and in coherence with basic concepts of wave-particle interactions [21, 36], this yields a kinetic interpretation of electron cyclotron resonances. Then, Paragraph 3.2.4 gives an overview of the conductivity tensor which has to be studied carefully.

3.2.1. A reduced system of equations. From (2.90)-(2.93), we can extract the equations composing (3.6). Since by assumption $\nabla_{\mathbf{p}}\phi \equiv 0$, the term coming with ε^{-1} in factor after substitution of (3.3) inside (2.91) furnishes:

$$(3.7) \quad \begin{aligned} \partial_{\mathbf{t}}\phi \partial_{\theta}\mathcal{F}_{0,1} + \frac{1}{\langle r \rangle} O(\mathbf{x}) p \cdot \nabla_{\mathbf{x}}\phi \partial_{\theta}\mathcal{F}_{0,1} + \frac{\mathbf{b}_e(\mathbf{x})}{\langle r \rangle} \partial_{\omega}\mathcal{F}_{0,1} \\ - 2 G_1^{\text{d}}(\Psi(\rho, z)) \partial_r F_1^{\text{d}}(r^2) p \cdot \mathcal{E}_0 = 0. \end{aligned}$$

For $\alpha \in \{2, \dots, N\}$, the equations inside (2.90) give rise to:

$$(3.8) \quad \partial_{\mathbf{t}}\phi \partial_{\theta}\mathcal{F}_{0,\alpha} + 2 G_{\alpha}^{\text{d}}(\Psi(\rho, z)) \partial_r F_{\alpha}^{\text{d}}(r^2) p \cdot \mathcal{E}_0 = 0.$$

The Maxwell's equations (2.92) provide:

$$(3.9a) \quad \partial_{\mathbf{t}}\phi \partial_{\theta}\mathcal{B}_0 + {}^t O(\mathbf{x}) \nabla_{\mathbf{x}}\phi \times \partial_{\theta}\mathcal{E}_0 = 0,$$

$$(3.9b) \quad \partial_{\mathbf{t}}\phi \partial_{\theta}\mathcal{E}_0 - {}^t O(\mathbf{x}) \nabla_{\mathbf{x}}\phi \times \partial_{\theta}\mathcal{B}_0 = \mathcal{J}_1(\mathcal{F}_{0,1}(\mathbf{t}, \mathbf{x}, \cdot)),$$

together with:

$$(3.10) \quad {}^t O(\mathbf{x}) \nabla_{\mathbf{x}}\phi \cdot \partial_{\theta}\mathcal{B}_0 = 0, \quad {}^t O(\mathbf{x}) \nabla_{\mathbf{x}}\phi \cdot \partial_{\theta}\mathcal{E}_0 = -\rho(\mathcal{F}_{0,1}(\mathbf{t}, \mathbf{x}, \cdot)).$$

In view of (2.93), with $dp = r^2 d\varpi d\omega dr$, we find that:

$$\mathcal{J}_1(\mathcal{F}_{0,1}) := \int_{\mathbb{R}^3} \frac{p}{\langle r \rangle} \mathcal{F}_{0,1} dp, \quad \rho(\mathcal{F}_{0,1}) = \int_{\mathbb{R}^3} \mathcal{F}_{0,1} dp.$$

Consider the expansion in Fourier series of $\mathcal{U}_0(\mathbf{t}, \mathbf{y}, \cdot)$, as in (3.4). The situation under study is dispersive. After adjusting ϕ in order to obtain $\mathcal{G}_{-1}^l \equiv 0$ (and therefore $\bar{\mathcal{G}}_{-1}^{-l} \equiv 0$) for some $l \in \mathbb{Z}^*$, the other conditions $\mathcal{G}_{-1}^{\ell} \equiv 0$ (with $\ell \neq |l|$) are in general not verified (except for the trivial choice $\mathcal{U}_0^{\ell} \equiv 0$). This is why, at leading order, only one Fourier coefficient will be switched on.

Assumption 3.2. [presence of a non-trivial monochromatic electromagnetic oscillation] There is some non-zero integer $l \in \mathbb{N}^*$ such that:

$$(3.11) \quad (E_0^l, B_0^l) \equiv (\bar{E}_0^{-l}, \bar{B}_0^{-l}) \neq 0, \quad \mathcal{U}_0^{\ell} \equiv 0, \quad \forall \ell \in \mathbb{Z} \setminus \{-l, l\}.$$

With $l \in \mathbb{Z}^*$ as in Assumption 3.2, introduce:

$$\tau := l \partial_t \phi(\mathbf{t}, \mathbf{x}) \in \mathbb{R}, \quad \xi := l {}^t O(\mathbf{x}) \nabla_{\mathbf{x}} \phi(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^3.$$

Then, from equations (3.7) and (3.8), we can extract:

$$(3.12) \quad \left[i \tau + i \frac{1}{\langle r \rangle} p \cdot \xi + \frac{\mathbf{b}_e}{\langle r \rangle} \partial_\omega \right] \mathcal{F}_{0,1}^l = \frac{2}{\langle r \rangle} G_1^d(\Psi(\rho, z)) \partial_r F_1^d(r^2) p \cdot E_0^l.$$

On the other hand, from equation (3.8), we get:

$$(3.13) \quad \forall \alpha \in \{2, \dots, N\}, \quad i \tau \mathcal{F}_{0,\alpha}^l = -2 G_\alpha^d(\Psi(\rho, z)) \partial_r F_\alpha^d(r^2) p \cdot E_0^l.$$

Moreover, the Maxwell's equations (3.9)-(3.10) reduce to:

$$(3.14a) \quad \tau B_0^l + \xi \times E_0^l = 0, \quad \xi \times B_0^l - \tau E_0^l = i \mathcal{J}_1(\mathcal{F}_{0,1}^l),$$

$$(3.14b) \quad \xi \cdot B_0^l = 0, \quad \xi \cdot E_0^l = i \rho(\mathcal{F}_{0,1}^l).$$

Lemma 3.1. *Fix $\tau \neq 0$ and assume that $(\mathcal{F}_{0,1}^l, E_0^l, B_0^l)$ satisfies (3.12) and (3.14a). Then, the two equations of (3.14b) are satisfied.*

Proof. Knowing that $\tau \neq 0$, the scalar product with ξ of the first equation in (3.14a) yields directly $\xi \cdot B_0^l = 0$. Using the second equation of (3.14a), we get:

$$(3.15) \quad -\tau \xi \cdot E_0^l = i \xi \cdot \mathcal{J}_1(\mathcal{F}_{0,1}^l).$$

Integrate (3.12) with respect to p in order to obtain:

$$(3.16) \quad \tau \rho(\mathcal{F}_{0,1}^l) + \xi \cdot \mathcal{J}_1(\mathcal{F}_{0,1}^l) = 0.$$

Combining (3.15) and (3.16), since $\tau \neq 0$, we get the second equation of (3.14b). \square

In view of Lemma 3.1, we can forget about the condition (3.14b). On the other hand, we can eliminate B_0^l from (3.14a) to retain:

$$(3.17) \quad [(\tau^2 - |\xi|^2) Id + \xi {}^t \xi] E_0^l = -i \tau \mathcal{J}_1(\mathcal{F}_0^l).$$

Observe that the functions $\mathcal{F}_{0,\alpha}^l$ with $\alpha \neq 1$ are not present at the level of (3.12)-(3.17). Knowing what E_0^l is, we can deduce the expressions $\mathcal{F}_{0,\alpha}^l$ from (3.13). The relation (3.13) just says that the presence of a non trivial electric field E_0^l is associated with prescribed oscillations at the level of the ions's kinetic distribution functions.

We now concentrate on the remaining system (3.12)-(3.17) on $(\mathcal{F}_{0,1}^l, E_0^l)$. To simplify the notations, we drop the subscript 1 (related to electrons). We use the notations G^d , F^d and \mathcal{F}_0^l instead of G_1^d , F_1^d and $\mathcal{F}_{0,1}^l$. The spherical coordinates which are associated with the direction $\xi \in \mathbb{R}^3$ are:

$$\xi = {}^t(\xi_1, \xi_2, \xi_3) = |\xi| (\cos \omega_\xi \sin \varpi_\xi, \sin \omega_\xi \sin \varpi_\xi, \cos \varpi_\xi) := (\xi_\perp \cos \omega_\xi, \xi_\perp \sin \omega_\xi, \xi_\parallel),$$

where $\xi_\perp := |\xi| \sin \varpi_\xi$ and $\xi_\parallel := |\xi| \cos \varpi_\xi$. Another preliminary step is to reduce the discussion to the case where $\omega_\xi = 0$ (or $\xi_2 = 0$). This can be done by rotation of both ξ and p . Select an orthogonal matrix $R \in SO(3)$ which is such that:

$$R \xi = (\xi_\perp, 0, \xi_\parallel), \quad \check{p} := R p = |p| (\cos(\omega - \omega_\xi) \sin \varpi, \sin(\omega - \omega_\xi) \sin \varpi, \cos \varpi).$$

Introduce:

$$(3.18) \quad \check{\mathcal{F}}_0^l(\mathbf{t}, \mathbf{x}, \check{p}) := \mathcal{F}_0^l(\mathbf{t}, \mathbf{x}, {}^t R \check{p}) \equiv \mathcal{F}_{0,1}^l(\mathbf{t}, \mathbf{x}, {}^t R \check{p}), \quad \check{E}_0^l(\mathbf{t}, \mathbf{x}) := R E_0^l(\mathbf{t}, \mathbf{x}).$$

Lemma 3.2. *The couple $(\check{\mathcal{F}}_0^l, \check{E}_0^l)$ is a solution to (3.12)-(3.17) with $\xi = (\xi_\perp, 0, \xi_\parallel)$.*

Proof. The equation (3.12) amounts to the same thing as:

$$\left[i\tau + i \frac{1}{\langle r \rangle} R p \cdot R \xi + \frac{\mathbf{b}_e}{\langle r \rangle} \partial_\omega \right] \mathcal{F}_0^l(\mathbf{t}, \mathbf{x}, p) = \frac{2}{\langle r \rangle} G^d(\Psi(\rho, z)) \partial_r F^d(r^2) R p \cdot R E_0^l.$$

Replace p by ${}^t R \check{p}$ to recover (3.12) for $(\check{\mathcal{F}}_0^l, \check{E}_0^l)(\mathbf{t}, \mathbf{x}, \check{p})$, this time with $\xi = (\xi_\perp, 0, \xi_\parallel)$. On the other hand, apply the matrix R to the left of (3.17) to find:

$$\left[(\tau^2 - |R\xi|^2) Id + (R\xi) {}^t(R\xi) \right] R E_0^l = -i\tau R \mathcal{J}_1(\mathcal{F}_0^l).$$

Now, to obtain (3.17) for $(\check{\mathcal{F}}_0^l, \check{E}_0^l)$, it suffices to remark that:

$$R \mathcal{J}_1(\mathcal{F}_0^l) = \int_{\mathbb{R}^3} \frac{\check{p}}{\langle |\check{p}| \rangle} \mathcal{F}_{0,1}(\mathbf{t}, \mathbf{x}, {}^t R \check{p}) d\check{p} = \mathcal{J}_1(\check{\mathcal{F}}_0^l). \quad \square$$

The system (3.12)-(3.17) will be studied with $\xi = (\xi_\perp, 0, \xi_\parallel)$. The general situation can be obtained by coming back from $(\check{\mathcal{F}}_0^l, \check{E}_0^l)$ to (\mathcal{F}_0^l, E_0^l) through (3.18). From now on, we will assume that $\omega_\xi = 0$.

