A stabilised finite element method for a time-dependent problem solved using a fictitious domain method

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A STABILISED FINITE ELEMENT METHOD FOR A TIME-DEPENDENT PROBLEM SOLVED USING A FICTITIOUS DOMAIN METHOD

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Abstract. In this work we present and analyse a stabilised finite element method for a fictitious domain formulation of a transient heat equation posed on an irregular domain. First, the problem is reformulated as a variational problem with constraints in a larger, simpler domain. In order to have freedom to choose the spaces for primal variable and Lagrange multiplier, we introduce a stabilising term whose consistency error can be shown to be of optimal order. We show that the fully discrete problem is stable and enjoys optimal convergence under the extra assumption that the initial condition has been chosen appropriately. This requirement is not linked to the stabilised character of the method, as the numerical experiments confirm.

September 25, 2017
Key words : Fictitious domain method; stabilised finite element method; time-dependent problem.

1. Introduction

This work deals with the approximation of problems posed on irregular domains. The approximation of such problems is challenging since, at least for standard finite element methods, the mesh needs to resolve the geometrical features of the domain, which may lead to highly refined, or highly irregular, meshes if the domain is complicated. To avoid this restriction, several approaches have been advocated over the years. For example, cut elements as in [10, 11], composite finite element method as in [19] or ideas related to XFEM approaches in [24, 20]. Further approaches include the fat boundary method [23, 5], non-boundary fitted meshes (see, e.g., [25]), or distributed Lagrange multiplier approaches [8, 7].

In this paper we apply the fictitious domain method for a heat equation posed on an complicated spatial domain. This approach consists on introducing a larger, and simpler, domain that includes the original physical domain where the problem is posed, and replace the original partial differential equation by a mixed problem involving an extension of the solution to the fictitious domain, and a Lagrange multiplier defined on the physical boundary. The analysis of this method for a steady-state Poisson equation was first given in [17]. In the original paper the stability and convergence of the method was proven for a piecewise linear discretisation of the primal variable, and a piecewise constant approximation of the Lagrange multiplier, for meshes that need to satisfy appropriate size restrictions. This restriction excluded, in particular, the possibility of using the intersection of the mesh for the primal variable and the physical boundary to define the finite element space for the Lagrange multiplier. To avoid this restriction, in [3] a LPS-inspired stabilised method was proposed and tested.

Most, if not all, the above-mentioned references deal with the steady-state problem. Fewer works have dealt with the time-dependent one, especially considering the analysis of the resulting method (see, e.g. [1, 14, **?**]). This lack of analysis is especially true for the case in which stabilised finite element methods are used for the space discretisation. The stabilisation of finite element methods for time dependent problems has been the object of many studies over the years. The motivation for this has been that, for the case of mixed problems, such as the Stokes equations,

unexpected results have been observed when the spatial and time discretisation parameters are not chosen carefully. More precisely, for different stabilised finite element methods the fully discrete problem can be proven stable only if $\delta t \geq Ch^2$, where h and δt stand for the mesh width and time step, respectively (see [6, 12, 13, 2]). Later, in [9], this restriction on the time step and mesh size was removed using symmetric stabilisation and, more importantly, a judicious choice of initial condition (see also [22] for a more recent study concerning the PSPG method).

Based on the discussion from the above paragraphs, in this paper we follow closely the approach given in [9], and present the stability and convergence analyses of both stabilised, and inf-sup stable, finite element methods for the time dependent heat equation approximated by a fictitious domain method. For both cases, unconditional stability can be proven if the initial condition is chosen appropriately. In fact, even if in this work we detail the analysis for the stabilised case, very similar proofs, leading to essentially the same results, can be carried out for the inf-sup stable setting. We have detailed the main differences throughout the manuscript.

The rest of the manuscript is organised as follows. In Section 2, we introduce the problem under consideration and some useful notation. In Section 3, we analise the stability of the fully discrete problems, and the convergence of the methods is proved in Section 4. Some numerical results are presented in Section 5 and a conclusion in Section 6.

2. Problem setting

We now present notation that will be used on what follows. We adopt the standard notation for Sobolev spaces (aligned with, e.g., [15]). In particular, for $D \subseteq \mathbb{R}^2$, $H^1(D)$ ($H^1_0(D)$) will denote the space of (generalised) functions of $L^2(D)$ with first derivatives also belonging to $L^2(D)$ (and that vanish on ∂D). The inner f on $L^2(D)$ is denoted by $(\cdot,\cdot)_D$, its associated norm is denoted by $\|\cdot\|_{0,D}$, and, for $m \geq 1$, the norm (seminorm) in $H^m(D)$ is denoted by $\|\cdot\|_{m,\Omega}$ ($|\cdot|_{m,\Omega}$). We keep the same notation for vector-valued functions. The space of traces of functions of $H^1(D)$ on ∂D is denoted by $H^{\frac{1}{2}}(\partial D)$, provided with the trace norm

$$\|\mu\|_{\frac{1}{2},\partial D} = \inf_{\substack{v\in H^1(D)\\ v|_{\partial D}=\mu}} \|v\|_{1,D}.$$

The dual of $H^{\frac{1}{2}}(\partial D)$, with respect to the $L^2(\partial D)$ inner product, is denoted by $H^{-\frac{1}{2}}(\partial D)$, and the duality pairing between them is denoted by $\langle\cdot,\cdot\rangle_{\partial D}$. The corresponding dual norm is denoted by $\|\cdot\|_{-\frac{1}{2},\partial D}$.

Let ω be an open bounded domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary $\gamma := \partial\omega$. We consider $\gamma = \bigcup_{i=1}^{M} \gamma_i$, where γ_i are the M smooth components of γ, that is, if γ has corners then $\gamma_i \cap \gamma_{i+1}$ will be its corner points (see Fig. 1).

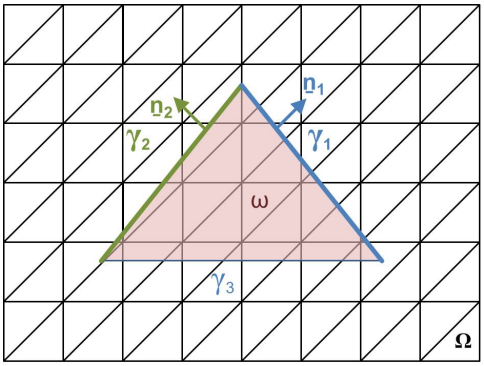

FIGURE 1. Example of γ where $M = 3$.

The following space will be needed throughout the manuscript:

$$\mathcal{H} := \prod_{i=1}^{M} H^{\frac{1}{2}}(\gamma_i) \quad \text{with norm} \quad \|\mu\|_{\mathcal{H}} := \sum_{i=1}^{M} \|\mu\|_{\frac{1}{2},\gamma_i}.$$

Let $T > 0$ be a final time. The problem of interest in this work reads as follows:
find $\hat{u} : \omega \times [0,T) \longrightarrow \mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} \partial_t \hat{u} - \Delta \hat{u} &=& \hat{f} \quad \text{in} \quad \omega \times (0,T), \\ \hat{u} &=& g \quad \text{on} \quad \gamma \times (0,T), \\ \hat{u}(x,0) &=& \hat{u}_0(x) \quad \text{in} \quad \omega, \end{array} \right. \tag{2.1}$$

where $\hat{f} : \omega \times [0,T] \longrightarrow \mathbb{R}$, $g : \gamma \times [0,T] \longrightarrow \mathbb{R}$, $\hat{u}_0 : \omega \longrightarrow \mathbb{R}$ stand for source term, and boundary and initial condition, respectively. We will assume that $\hat{u}_0$ and g are compatible in the sense that $\hat{u}_0|_\gamma = g(\cdot, 0)$.

To introduce the fictitious domain formulation for this problem, we first define Ω as an open bounded set such that $\omega \subseteq \Omega$ (Ω should be "simpler" than ω), and extensions $f : \Omega \longrightarrow \mathbb{R}, u_0 : \Omega \longrightarrow \mathbb{R}$ such that $f|_\omega = \hat{f}, u_0|_\omega = \hat{u}_0$. We now rewrite this problem in an equivalent way following the approach presented in [17]. Defining

$$a(u,v) = (\nabla u, \nabla v)_\Omega \quad \text{and} \quad b(\lambda, v) = \langle \lambda, v \rangle_\gamma,$$

then the following equivalent weak form for (2.1) can be written:

Find $(u, \lambda) \in \{H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))\} \times L^\infty(0,T; H^{-\frac{1}{2}}(\gamma))$ such that $u(x,0) = u_0(x)$, and

$$\left\{ \begin{array}{rcl} (\partial_t u, v)_\Omega + a(u,v) - b(\lambda, v) &=& (f,v)_\Omega \\ b(\mu, u) &=& b(\mu, g), \end{array} \right. \tag{2.2}$$

for all $v \in H_0^1(\Omega)$, $\mu \in H^{-\frac{1}{2}}(\gamma)$, and almost all $t \in (0,T)$. Problems (2.1) and (2.2) are linked by the fact that if (u, λ) satisfies (2.2), then $u|_\omega$ satisfies (2.1) and λ coincides with the jump of the normal derivative of u on γ (see [18, 17] for details).

We now describe the discretisation strategies. For the time discretisation, we use the implicit Euler method (although the same analysis can be extended to more involved schemes). Let $N \in \mathbb{N}_0$ be given. We consider a uniform partition $\{(t_n, t_{n+1}]\}_{0 \leqslant n \leqslant N-1}$, with $t_n = n\delta t$, of the time interval $[0,T]$ with time-step size $\delta t = \frac{T}{N}$. For a time-dependent function v, we will define

its approximate time derivative Dv^{m+1} as follows

$$Dv^{m+1} := \frac{v(\cdot, t^{m+1}) - v(\cdot, t^m)}{\delta t}.$$

For space discretisation, let $\{\mathcal{T}_h\}_{h>0}$ denote a shape-regular family of triangulations of the domain Ω. For each triangulation $\mathcal{T}_h$ the subscript h refers to the level of refinement of the triangulation, defined by

$$h = \max_{K \in \mathcal{T}_h} h_K,$$

where $h_K = \operatorname{diam}(K)$. Let γ_h and $\gamma_{\tilde{h}}$ be two partitions of γ such that the vertices of $\gamma_{\tilde{h}}$ are also vertices of γ_h, with edges $\tilde{e}$ satisfying the following (see [17]): there exists $C > 0$ (independent of h) such that $3h \leqslant |\tilde{e}| \leqslant Ch$, for all $\tilde{e} \in \gamma_{\tilde{h}}$. We suppose that for all $\tilde{e} \in \gamma_{\tilde{h}}$, $\operatorname{card}\{e \in \gamma_h : e \subset \tilde{e}\} \leqslant C$, where $C > 0$ is independent of $\tilde{e}$ and h.

Remark 2.1. *We have left the choice of meshes on γ open. In particular, γ_h can be taken as the partition of γ induced by $\mathcal{T}_h$. This is, the collection of edges e such that their end points are the intersections of γ with the edges of the triangulation $\mathcal{T}_h$, plus the angular points of γ (see Fig. 2). Then, $\gamma_{\tilde{h}}$ can be built by gathering several contiguous elements of γ_h until the restriction $|\tilde{e}| \geq 3h$ is satisfied.*

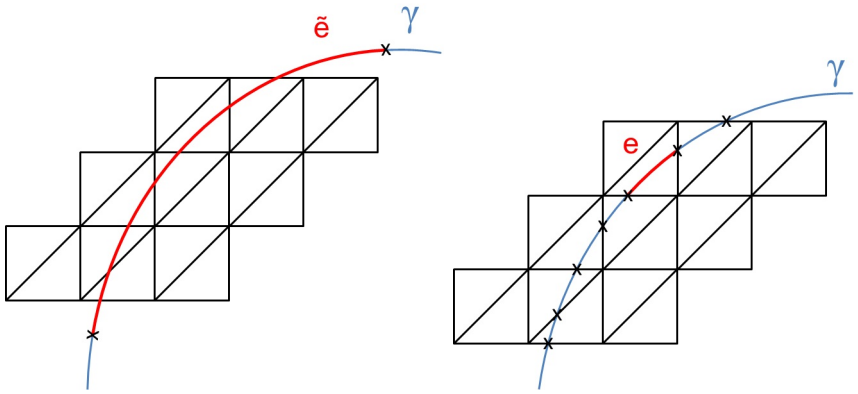

FIGURE 2. Example of meshes on γ.