3.2.2. Fourier analysis through the Jacobi-Anger identity. Define the scalar function:

$$(3.19) \quad \zeta \equiv \zeta(\mathbf{x}, r, \varpi, \xi_\perp) := r \xi_\perp \sin \varpi \mathbf{b}_e(\mathbf{x})^{-1}.$$

Knowing that $\omega_\xi = 0$, the equation (3.12) is translated into:

$$(3.20) \quad \left[i\tau + i \frac{\mathbf{b}_e \zeta}{\langle r \rangle} \cos \varpi + i \frac{r \xi_\parallel \cos \varpi}{\langle r \rangle} + \frac{\mathbf{b}_e}{\langle r \rangle} \partial_\omega \right] \mathcal{F}_0^l = \frac{2}{\langle r \rangle} G^d(\Psi(\rho, z)) \partial_r F^d(r^2) p \cdot E_0^l.$$

This can be viewed as a first order differential equation with respect to $\omega \in \mathbb{T}$, where the variables \mathbf{x} , ϖ , r , τ and ξ play the part of parameters. Now, we want to remove the variable coefficient in ω from the differential operator which in the equation (3.20) is inside brackets. This means concretely to eliminate the presence of "cos ω ". This can be achieved by replacing \mathcal{F}_0^l by:

$$(3.21) \quad \mathfrak{F}_0^l(\mathbf{t}, \mathbf{x}, p) := \exp(i \zeta \sin \omega) \mathcal{F}_0^l(\mathbf{t}, \mathbf{x}, p).$$

Then, the equation (3.20) becomes:

$$(3.22) \quad \left[i\tau + \frac{i r \xi_\parallel \cos \varpi}{\langle r \rangle} + \frac{\mathbf{b}_e}{\langle r \rangle} \partial_\omega \right] \mathfrak{F}_0^l = \frac{2}{\langle r \rangle} G^d(\Psi(\rho, z)) \partial_r F^d(r^2) p \cdot E_0^l \exp(i \zeta \sin \omega).$$

By this way, the discussion is reduced to the study of a linear differential equation in ω with constant coefficients. The counterpart is that all the Fourier coefficients (with respect to ω) of the right hand side of (3.22) are non zero. But now, we can solve (3.22) through a Fourier analysis in ω . To this end, decompose $\mathfrak{F}_0^l(\cdot)$ according to:

$$\mathfrak{F}_0^l(\mathbf{t}, \mathbf{x}, \varpi, \omega, r) = \sum_{m \in \mathbb{Z}} \mathfrak{F}_0^{l,m}(\mathbf{t}, \mathbf{x}, \varpi, r) e^{i m \omega}.$$

Lemma 3.3. *The condition (3.22) is satisfied if and only if, for all $m \in \mathbb{Z}$, we have:*

$$(3.23) \quad i [\tau + \tau_m] \mathfrak{F}_0^{l,m} = \frac{2r}{\langle r \rangle} G^d(\Psi(\rho, z)) \partial_r F^d(r^2) \begin{pmatrix} m \zeta^{-1} J_m(\zeta) \sin \varpi \\ -i J'_m(\zeta) \sin \varpi \\ J_m(\zeta) \cos \varpi \end{pmatrix} \cdot E_0^l,$$

where:

$$(3.24) \quad \tau_m(\mathbf{x}, p, \xi) \equiv \tau_m(\mathbf{x}, r, \varpi, \xi) := \langle r \rangle^{-1} (r \xi_{\parallel} \cos \varpi + m \mathbf{b}_e) = -\tau_{-m}(\mathbf{x}, r, \pi - \varpi, \xi).$$

Proof. It suffices to compute the Fourier coefficient in ω of the right hand side of (3.22). To do so, recall the **Jacobi-Anger** identity:

$$(3.25) \quad \exp(i \zeta \sin \omega) = \sum_{m \in \mathbb{Z}} J_m(\zeta) e^{i m \omega}, \quad \forall (\zeta, \omega) \in \mathbb{R} \times \mathbb{T},$$

where $J_m(\cdot)$ denotes the m -th Bessel function of the first kind. The formula (3.23) is a consequence of (3.25) together with the **wellknown relations**:

$$(3.26a) \quad J_{m+1}(\zeta) + J_{m-1}(\zeta) = 2 m \zeta^{-1} J_m(\zeta),$$

$$(3.26b) \quad J_{m+1}(\zeta) - J_{m-1}(\zeta) = -2 J'_m(\zeta). \quad \square$$

3.2.3. Kinetic interpretation of *electron cyclotron resonances*. Recall the definition (3.24) of τ_m . The expression $\tau + \tau_m$ depends on the position $\mathbf{x} \in \Omega$, the momentum $p \in \mathbb{R}^3$ with norm $r \in \mathbb{R}_+$ and pitch angle $\varpi \in [0, \pi]$, the direction $\xi \in \mathbb{R}^3$, the Fourier mode $m \in \mathbb{Z}$, and the time frequency $\tau \in \mathbb{R}$. The electrons gyrate with the local gyrofrequency $\mathbf{b}_e(\mathbf{x})$. On the other hand, their guiding centers move with the drift velocity $\mathbf{v}_G = r \cos \varpi e_3(\mathbf{x})$ where $e_3(\cdot)$ is the unit vector field pointing in the magnetic direction. Looking at ξ as a wave vector $\mathbf{k} \in \mathbb{R}^3$, we have $\mathbf{v}_G \cdot \mathbf{k} = r \xi_{\parallel} \cos \varpi$. Seen in this way, the function $\tau + \tau_m$ can be interpreted as a (relativistic) *gyroballistic dispersion function*.

The important role of $\tau + \tau_m$ results naturally from the preceding asymptotic analysis. In view of (3.23), as long as $\tau + \tau_m \neq 0$, the expression $\mathfrak{F}_0^{l,m}$ can easily be expressed in terms of E_0^l . However, difficulties arise when $\tau + \tau_m = 0$. Below, such special values are set aside.

Definition 3.1. [*notion of kinetic resonance*] Given $(\mathbf{x}, p, \xi, m) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{Z}$, the *resonant time frequency* is given by $-\tau_m(\mathbf{x}, r, \varpi, \xi)$. Equivalently, it is the time frequency τ satisfying the condition:

$$(3.27) \quad \langle r \rangle \tau + r \xi_{\parallel} \cos \varpi + m \mathbf{b}_e(\mathbf{x}) = 0.$$

In (3.27), the quantity $\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi$ can be viewed as a Doppler shifted frequency [21]. Since the right-hand term of (3.23) is divided by $\tau + \tau_m$ with $\tau + \tau_m \simeq 0$ near resonances, it can be said [36] that the interactions between the waves and the charged particles become strong when the particles sense the Doppler-shifted wave at its cyclotron frequency ($m = 1$) or its harmonics ($m \in \mathbb{Z}$ with $m \neq 1$). The special case $m = 0$ corresponds to the well-known Landau resonance.

Given $(\mathbf{x}, r, \varpi, \xi)$ and m , there exists obviously one and only one resonance, which is given by $\tau = -\tau_m(\mathbf{x}, r, \varpi, \xi) = 0$. Another issue is whether all values $\tau \in \mathbb{R}$ are resonant, and under what conditions.

Lemma 3.4. *[Infinite set of resonances] Fix $(\mathbf{x}, \xi) \in \Omega \times (\mathbb{R}^3 \setminus \{0\})$ as well as $\varpi \in [0, \pi]$ and $\tau \in \mathbb{R}$. There exists infinitely many $(m, r) \in \mathbb{Z} \times \mathbb{R}_+$ such that $\tau + \tau_m(\mathbf{x}, r, \varpi, \xi) = 0$.*

Proof. The condition $\tau + \tau_m = 0$ is equivalent to:

$$(3.28) \quad r^2 (\tau^2 - \xi_{\parallel}^2 \cos^2 \varpi) - 2 r m \mathbf{b}_e \xi_{\parallel} \cos \varpi + \tau^2 - m^2 \mathbf{b}_e^2 = 0.$$

- First, if $\tau^2 = \xi_{\parallel}^2 \cos^2 \varpi = 0$, just take $m = 0$. Then, any value $r \in \mathbb{R}_+$ can be selected.

- Secondly, if $\tau^2 = \xi_{\parallel}^2 \cos^2 \varpi \neq 0$, to solve (3.28), it suffices to obtain $r(m) \geq 0$ with:

$$r(m) = \frac{\xi_{\parallel}^2 \cos^2 \varpi - m^2 \mathbf{b}_e^2}{2 m \mathbf{b}_e \xi_{\parallel} \cos \varpi} = -\frac{m \mathbf{b}_e}{2 \xi_{\parallel} \cos \varpi} + O\left(\frac{1}{m}\right).$$

Now, either for $m \rightarrow -\infty$ or for $m \rightarrow +\infty$, we find that $r(m) \geq 0$.

- Thirdly, if $\tau = 0$ and $\xi_{\parallel}^2 \cos^2 \varpi \neq 0$, the second order polynomial (3.28) has a double root which is given by $r(m) = -m \mathbf{b}_e / (\xi_{\parallel} \cos \varpi)$. Again, either for $m \rightarrow -\infty$ or for $m \rightarrow +\infty$, we have $r(m) \geq 0$.

- Finally, assume that $\tau \neq 0$ and $\tau^2 \neq \xi_{\parallel}^2 \cos^2 \varpi$. The condition (3.28) is a quadratic equation to be solved for r , with discriminant $\Delta = 4 \tau^2 (\xi_{\parallel}^2 \cos^2 \varpi + m^2 \mathbf{b}_e^2 - \tau^2)$. Since, $\mathbf{b}_e(\mathbf{x}) > 0$, the number Δ is sure to become positive for all $m \in \mathbb{Z}$ sufficiently large. Then, the two roots $r_{\pm}(m)$ of (3.28) are:

$$r_{-}(m) := \frac{2 m \mathbf{b}_e \xi_{\parallel} \cos \varpi - \sqrt{\Delta}}{2 (\tau^2 - \xi_{\parallel}^2 \cos^2 \varpi)}, \quad r_{+}(m) := \frac{2 m \mathbf{b}_e \xi_{\parallel} \cos \varpi + \sqrt{\Delta}}{2 (\tau^2 - \xi_{\parallel}^2 \cos^2 \varpi)}.$$

In particular, when $|m|$ goes to infinity, there remains:

$$r_{\pm}(m) := -\frac{m \mathbf{b}_e}{|\xi_{\parallel} \cos \varpi| \mp |\tau|} + O\left(\frac{1}{m}\right).$$

For $m \rightarrow -\infty$, we find $r_{-}(m) \geq 0$, and the value $r_{-}(m)$ is suitable. Either for $m \rightarrow -\infty$ or for $m \rightarrow +\infty$, the selection of $r_{+}(m)$ is also relevant. \square

A number of differences between the cold case (see [8]) and the hot case (3.22) deserve to be emphasized. These aspects are commented below.