Associated to the partitions just described, we define the following finite element spaces:

$$V_h = \{v_h \in C^0(\bar{\Omega}) \cap H_0^1(\Omega) : v_h|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\},$$
$$\Lambda_h = \{\mu_h \in L^2(\gamma) : \mu_h|_e \in \mathbb{P}_0(e), \forall e \in \gamma_h\},$$
$$\Lambda_{\tilde{h}} = \{\mu_{\tilde{h}} \in L^2(\gamma) : \mu_{\tilde{h}}|_{\tilde{e}} \in \mathbb{P}_0(\tilde{e}), \forall \tilde{e} \in \gamma_{\tilde{h}}\}.$$

We observe that $\Lambda_{\tilde{h}} \subseteq \Lambda_h$, and also denote $W_h := V_h \times \Lambda_h$. Additionally, we introduce the Lagrange interpolation operator $i_h : C^0(\overline{\Omega}) \longrightarrow V_h$. Thanks to the hypothesis on $\mathcal{T}_h$ and $\gamma_{\tilde{h}}$, the pair $W_{\tilde{h}} := V_h \times \Lambda_{\tilde{h}}$ satisfies the following discrete inf-sup condition (see [17]): there exists $\beta > 0$, independent of h and $\tilde{h}$, such that

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\langle \mu_{\tilde{h}}, v_h \rangle_\gamma}{|v_h|_{1,\Omega}} \geq \beta \|\mu_{\tilde{h}}\|_{-\frac{1}{2},\gamma} \qquad \forall \mu_{\tilde{h}} \in \Lambda_{\tilde{h}}. \tag{2.3}$$

On the other hand, the pair $V_h \times \Lambda_h$ is not inf-sup stable. For this case, it is proven in [3] that there exist two constants $C, \tilde{\beta} > 0$, independent of h, such that

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\langle \mu_h, v_h \rangle_\gamma}{|v_h|_{1,\Omega}} + C\Big(\sum_{\tilde{e} \in \gamma_{\tilde{h}}} |\tilde{e}| \|\mu - \Pi_{\tilde{h}} \mu\|^2_{0,\tilde{e}} \Big)^{\frac{1}{2}} \geqslant \tilde{\beta} \|\mu_h\|_{-\frac{1}{2},\gamma} \qquad \forall \mu_h \in \Lambda_h, \tag{2.4}$$

where $\Pi_{\tilde{h}} : L^2(\gamma) \longrightarrow \Lambda_{\tilde{h}}$ is defined as $(\Pi_{\tilde{h}} \xi)|_{\tilde{e}} = |\tilde{e}|^{-1} (\xi, 1)_{\tilde{e}}$ for each $\tilde{e} \in \gamma_{\tilde{h}}$.

The main focus of this work is to approximate problem (2.2) using the space W_h, although some difficulties associated to this non inf-sup stable choice are shared by the inf-sup stable alternative $W_{\tilde{h}}$, and we will include comments on these similarities throughout the manuscript. Since we only have the weak inf-sup condition (2.4), stabilisation is needed. Then we introduce the bilinear form $j : L^2(\gamma) \times L^2(\gamma) \longrightarrow \mathbb{R}$ defined as

$$j(\mu, \xi) = \sum_{\tilde{e} \in \gamma_{\tilde{h}}} |\tilde{e}| (\mu - \Pi_{\tilde{h}} \mu, \xi - \Pi_{\tilde{h}} \xi)_{\tilde{e}}.$$

This bilinear form satisfies the following properties:

Symmetry:

$$j(\mu, \xi) = j(\xi, \mu) \quad \forall \mu, \xi \in L^2(\gamma); \tag{2.5}$$

Continuity:

$$|j(\mu, \xi)| \leq Ch \|\mu - \Pi_{\tilde{h}} \mu\|_{0,\gamma} \|\xi - \Pi_{\tilde{h}} \xi\|_{0,\gamma} \quad \forall \mu, \xi \in L^2(\gamma); \tag{2.6}$$

Weak consistency: There exists $C > 0$, independent of h, such that (cf. [15])

$$|j(\mu, \xi)| \leq \sum_{\tilde{e} \in \gamma_{\tilde{h}}} |\tilde{e}| \|\mu - \Pi_{\tilde{h}} \mu\|_{0,\tilde{e}} \|\xi - \Pi_{\tilde{h}} \xi\|_{0,\tilde{e}} \leq Ch^2 \|\mu\|_{\mathcal{H}} \|\xi\|_{\mathcal{H}} \quad \forall \mu, \xi \in \mathcal{H}. \tag{2.7}$$

We end this section by presenting the fully discrete methods to be analysed in this work. The fully discrete problems read as follows:

1) Mixed finite element method : Given a suitable approximation of $u_h^0 \in V_h$ of u_0, for $0 \leq n \leq N-1$, find $(u_h^{n+1}, \lambda_{\tilde{h}}^{n+1}) \in W_{\tilde{h}}$ such that

$$\left\{ \begin{array}{rcl} \frac{1}{\delta t}(u_h^{n+1} - u_h^n, v_h)_\Omega + a(u_h^{n+1}, v_h) - b(\lambda_{\tilde{h}}^{n+1}, v_h) & = & (f(t_{n+1}), v_h)_\Omega \\ b(\mu_{\tilde{h}}, u_h^{n+1}) & = & b(\mu_{\tilde{h}}, g), \end{array} \right. \tag{2.8}$$

for all $(v_h, \mu_{\tilde{h}}) \in W_{\tilde{h}}$.

2) Stabilised method : Given a suitable approximation of $u_h^0 \in V_h$ of u_0, for $0 \leq n \leq N-1$, find $(u_h^{n+1}, \lambda_h^{n+1}) \in W_h$ such that

$$\left\{ \begin{array}{rcl} \frac{1}{\delta t}(u_h^{n+1} - u_h^n, v_h)_\Omega + a(u_h^{n+1}, v_h) - b(\lambda_h^{n+1}, v_h) & = & (f(t_{n+1}), v_h)_\Omega \\ b(\mu_h, u_h^{n+1}) + j(\lambda_h^{n+1}, \mu_h) & = & b(\mu_h, g), \end{array} \right. \tag{2.9}$$

for all $(v_h, \mu_h) \in W_h$.

3. Stability analysis

In this section, we analise the stability of problem (2.9). One of our main tools is the following Ritz projection $S_h : W \longrightarrow W_h$ defined as follows: for each $(w, \xi) \in W$, the projection $S_h(w, \xi) = (P_h(w, \xi), R_h(w, \xi)) \in W_h$ is the unique solution of

$$\left\{ \begin{array}{rcl} a(P_h(w, \xi), v_h) - b(R_h(w, \xi), v_h) & = & a(w, v_h) - b(\xi, v_h) \\ b(\mu_h, P_h(w, \xi)) + j(R_h(w, \xi), \mu_h) & = & b(\mu_h, w), \end{array} \right. \tag{3.1}$$

for all $(v_h, \mu_h) \in W_h$. The well-posedness of (3.1) has been proven in [3]. Moreover, defining the norm

$$|||(v_h, \mu_h)|||_h^2 = |v_h|_{1,\Omega}^2 + j(\mu_h, \mu_h),$$

then the following stability and approximation results hold (see [3]):

$$|||(P_h(w,\xi), R_h(w,\xi))|||_h^2 \leq C \left(|w|_{1,\Omega}^2 + \|\xi\|_{-\frac{1}{2},\gamma}^2 \right), \tag{3.2}$$

and, if $(w, \xi) \in H^2(\Omega) \times \mathcal{H}$, then there exists $C > 0$ independent of h such that

$$\|\xi - R_h(w,\xi)\|_{-\frac{1}{2},\gamma} + |||(w - P_h(w,\xi), \xi - R_h(w,\xi))|||_h \leq Ch \left(|w|_{2,\Omega} + \|\xi\|_{\mathcal{H}} \right). \tag{3.3}$$

We now move onto analysis of stability. For this we will only assume that the initial condition u_h^0 is given by any $H_0^1(\Omega)$-stable approximation of u_0. With this choice, stability can only be proved up to a term involving the time derivative of the discrete solution. This undesired behaviour will then be corrected by changing the choice of initial condition.

Lemma 3.1. *Let us suppose that $\|u_h^0\|_{1,\Omega} \leq C|u_0|_{1,\Omega}$ and let $\{(u_h^n, \lambda_h^n)\}_{n=1}^N$ be a solution of the fully discrete problem (2.9). Then there exists $C > 0$, independent of h and δt, such that the following estimate holds for $1 \leq n \leq N$:*

$$\begin{aligned} &\|u_h^n\|_{0,\Omega}^2 + \sum_{m=0}^{n-1} \delta t |||(u_h^{m+1}, \lambda_h^{m+1})|||_h^2 \\ &\leq C|u_0|_{1,\Omega}^2 + C \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|Du_h^{m+1}\|_{0,\Omega}^2 \right), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma}^2 \\ &\leq \frac{C}{\tilde{\beta}^2} \left(|u_0|_{1,\Omega}^2 + \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|Du_h^{m+1}\|_{0,\Omega}^2 \right) \right), \end{aligned} \tag{3.5}$$

where $\tilde{\beta}$ is the constant from (2.4).