◦ *Cold situation.* In [8], only three Fourier coefficients ($F_0^{l,-1}$, $F_0^{l,0}$ and $F_0^{l,1}$) were involved, and the set of resonances (namely $\tau = 0$ and $\tau = \pm \mathbf{b}_e$) was finite, simple and localized in the usual (time-space) cotangent space.

◦ *Hot situation.* When dealing with (3.22), all Fourier coefficients $\mathfrak{F}_0^{l,m}$ (with $m \in \mathbb{Z}$) make some non trivial contribution. Moreover, the structure of resonances is much more complicated. It is based on kinetic features, in the sense that the velocity p plays a role through special choices of ϖ and r . As revealed by Lemma 3.4, all values of τ are affected (for some p) by a (kinetic) resonance.

Technically, the implementation of all the coefficients $\mathfrak{F}_0^{l,m}$ comes from the change (3.21). In practice, this reveals the impact on the dispersion relations of fast (hot) beams of particles, as it can be achieved through the advection term $(\mathbf{v} \cdot \nabla_{\mathbf{x}})f$. For the same reasons, in the hot case, the time resonant frequency τ_m has come to depend on (r, ϖ) . In some ways, the velocities \mathbf{v} contribute to resonances that are dispatched in a continuum of time-space frequencies, instead of being focused on special positions.

The above discussion reveals that the resonances come from the interaction between an electromagnetic wave represented in (3.23) by $E_0^l(\mathbf{t}, \mathbf{x})$ and a population of particles that is associated with $\mathfrak{F}_0^{l,m}(\mathbf{t}, \mathbf{x}, \varpi, r)$, where (m, r) is adjusted as in Lemma 3.4. This is consistent with the basic concepts of wave-particle interactions in collisionless plasmas [36]. In the hot case, as a consequence of the kinetic aspects, the approach of [8] cannot be implemented. Another method must be found. This starts in the next Paragraph 3.2.4 with formal computations. This continues in Section 3.3 with a rigorous work of justification.

3.2.4. *Formal resolution of the system (3.12)-(3.17).* In view of (3.21)-(3.22), the density coefficient \mathcal{F}_0^l can be viewed as a linear function of E_0^l through:

$$(3.29) \quad \mathcal{F}_0^l(\mathbf{t}, \mathbf{x}, p) = \exp(-i\zeta \sin \varpi) \mathfrak{F}_0^l(\mathbf{t}, \mathbf{x}, p) = V(\mathbf{t}, \mathbf{x}, p, \tau, \xi) \cdot E_0^l(\mathbf{t}, \mathbf{x}),$$

where $V(\cdot)$ is the vector valued function which can be obtained by inverting the relations contained in (3.23). This furnishes:

$$(3.30) \quad V(\mathbf{t}, \mathbf{x}, p, \tau, \xi) \equiv V(\mathbf{x}, r, \varpi, \tau, \xi_{\perp}, \xi_{\parallel}) := \frac{2r}{\langle r \rangle} G^d(\Psi(\rho, z)) \partial_r F^d(r^2) \\ \times \sum_{(m,n) \in \mathbb{Z}^2} \frac{J_n(\zeta)}{i(\tau + \tau_m)} \begin{pmatrix} m \zeta^{-1} J_m(\zeta) \sin \varpi \\ -i J'_m(\zeta) \sin \varpi \\ J_m(\zeta) \cos \varpi \end{pmatrix} e^{i(m-n)\varpi}.$$

Due to the factor $(\tau + \tau_m)^{-1}$, it must be clear that the formula (3.30) has no sense at resonant time frequencies. Still, the relation (3.29) with $V(\cdot)$ as in (3.30) can be exploited in order to express $\mathcal{J}_1(\mathcal{F}_0^l)$ in terms of E_0^l .

Lemma 3.5. *The vector $\mathcal{J}_1(\mathcal{F}_0^l)$ can be determined through $\mathcal{J}_1(\mathcal{F}_0^l) = \sigma(\mathbf{x}, \tau, \xi) E_0^l$ where the conductivity tensor $\sigma(\cdot)$ is given by:*

$$(3.31) \quad \sigma(\mathbf{x}, \tau, \xi) := -4\pi i G^d(\Psi(\rho, z)) \sum_{n \in \mathbb{Z}} \int_0^{+\infty} \int_0^{\pi} \frac{r^4}{\langle r \rangle^2} \frac{\partial_r F^d(r^2)}{(\tau + \tau_n)} T_n dr d\varpi.$$

At the level of (3.31), with ζ and τ_n as in (3.19) and (3.24), the " T_n " symbol stands for the skew-symmetric matrix:

$$T_n := \begin{pmatrix} \frac{n^2 J_n^2(\zeta)}{\zeta^2} \sin^3 \varpi & \frac{i n J_n(\zeta) J'_n(\zeta)}{\zeta} \sin^3 \varpi & \frac{n J_n^2(\zeta)}{\zeta} \cos \varpi \sin^2 \varpi \\ -\frac{i n J_n(\zeta) J'_n(\zeta)}{\zeta} \sin^3 \varpi & (J'_n(\zeta))^2 \sin^3 \varpi & -i J_n(\zeta) J'_n(\zeta) \cos \varpi \sin^2 \varpi \\ \frac{n J_n^2(\zeta)}{\zeta} \cos \varpi \sin^2 \varpi & i J_n(\zeta) J'_n(\zeta) \cos \varpi \sin^2 \varpi & J_n^2(\zeta) \cos^2 \varpi \sin \varpi \end{pmatrix}.$$

A first remark is about the general form of (3.31). There are some similarities with models already proposed. For instance, just replace v_\perp and v_\parallel respectively by $r \sin \varpi$ and $r \cos \varpi$ in the formulas (3.4) and (3.5) of [12]. The novelty here is the relativistic context and the influence of the magnetic surfaces (through the function Ψ). In addition, the formula (3.31) has been justified through a rigorous approach, from basic principles. At this stage, the definition (3.31) of $\sigma(\cdot)$ is only formal. Indeed, the denominator $\tau + \tau_n$ vanishes at the resonances. At first sight, nothing guarantees that the improper integrals of (3.31) converge. On the other hand, in (3.31), the convergence (with respect to $n \in \mathbb{N}$) of the series could be problematic.

Proof. Recall that:

$$\mathcal{J}_1(\mathcal{F}_0^l) = \int_0^{+\infty} \int_0^\pi \int_{-\pi}^\pi \frac{r^3}{\langle r \rangle} \begin{pmatrix} \cos \omega \sin \varpi \\ \sin \omega \sin \varpi \\ \cos \varpi \end{pmatrix} \mathcal{F}_0^l(\mathbf{t}, \mathbf{x}, \varpi, \omega, r) \sin \varpi \, dr \, d\varpi \, d\omega.$$

With (3.26), (3.29) and (3.30), we get the result by direct calculation. \square

Applying Lemma 3.5, the system (3.17) reduces to:

$$(3.32) \quad \mathfrak{N}(\mathbf{x}, \tau, \xi) E_0^l = 0, \quad \mathfrak{N}(\mathbf{x}, \tau, \xi) := \xi^t \xi + (\tau^2 - |\xi|^2) Id + i \tau \sigma(\mathbf{x}, \tau, \xi).$$

We now define a notion of characteristic variety associated with hot magnetized plasmas.

Definition 3.2 (characteristic variety). *The characteristic variety which is associated with hot magnetized plasmas is the subset \mathcal{V} of the cotangent bundle T^*M composed of:*

$$(3.33) \quad \mathcal{V} := \{(\mathbf{t}, \mathbf{x}, \tau, \xi) \in [0, T] \times \Omega \times (\mathbb{R}^4 \setminus \{0\}); \det \mathfrak{N}(\mathbf{x}, \tau, \xi) = 0\}.$$

The relation $\det \mathfrak{N}(\mathbf{x}, \tau, \xi) = 0$ depends on $\mathbf{x} \in \Omega$, on $\tau \in \mathbb{R}$, on $|\xi| \in \mathbb{R}_+$ and on the angle $\varpi_\xi \in [0, \pi]$. But, it neither involves the time $\mathbf{t} \in [0, T]$ nor implies the angle $\omega_\xi \in [0, 2\pi]$. By the way, note that $(\tau, \xi) \in \mathcal{V}$ implies $(-\tau, -\xi) \in \mathcal{V}$. On the other hand, for some subset $V(\mathbf{x}, \varpi)$ of the half-space $\mathbb{R} \times \mathbb{R}_+$, we have:

$$(3.34) \quad \mathcal{V} := \{(\mathbf{t}, \mathbf{x}, \tau, \xi) \in [0, 1] \times \Omega \times (\mathbb{R}^4 \setminus \{0\}); (\tau, |\xi|) \in V(\mathbf{x}, \varpi_\xi)\}.$$

Locally, in the neighbourhood of a regular point of \mathcal{V} , the characteristic variety can be parameterized as follows:

$$\mathcal{V} = \{(\mathbf{x}, \tau, \xi) \in \Omega \times (\mathbb{R}^4 \setminus \{0\}); D_M(\mathbf{x}, \tau, \xi) = 0\},$$

where $D_M(\cdot)$ stands for the dispersion relation of electromagnetic waves. Note that the gyroballistic dispersion function $D_m = \tau + \tau_m$ is constitutive of the definition (3.31), and therefore of $D_M(\cdot)$. There have been many interpretations of wave-particle interaction. A way of doing things is to suppose that the dispersion relation $D_M(\cdot)$ is given *a priori*. Then, it is to locate the positions (\mathbf{x}, τ, ξ) where $D_M(\mathbf{x}, \tau, \xi) = 0$ and $D_m(\mathbf{x}, p, \tau, \xi) = 0$ (for some p), and to consider that it is where the exchanges of energy take place through *gyroresonant wave conversion* [9]. We adopt here a different approach since the effects of $D_m(\cdot)$ are directly incorporated inside $D_M(\cdot)$. As a byproduct, when defining $D_M(\cdot)$, we are faced with new difficulties (of convergence and summability), which are solved in the next Part 3.3. The questions about active power transfer will not be investigated in this article. It is the next step, related to the transport equations on the amplitudes.

3.3. Analysis of the conductivity tensor $\sigma(\cdot)$. In the cold case [8], exact algebraic dispersion relations are available. In contrast, in the hot case, the formula (3.31) cannot be solved analytically. Due to the resonances, it is even not clear whether the matrix $\sigma(\cdot)$ makes sense. In most texts on plasma physics, this last difficulty is simply avoided by working in the upper half of the complex plane, that is with $Im \tau > 0$. In this Part 3.3, we will define the coefficients of $\sigma(\cdot)$ for $\tau \in \mathbb{R}$. This means to verify that the improper integrals involved at the level of (3.31) are convergent for all $n \in \mathbb{Z}$, and also that (3.31) gives rise to a convergent series with respect to $n \in \mathbb{N}$. To our knowledge, the following analysis is original. We will consider the general situation, that is when $\xi_{\parallel} \neq 0$ and $\xi_{\perp} \neq 0$. The discussion starts in Paragraph 3.3.1 with a change of variables allowing to transform (3.31) in a usable way. This allows to highlight the role of the Hilbert transform, which is introduced in Paragraph 3.3.2.

3.3.1. *A change of variables.* Consider the transformation:

$$\Phi : \begin{array}{l}]0, +\infty[\times]0, \pi[\longrightarrow D \\ (r, \varpi) \longmapsto (y, z) := (r \cos \varpi, \langle r \rangle), \end{array}$$

where:

$$D := \left\{ z \in]1, +\infty[, y \in]-\sqrt{z^2 - 1}, \sqrt{z^2 - 1}[\right\}.$$

The Jacobian of the transformation Φ is given by

$$(3.35) \quad J_{\Phi}(\Phi^{-1}(y, z)) := z^{-1} \sqrt{z^2 - 1} \sqrt{z^2 - 1 - y^2}.$$

On the other hand, with $F^d(\cdot)$ as in (2.60) and $F^d(\cdot)$ as indicated below, we have:

$$(3.36) \quad 2 (\partial_r F^d)(r^2) = z^{-1} \partial_z F^d(z), \quad F^d(z) := F^d(z^2 - 1) = F^d(r^2).$$

Using (3.35) and (3.36), the matrix $\sigma(\cdot)$ can be rewritten:

$$(3.37) \quad \sigma(\mathbf{x}, \tau, \xi) := -2\pi i G^d(\Psi(\rho, z)) \times \sum_{m \in \mathbb{Z}} \int_1^{+\infty} \left(\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \frac{\mathcal{T}_m(y, z)}{\tau z + \xi_{\parallel} y + m \mathbf{b}_e} dy \right) \frac{\partial_z F^d(z)}{z} dz.$$

In (3.37), $\mathcal{T}_m(y, z)$ is the skew-symmetric matrix whose coefficients are given by:

$$(3.38a) \quad \mathcal{T}_m^{1,1} := \frac{m^2 J_m^2(\zeta)}{\zeta^2} (z^2 - 1 - y^2), \quad \mathcal{T}_m^{1,2} := \frac{i m J_m(\zeta) J'_m(\zeta)}{\zeta} (z^2 - 1 - y^2),$$

$$(3.38b) \quad \mathcal{T}_m^{1,3} := \frac{m J_m^2(\zeta)}{\zeta} y \sqrt{z^2 - 1 - y^2}, \quad \mathcal{T}_m^{2,1} := \frac{-i m J_m(\zeta) J'_m(\zeta)}{\zeta} (z^2 - 1 - y^2),$$

$$(3.38c) \quad \mathcal{T}_m^{2,2} := J'_m(\zeta)^2 (z^2 - 1 - y^2), \quad \mathcal{T}_m^{2,3} := -i J_m(\zeta) J'_m(\zeta) y \sqrt{z^2 - 1 - y^2},$$

$$(3.38d) \quad \mathcal{T}_m^{3,1} := \frac{m J_m^2(\zeta)}{\zeta} y \sqrt{z^2 - 1 - y^2}, \quad \mathcal{T}_m^{3,2} := i J_m(\zeta) J'_m(\zeta) y \sqrt{z^2 - 1 - y^2},$$

$$(3.38e) \quad \mathcal{T}_m^{3,3} := J_m^2(\zeta) y^2.$$

In (3.38), the symbol ζ must be viewed as a function of (y, z) as indicated below:

$$(3.39) \quad \zeta = \xi_{\perp} \mathbf{b}_e(\mathbf{x})^{-1} \sqrt{z^2 - 1 - y^2}.$$

3.3.2. *The Hilbert transform.* Consider the improper integral which at the level of (3.37) is inside brackets, that is:

$$\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \frac{\mathcal{T}_m(y, z)}{\tau z + \xi_{\parallel} y + m \mathbf{b}_e} dy = \int_{\mathbb{R}} \frac{\mathcal{T}_m(y, z)}{\tau z + \xi_{\parallel} y + m \mathbf{b}_e} \mathbb{1}_{]-\sqrt{z^2-1}, \sqrt{z^2-1}[}(y) dy.$$

Changing y into \tilde{y} with $\tau z + \xi_{\parallel} y + m \mathbf{b}_e = \xi_{\parallel} \tilde{y}$, this becomes:

$$(3.40) \quad \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \frac{\mathcal{T}_m(y, z)}{\tau z + \xi_{\parallel} y + m \mathbf{b}_e} dy = \frac{1}{\xi_{\parallel}} \mathcal{I}_0(z), \quad \mathcal{I}_0(z) := \int_{\mathbb{R}} \frac{\mathbf{T}_m(\tilde{y}, z)}{\tilde{y}} d\tilde{y},$$

where we have introduced:

$$(3.41) \quad \mathbf{T}_m(y, z) := \mathcal{T}_m\left(y - \frac{(\tau z + m \mathbf{b}_e)}{\xi_{\parallel}}, z\right) \mathbb{1}_{]-\sqrt{z^2-1}, \sqrt{z^2-1}[}\left(y - \frac{(\tau z + m \mathbf{b}_e)}{\xi_{\parallel}}\right).$$

We have to specify a prescription for dealing with the singular denominator in the integral on the right hand side of (3.40). To this end, a standard procedure is to push the singularity above the real \tilde{y} -axis, in order to recover a well-defined contour integral. Given a small parameter $\eta \in \mathbb{R}_+^*$, this amounts to look at:

$$\tilde{\mathcal{I}}_{\eta}(z) := \int_{\mathbb{R}} \frac{\mathbf{T}_m(\tilde{y}, z)}{\tilde{y} - i\eta} d\tilde{y}, \quad \tilde{\mathcal{I}}_0(z) := \lim_{\eta \rightarrow 0^+} \tilde{\mathcal{I}}_{\eta}(z).$$

The **Plemelj formula** allows to define $\tilde{\mathcal{I}}_0(z)$ as a complex number whose real part $\mathcal{I}_0(z)$ is:

$$(3.42) \quad \mathcal{I}_0(z) := \operatorname{Re} \tilde{\mathcal{I}}_0(z) = \mathcal{H}((\mathbf{T}_m(\cdot, z))(0)),$$

where the linear operator \mathcal{H} is determined as indicated below.

Definition 3.3 (Hilbert transform). Given $f : \mathbb{R} \rightarrow \mathbb{R}$ in the Schwartz space $\mathcal{S}(\mathbb{R})$, the *Hilbert transform* $\mathcal{H}f(\cdot)$ of $f(\cdot)$ is the function given by:

$$(3.43) \quad \mathcal{H}f(x) := \lim_{\varepsilon \rightarrow 0^+} \mathcal{H}_{\varepsilon}f(x), \quad \mathcal{H}_{\varepsilon}f(x) := \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y} dy.$$

The quantity $\mathcal{I}_0(z)$ is usually associated with a *phase*, whereas the imaginary part of $\tilde{\mathcal{I}}_0(z)$ could be interpreted as inducing some damping or amplification effect on the *amplitude* of the waves. Now, the perspective of our WKB hierarchy is to look successively at the terms with different powers of ε in factor. In this process, the *eikonal equation* (ε^{-1}) is listed first, whereas the transport equation (ε^0) is displayed after. From this standpoint, the primary step is to identify the *geometry* of the phase. In this respect, the definition (3.42) with \mathbf{T}_m as in (3.41) is what comes first. This is why it is selected in what follows. Of course alternative definitions of $\mathcal{I}_0(z)$ may be considered. But they will be ignored here, because the aim of this article is to focus on *dispersion relations*. Furthermore, as will be seen below, the formula (3.42) makes perfectly sense.

As a matter of fact, the Hilbert transform of the function $f(\cdot)$ is well-defined provided the integral (3.43) exists as a Cauchy principal value. Intuitively, the contributions related to the negative and positive values of \tilde{y} can compensate when determining $\mathcal{I}_0(z)$ through the integral of (3.40). On the one hand, the function $\mathbf{T}_m(\cdot, z)$ of (3.41) has compact support. It is piecewise continuously differentiable, with two possible jumps. On the other hand, the operator \mathcal{H} is an isometry on $L^2(\mathbb{R})$. It also maps bounded functions to the Banach space of bounded mean oscillation classes, denoted by $BMO(\mathbb{R})$. Since $\mathbf{T}_m(\cdot, z) \in (L^2 \cap L^\infty)(\mathbb{R})$ both arguments can be used to define $\mathcal{I}_0(z)$. Coming back to the initial formulation (3.37), with (3.40) and (3.42) in mind, we can rewrite (3.37) in the form:

$$(3.44) \quad \sigma(\mathbf{x}, \tau, \xi) := \frac{2\pi i G^d(\Psi(\rho, z))}{\xi_{\parallel}} \int_1^{+\infty} \mathcal{H}(\mathbf{T}(\cdot, z))(0) \frac{\partial_z F^d}{z} dz,$$

where:

$$(3.45) \quad \mathbf{T}(y, z) := \sum_{m \in \mathbb{Z}} \mathbf{T}_m(y, z),$$

The discussion about a rigorous definition of the dielectric tensor $\sigma(\cdot)$ is not finished. Supplementary estimates on $\mathcal{H}(\mathbf{T}(\cdot, z))(0)$ are needed to be sure that, inside (3.44), the integral with respect to z is convergent. A functional framework that is suitable for that purpose is exhibited below.