Proof. First, using (2.4) and (2.9) with $\mu_h = 0$, and the Cauchy Schwarz and Poincaré inequalities, we get

$$\begin{aligned} \tilde{\beta}\|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma} &\leq \sup_{v_h \in V_h \setminus \{0\}} \frac{b(\lambda_h^{m+1}, v_h)}{|v_h|_{1,\Omega}} + Cj(\lambda_h^{m+1}, \lambda_h^{m+1})^{\frac{1}{2}} \\ &= \sup_{v_h \in V_h \setminus \{0\}} \frac{(f(t_{m+1}), v_h)_\Omega - a(u_h^{m+1}, v_h) - (Du_h^{m+1}, v_h)_\Omega}{|v_h|_{1,\Omega}} + Cj(\lambda_h^{m+1}, \lambda_h^{m+1})^{\frac{1}{2}} \\ &\leq C\|f(t_{m+1})\|_{0,\Omega} + |u_h^{m+1}|_{1,\Omega} + C\|Du_h^{m+1}\|_{0,\Omega} + Cj(\lambda_h^{m+1}, \lambda_h^{m+1})^{\frac{1}{2}}. \end{aligned}$$

Squaring, multiplying by δt and adding over $0 \le m \le n-1$ leads to

$$\begin{aligned}
&\sum_{m=0}^{n-1} \delta t \, \|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma} \\
&\le \frac{C}{\widetilde{\beta}^2} \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|^2_{0,\Omega} + |u_h^{m+1}|^2_{1,\Omega} + \left\|Du_h^{m+1}\right\|^2_{0,\Omega} + j(\lambda_h^{m+1}, \lambda_h^{m+1}) \right) \\
&= \frac{C}{\widetilde{\beta}^2} \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|^2_{0,\Omega} + \|Du_h^{m+1}\|^2_{0,\Omega} + \left|\left|\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\right|\right|_h^2 \right). \qquad (3.6)
\end{aligned}$$

Now, to prove (3.4), we take $v_h = u_h^{m+1}$, $\mu_h = \lambda_h^{m+1}$ in (2.9) and get

$$\left(\frac{u_h^{m+1} - u_h^m}{\delta t}, u_h^{m+1} \right)_\Omega + a(u_h^{m+1}, u_h^{m+1}) + j(\lambda_h^{m+1}, \lambda_h^{m+1}) = (f(t_{m+1}), u_h^{m+1})_\Omega + b(\lambda_h^{m+1}, g(t_{m+1})).$$

Using the equality $(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$, Cauchy Schwarz and Poincaré inequalities in the above expression, we get

$$\begin{aligned}
&\frac{1}{2\delta t}\Big(\|u_h^{m+1}\|^2_{0,\Omega} - \|u_h^m\|^2_{0,\Omega} \Big) + \frac{1}{2\delta t} \|u_h^{m+1} - u_h^m\|^2_{0,\Omega} + \left|\left|\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\right|\right|_h^2 \\
&= (f(t_{m+1}), u_h^{m+1}) + b(\lambda_h^{m+1}, g(t_{m+1})) \\
&\le C \|f(t_{m+1})\|_{0,\Omega} |u_h^{m+1}|_{1,\Omega} + \|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma} \|g(t_{m+1})\|_{\frac{1}{2},\gamma}. \qquad (3.7)
\end{aligned}$$

Now, after multiplying by $2\delta t$, adding over $0 \le m \le n-1$ and using Young's inequality, we obtain

$$\begin{aligned}
&\|u_h^n\|^2_{0,\Omega} - \|u_h^0\|^2_{0,\Omega} + \sum_{m=0}^{n-1} \|u_h^{m+1} - u_h^m\|^2_{0,\Omega} + 2 \sum_{m=0}^{n-1} \delta t \left|\left|\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\right|\right|_h^2 \\
&\le C \sum_{m=0}^{n-1} \delta t \|f(t_{m+1})\|_{0,\Omega} |u_h^{m+1}|_{1,\Omega} + 2 \sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma} \|g(t_{m+1})\|_{\frac{1}{2},\gamma} \\
&\le C \sum_{m=0}^{n-1} \delta t \|f(t_{m+1})\|^2_{0,\Omega} + \sum_{m=0}^{n-1} \delta t |u_h^{m+1}|^2_{1,\Omega} + \epsilon \sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma} + \frac{1}{\epsilon} \sum_{m=0}^{n-1} \delta t \|g(t_{m+1})\|^2_{\frac{1}{2},\gamma},
\end{aligned}$$

where $\epsilon > 0$ will be chosen later. Then as $|u_h^{m+1}|^2_{1,\Omega} \le \left|\left|\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\right|\right|_h^2$, we rearrange terms and arrive at

$$\begin{aligned}
&\|u_h^n\|^2_{0,\Omega} + \sum_{m=0}^{n-1} \|u_h^{m+1} - u_h^m\|^2_{0,\Omega} + \sum_{m=0}^{n-1} \delta t \left|\left|\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\right|\right|_h^2 \\
&\le \|u_h^0\|^2_{0,\Omega} + C \sum_{m=0}^{n-1} \delta t \|f(t_{m+1})\|^2_{0,\Omega} + \epsilon \sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma} + \frac{1}{\epsilon} \sum_{m=0}^{n-1} \delta t \|g(t_{m+1})\|^2_{\frac{1}{2},\gamma} \\
&\le \|u_h^0\|^2_{0,\Omega} + C \sum_{m=0}^{n-1} \delta t \|f(t_{m+1})\|^2_{0,\Omega} + \frac{1}{\epsilon} \sum_{m=0}^{n-1} \delta t \|g(t_{m+1})\|^2_{\frac{1}{2},\gamma} \\
&+ \epsilon \frac{C}{\widetilde{\beta}^2} \sum_{m=0}^{n-1} \delta t \Big(\left|\left|\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\right|\right|_h^2 + \|Du_h^{m+1}\|^2_{0,\Omega} + \|f(t_{m+1})\|^2_{0,\Omega} \Big).
\end{aligned}$$

Then, taking $\epsilon = \frac{\tilde{\beta}^2}{2C}$ in the above inequality, and rearranging, gives

$$\begin{aligned}
&\|u_h^n\|_{0,\Omega}^2 + \sum_{m=0}^{n-1} \|u_h^{m+1} - u_h^m\|_{0,\Omega}^2 + \sum_{m=0}^{n-1} \delta t \left|\!\left|\!\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\!\right|\!\right|_h^2 \\
&\leq C\|u_h^0\|_{0,\Omega}^2 + C \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|Du_h^{m+1}\|_{0,\Omega}^2 \right),
\end{aligned}$$

which proves (3.4) applying the inequality $|u_h^0|_{1,\Omega} \leq C|u_0|_{1,\Omega}$. We obtain the estimate (3.5) inserting (3.4) into (3.6). □

Remark 3.2. *It is interesting to notice that, for $g = 0$, a stability result stronger than* (3.4) *can be proved. More precisely, it would not have the term involving Du_h^{m+1} in its right-hand side. In fact, that term only appears in the estimate due to the calculations leading to* (3.7)*. In there it can be noticed that, if $g = 0$, then there is no need to bound the norm of λ_h^{m+1} to have a bound for u_h^{m+1}, and then stability is proved without any further hypothesis.*

We now bound the time derivative Du_h^{m+1} using a special initial condition, and an extra regularity assumption for g.

Theorem 3.1. *Let $\{(u_h^n, \lambda_h^n)\}_{n=1}^N$ be the solution of the fully discrete problem* (2.9) *where we consider $u_h^0 = P_h(u_0, 0)$. Let us also assume $\partial_t g \in C^0(0, T; H^{\frac{1}{2}}(\gamma))$. Then for all $1 \leq n \leq N$ the following estimate holds*

$$\begin{aligned}
&\|u_h^n\|_{0,\Omega}^2 + \sum_{m=0}^{n-1} \delta t \left|\!\left|\!\left| (u_h^{m+1}, \lambda_h^{m+1}) \right|\!\right|\!\right|_h^2 \\
&\leq C|u_0|_{1,\Omega}^2 + C \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|\partial_t g\|^2_{L^\infty(t_m, t_{m+1}; H^{\frac{1}{2}}(\gamma))} \right),
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma} \\
&\leq\ C|u_0|_{1,\Omega}^2 + C \sum_{m=0}^{n-1} \delta t \left(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|\partial_t g\|^2_{L^\infty(t_n, t_{n+1}; H^{\frac{1}{2}}(\gamma))} \right).
\end{aligned} \tag{3.9}$$

Proof. Based on the previous lemma, it is enough to prove that

$$\begin{aligned}
&\sum_{m=0}^{n-1} \delta t \|Du_h^{m+1}\|_{0,\Omega}^2 + |\!|\!|(u_h^n, \lambda_h^n)|\!|\!|_h^2 \\
&\leq \Big(C\,|u_0|_{1,\Omega}^2 + \sum_{m=0}^{n-1} \delta t \big(\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\epsilon} \|\partial_t g\|^2_{L^\infty((t_m, t_{m+1}), H^{\frac{1}{2}}(\gamma))} + \epsilon \|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma} \big) \Big),
\end{aligned} \tag{3.10}$$

and then choose $\epsilon > 0$ small enough such that the term involving $\|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma}$ can be moved to the LHS of (3.5).

For $0 \leq m \leq N-1$, taking $v_h = Du_h^{m+1}$ and $\mu_h = 0$ in (2.9) and using the Cauchy Schwarz inequality, we have

$$\begin{aligned}&\|Du_h^{m+1}\|_{0,\Omega}^2 + a(u_h^{m+1}, Du_h^{m+1}) - b(\lambda_h^{m+1}, Du_h^{m+1}) = (f(t_{m+1}), Du_h^{m+1})_\Omega \\ &\leq \|f(t_{m+1})\|_{0,\Omega}\|Du_h^{m+1}\|_{0,\Omega} \leq \frac{1}{2}\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{2}\|Du_h^{m+1}\|_{0,\Omega}^2,\end{aligned}$$

which implies

$$\frac{1}{2}\|Du_h^{m+1}\|_{0,\Omega}^2 + a(u_h^{m+1}, Du_h^{m+1}) - b(\lambda_h^{m+1}, Du_h^{m+1}) \leq \frac{1}{2}\|f(t_{m+1})\|_{0,\Omega}^2. \tag{3.11}$$

On the other hand, for $1 \leq m \leq N-1$, testing (2.9) at the time levels $m+1$ and m with $v_h = 0$ and $\mu_h = \lambda_h^{m+1}$ gives

$$\begin{aligned}b(\lambda_h^{m+1}, g(t_{m+1})) - b(\lambda_h^{m+1}, u_h^{m+1}) &= j(\lambda_h^{m+1}, \lambda_h^{m+1}), \\ b(\lambda_h^{m+1}, g(t_m)) - b(\lambda_h^{m+1}, u_h^m) &= j(\lambda_h^m, \lambda_h^{m+1}).\end{aligned} \tag{3.12}$$

Therefore, by subtracting these equalities, dividing by δt and using the bilinearity of $j(.,.)$, we obtain

$$-b(\lambda_h^{m+1}, Du_h^{m+1}) = j(D\lambda_h^{m+1}, \lambda_h^{m+1}) - \frac{1}{\delta t} b(\lambda_h^{m+1}, g(t_{m+1}) - g(t_m)), \tag{3.13}$$

for $1 \leq m \leq N-1$. It then follows from (3.11) and the mean value inequality (see, e.g., [4, Thorme 1.7, p.20]) that

$$\begin{aligned}&\frac{1}{2}\|Du_h^{m+1}\|_{0,\Omega}^2 + a(u_h^{m+1}, Du_h^{m+1}) + j(D\lambda_h^{m+1}, \lambda_h^{m+1}) \\ &\leq \frac{1}{2}\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\delta t} b(\lambda_h^{m+1}, g(t_{m+1}) - g(t_m)) \\ &\leq \frac{1}{2}\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\delta t}\|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma}\|g(t_{m+1}) - g(t_m)\|_{\frac{1}{2},\gamma} \\ &\leq \frac{1}{2}\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\delta t}\|\lambda_h^{m+1}\|_{-\frac{1}{2},\gamma}\delta t\|\partial_t g\|_{L^\infty((t_m,t_{m+1}),H^{\frac{1}{2}}(\gamma))} \\ &\leq \frac{1}{2}\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\epsilon}\|\partial_t g\|^2_{L^\infty((t_m,t_{m+1}),H^{\frac{1}{2}}(\gamma))} + \epsilon\|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma}.\end{aligned} \tag{3.14}$$

On the other hand, using the symmetry and bilinearity of $a(.,.)$ and $j(.,.)$, and, once again, $a(a-b) = \frac{a^2}{2} - \frac{b^2}{2} + \frac{(a-b)^2}{2}$, gives

$$\begin{aligned}a(u_h^{m+1}, Du_h^{m+1}) &= \frac{1}{2}Da(u_h^{m+1}, u_h^{m+1}) + \frac{\delta t}{2}a(Du_h^{m+1}, Du_h^{m+1}), \\ j(\lambda_h^{m+1}, D\lambda_h^{m+1}) &= \frac{1}{2}Dj(\lambda_h^{m+1}, \lambda_h^{m+1}) + \frac{\delta t}{2}j(D\lambda_h^{m+1}, D\lambda_h^{m+1}).\end{aligned}$$