Lemma 3.6. *Fix $x \in \mathbb{R}$ and $\eta > 0$. Assume that the function $f(\cdot)$ is in $L^1(\mathbb{R})$ and that it is Lipschitz on the interval $[x - \eta, x + \eta]$ with Lipschitz constant:*

$$\|f\|_{Lip([x-\eta, x+\eta])} := \sup_{x-\eta \leq s < t \leq x+\eta} \frac{|f(s) - f(t)|}{|s - t|}.$$

Then, the improper integral $\mathcal{H}f(x)$ is well-defined. Moreover, we have:

$$(3.46) \quad |\mathcal{H}f(x)| \leq 2\eta \|f\|_{Lip([x-\eta, x+\eta])} + \eta^{-1} \|f\|_{L^1(\mathbb{R})}.$$

Proof. The expression $\mathcal{H}_\varepsilon f(x)$ makes sense and we have $|\mathcal{H}_\varepsilon f(x)| \leq \varepsilon^{-1} \|f\|_{L^1(\mathbb{R})}$. Using the fact that t^{-1} is an odd function, we can split the integral of (3.43) according to:

$$\mathcal{H}_\varepsilon f(x) = \int_{\varepsilon < |t| < \eta} \frac{f(x-t) - f(x)}{t} dt + \int_{|t| \geq \eta} \frac{f(x-t)}{t} dt.$$

Then a rough estimate gives rise to:

$$|\mathcal{H}_\varepsilon f(x)| \leq \int_{\varepsilon < |t| < \eta} \|f\|_{Lip([x-\eta, x+\eta])} dt + \eta^{-1} \int_{|t| \geq \eta} |f(x-t)| dt.$$

By passing to the limit ($\varepsilon \rightarrow 0$), we easily get the result from Lemma 3.6. \square

The series (3.45) is in fact finite, with a number of terms depending on y and z . For $m \in \mathbb{Z}$ and $1 \leq i, j \leq 3$, we denote by $\mathbf{T}_m^{i,j}$ the coefficient of index (i, j) inside \mathbf{T}_m . The matter in the next Paragraphs 3.3.3 and 3.3.4 is to show that, at least for $(m, i, j) \neq (0, 3, 3)$, the functions $\mathbf{T}_m^{i,j}(\cdot, z)$ satisfy the assumptions of Lemma 3.6 for $x = 0$ and $\eta = 1$. The more singular case of $\mathbf{T}_0^{3,3}$ is handled separately. It is addressed in Paragraph 3.3.4.

3.3.3. *Lipschitz estimates.* We focus here on the Lipschitz condition of Lemma 3.6.

Proposition 3.1. *For all $(m, i, j) \in \mathbb{Z} \times \llbracket 1, 3 \rrbracket^2 \setminus \{(0, 3, 3)\}$ and $z \in]1, +\infty[$, the function $\mathbf{T}_m^{i,j}(\cdot, z)$ is Lipschitz on \mathbb{R} . For all $(i, j) \in \llbracket 1, 3 \rrbracket^2 \setminus \{(3, 3)\}$, there exists a polynomial $P^{i,j}(\cdot) \in \mathbb{R}[X]$ such that:*

$$(3.47) \quad \forall z \in]1, +\infty[, \quad \sum_{m \in \mathbb{Z}} \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{Lip([-1,1])} \leq P^{i,j}(\sqrt{z^2 - 1}).$$

There exists also a polynomial $P^{3,3}(\cdot) \in \mathbb{R}[X]$ such that:

$$(3.48) \quad \forall z \in]1, +\infty[, \quad \sum_{m \in \mathbb{Z}^*} \|\mathbf{T}_m^{3,3}(\cdot, z)\|_{Lip([-1,1])} \leq P^{3,3}(\sqrt{z^2 - 1}).$$

Proof. Fix $z \in]1, +\infty[$ and $(m, i, j) \in \mathbb{Z} \times \llbracket 1, 3 \rrbracket^2 \setminus \{(0, 3, 3)\}$. Then, the function $\mathbf{T}_m^{i,j}(\cdot, z)$ is Lipschitz on \mathbb{R} if and only if the function $\mathcal{T}_m^{i,j}(\cdot, z) \mathbb{1}_{] -\sqrt{z^2-1}, \sqrt{z^2-1}[}(\cdot)$ is Lipschitz on \mathbb{R} . This function is clearly \mathcal{C}^1 on $\mathbb{R} \setminus \{\pm\sqrt{z^2 - 1}\}$. Difficulties can arise at $\pm\sqrt{z^2 - 1}$, due to the Heaviside step function. The idea is to compensate for this by the behaviour (vanishing when $m \neq 0$) of the functions $J_m(\cdot)$ or by the $\sqrt{z^2 - 1 - y^2}$ factor that appears in almost all the coefficients of \mathcal{T}_m . The Taylor expansions near $\zeta = 0$ of the **Bessel functions** are:

$$(3.49a) \quad J_0(\zeta) \underset{\zeta \rightarrow 0}{=} 1 + O(\zeta^2), \quad J_{\pm 1}(\zeta) \underset{\zeta \rightarrow 0}{=} \pm \frac{1}{2} \zeta + O(\zeta^2),$$

$$(3.49b) \quad J_n(\zeta) \underset{\zeta \rightarrow 0}{=} O(\zeta^2), \quad \forall n \in \mathbb{Z} \setminus \{0, \pm 1\}.$$

It follows that, except for $(m, i, j) = (0, 3, 3)$, all the coefficients $\mathcal{T}_m^{i,j}(\cdot, z)$ of (3.38) vanish at the points $\pm\sqrt{z^2 - 1}$. The functions $\mathbf{T}_m^{i,j}(\cdot, z)$ are therefore continuous on \mathbb{R} . To see why the functions $\mathbf{T}_m^{i,j}(\cdot, z)$ are Lipschitz and why we have both (3.47) and (3.48), we need to be more specific. We first establish the Lipschitz property.

Lemma 3.7. *Fix $(m, i, j) \in \mathbb{Z} \times \llbracket 1, 3 \rrbracket^2 \setminus \{(0, 3, 3)\}$ and $z \in]1, +\infty[$. Then:*

$$(3.50) \quad \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{Lip(\mathbb{R})} < +\infty.$$

Proof. Recall (3.39), and retain that $\sqrt{z^2 - 1 - y^2} \partial_y \zeta = -\xi_\perp \mathbf{b}_e(\mathbf{x})^{-1} y$ with $\xi_\perp \neq 0$. For $|y| < \sqrt{z^2 - 1}$, compute:

$$(3.51a) \quad \partial_y \mathcal{T}_m^{1,1}(y, z) = -2 m^2 y \frac{J_m(\zeta) J'_m(\zeta)}{\zeta},$$

$$(3.51b) \quad \partial_y \mathcal{T}_m^{1,2}(y, z) = -i m y \left[J'_m(\zeta)^2 + J_m(\zeta) J''_m(\zeta) + \frac{J_m(\zeta) J'_m(\zeta)}{\zeta} \right],$$

$$(3.51c) \quad \partial_y \mathcal{T}_m^{1,3}(y, z) = \frac{m \mathbf{b}_e}{\xi_\perp} J_m^2(\zeta) - \frac{2 m \xi_\perp}{\mathbf{b}_e} y^2 \frac{J_m(\zeta) J'_m(\zeta)}{\zeta},$$

$$(3.51d) \quad \partial_y \mathcal{T}_m^{2,2}(y, z) = -2 y \zeta J'_m(\zeta) J''_m(\zeta) - 2 y J'_m(\zeta)^2,$$

$$(3.52a) \quad \begin{aligned} \partial_y \mathcal{T}_m^{2,3}(y, z) &= -i \frac{\mathbf{b}_e}{\xi_\perp} \zeta J_m(\zeta) J'_m(\zeta) \\ &\quad + i \frac{\xi_\perp}{\mathbf{b}_e} y^2 \left[J'_m(\zeta)^2 + J_m(\zeta) J''_m(\zeta) + \frac{J_m(\zeta) J'_m(\zeta)}{\zeta} \right], \end{aligned}$$

$$(3.52b) \quad \partial_y \mathcal{T}_m^{3,3}(y, z) = 2 y \left[J_m^2(\zeta) - y^2 \frac{\xi_\perp^2}{\mathbf{b}_e^2} \frac{J_m(\zeta) J'_m(\zeta)}{\zeta} \right].$$

Since the matrix $\partial_y \mathcal{T}_m$ is skew-symmetric, we have also:

$$\partial_y \mathcal{T}_m^{2,1} = \partial_y \bar{\mathcal{T}}_m^{1,2}, \quad \partial_y \mathcal{T}_m^{3,1} = \partial_y \bar{\mathcal{T}}_m^{1,3}, \quad \partial_y \mathcal{T}_m^{3,2} = \partial_y \bar{\mathcal{T}}_m^{2,3}.$$

As long as $m \neq 0$, in view of (3.49a) and (3.49b), all the quantities $\partial_y \mathcal{T}_m^*$ remain bounded when ζ goes to zero, or equivalently when y goes to $(\mp \sqrt{z^2 - 1}) \pm$. When $m = 0$, the same applies because $J'_0(\zeta) = O(\zeta)$. Thus, the following one sided limits exist and are finite:

$$(3.53) \quad \lim_{y \rightarrow (\mp \sqrt{z^2 - 1}) \pm} \partial_y \mathcal{T}_m^{i,j}(y, z) < +\infty.$$

On the other hand, from the definition (3.41), we can infer that:

$$(3.54) \quad \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{Lip(\mathbb{R})} = \|\partial_y \mathcal{T}_m^{i,j}(\cdot, z)\|_{L^\infty(]-\sqrt{z^2-1}, \sqrt{z^2-1}[)}.$$

Since the function $\mathcal{T}_m^{i,j}(\cdot, z)$ is of class \mathcal{C}^1 on the interval $]-\sqrt{z^2-1}, \sqrt{z^2-1}[$, combining (3.53) and (3.54), we get (3.7). \square

To complete the proof of Lemma 3.1, we have to show the upper bounds (3.47) and (3.48). To this end, an important argument is that only a finite number of $m \in \mathbb{Z}$ contribute in the series (3.47) and (3.48). As a matter of fact, for $m \in \mathbb{Z}$ sufficiently large, namely:

$$(3.55) \quad |m| \geq M(z) := \mathbf{b}_e^{-1} \left(|\xi_\parallel| (1 + \sqrt{z^2 - 1}) + |\tau| z \right),$$

we have:

$$\forall y \in [-1, 1], \quad |y - \xi_\parallel^{-1} (\tau z + m \mathbf{b}_e)| \geq \sqrt{z^2 - 1}.$$

Retain that

$$\forall |m| \geq M(z), \quad \forall y \in [-1, 1], \quad \mathbf{T}_m^{i,j}(y, z) = 0.$$

Therefore, for all $(i, j) \in \llbracket 1, 3 \rrbracket^2 \setminus \{(3, 3)\}$, we have:

$$(3.56) \quad \begin{aligned} \sum_{m \in \mathbb{Z}} \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{Lip([-1, 1])} &= \sum_{|m| \leq M(z)} \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{Lip([-1, 1])} \\ &\leq \sum_{|m| \leq M(z)} \|\partial_y \mathcal{T}_m^{i,j}(\cdot, z)\|_{L^\infty(]-\sqrt{z^2-1}, \sqrt{z^2-1}[)}. \end{aligned}$$

Similarly:

$$(3.57) \quad \sum_{m \in \mathbb{Z}^*} \|\mathbf{T}_m^{3,3}(\cdot, z)\|_{Lip([-1, 1])} \leq \sum_{0 < |m| \leq M(z)} \|\partial_y \mathcal{T}_m^{3,3}(\cdot, z)\|_{L^\infty(]-\sqrt{z^2-1}, \sqrt{z^2-1}[)}.$$

For all $m \in \mathbb{N}$, the Bessel function of the first kind can be defined over the integral:

$$(3.58) \quad J_{-m}(\zeta) = (-1)^m J_m(\zeta), \quad J_m(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\zeta \sin t - m t) dt.$$

From (3.58) for $k = 0$ and from (3.26b) for $k \in \{1, 2\}$, we easily get:

$$(3.59) \quad \forall m \in \mathbb{Z}, \quad \forall \zeta \in \mathbb{R}, \quad \forall k \in \{0, 1, 2\}, \quad |J_m^{(k)}(\zeta)| \leq 1.$$

Then, using (3.26a), we can obtain:

$$(3.60) \quad \forall m \in \mathbb{Z}^*, \quad \forall \zeta \in \mathbb{R}, \quad \left| \zeta^{-1} J_m(\zeta) \right| \leq 1.$$

For $m = 0$, looking at (3.58), we can assert that:

$$(3.61) \quad \left| \zeta^{-1} J'_0(\zeta) \right| \leq 1.$$

Combining (3.58), (3.59), (3.60) and (3.61) at the level of (3.52), we see that:

$$|\partial_y \mathcal{T}_m^{i,j}(y, z)| \leq C (1 + \zeta^2 + m^2 + |y|^3),$$

where the constant C is uniform with respect to the variables (ζ, m, y) . On the domains under consideration, we have:

$$|y| \leq \sqrt{z^2 - 1}, \quad |\zeta| \leq |\xi_\perp| \mathbf{b}_e(\mathbf{x})^{-1} \sqrt{z^2 - 1}, \quad |m| \leq |M(z)| \leq C (1 + \sqrt{z^2 - 1})^2.$$

Thus, looking at (3.56), we can recover (3.47) and (3.48). \square

3.3.4. L^1 -estimates. We focus here on the L^1 -bound required by Lemma 3.6.

Proposition 3.2. *Fix $(m, i, j) \in \mathbb{Z} \times \llbracket 1, 3 \rrbracket^2$ and $z \in]1, +\infty[$. Then, $\mathbf{T}_m^{i,j}(\cdot, z) \in L^1(\mathbb{R})$. More precisely, we have:*

$$(3.62) \quad \forall z \in]1, +\infty[, \quad \sum_{m \in \mathbb{Z}} \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{L^1(\mathbb{R})} \leq 2 (z^2 - 1)^{3/2}.$$

Proof. The function $\mathbf{T}_m^{i,j}(\cdot, z)$ is bounded on \mathbb{R} with compact support. It is therefore integrable on \mathbb{R} . Moreover:

$$(3.63) \quad \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{L^1(\mathbb{R})} = \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \left| \mathcal{T}_m^{i,j}(y, z) \right| dy.$$

Observe that the matrix T_m has the following structure:

$$T_n = \begin{pmatrix} a^2 \sin^3 \varpi & i a b \sin^3 \varpi & a c \cos \varpi \sin^2 \varpi \\ -i a b \sin^3 \varpi & b^2 \sin^3 \varpi & -i b c \cos \varpi \sin^2 \varpi \\ a c \cos \varpi \sin^2 \varpi & i b c \cos \varpi \sin^2 \varpi & c^2 \cos^2 \varpi \sin \varpi \end{pmatrix},$$

with $(a, b, c) = (m \zeta^{-1} J_m(\zeta), J'_m(\zeta), J_m(\zeta))$. Thus, using the elementary case of Young's inequality, the off-diagonal coefficients can be controlled by the diagonal ones:

$$(3.64) \quad \forall (i, j) \in \llbracket 1, 3 \rrbracket^2, \quad \|\mathbf{T}_m^{i,j}(\cdot, z)\|_{L^1(\mathbb{R})} \leq 2^{-1} \left(\|\mathbf{T}_m^{i,i}(\cdot, z)\|_{L^1(\mathbb{R})} + \|\mathbf{T}_m^{j,j}(\cdot, z)\|_{L^1(\mathbb{R})} \right).$$

To get (3.62), it suffices to prove the result for $\|\mathbf{T}_m^{i,i}(\cdot, z)\|_{L^1(\mathbb{R})}$ with $1 \leq i \leq 3$. To do so, recall the following addition theorem on the Bessel functions of the first kind ([37], p.358):

$$(3.65) \quad \forall (r, s, \theta) \in \mathbb{R}^2 \times \mathbb{T}, \quad \sum_{m \in \mathbb{Z}} J_m(r) J_m(s) e^{-im\theta} = J_0\left(\sqrt{r^2 + s^2 - 2rs \cos \theta}\right).$$

Looking at (3.65) for $r = s$ and $\theta = 0$, taking two derivatives of (3.65) with respect to θ at $r = s$ and $\theta = 0$, and then applying ∂_r and ∂_s at $r = s$ and $\theta = 0$, we find successively:

$$(3.66) \quad \forall r \in \mathbb{R}, \quad \sum_{m \in \mathbb{Z}} J_m^2(r) = 1, \quad \sum_{m \in \mathbb{Z}} m^2 J_m^2(r) = \frac{r^2}{2}, \quad \sum_{m \in \mathbb{Z}} J'_m(r)^2 = \frac{1}{2}.$$

Combining (3.38) and (3.66), we get:

$$\forall i \in \llbracket 1, 3 \rrbracket, \quad \forall y \in] -\sqrt{z^2 - 1}, \sqrt{z^2 - 1}[, \quad \sup_{|y| \leq \sqrt{z^2 - 1}} \sum_{m \in \mathbb{Z}} \left| \mathcal{T}_m^{i,i}(y, z) \right| \leq z^2 - 1.$$

In view of (3.63), this estimate leads directly to (3.62). \square

3.3.5. Study of the most singular coefficient. The expression $\mathbf{T}_0^{3,3}(y, z)$ does contribute at the level of (3.44). However, it cannot be treated as before. As a matter of fact, since $J_0(0) = 1$, for all $z > 1$, the function issued from (3.38e) and (3.41), say:

$$y \mapsto \mathbf{T}_0^{3,3}\left(y + \frac{\tau z}{\xi_{\parallel}}, z\right) = y^2 J_0^2\left(\frac{\xi_{\perp}}{\mathbf{b}_e} \sqrt{z^2 - 1 - y^2}\right) \mathbb{1}_{]-\sqrt{z^2 - 1}, \sqrt{z^2 - 1}[}(y),$$

is discontinuous at $y = \pm\sqrt{z^2 - 1}$. Thus, another argument must be put forward.

Lemma 3.8. *Except possibly for two values of $z \in [1, +\infty[$, the expression $\mathcal{H}(\mathbf{T}_0^{3,3}(\cdot, z))(0)$ makes sense. Moreover, it is locally integrable with respect to $z \in [1, +\infty[$.*

Proof. The idea is to remove from $\mathbf{T}_0^{3,3}(\cdot, z)$ the singular part. To this end, introduce the following decomposition of $\mathbf{T}_0^{3,3}(y, z)$, which reveals the auxiliary function $\mathfrak{T}_0^{3,3}(y, z)$:

$$(3.67) \quad \mathfrak{T}_0^{3,3}(y, z) := \mathbf{T}_0^{3,3}(y, z) - (z^2 - 1) \mathbb{1}_{]-\sqrt{z^2 - 1}, \sqrt{z^2 - 1}[}\left(y - \frac{\tau z}{\xi_{\parallel}}\right).$$

In view of (3.49a), the function $\mathfrak{T}_0^{3,3}(\cdot, z)$ is continuous on \mathbb{R} with compact support. The information (3.52b), (3.59) and (3.61) indicates that $\partial_y \mathcal{T}_0^{3,3}(\cdot, z)$ is bounded on the interval $]-\sqrt{z^2 - 1}, \sqrt{z^2 - 1}[$. It follows that $\mathfrak{T}_0^{3,3}(\cdot, z)$ is Lipschitz on \mathbb{R} . Moreover, there exists a polynomial $P_0^{3,3}(\cdot) \in \mathbb{R}[X]$ such that:

$$(3.68) \quad \forall z \in [1, +\infty[, \quad \|\mathfrak{T}_0^{3,3}(\cdot, z)\|_{Lip([-1, 1])} \leq P_0^{3,3}(\sqrt{z^2 - 1}).$$

From Lemma 3.2, we know that $\mathfrak{T}_0^{3,3}(\cdot, z)$ belongs to $L^1(\mathbb{R})$. To define $\mathcal{H}(\mathfrak{T}_0^{3,3}(\cdot, z))(0)$, we can apply Lemma 3.6. We now turn to the remaining (more singular) part on the right hand side of (3.67). The Hilbert transform of the characteristic function $\mathbb{1}_{[a, b]}(\cdot)$ is:

$$\forall t \in \mathbb{R} \setminus \{a, b\}, \quad \mathcal{H}(\mathbb{1}_{[a, b]}(\cdot))(t) = \ln \left| \frac{t - a}{t - b} \right|.$$

Thus, for almost every $z \in]1, +\infty[$, we have:

$$(3.69) \quad \mathcal{H}\left((z^2 - 1) \mathbf{1}_{]-\sqrt{z^2-1}, \sqrt{z^2-1}[} \left(\cdot - \frac{\tau z}{\xi_{\parallel}}\right)\right)(0) = (z^2 - 1) \ln \left| \frac{\xi_{\parallel}^{-1} \tau z + \sqrt{z^2 - 1}}{\xi_{\parallel}^{-1} \tau z - \sqrt{z^2 - 1}} \right|.$$

More precisely, (3.69) is well-defined for all $z \in]1, +\infty[$ when $|\tau| > \sqrt{2} |\xi_{\parallel}|$. On the contrary, when $|\tau| \leq \sqrt{2} |\xi_{\parallel}|$, only the two values $z_{\pm} = \pm \xi_{\parallel}^{-1} (\tau^2 - \xi_{\parallel}^2)^{1/2}$ can be problematic. However, the logarithmic behaviour near the (potentially) singular points z_{\pm} is compatible with L_{loc}^1 -estimates. Retain also that the asymptotic behaviour of (3.69) when $z \rightarrow \pm\infty$ is controlled by $C(z^2 - 1)$ for some constant $C \in \mathbb{R}_+$. \square

Lemma 3.9. *Assume that $\xi_{\parallel} \neq 0$ and $\xi_{\perp} \neq 0$. Then, the conductivity tensor $\sigma(\mathbf{x}, \tau, \xi)$ is well-defined through (3.44).*

Proof. It suffices to show that the Lebesgue integral inside (3.44) makes sense and is finite. For $(i, j) \neq (3, 3)$, applying (3.46) with Lemmas 3.1 and 3.2, we obtain that $\mathcal{H}(\mathbf{T}^{i,j}(\cdot, z))(0)$ is locally bounded with respect to $z \in]1, +\infty[$. It is therefore locally integrable. In view of (3.48) and Lemma 3.8, the coefficient $\mathcal{H}(\mathbf{T}^{3,3}(\cdot, z))(0)$ is also locally integrable. Now, recall that $F^d(\cdot) \in \mathcal{S}(\mathbb{R}_+; \mathbb{R})$. Thus, the controls (3.47), (3.48), (3.62) and (3.68), by at most some polynomial growth in z is compensated by the rapid decreasing of $\partial_z F^d(\cdot)$. The integrand inside (3.44) is in $L^1(]1, +\infty[)$. \square

3.4. Interesting case studies. It was shown in Section 3.3 that the matrix $\sigma(\cdot)$ is well-defined for all ξ such that $\xi_{\parallel} \neq 0$ and $\xi_{\perp} \neq 0$. The case of a parallel propagation ($\xi_{\perp} = 0$) is considered in Paragraph 3.4.1. The case of a perpendicular propagation ($\xi_{\parallel} = 0$) is studied in Paragraph 3.4.2. Looking at the number of resonances to be taken into account, that is the number of nonzero terms in the sum (3.31), the following may be noted. There is a growing complexity when passing from the parallel case (involving $m = -1, m = 0$ and $m = 1$), to the generic case of Section 3.3 (implying a finite number of m), up to the perpendicular case (giving rise to the selection of an infinite number of $m \in \mathbb{Z}$).