Hence, (3.14) becomes

$$\begin{aligned}&\|Du_h^{m+1}\|_{0,\Omega}^2 + D(a(u_h^{m+1}, u_h^{m+1}) + j(\lambda_h^{m+1}, \lambda_h^{m+1})) \\ &\leq \|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\epsilon}\|\partial_t g\|^2_{L^\infty((t_m,t_{m+1}),H^{\frac{1}{2}}(\gamma))} + \epsilon\|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma},\end{aligned}$$

for $1 \leq m \leq N-1$. After multiplication by δt and summation over $1 \leq m \leq n-1$, it follows that

$$\begin{aligned}
&\sum_{m=1}^{n-1} \delta t \|Du_h^{m+1}\|_{0,\Omega}^2 + |||(u_h^n, \lambda_h^n)|||_h^2 \\
&\leq |||(u_h^1, \lambda_h^1)|||_h^2 + \sum_{m=1}^{n-1} \delta t \left(\|f(t_{m+1})\|_{0,\Omega}^2 + \frac{1}{\epsilon} \|\partial_t g\|^2_{L^\infty((t_m,t_{m+1}),H^{\frac{1}{2}}(\gamma))} + \epsilon \|\lambda_h^{m+1}\|^2_{-\frac{1}{2},\gamma} \right).
\end{aligned} \tag{3.15}$$

Since the initial approximation of u is given in terms of the Ritz-projection, $u_h^0 = P_h(u_0, 0)$, by setting $\lambda_h^0 = R_h(u_0, 0)$ it follows that (3.13) also holds for $m = 0$. That is

$$-b(\lambda_h^1, Du_h^1) = j(D\lambda_h^1, \lambda_h^1) - \frac{1}{\delta t} b(\lambda_h^1, g(t_1) - g(t_0)). \tag{3.16}$$

Next, taking, for $n = 0$, $v_h = Du_h^1$, $\mu_h = 0$ in (2.9) and multiplying by $2\delta t$, we get

$$\delta t \|Du_h^1\|_{0,\Omega}^2 + a(u_h^1, u_h^1) - a(u_h^0, u_h^0) \leq \delta t \|f(t_1)\|_{0,\Omega}^2 + 2\delta t b(\lambda_h^1, Du_h^1),$$

which, after using (3.16) leads to

$$\begin{aligned}
&\delta t \|Du_h^1\|_{0,\Omega}^2 + |u_h^1|_{1,\Omega}^2 \\
&\leq |u_h^0|_{1,\Omega}^2 + \delta t \|f(t_1)\|_{0,\Omega}^2 + 2b(\lambda_h^1, g(t_1) - g(t_0)) - 2\delta t j(D\lambda_h^1, \lambda_h^1) \\
&= |u_h^0|_{1,\Omega}^2 + \delta t \|f(t_1)\|_{0,\Omega}^2 + 2b(\lambda_h^1, g(t_1) - g(t_0)) - 2j(\lambda_h^1, \lambda_h^1) + 2j(\lambda_h^0, \lambda_h^1) \\
&\leq |u_h^0|_{1,\Omega}^2 + \delta t \|f(t_1)\|_{0,\Omega}^2 + 2b(\lambda_h^1, g(t_1) - g(t_0)) - j(\lambda_h^1, \lambda_h^1) + j(\lambda_h^0, \lambda_h^0),
\end{aligned}$$

and then, rearranging and using a similar argument to bound the time derivative of g, we get

$$\begin{aligned}
&\delta t \|Du_h^1\|_{0,\Omega}^2 + |||(u_h^1, \lambda_h^1)|||_h^2 \\
&\leq |||(u_h^0, \lambda_h^0)|||_h^2 + \delta t \|f(t_1)\|_{0,\Omega}^2 + 2b(\lambda_h^1, g(t_1) - g(t_0)) \\
&\leq |||(u_h^0, \lambda_h^0)|||_h^2 + \delta t \|f(t_1)\|_{0,\Omega}^2 + 2\delta t \|\lambda_h^1\|_{-\frac{1}{2},\gamma} \|\partial_t g\|_{L^\infty((t_m,t_{m+1}),H^{\frac{1}{2}}(\gamma))} \\
&\leq |||(u_h^0, \lambda_h^0)|||_h^2 + \delta t \|f(t_1)\|_{0,\Omega}^2 + \epsilon \delta t \|\lambda_h^1\|^2_{-\frac{1}{2},\gamma} + \frac{\delta t}{\epsilon} \|\partial_t g\|^2_{L^\infty((t_m,t_{m+1}),H^{\frac{1}{2}}(\gamma))}.
\end{aligned} \tag{3.17}$$

Then we get the estimate (3.10) adding (3.17) to (3.15) and using the stability of the Ritz-projection (3.2), $|||(u_h^0, \lambda_h^0)|||_h^2 \leq C|u_0|_{1,\Omega}^2$.

Estimate (3.9) is then proven taking $\epsilon = \frac{\tilde{\beta}}{2C}$ in (3.10) where C is the constant from (3.5). The estimate (3.8) is obtained in an analogous way. □

3.1. **The inf-sup stable case.** In this section we briefly summerise the differences of the approach in the case the inf-sup stable space $W_{\tilde{h}}$ is used for space discretisation. Our main goal is to show that the stability deficiencies of the stabilised method are not due to the stabilisation, but rather to the incorrect approximation of the initial condition. The proofs are very similar to the ones presented on the last section, just by setting $j(\cdot, \cdot) = 0$, and that is why we only sketch them here. The first step is to modify the definition of the Ritz-projection to accommodate it to this case. For the present case the Ritz projector is defined as follows:

$$S_{\tilde{h}} : W \longrightarrow W_{\tilde{h}},$$

where $W := H_0^1(\Omega)\times H^{-\frac{1}{2}}(\gamma)$. For each $(w,\xi)\in W$, the projection $S_{\tilde{h}}(w,\xi) = (P_{\tilde{h}}(w,\xi), R_{\tilde{h}}(w,\xi)) \in W_{\tilde{h}}$ is defined as the unique solution of

$$\left\{\begin{array}{rcl} a(P_{\tilde{h}}(w,\xi),v_h) - b(R_{\tilde{h}}(w,\xi),v_h) &=& a(w,v_h) - b(\xi,v_h) \\ b(\mu_{\tilde{h}}, P_{\tilde{h}}(w,\xi)) &=& b(\mu_{\tilde{h}}, w), \end{array}\right. \tag{3.18}$$

for all $(v_h,\mu_{\tilde{h}})\in W_{\tilde{h}}$. Problem (3.18) is well-posed thanks to the inf-sup condition (2.3). Moreover, defining the norm

$$\|(v,\mu)\|_W^2 := |v|_{1,\Omega}^2 + \|\mu\|_{-\frac{1}{2},\gamma}^2,$$

then the following stability and approximation results hold (see [17]): There exists $C>0$, independent of h, such that

$$\|P_{\tilde{h}}(w,\xi)\|_W^2 \leqslant C\big(|w|_{1,\Omega}^2 + \|\xi\|_{-\frac{1}{2},\gamma}^2\big), \tag{3.19}$$

and if $(w,\xi)\in H^2(\Omega)\times\mathcal{H}$, then there exist $C>0$, independent of h, such that

$$|w - P_{\tilde{h}}(w,\xi)|_{1,\Omega} + \|\xi - R_{\tilde{h}}(w,\xi)\|_{-\frac{1}{2},\gamma} \leq Ch(|w|_{2,\Omega} + \|\xi\|_{\mathcal{H}}). \tag{3.20}$$

When analysing the method (2.8), the equivalent of (3.4) and (3.5) read:

$$\begin{aligned} &\|u_h^n\|_{0,\Omega}^2 + \sum_{m=0}^{n-1}\Big(\delta t|u_h^{m+1}|_{1,\Omega}^2 + \|u_h^{m+1}-u_h^m\|_{0,\Omega}^2\Big) \\ &\leq C|u_0|_{1,\Omega}^2 + C\sum_{m=0}^{n-1}\delta t\Big(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|Du_h^{m+1}\|_{0,\Omega}^2\Big), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\sum_{m=0}^{n-1}\delta t\|\lambda_{\tilde{h}}^{m+1}\|_{-\frac{1}{2},\gamma}^2 \\ &\leq C|u_0|_{1,\Omega}^2 + \frac{C}{\beta^2}\sum_{m=0}^{n-1}\delta t\Big(\|f(t_{m+1})\|_{0,\Omega}^2 + \|g(t_{m+1})\|_{\frac{1}{2},\gamma}^2 + \|Du_h^{m+1}\|_{0,\Omega}^2\Big), \end{aligned} \tag{3.22}$$

where β is the constant from (2.3). Then, if u_h^0 is supposed to be the corresponding Ritz-projection of $(u_0,0)$, the following estimate can be derived as in (3.17):

$$\delta t b(\lambda_{\tilde{h}}^1, Du_h^1) = \delta t\langle\lambda_{\tilde{h}}^1, \frac{u_h^1-u_h^0}{\delta t}\rangle_\gamma \leq \frac{1}{2\epsilon}\delta t\|\partial_t g\|_{L^\infty((0,t_1),H^{\frac{1}{2}}(\gamma))}^2 + C|u_0|_{1,\Omega}^2 + C\epsilon\delta t\|\lambda_{\tilde{h}}^1\|_{-\frac{1}{2},\gamma}^2, \tag{3.23}$$

where $\epsilon>0$ will be chosen small enough to move the term $\delta t\|\lambda_{\tilde{h}}^1\|_{-\frac{1}{2},\gamma}^2$ to the left-hand side. Then, following as in the proof of Theorem 3.1, a stability result independent of h and δt can be obtained.

We finish this section by remarking that, if $i_h(u_0)$ (the Lagrange interpolate of u_0) is chosen as the initial condition, a bound similar to (3.23) can be obtained under the extra hypothesis that $\delta t \geq h^2$. In fact, supposing $u_0\in H^2(\Omega)$, and using the approximation properties of i_h we arrive at

$$
\begin{aligned}
\delta t b(\lambda_{\tilde{h}}^1, Du_h^1) &= \delta t \Big\langle \lambda_{\tilde{h}}^1, \frac{u_h^1 - u_h^0}{\delta t} \Big\rangle_\gamma \\
&= \langle \lambda_{\tilde{h}}^1, u_h^1 \rangle_\gamma - \langle \lambda_{\tilde{h}}^1, u_h^0 \rangle_\gamma \\
&= \langle \lambda_{\tilde{h}}^1, g(t_1) - g(t_0) \rangle_\gamma + \langle \lambda_{\tilde{h}}^1, g(t_0) - i_h(u_0) \rangle_\gamma \\
&\leq \frac{C}{\epsilon} \delta t \|\partial_t g\|^2_{L^\infty((0,t_1),H^{\frac{1}{2}}(\gamma))} + \frac{\epsilon}{2} \delta t \|\lambda_{\tilde{h}}^1\|^2_{-\frac{1}{2},\gamma} + Ch|u_0|_{2,\Omega} \|\lambda_{\tilde{h}}^1\|_{-\frac{1}{2},\gamma} \\
&\leq \frac{C}{\epsilon} \delta t \|\partial_t g\|^2_{L^\infty((0,t_1),H^{\frac{1}{2}}(\gamma))} + \frac{\epsilon}{2} \delta t \|\lambda_{\tilde{h}}^1\|^2_{-\frac{1}{2},\gamma} + C\frac{1}{2\epsilon} |u_0|^2_{2,\Omega} + \frac{\epsilon}{2} h^2 \|\lambda_{\tilde{h}}^1\|^2_{-\frac{1}{2},\gamma} \\
&\leq \frac{C}{\epsilon} \delta t \|\partial_t g\|^2_{L^\infty((0,t_1),H^{\frac{1}{2}}(\gamma))} + \frac{C}{2\epsilon} |u_0|^2_{2,\Omega} + \frac{1}{2} \epsilon (h^2 + \delta t) \|\lambda_{\tilde{h}}^1\|^2_{-\frac{1}{2},\gamma},
\end{aligned}
$$

for any $\epsilon > 0$. By supposing $h^2 \leq \delta t$ and choosing $\epsilon < 0$ small enough we get a stability result similar to the one given in Theorem 3.1.