3.4.1. Parallel propagation ($\xi_{\perp} = 0$ and $\xi_{\parallel} \neq 0$). When $\xi_{\perp} = 0$, as indicated in (3.39), we simply find $\zeta = 0$. Now, in many formulas such as (3.30) or (3.38), the term ζ^{-1} appears in factor. However, this singularity is compensated by $m J_m(\zeta)$. Indeed, the relation (3.26a) is valid for all values of $m \in \mathbb{Z}$ and $\zeta \in \mathbb{R}$. The function $\chi_m : \zeta \mapsto m \zeta^{-1} J_m(\zeta)$ can be extended to some analytic function on \mathbb{R} satisfying:

$$(3.70) \quad \chi_{\pm 1}(0) = \frac{1}{2}, \quad \chi_m(0) = 0, \quad \forall m \in \mathbb{Z} \setminus \{-1, +1\}.$$

It follows that the matrices T_n of Lemma 3.5 are zero except if $n \in \{-1, 0, +1\}$. We find:

$$(3.71) \quad \begin{aligned} \sigma(\mathbf{x}, \tau, \xi) &= -4\pi i G^d(\Psi(\rho, z)) \sum_{|n| \leq 1} \int_0^{+\infty} \int_0^{\pi} \frac{r^4 \partial_r F^d(r^2)}{\langle r \rangle^2 (\tau + \tau_n)} T_n dr d\varpi. \\ &= -4\pi i G^d(\Psi(\rho, z)) \int_0^{+\infty} \int_0^{\pi} \frac{r^4 \partial_r F^d(r^2)}{\langle r \rangle} T(\text{par}) dr d\varpi. \end{aligned}$$

In (3.71), the matrix $T(\text{par})$ is obtained by summing up what comes from T_{-1} , T_0 and T_1 :

$$T(\text{par}) := \begin{pmatrix} \frac{(\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi) \sin^3 \varpi}{2 \left[(\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi)^2 - \mathbf{b}_e^2 \right]} & \frac{-i \mathbf{b}_e \sin^3 \varpi}{2 \left[(\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi)^2 - \mathbf{b}_e^2 \right]} & 0 \\ \frac{i \mathbf{b}_e \sin^3 \varpi}{2 \left[(\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi)^2 - \mathbf{b}_e^2 \right]} & \frac{(\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi) \sin^3 \varpi}{2 \left[(\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi)^2 - \mathbf{b}_e^2 \right]} & 0 \\ 0 & 0 & \frac{\cos^2 \varpi \sin \varpi}{\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi} \end{pmatrix}.$$

When $\xi_{\perp} = 0$, the condition $\det \mathfrak{N}(\mathbf{x}, \tau, \xi) = 0$ reduces to $\Lambda_L \Lambda_T^- \Lambda_T^+ = 0$. To simplify, introduce $\kappa \equiv \kappa(\mathbf{x}) := 4 \pi G^{\text{d}}(\Psi(\rho, z))$. The connected components of \mathcal{V} can be determined by looking at the three conditions:

$$(3.72a) \quad \Lambda_L(\mathbf{x}, \tau, \xi_{\parallel}) := \tau^2 + \kappa \tau \int_0^{+\infty} \int_0^{\pi} \frac{r^4 \partial_r F^{\text{d}}(r^2) \cos^2 \varpi \sin \varpi}{\langle r \rangle (\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi)} dr d\varpi = 0,$$

$$(3.72b) \quad \Lambda_T^-(\mathbf{x}, \tau, \xi_{\parallel}) := \tau^2 - \xi_{\parallel}^2 - \kappa \tau \int_0^{+\infty} \int_0^{\pi} \frac{r^4 \partial_r F^{\text{d}}(r^2) \sin^3 \varpi}{2 \langle r \rangle [\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi - \mathbf{b}_e]} dr d\varpi = 0,$$

$$(3.72c) \quad \Lambda_T^+(\mathbf{x}, \tau, \xi_{\parallel}) := \tau^2 - \xi_{\parallel}^2 - \kappa \tau \int_0^{+\infty} \int_0^{\pi} \frac{r^4 \partial_r F^{\text{d}}(r^2) \sin^3 \varpi}{2 \langle r \rangle [\langle r \rangle \tau + r \xi_{\parallel} \cos \varpi + \mathbf{b}_e]} dr d\varpi = 0.$$

a) *Transverse waves.* We find two components $V_T^+(\mathbf{x}, 0)$ and $V_T^-(\mathbf{x}, 0)$ which are:

$$(3.73a) \quad V_T^-(\mathbf{x}, 0) = \left\{ (\tau, |\xi|) \in \mathbb{R} \times \mathbb{R}_+ ; \Lambda_T^-(\mathbf{x}, \tau, |\xi_{\parallel}|) = 0 \right\},$$

$$(3.73b) \quad V_T^+(\mathbf{x}, 0) = \left\{ (\tau, |\xi|) \in \mathbb{R} \times \mathbb{R}_+ ; \Lambda_T^+(\mathbf{x}, \tau, |\xi_{\parallel}|) = 0 \right\}.$$

b) *Longitudinal waves.* We find only one component:

$$(3.74) \quad V_L(\mathbf{x}, 0) = \left\{ (\tau, |\xi|) \in \mathbb{R} \times \mathbb{R}_+ ; \Lambda_L(\mathbf{x}, \tau, |\xi_{\parallel}|) = 0 \right\}.$$

3.4.2. *Perpendicular propagation* ($\xi_{\parallel} = 0$ and $\xi_{\perp} \neq 0$). The content of Paragraph 3.3.1 is still valid in the case $\xi_{\parallel} = 0$. However, the interpretation of $\sigma(\cdot)$ as in (3.44) does not hold. To get around this difficulty, the idea is to use Fubini's theorem at the level of (3.37) in order to recognize some Hilbert transform issued from an integration with respect to z (instead of y). The equation (3.37) then becomes:

$$(3.75) \quad \begin{aligned} \sigma(\mathbf{x}, \tau, (\xi_{\perp}, 0, 0)) &:= -2 \pi i G^{\text{d}}(\Psi(\rho, z)) \\ &\times \sum_{m \in \mathbb{Z}} \int_1^{+\infty} \left(\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \frac{\mathcal{T}_m(y, z)}{\tau z + m \mathbf{b}_e} dy \right) \frac{\partial_z F^{\text{d}}(z)}{z} dz. \end{aligned}$$

In other words:

$$(3.76) \quad \sigma(\mathbf{x}, \tau, (\xi_\perp, 0, 0)) = 2\pi i G^d(\Psi(\rho, z)) \frac{1}{\tau} \sum_{m \in \mathbb{Z}} \mathcal{H}(\mathcal{T}_m) \left(-\frac{m \mathbf{b}_e}{\tau} \right),$$

where:

$$(3.77) \quad \mathcal{T}_m(z) := \mathbf{1}_{]1, +\infty[}(z) \frac{\partial_z F^d(z)}{z} \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \mathcal{T}_m(y, z) dy,$$

and $\mathcal{T}_m(y, z)$ is the skew-symmetric matrix given by (3.38).

Proposition 3.3. *For all $m \in \mathbb{Z}$, the function $\mathcal{T}_m(\cdot)$ is Lipschitz on \mathbb{R} . Moreover, there exists a constant $C > 0$ such that:*

$$(3.78) \quad \forall m \in \mathbb{Z}, \quad \|\partial_z \mathcal{T}_m\|_\infty := \sup_{z \in \mathbb{R}} \sup_{1 \leq i, j \leq 3} |\partial_z \mathcal{T}_m^{i,j}(z)| \leq C (m^2 + 1).$$

Proof. Fix $m \in \mathbb{Z}$ and $(i, j) \in \llbracket 1, 3 \rrbracket^2$. Then:

$$(3.79) \quad \mathcal{T}_m^{i,j}(z) = \mathbf{1}_{]1, +\infty[}(z) \frac{\partial_z F^d(z)}{z} \int_{-1}^1 \sqrt{z^2-1} \mathcal{T}_m^{i,j}(\sqrt{z^2-1} y, z) dy.$$

Note that $\mathcal{T}_m^{i,j}(1-) = \mathcal{T}_m^{i,j}(1+) = 0$. The function $\mathcal{T}_m^{i,j}(\cdot)$ is therefore continuous on \mathbb{R} . To prove Proposition 3.3, it suffices now to prove Lemma 3.10 below. \square

Lemma 3.10. *For all $(m, i, j) \in \mathbb{Z} \times \llbracket 1, 3 \rrbracket^2$ and $z \in]1, +\infty[$, the derivative $\partial_z \mathcal{T}_m^{i,j}(z)$ exists. Moreover, we have:*

$$(3.80) \quad \exists C \in \mathbb{R}_+; \quad \forall m \in \mathbb{Z}, \quad \sup_{z \in]1, +\infty[} \sup_{1 \leq i, j \leq 3} |\partial_z \mathcal{T}_m^{i,j}(z)| \leq C (m^2 + 1).$$

Proof. Recall (3.79). Consider the auxiliary function:

$$(3.81) \quad \begin{aligned} \mathfrak{f}_m^{i,j} :]1, +\infty[\times] -1, 1[&\longrightarrow \mathbb{R} \\ (z, y) &\longmapsto \mathfrak{f}_m^{i,j}(z, y) := \sqrt{z^2-1} \mathcal{T}_m^{i,j}(\sqrt{z^2-1} y, z), \end{aligned}$$

so that:

$$(3.82) \quad \mathcal{T}_m^{i,j}(z) = \mathbf{1}_{]1, +\infty[}(z) \frac{\partial_z F^d(z)}{z} \int_{-1}^1 \mathfrak{f}_m^{i,j}(z, y) dy.$$

With $y = \sqrt{z^2-1} y$, we find:

$$(3.83) \quad \zeta \equiv \zeta(z, y) = \xi_\perp \mathbf{b}_e(\mathbf{x})^{-1} \sqrt{z^2-1} \sqrt{1-y^2},$$

as well as:

$$(3.84a) \quad \mathfrak{f}_m^{1,1}(z, y) = \frac{\mathbf{b}_e^2}{\xi_\perp^2} m^2 J_m^2(\zeta) \sqrt{z^2-1},$$

$$(3.84b) \quad \mathfrak{f}_m^{1,2}(z, y) = \frac{\mathbf{b}_e}{\xi_\perp} i m J_m(\zeta) J'_m(\zeta) (z^2-1) \sqrt{1-y^2},$$

$$(3.84c) \quad \mathfrak{f}_m^{1,3}(z, y) = \frac{\mathbf{b}_e}{\xi_\perp} m J_m^2(\zeta) (z^2-1) y,$$

$$(3.85a) \quad f_m^{2,2}(z, y) = J'_m(\zeta)^2 (z^2 - 1)^{3/2} (1 - y^2),$$

$$(3.85b) \quad f_m^{2,3}(z, y) = -i J_m(\zeta) J'_m(\zeta) (z^2 - 1)^{3/2} y \sqrt{1 - y^2},$$

$$(3.85c) \quad f_m^{3,3}(z, y) = J_m^2(\zeta) (z^2 - 1)^{3/2} y^2.$$

Fix $(m, i, j) \in \mathbb{Z} \times \llbracket 1, 3 \rrbracket^2$. We will drop the reference to (m, i, j) when it is not necessary to mention it. On the interval $]1, +\infty[$, the function $z \mapsto z^{-1} \partial_z F^d(z)$ is of class C^∞ . Thus, at the level of (3.82), it suffices to look at the parameter-dependent integral:

$$]1, +\infty[\ni z \mapsto \int_{-1}^1 f_m^{i,j}(z, y) dy.$$

In order to be able to apply the Leibniz's rule for differentiation under the integral sign, we can check the following conditions:

- (i) For all $z > 1$, the function $y \mapsto f(z, y)$ is integrable on $] - 1, 1[$;
- (ii) For all $y \in] - 1, 1[$, the function $z \mapsto f(z, y)$ is of class C^1 on $]1, +\infty[$;
- (iii) The function $(z, y) \mapsto \partial_z f(z, y)$ is bounded on every set of the form $K \times] - 1, 1[$, where K is a compact subset of $]1, +\infty[$.

Below, we perform the verification work step by step.

(i) Fix $z \in]1, +\infty[$. In view of (3.38) and (3.39), the function $\mathcal{T}_m^{i,j}(\cdot, z)$ is continuous on the interval $] - 1, 1[$. In view of (3.81), this also holds true for $f_m^{i,j}(z, \cdot)$.

(ii) Fix $y \in] - 1, 1[$. On the one hand, the function $\zeta(\cdot, y)$ is smooth. On the other hand, the Bessel function $J_m(\cdot)$ is analytic on \mathbb{R} . In view of (3.85), the function $z \mapsto f(z, y)$ is therefore of class C^1 on $]1, +\infty[$.

(iii) We have to compute $D_m^{i,j}(z, y) := \partial_z f_m^{i,j}(z, y)$. Since the matrix D is skew-symmetric, it is enough to compute $D_m^{i,j}$ when $i \leq j$. Use (3.83) and (3.85) in order to obtain:

$$(3.86a) \quad D_m^{1,1}(z, y) = \frac{\mathbf{b}_e^2}{\xi_\perp^2} \frac{z}{\sqrt{z^2 - 1}} m^2 J_m^2(\zeta) + 2 \frac{\mathbf{b}_e}{\xi_\perp} z m^2 J_m(\zeta) J'_m(\zeta) \sqrt{1 - y^2},$$

$$(3.86b) \quad D_m^{1,2}(z, y) = 2 i \frac{\mathbf{b}_e}{\xi_\perp} z m J_m(\zeta) J'_m(\zeta) \sqrt{1 - y^2} \\ + i z \sqrt{z^2 - 1} m [J'_m(\zeta)^2 + J_m(\zeta) J''_m(\zeta)] (1 - y^2),$$

$$(3.86c) \quad D_m^{1,3}(z, y) = 2 \frac{\mathbf{b}_e}{\xi_\perp} z m J_m^2(\zeta) y \\ + 2 z \sqrt{z^2 - 1} m J_m(\zeta) J'_m(\zeta) y \sqrt{1 - y^2},$$

$$(3.86d) \quad D_m^{2,2}(z, y) = 3 z \sqrt{z^2 - 1} J'_m(\zeta)^2 (1 - y^2) \\ + 2 \frac{\xi_\perp}{\mathbf{b}_e} z (z^2 - 1) J'_m(\zeta) J''_m(\zeta) (1 - y^2)^{3/2},$$

$$\begin{aligned}
(3.87a) \quad D_m^{2,3}(z, y) &= -3 i z \sqrt{z^2 - 1} J_m(\zeta) J'_m(\zeta) y \sqrt{1 - y^2} \\
&\quad - i \frac{\xi_\perp}{\mathbf{b}_e} z (z^2 - 1) [J'_m(\zeta)^2 + J_m(\zeta) J''_m(\zeta)] y (1 - y^2), \\
(3.87b) \quad D_m^{3,3}(z, y) &= 3 z \sqrt{z^2 - 1} J_m^2(\zeta) y^2 \\
&\quad + 2 \frac{\xi_\perp}{\mathbf{b}_e} z (z^2 - 1) J_m(\zeta) J'_m(\zeta) y^2 \sqrt{1 - y^2}.
\end{aligned}$$

All these functions $D_m^{i,j}(\cdot)$ are clearly bounded on $K \times]-1, 1[$, at least if K is compact which is contained in $]1, +\infty[$. Since the three conditions (i), (ii) and (iii) are satisfied, we can assert that:

$$(3.88) \quad \partial_z \left(\int_{-1}^1 \mathfrak{f}_m^{i,j}(z, y) dy \right) = \int_{-1}^1 D_m^{i,j}(z, y) dy.$$

It follows that:

$$(3.89) \quad \exists C \in \mathbb{R}_+; \quad \forall z \in]1, +\infty[, \quad |\partial_z \mathcal{F}_m^{i,j}(z)| \leq C \sup_{(z,y) \in]1, +\infty[\times]-1, 1[} |D_m^{i,j}(z, y)|.$$

Exploit (3.59) to control the $J_m^{(k)}(\zeta)$ uniformly with respect to $m \in \mathbb{Z}$. In view of (3.87), as indicated in (3.80), the growth in m is at most m^2 . When z goes to $1+$, the only term which may be problematic is $D_m^{1,1}(z, y)$, see (3.86a). For $m = 0$, there is nothing to do. For $|m| = 1$, exploit (3.49a) together with (3.83) to see that $|D_m^{1,1}(z, y)|$ remains bounded. For $|m| > 1$, the expression $|D_m^{1,1}(z, y)|$ simply tends to zero when z goes to $1+$. By this way, we recover (3.80). \square

Lemma 3.11. *For all $m \in \mathbb{Z}$ and for all $(i, j) \in \llbracket 1, 3 \rrbracket^2$, the functions $\mathcal{F}_m^{i,j}(\cdot)$ is in $L^1(\mathbb{R})$. For all $(i, j) \in \llbracket 1, 3 \rrbracket^2$, there exists a polynomial $Q^{i,j} \in \mathbb{R}[X]$ such that:*

$$(3.90) \quad \sum_{m \in \mathbb{Z}} m^4 \|\mathcal{F}_m^{i,j}\|_{L^1(\mathbb{R})} \leq \int_1^{+\infty} Q^{i,j}(\sqrt{z^2 - 1}) \partial_z F^d(z) dz < +\infty.$$

Proof. Use again (3.64) to reduce the discussion to the case $i = j$. At the level of (3.65), apply the derivatives ∂_θ , ∂_r and ∂_s . Then, take $r = s$ and $\theta = 0$. By this way, we can obtain the existence of polynomials $(P_i)_{1 \leq i \leq 3} \in \mathbb{R}[X]^3$ such that, for all $r \in \mathbb{R}$ and $j \in \{2, 3\}$:

$$(3.91) \quad \sum_{m \in \mathbb{Z}} m^{2j} J_m^2(r) = P_j(r), \quad \sum_{m \in \mathbb{Z}} m^4 (J'_m(r))^2 = P_4(r).$$

Combining (3.82) and (3.85), we can see that:

$$\exists C \in \mathbb{R}_+; \quad |\mathcal{F}_m^{i,j}(z)| \leq C \mathbb{1}_{]1, +\infty[}(z) \frac{|\partial_z F^d(z)|}{z} (1 + m^3 + z^3).$$

Knowing that $F^d(\cdot) \in \mathcal{S}(\mathbb{R})$, this gives $\mathcal{F}_m^{i,j}(\cdot) \in L^1(\mathbb{R})$. Then, it suffices to exploit (3.91) together with (3.85) in the sum of (3.90) in order to recover (3.90). \square

In view of Lemmas 3.3 and 3.11, for all $\eta \in \mathbb{R}_+^*$, we can apply Lemma 3.6. This yields:

$$(3.92) \quad |\mathcal{H}(\mathcal{F}_m^{i,j})(-\frac{m \mathbf{b}_e}{\tau})| \leq 2 \eta \|\partial_z \mathcal{F}_m^{i,j}\|_\infty + \eta^{-1} \|\mathcal{F}_m^{i,j}\|_{L^1(\mathbb{R})}.$$

In (3.92), select $\eta = (m^4 + 1)^{-1}$. Take the sum over $m \in \mathbb{Z}$. In view of (3.78) and (3.90), the corresponding series is absolutely convergent. Coming back to (3.75), this implies that the matrix $\sigma(\mathbf{x}, \tau, (\xi_\perp, 0, 0))$ is well-defined.

3.4.3. *Perspectives.* It would be interesting to study \mathcal{V} through numerical computations. This would allow to produce concrete representations of \mathcal{V} , similar to the ones obtained in [8]. Note that there are some recent works dealing with the numerical aspects [34, 39] but they are restricted to the rough case where the external magnetic field is constant and where the Maxwell-Boltzmann distribution function depends only on p .

In the same way, our model (3.31) can help to improve computations in reflectometry [19]. Indeed, it allows to detect some relevant impacts induced by the spatial variations of the external magnetic field and of the equilibrium distribution function.

Besides, our analysis is a prerequisite for further mathematical developments. It would be interesting to complete the WKB analysis for times $t \sim 1$, and beyond for times $t \sim \varepsilon^{-1}$. From a physics point of view, questions about wave-particle interactions [21, 36], anomalous transport [6], or confinement properties could thereby benefit from new perspectives [7].

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