4. Convergence analysis

In this section we prove optimal order error estimates for the fully discrete method (2.9). We start by presenting the following result on consistency. Its proof is direct verification.

Lemma 4.1. *Let (u, λ) be the solution of (2.1) and let $\{(u_h^n, \lambda_h^n)\}_{0 \leq n \leq N}$ be the solution of (2.9). Assume that $u \in \mathcal{C}^0(0,T; H_0^1(\Omega))$ and $\lambda \in \mathcal{C}^0(0,T; H^{-\frac{1}{2}}(\gamma))$. Then, for $0 \leq n \leq N-1$, the following holds*

$$
\begin{aligned}
&(Du(t_{n+1}) - Du_h^{n+1}, v_h)_\Omega + a(u(t_{n+1}) - u_h^{n+1}, v_h) - b(\lambda(t_{n+1}) - \lambda_h^{n+1}, v_h) \\
&+ b(\mu_h, u(t_{n+1}) - u_h^{n+1}) = j(\lambda_h^{n+1}, \mu_h) + (Du(t_{n+1}) - \partial_t u(t_{n+1}), v_h)_\Omega,
\end{aligned}
$$

for all $(v_h, \mu_h) \in W_h$.

We first show an estimate for the errors in a norm including the $|||.|||_h$ norm in space. It is interesting to note that this result is optimal independently of the choice of the initial condition.

Theorem 4.1. *Let us assume that $u \in H^1(0,T; H^2(\Omega)) \cap H^2(0,T; L^2(\Omega)) \cap C^0(0,T; H^2(\Omega))$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\lambda \in H^1(0,T; \mathcal{H})$, and set $u_h^0 \in V_h$ as $u_h^0 = i_h(u)$. Then the following estimate holds for $1 \leq n \leq N$:*

$$
\begin{aligned}
&\|u_h^n - u(t_n)\|^2_{0,\Omega} + \sum_{m=0}^{n-1} \delta t \big|\!\big|\!\big| (u_h^{m+1} - u(t_{m+1}), \lambda_h^{m+1} - \lambda(t_{m+1})) \big|\!\big|\!\big|_h^2 \\
&\leq Ch^2 \Big(\|u_0\|^2_{2,\Omega} + \|\lambda(0)\|^2_{\mathcal{H}} \Big) + Ch^2 \delta t^2 \left(\|\partial_t u\|^2_{L^2(0,t_n;H^2(\Omega))} + \|\partial_t \lambda\|^2_{L^2(0,t_n;\mathcal{H})} \right) \\
&+ C \left(\delta t^2 \|\partial_{tt} u\|^2_{L^2(0,t_n;L^2(\Omega))} + h^2 \|\lambda\|^2_{C^0(t_1,t_n;\mathcal{H})} + h^2 \|u\|^2_{C^0(t_1,t_n;H^2(\Omega))} \right),
\end{aligned}
$$

with $C > 0$ a positive constant independent of h and δt.

Proof. As usual, for $m = 0, ..., N$, we decompose the error into interpolation and discrete errors as follows

$$
u(t_m) - u_h^m = \underbrace{u(t_m) - P_h(u(t_m), \lambda(t_m))}_{\theta_\pi^m} + \underbrace{P_h(u(t_m), \lambda(t_m)) - u_h^m}_{\theta_h^m} := \theta_\pi^m + \theta_h^m, \tag{4.1}
$$

$$\lambda(t_m) - \lambda_h^m = \underbrace{\lambda(t_m) - R_h(u(t_m), \lambda(t_m))}_{y_\pi^m} + \underbrace{R_h(u(t_m), \lambda(t_m)) - \lambda_h^m}_{y_h^m} := y_\pi^m + y_h^m, \tag{4.2}$$

where (P_h, R_h) is defined in (3.1). To ease the notation, we note, for $m = 0, \ldots, N$,

$$\hat{u}^m = P_h(u(t_m), \lambda(t_m)) \quad \text{and} \quad \hat{\lambda}^m = R_h(u(t_m), \lambda(t_m)).$$

The term θ_π^{m+1} can be bounded using (3.20). In order to estimate θ_h^{m+1}, using the definition of the bilinear form $a(.,.) + j(.,.)$ and

$$(Du_h^{m+1}, u_h^{m+1}) = \frac{1}{2} D\|u_h^{m+1}\|_{0,\Omega}^2 + \frac{1}{2\delta t}\|u_h^{m+1} - u_h^m\|_{0,\Omega}^2,$$

we get

$$\begin{aligned}
&\frac{1}{2} D\|\theta_h^{m+1}\|_{0,\Omega}^2 + |||(\theta_h^{m+1}, y_h^{m+1})|||_h^2 \\
&\leq (D\theta_h^{m+1}, \theta_h^{m+1})_\Omega + |||(\theta_h^{m+1}, y_h^{m+1})|||_h^2 \\
&= \underbrace{(D\theta_h^{m+1}, \theta_h^{m+1})_\Omega + a(\theta_h^{m+1}, \theta_h^{m+1}) + b(y_h^{m+1}, \theta_h^{m+1}) - b(y_h^{m+1}, \theta_h^{m+1}) + j(y_h^{m+1}, y_h^{m+1})}_{T_1^{m+1}}.
\end{aligned} \tag{4.3}$$

Now, using (4.1)-(4.2), we have

$$\begin{aligned}
T_1^{m+1} &= -(D\theta_\pi^{m+1}, \theta_h^{m+1})_\Omega - a(\theta_\pi^{m+1}, \theta_h^{m+1}) + a(u(t_{m+1}) - u_h^{m+1}, \theta_h^{m+1}) \\
&+ j(\hat{\lambda}^{m+1}, y_h^{m+1}) - j(\lambda_h^{m+1}, y_h^{m+1}) + (Du(t_{m+1}) - Du_h^{m+1}, \theta_h^{m+1})_\Omega \\
&- b(\hat{\lambda}^{m+1} - \lambda(t_{m+1}), \theta_h^{m+1}) + b(y_h^{m+1}, \hat{u}^{m+1} - u(t_{m+1})) \\
&- b(\lambda(t_{m+1}) - \lambda_h^{m+1}, \theta_h^{m+1}) + b(y_h^{m+1}, u(t_{m+1}) - u_h^{m+1}).
\end{aligned}$$

By the modified Galerkin orthogonality (see Lemma 4.1), this expression reduces to

$$\begin{aligned}
T_1^{m+1} &= -(D\theta_\pi^{m+1}, \theta_h^{m+1})_\Omega + (Du(t_{m+1}) - \partial_t u(t_{m+1}), \theta_h^{m+1})_\Omega - a(\theta_\pi^{m+1}, \theta_h^{m+1}) \\
&+ j(\hat{\lambda}^{m+1}, y_h^{m+1}) + b(y_\pi^{m+1}, \theta_h^{m+1}) - b(y_h^{m+1}, \theta_\pi^{m+1}) \\
&= I + II + III + IV + V.
\end{aligned} \tag{4.4}$$

We now bound the above right-hand side term by term using the Cauchy Schwarz and the Poincaré inequalities, (3.3), and properties of $j(.,.)$ (cf. (2.6)-(2.7)):

$$I \leq C \left(\|D\theta_\pi^{m+1}\|_{0,\Omega} + \|Du(t_{m+1}) - \partial_t u(t_{m+1})\|_{0,\Omega} \right) |||(\theta_h^{m+1}, y_h^{m+1})|||_h, \tag{4.5}$$

$$II \leq |a(\theta_\pi^{m+1}, \theta_h^{m+1})| \leq |\theta_\pi^{m+1}|_{1,\Omega} |||(\theta_h^{m+1}, y_h^{m+1})|||_h, \tag{4.6}$$

$$\begin{aligned} III &\leq j(\hat{\lambda}^{m+1}, \hat{\lambda}^{m+1})^{\frac{1}{2}} j(y_h^{m+1}, y_h^{m+1})^{\frac{1}{2}} \leq j(\hat{\lambda}^{m+1}, \hat{\lambda}^{m+1})^{\frac{1}{2}} |||(\theta_h^{m+1}, y_h^{m+1})|||_h \\ &\leq \{ j(\lambda(t_{m+1}) - \hat{\lambda}^{m+1}, \lambda(t_{m+1}) - \hat{\lambda}^{m+1})^{\frac{1}{2}} + j(\lambda(t_{m+1}), \lambda(t_{m+1}))^{\frac{1}{2}} \} |||(\theta_h^{m+1}, y_h^{m+1})|||_h \\ &\leq Ch\{ |u(t_{m+1})|_{2,\Omega} + \|\lambda(t_{m+1})\|_{\mathcal{H}} \} |||(\theta_h^{m+1}, y_h^{m+1})|||_h, \end{aligned} \tag{4.7}$$

$$\begin{aligned} IV &= \langle y_\pi^{m+1}, \theta_h^{m+1} \rangle_\gamma \\ &\leq \|\lambda(t_{m+1}) - \hat{\lambda}^{m+1}\|_{-\frac{1}{2},\gamma} \|\theta_h^{m+1}\|_{\frac{1}{2},\gamma} \\ &\leq Ch\big\{ |u(t_{m+1})|_{2,\Omega} + \|\lambda(t_{m+1})\|_{\mathcal{H}} \big\} \|\theta_h^{m+1}\|_{1,\Omega} \\ &\leq Ch\big\{ |u(t_{m+1})|_{2,\Omega} + \|\lambda(t_{m+1})\|_{\mathcal{H}} \big\} |||(\theta_h^{m+1}, y_h^{m+1})|||_h, \end{aligned} \tag{4.8}$$

$$\begin{aligned} V &= -b(y_h^{m+1}, \theta_\pi^{m+1}) = -\langle y_h^{m+1}, \theta_\pi^{m+1} \rangle_\gamma = j(\hat{\lambda}^{m+1}, y_h^{m+1}) \\ &= j(\hat{\lambda}^{m+1} - \lambda(t_{m+1}), y_h^{m+1}) + j(\lambda(t_{m+1}), y_h^{m+1}) \\ &\leq \Big(j(\hat{\lambda}^{m+1} - \lambda(t_{m+1}), \hat{\lambda}^{m+1} - \lambda(t_{m+1})) + j(\lambda(t_{m+1}), \lambda(t_{m+1})) \Big)^{\frac{1}{2}} j(y_h^{m+1}, y_h^{m+1})^{\frac{1}{2}} \\ &\leq Ch(|u(t_{m+1})|_{2,\Omega} + \|\lambda(t_{m+1})\|_{\mathcal{H}}) |||(\theta_h^{m+1}, y_h^{m+1})|||_h. \end{aligned} \tag{4.9}$$

Then collecting (4.4)-(4.9) and applying (3.3) once again, we get

$$\begin{aligned} T_1^{m+1} &\leq C \underbrace{\Big(\|Du(t_{m+1}) - \partial_t u(t_{m+1})\|_{0,\Omega} + \|D\theta_\pi^{m+1}\|_{0,\Omega} \Big)}_{T_2^{m+1}} |||(\theta_h^{m+1}, y_h^{m+1})|||_h \\ &+ Ch\Big(|u(t_{m+1})|_{2,\Omega} + \|\lambda(t_{m+1})\|_{\mathcal{H}} \Big) |||(\theta_h^{m+1}, y_h^{m+1})|||_h. \end{aligned} \tag{4.10}$$

We bound separately both terms included in T_2^{m+1}. For the first one, a Taylor expansion gives

$$u(t_m) = u(t_{m+1}) - \delta t \partial_t u(t_{m+1}) - \frac{1}{2} \int_{t_m}^{t_{m+1}} \partial_{tt} u(s)(t_m - s) ds,$$

which implies

$$\begin{aligned} &\left| \frac{u(t_{m+1}) - u(t_m)}{\delta t} - \partial_t u(t_{m+1}) \right| \leq \frac{1}{2\delta t} \int_{t_m}^{t_{m+1}} |\partial_{tt} u(s)||(t_m - s)| ds \\ &\leq \frac{1}{2} \int_{t_m}^{t_{m+1}} |\partial_{tt} u(s)| ds \leq \frac{1}{2} \left(\int_{t_m}^{t_{m+1}} 1^2 ds \right)^{\frac{1}{2}} \left(\int_{t_m}^{t_{m+1}} |\partial_{tt} u(s)|^2 ds \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \delta t^{\frac{1}{2}} \|\partial_{tt} u\|_{L^2(t_m, t_{m+1})}. \end{aligned}$$

Thus,

$$\left\|\frac{u(t_{m+1})-u(t_m)}{\delta t}-\partial_t u(t_{m+1})\right\|_{0,\Omega}^2=\int_\Omega\left|\frac{u(t_{m+1})-u(t_m)}{\delta t}-\partial_t u(t_{m+1})\right|^2dx$$

$$\leq\frac{1}{4}\delta t\int_\Omega\|\partial_{tt}u\|^2_{L^2(t_m,t_{m+1})}dx=\frac{\delta t}{4}\|\partial_{tt}u\|^2_{L^2(t_m,t_{m+1};L^2(\Omega))}. \tag{4.11}$$

For the second term we have, using (3.20) once again and a Taylor expansion,

$$\begin{aligned}\|D\theta_\pi^{m+1}\|_{0,\Omega}^2&=\left\|\frac{\theta_\pi^{m+1}-\theta_\pi^m}{\delta t}\right\|_{0,\Omega}^2\\&=\left\|\frac{u(t_{m+1})-P_{\tilde h}(u(t_{m+1}),\lambda(t_{m+1}))-u(t_m)+P_{\tilde h}(u(t_m),\lambda(t_m))}{\delta t}\right\|_{0,\Omega}^2\\&=\left\|\frac{u(t_{m+1})-u(t_m)}{\delta t}-P_{\tilde h}\left(\frac{u(t_{m+1})-u(t_m)}{\delta t},\frac{\lambda(t_{m+1})-\lambda(t_m)}{\delta t}\right)\right\|_{0,\Omega}^2\\&\leq Ch^2\Big(\left|\frac{u(t_{m+1})-u(t_m)}{\delta t}\right|_{2,\Omega}^2+\left\|\frac{\lambda(t_{m+1})-\lambda(t_m)}{\delta t}\right\|_{\mathcal H}^2\Big)\\&\leq Ch^2\delta t\left(\|\partial_t u\|^2_{L^2(t_m,t_{m+1};H^2(\Omega))}+\|\partial_t\lambda\|^2_{L^2(t_m,t_{m+1};\mathcal H)}\right).\end{aligned} \tag{4.12}$$

Thus, from (4.10) and using Young's inequality, it follows that

$$\begin{aligned}T_1^{m+1}\leq{}&\frac{1}{2}|||(\theta_h^{m+1},y_h^{m+1})|||_h^2+C\Big(\delta t\|\partial_{tt}u\|^2_{L^2(t_m,t_{m+1};L^2(\Omega))}\\&+h^2\delta t\big(\|\partial_t u\|^2_{L^2(t_m,t_{m+1};H^2(\Omega))}+\|\partial_t\lambda\|^2_{L^2(t_m,t_{m+1};\mathcal H)}\big)\\&+h^2(|u(t_{m+1})|^2_{2,\Omega}+\|\lambda(t_{m+1})\|^2_{\mathcal H})\Big).\end{aligned}$$

By inserting this expression into (4.3), multiplying the resulting expression by $2\delta t$, and adding over $0\leq m\leq n-1$, we obtain

$$\begin{aligned}&\|\theta_h^n\|^2_{0,\Omega}+\sum_{m=0}^{n-1}\delta t|||(\theta_h^{m+1},y_h^{m+1})|||_h^2\\&\leq\|\theta_h^0\|^2_{0,\Omega}+C\Big(\frac{\delta t^2}{2}\|\partial_{tt}u\|^2_{L^2(0,t_n;L^2(\Omega))}+Ch^2\delta t^2\Big(\|\partial_t u\|^2_{L^2(0,t_n;H^2(\Omega))}+\|\partial_t\lambda\|^2_{L^2(0,t_n;\mathcal H)}\Big)\\&\quad+\sum_{m=0}^{n-1}h^2\delta t\ \big(|u(t_{m+1})|^2_{2,\Omega}+\|\lambda(t_{m+1})\|^2_{\mathcal H}\big)\ \Big).\end{aligned}$$

It only remains to bound $\|\theta_h^0\|_{0,\Omega}$. For this, we use the approximation properties of i_h (cf. [15]) and of R_h (see (3.3)) and get

$$\begin{aligned}\|\theta_h^0\|=\|P_h(u_0,\lambda(0))-i_h(u_0)\|_{0,\Omega}&\leq\|P_h(u_0,\lambda(0))-u_0\|_{0,\Omega}+\|u_0-i_h(u_0)\|_{0,\Omega}\\&\leq Ch(|u_0|_{2,\Omega}+\|\lambda(0)\|_{\mathcal H}),\end{aligned}$$

which proves the result. □

The next result states an estimate for the error in a norm involving the $H^{-\frac{1}{2}}(\gamma)$-norm of the Lagrange multiplier λ. This estimate is optimal, up to a term that needs the right choice of initial condition, and that will be treated at a later stage.

Lemma 4.2. *Let us assume that $u \in H^1(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega)) \cap C^0(0,T;H^2(\Omega))$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\lambda \in H^1(0,T;\mathcal{H})$, and set $u_h^0 \in V_h$ as $u_h^0 = i_h(u_0)$. Then the following estimate holds for $1 \leq n \leq N$:*

$$\begin{aligned}
&\sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1} - \lambda(t_{m+1})\|_{-\frac{1}{2},\gamma}^2 \leq Ch^2 \|\lambda\|_{C^0(t_1,t_n;\mathcal{H})}^2 \\
&+ C\Big(h^2 \Big(\|u_0\|_{2,\Omega}^2 + \|\lambda(0)\|_{\mathcal{H}}^2 \Big) + h^2 \delta t^2 \Big(\|\partial_t u\|_{L^2(0,t_n;H^2(\Omega))}^2 + \|\partial_t \lambda\|_{L^2(0,t_n;\mathcal{H})}^2 \Big) \\
&+ C\Big(\delta t^2 \|\partial_{tt} u\|_{L^2(0,t_n;L^2(\Omega))}^2 + h^2 \|\lambda\|_{C^0(t_1,t_n;\mathcal{H})}^2 + h^2 \|u\|_{C^0(t_1,t_n;H^2(\Omega))}^2 \Big) \\
&+ \sum_{m=0}^{n-1} \delta t \|\partial_t u(t_{m+1}) - D u_h^{m+1}\|_{0,\Omega}^2 \Big),
\end{aligned}$$

with $C > 0$ a positive constant independent of h and δt.

Proof. We adopt the notations used in the proof of Theorem 4.1. Let $n \in \{1, ..., N\}$ and $m \in \{0, ..., n-1\}$. Then, (2.4) gives

$$\tilde{\beta} \|y_h^{m+1}\|_{-\frac{1}{2},\gamma} \leq \sup_{v_h \in V_h \setminus \{0\}} \frac{b(y_h^{m+1}, v_h)}{|v_h|_{1,\Omega}} + C j(y_h^{m+1}, y_h^{m+1})^{\frac{1}{2}}.$$

From (4.2), we have

$$b(y_h^{m+1}, v_h) = -b(y_\pi^{m+1}, v_h) + b(\lambda(t_{m+1}) - \lambda_h^{m+1}, v_h).$$

The first term can be bounded using the continuity of $b(.,.)$ and the trace and Poincaré inequalities, which yields

$$b(y_\pi^{m+1}, v_h) \leq C \|y_\pi^{m+1}\|_{-\frac{1}{2},\gamma} |v_h|_{1,\Omega}.$$

On the other hand, using the modified Galerkin orthogonality (see Lemma 4.1) with $\mu_h = 0$, we have

$$\begin{aligned}
&b(\lambda(t_{m+1}) - \lambda_h^{m+1}, v_h) = a(u(t_{m+1}) - u_h^{m+1}, v_h) + (\partial_t u(t_{m+1}) - D u_h^{m+1}, v_h)_\Omega \\
&\leq C \left|\!\left|\!\left| (u(t_{m+1}) - u_h^{m+1}, 0) \right|\!\right|\!\right|_h |v_h|_{1,\Omega} + \|\partial_t u(t_{m+1}) - D u_h^{m+1}\|_{0,\Omega} |v_h|_{1,\Omega}.
\end{aligned}$$

As a result, from the above estimates, we arrive at

$$\begin{aligned}
\tilde{\beta} \|y_h^{m+1}\|_{-\frac{1}{2},\gamma} &\leq \sup_{v_h \in V_h \setminus \{0\}} \frac{|-b(y_\pi^{m+1}, v_h) + b(\lambda(t_{m+1}) - \lambda_h^{m+1}, v_h)|}{|v_h|_{1,\Omega}} + C \left|\!\left|\!\left| (0, y_h^{m+1}) \right|\!\right|\!\right|_h \\
&\leq C\Big(\|y_\pi^{m+1}\|_{-\frac{1}{2},\gamma} + \left|\!\left|\!\left| (u(t_{m+1}) - u_h^{m+1}, y_h^{m+1}) \right|\!\right|\!\right|_h + \|\partial_t u(t_{m+1}) - D u_h^{m+1}\|_{0,\Omega} \Big).
\end{aligned}$$

Therefore, adding up from $m = 0$, to $n-1$, we obtain

$$\begin{aligned}
&\tilde{\beta}^2 \sum_{m=0}^{n-1} \delta t \|y_h^{m+1}\|_{-\frac{1}{2},\gamma}^2 \\
&\leq C \sum_{m=0}^{n-1} \delta t \Big(\|y_\pi^{m+1}\|_{-\frac{1}{2},\gamma}^2 + \left|\!\left|\!\left| (u(t_{m+1}) - u_h^{m+1}, y_h^{m+1}) \right|\!\right|\!\right|_h^2 + \|\partial_t u(t_{m+1}) - D u_h^{m+1}\|_{0,\Omega}^2 \Big).
\end{aligned}$$

We then conclude using the error estimate for u from Theorem 4.1, and the triangle inequality. □

Finally, to present an optimal error estimate, we show that, under the hypothesis that the initial condition is well-chosen, the term involving $\|\partial_t u - Du_h^{n+1}\|_{0,\Omega}$ can be bounded in an optimal way.

Theorem 4.2. *Let us assume the hypotheses of Lemma 4.2, but now suppose that the initial condition is given by $u_h^0 := P_h(u_0, 0)$. Then, for $1 \le n \le N$, the following error estimate holds*

$$\begin{aligned}\sum_{m=0}^{n-1} \delta t \|\lambda_h^{m+1} - \lambda(t_{m+1})\|^2_{-\frac{1}{2},\gamma} &\le Ch^2 \|\lambda\|^2_{C^0(t_1,t_n;\mathcal{H})} + Ch^2\|u_0\|^2_{2,\Omega} + Ch^2\|\lambda(0)\|^2_{\mathcal{H}} \\ &+ Ch^2\delta t^2 \left(\|\partial_t u\|^2_{L^2(0,t_n;H^2(\Omega))} + \|\partial_t \lambda\|^2_{L^2(0,t_n;\mathcal{H})}\right) \\ &+ C\Big(\delta t^2 \|\partial_{tt} u\|^2_{L^2(0,t_n;L^2(\Omega))} + h^2\|\lambda\|^2_{C^0(t_1,t_n;\mathcal{H})} + h^2\|u\|^2_{C^0(t_1,t_n;H^2(\Omega))}\Big).\end{aligned}$$

Proof. Based on the previous lemma, we just need to prove

$$\begin{aligned}&\sum_{m=0}^{n-1} \delta t \|\partial_t u(t_{m+1}) - Du_h^{m+1}\|^2_{0,\Omega} + \left|\!\left|\!\left|(\hat{u}^n - u_h^n, \hat{\lambda}^n - \lambda_h^n)\right|\!\right|\!\right|^2_h \\ &\le C\Big(\delta t^2 \|\partial_{tt} u\|^2_{L^2(0,t_n;L^2(\Omega))} + h^2\delta t^2 \left(\|\partial_t u\|^2_{L^2(0,t_n;H^2(\Omega))} + \|\partial_t \lambda\|^2_{L^2(0,t_n;\mathcal{H})}\right) + Ch^2\|\lambda(0)\|^2_{\mathcal{H}}.\end{aligned} \tag{4.13}$$

We recall that $\hat{u}^s = P_h(u(t_s), \lambda(t_s))$ and $\hat{\lambda}^s = R_h(u(t_s), \lambda(t_s))$. Once again, we decompose the error as in (4.1)-(4.2). Using the triangle inequality, we have

$$\begin{aligned}&\sum_{m=0}^{n-1} \delta t \|\partial_t u(t_{m+1}) - Du_h^{m+1}\|^2_{0,\Omega} = \sum_{m=0}^{n-1} \delta t \|\partial_t u(t_{m+1}) - Du(t_{m+1}) + D\theta_\pi^{m+1} + D\theta_h^{m+1}\|^2_{0,\Omega} \\ &\le C\sum_{m=0}^{n-1} \delta t \Big(\|\partial_t u(t_{m+1}) - Du(t_{m+1})\|^2_{0,\Omega} + \|D\theta_\pi^{m+1}\|^2_{0,\Omega} + \|D\theta_h^{m+1}\|^2_{0,\Omega}\Big).\end{aligned}$$

The first and second terms have already been bounded in (4.11)-(4.12). For the third term, we use the modified Galerkin orthogonality (see Lemma 4.1) with $\mu_h = 0$ and the definition of the Ritz projection (3.1) to obtain

$$\begin{aligned}&\|D\theta_h^{m+1}\|^2_{0,\Omega} + a(\theta_h^{m+1}, D\theta_h^{m+1}) - b(y_h^{m+1}, D\theta_h^{m+1}) \\ &= -(D\theta_\pi^{m+1}, D\theta_h^{m+1})_\Omega - a(\theta_\pi^{m+1}, D\theta_h^{m+1}) + b(y_\pi^{m+1}, D\theta_h^{m+1}) \\ &+ (Du(t_{m+1}) - \partial_t u(t_{m+1}), D\theta_h^{m+1})_\Omega \\ &= -(D\theta_\pi^{m+1}, D\theta_h^{m+1})_\Omega + (Du(t_{m+1}) - \partial_t u(t_{m+1}), D\theta_h^{m+1})_\Omega.\end{aligned}$$

Next, Young's inequality yields

$$\begin{aligned}&\frac{1}{2}\|D\theta_h^{n+1}\|^2_{0,\Omega} + a(\theta_h^{n+1}, D\theta_h^{n+1}) - b(y_h^{n+1}, D\theta_h^{n+1}) \\ &\le \|D\theta_\pi^{n+1}\|^2_{0,\Omega} + \|Du(t_{n+1}) - \partial_t u(t_{n+1})\|^2_{0,\Omega}.\end{aligned} \tag{4.14}$$

In addition, using $v_h = 0$ in (3.1), for $0 \le m \le n$, gives

$$b\big(\mu_h, \hat{u}^m\big) = -j(\hat{\lambda}^m, \mu_h) + b(\mu_h, g(t_m)). \tag{4.15}$$

On the other hand, testing (2.9) at the level $m (0 \le m \le n)$ with $v_h = 0$ we obtain

$$b(\mu_h, u_h^m) = -j(\lambda_h^m, \mu_h) + b(\mu_h, g(t_m)), \tag{4.16}$$

where this is valid for $m = 0$ thanks to the choice $u_h^0 = P_h(u_0, 0)$, and where $\lambda_h^0 = R_h(u_0, 0)$. As a result, from (4.15) - (4.16), we get

$$b(\mu_h, \theta_h^m) = -j(y_h^m, \mu_h),$$

for all $\mu_h \in \Lambda_h$ and $0 \leq m \leq N$. Therefore, for $0 \leq m \leq N-1$, the following holds

$$b(y_h^{m+1}, D\theta_h^{m+1}) = -j(Dy_h^{m+1}, y_h^{m+1}). \tag{4.17}$$

On the other hand, using the symmetry of a and j, we get

$$\begin{aligned} a(\theta_h^{m+1}, D\theta_h^{m+1}) &= \frac{1}{2} Da(\theta_h^{m+1}, \theta_h^{m+1}) + \frac{\delta t}{2} a(D\theta_h^{m+1}, D\theta_h^{m+1}), \\ j(y_h^{m+1}, Dy_h^{m+1}) &= \frac{1}{2} Dj(y_h^{m+1}, y_h^{m+1}) + \frac{\delta t}{2} j(Dy_h^{m+1}, Dy_h^{m+1}). \end{aligned}$$

Then, replacing (4.17) and the last set of equalities in (4.14), we get

$$\begin{aligned} &\frac{1}{2}\|D\theta_h^{m+1}\|_{0,\Omega}^2 + \frac{1}{2} D(a(\theta_h^{m+1}, \theta_h^{m+1}) + j(y_h^{m+1}, y_h^{m+1})) \\ &\leq \|D\theta_\pi^{m+1}\|_{0,\Omega}^2 + \|Du(t_{m+1}) - \partial_t u(t_{m+1})\|_{0,\Omega}^2. \end{aligned}$$

Thus, multiplication by $2\delta t$ and summation over $0 \leq m \leq n-1$, leads to

$$\begin{aligned} &\sum_{m=0}^{n-1} \delta t \|D\theta_h^{m+1}\|_{0,\Omega}^2 + |||(\theta_h^m, y_h^m)|||_h^2 \\ &\leq \left|||(\theta_h^0, y_h^0)\right|||_h^2 + C \sum_{m=0}^{n-1} \delta t \Big(\|D\theta_\pi^{m+1}\|_{0,\Omega}^2 + \|Du(t_{m+1}) - \partial_t u(t_{m+1})\|_{0,\Omega}^2 \Big). \end{aligned}$$

The only term that has not been bounded is the first one. To bound it, we use the linearity of the Ritz projection to get

$$\theta_h^0 = u_h^0 - P_h(u(0), \lambda(0)) = P_h(u(0), 0) - P_h(u(0), \lambda(0)) = -P_h(0, \lambda(0)),$$

$$y_h^0 = \lambda_h^0 - R_h(u(0), \lambda(0)) = R_h(u(0), 0) - R_h(u(0), 0) - R_h(0, \lambda(0)) = -R_h(0, \lambda(0)).$$

Hence,

$$\left|||(\theta_h^0, y_h^0)|||_h = |||(P_h(0, \lambda(0)), R_h(0, \lambda(0)))|||_h.$$

The use of (3.3) gives

$$|P_h(0, \lambda(0))|_{1,\Omega} = |P_h(0, \lambda(0)) - 0|_{1,\Omega} \leq Ch\Big(|0|_{2,\Omega} + \|\lambda(0)\|_{\mathcal{H}} \Big) = Ch\|\lambda(0)\|_{\mathcal{H}},$$

and

$$\begin{aligned} &j(R_h(0, \lambda(0)), R_h(0, \lambda(0))) = j(R_h(0, \lambda(0)) - \lambda(0), R_h(0, \lambda(0))) + j(\lambda(0), R_h(0, \lambda(0))) \\ &\leq C\Big(j(R_h(0, \lambda(0)) - \lambda(0)), R_h(0, \lambda(0)) - \lambda(0)) + j(\lambda(0), \lambda(0)) \Big)^{\frac{1}{2}} \Big(j(R_h(0, \lambda(0)), R_h(0, \lambda(0)) \Big)^{\frac{1}{2}} \\ &\leq Ch\|\lambda(0)\|_{\mathcal{H}} \Big(j(R_h(0, \lambda(0)), R_h(0, \lambda(0)) \Big)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\left|||(\theta_h^0, y_h^0)|||_h^2 = |||(P_h(0, \lambda(0)), R_h(0, \lambda(0)))|||_h^2 \leq Ch^2 \|\lambda(0)\|_{\mathcal{H}}^2,$$

which, after using (4.11) and (4.12) leads to

$$\sum_{m=0}^{n-1} \delta t \|D\theta_h^{m+1}\|_{0,\Omega}^2 + |||(\theta_h^n, y_h^n)|||_h^2$$
$$\leq C\Big(h^2\|\lambda(0)\|_{\mathcal{H}}^2 + \delta t^2\|\partial_{tt}u\|_{L^2(0,t_n;L^2(\Omega))}^2 + h^2\delta t^2\big(\|\partial_t u\|_{L^2(0,t_n;H^2(\Omega))}^2 + \|\partial_t\lambda\|_{L^2(0,t_n;\mathcal{H})}^2\big)\Big)$$

for $1 \leq n \leq N$. This proves (4.13), and then the proof is finished. □

Remark 4.3. *The inf-sup stable case. Again, following very similar arguments, optimal error estimates can be obtained for the method* (2.8) *under the assumption* $u_h^0 = P_h(u(0), 0)$.

Remark 4.4. *We finish this section by remarking that the hypotheses made on the exact solution, namely, that* $u(\cdot, t) \in H^2(\Omega)$ *and* $\lambda(\cdot, t) \in \mathcal{H}$, *are made just for simplicity. In fact, proceeding as above, and optimal,* $O(h^s)$ *error estimate can be proven for the, more realistic case, in which we suppose that* $u(\cdot, t) \in H^{1+s}(\Omega)$ *and* $\lambda(\cdot, t) \in H^s(\gamma_i), i = 1, \ldots, M$, *for some* $s \in (0, 1)$. *This lower regularity is typical of problems solved with a fictitious domain method. In fact, the construction of an extension of the datum* f *in such a way that the solution of* (2.2) *belongs to* $H^2(\Omega) \times \mathcal{H}$ *at every time* t, *appart from some preliminary results in* [16], *remains, up to our best knowledge, an open problem.*

5. Numerical studies

In this section, we report the results of numerical experiments that support the analytical results of Sections 3 and 4. We present computations demonstrating the optimal convergence using the space discretisation and the time discretisation defined in Section 2. We also verify numerically that the choice of the initial condition $u_h^0 = P_h(u_0, 0)$ guarantees a uniform approximation of λ, thus confirming the results in Theorem 4.2 and the analogous result for the inf-sup stable case. It is also discussed that for other choices of discrete initial condition the results for small time steps are not stable in the sense that the error blows up as the time step size diminishes. All computations have been performed using $FreeFem++$ [21].

We consider problem (2.1) with $\omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and its reformulation (2.2) using $\Omega = (-2.4, 4) \times (-2, 2)$. We have taken $T = 1$.

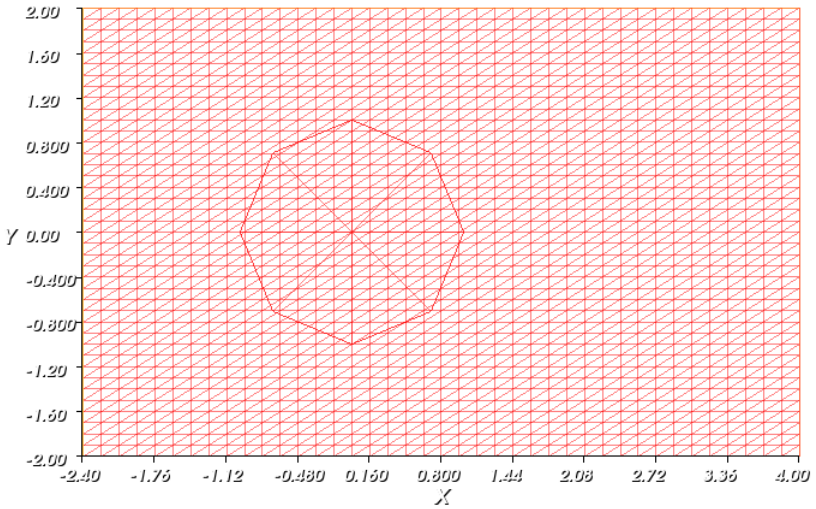

FIGURE 3. Meshes for inf-sup stable case when $n = 1$.

We have tested two examples with known analytical solution. We first choose f in $\Omega \times (0,T)$ such that the exact solution of (2.2) is given by

$$u_1(x,t) = e^t(x^2+y^2-1)\,, \tag{5.1}$$

thus giving $g = 0$ on $\gamma \times (0,T)$. Also, we have chosen another case where $g \neq 0$ on γ given by

$$u_2(x,t) = e^t(x^2+xy). \tag{5.2}$$

We have discretised this problem using a sequence of uniform meshes. The starting point is the mesh depicted in Fig. 3. To build the meshes, an integer number n is given. Then Ω is divided horizontally into $20 \cdot 2^n$ segments and vertically into $20 \cdot 2^n$ segments. The resulting quadrilateral mesh is then divided into triangles to form the mesh in Fig. 3 (where $n = 1$ is depicted). To build the mesh on γ, (i.e., $\gamma_{\tilde{h}}$), we divide γ into $4 \cdot 2^n$ curved segments. The pair of meshes $\mathcal{T}_h \times \gamma_{\tilde{h}}$ is used to implement the inf-sup stable method (2.8). For the stabilised method (2.9) we use $\mathcal{T}_h \times \gamma_h$, where the mesh γ_h is obtained after dividing each segment $\tilde{e}$ of $\gamma_{\tilde{h}}$ into 4 equally spaced curved segments.

For both test problems we have $\lambda = 0$. To measure convergence, we have computed the norms

$$\|u - u_{\tilde{h},\delta t}\|_*^2 := \|u_h^N - u(t_N)\|_{0,\Omega}^2 + \sum_{m=0}^{N-1} \delta t |u_h^{m+1} - u(t_{m+1})|_{1,\Omega}^2\,,$$
$$\|\lambda - \lambda_{\tilde{h},\delta t}\|_{**}^2 := \sum_{m=0}^{N-1} \delta t \|\lambda_{\tilde{h}}^{m+1} - \lambda(t_{m+1})\|_{0,\gamma}^2\,,$$

for Method (2.8), and

$$\|(u - u_{h,\delta t}, \lambda - \lambda_{h,\delta t})\|_+^2 := \|u_h^N - u(t_N)\|_{0,\Omega}^2 + \sum_{m=0}^{N-1} \delta t \left|\!\left|\!\left| (u_h^{m+1} - u(t_{m+1}), \lambda_h^{m+1} - \lambda(t_{m+1})) \right|\!\right|\!\right|_h^2,$$
$$\|\lambda - \lambda_{h,\delta t}\|_{++}^2 := \sum_{m=0}^{N-1} \delta t \|\lambda_h^{m+1} - \lambda(t_{m+1})\|_{0,\gamma}^2,$$

for the stabilised method (2.9). To balance the space and time discretisation error, we have chosen $\delta t = h$ in our experiments. The numerical results are depicted in Figs. 4- 7 and measured the behavior of the corresponding norm with respect to h and δt. We observe that all errors tend to zero as predicted by the theory. This behavior is independent of the choice of the initial conditions. In particular, we observe optimal convergence of all variables using both $u_h^0 = i_h(u_0)$ and $u_h^0 = P_h(u_0, 0)$.

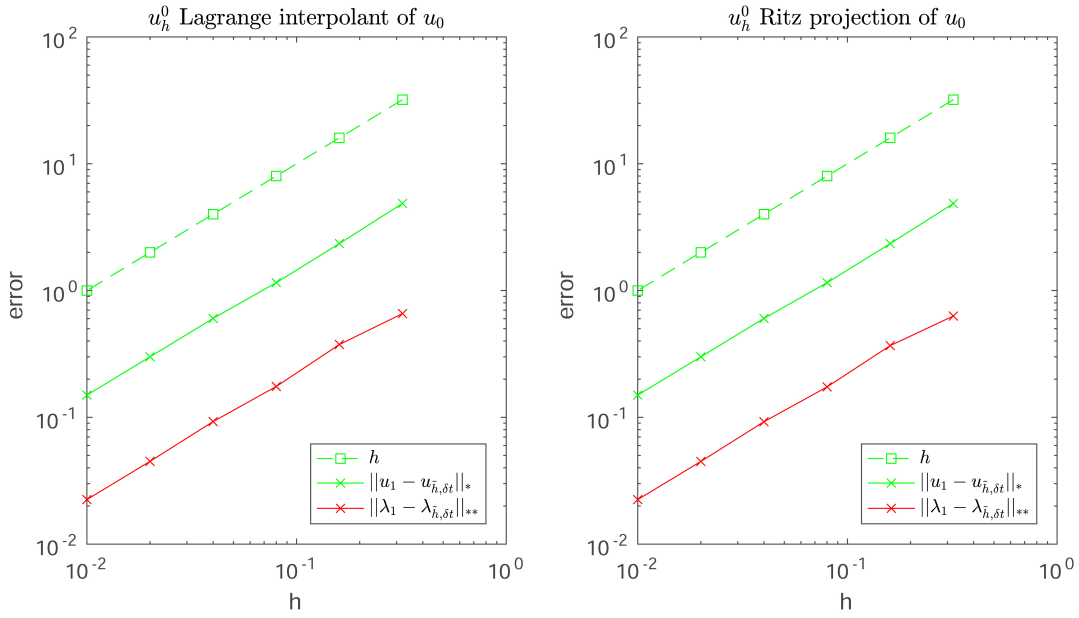

FIGURE 4. Convergence history of (2.8) for Solution (5.1).

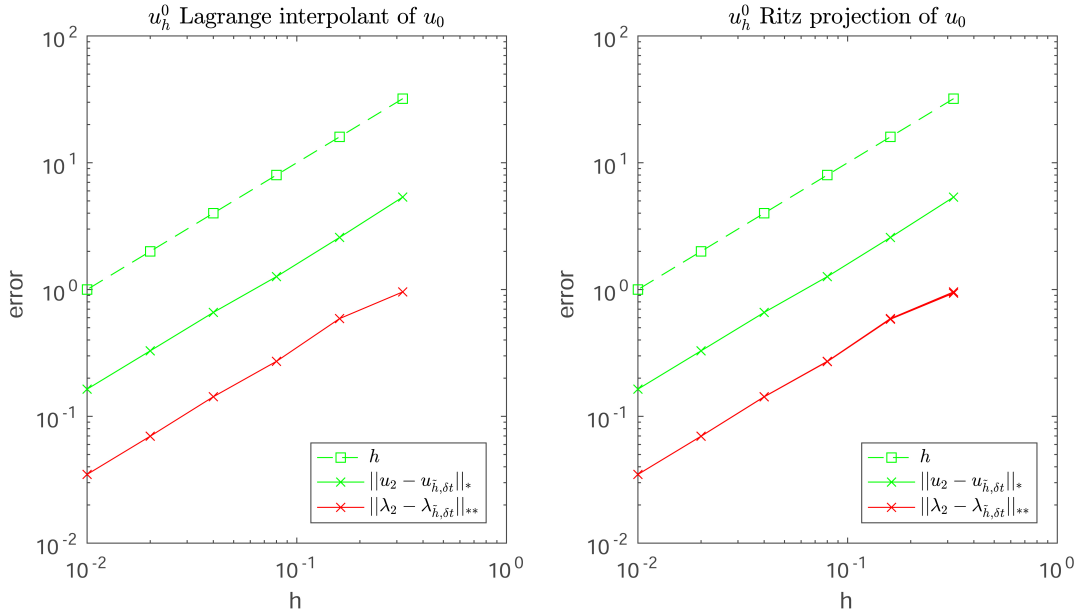

FIGURE 5. Convergence history of (2.8) for Solution (5.2).

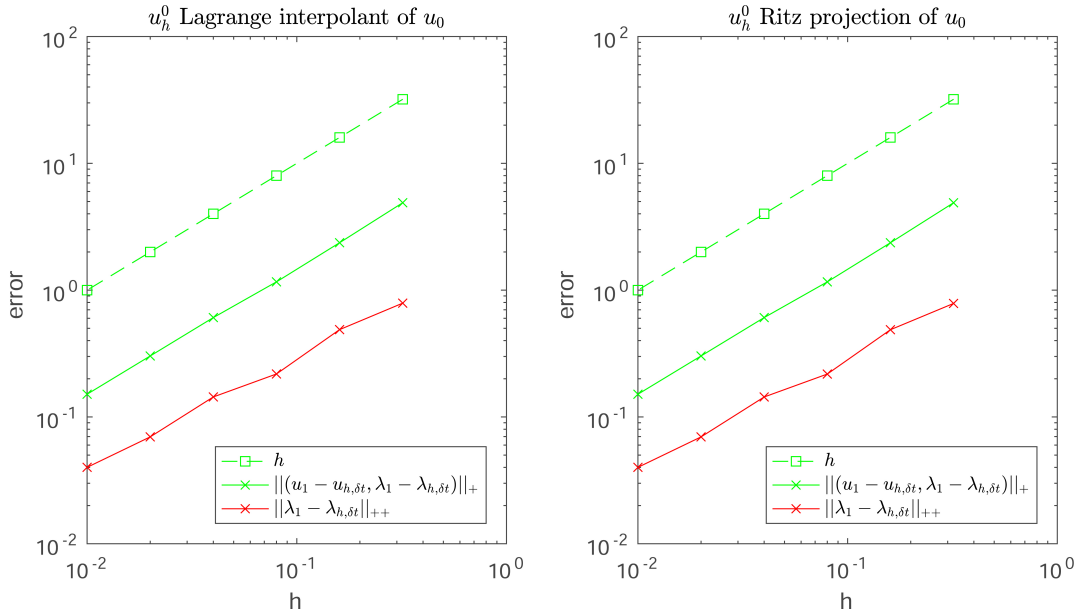

FIGURE 6. Convergence history of (2.9) for Solution (5.1).

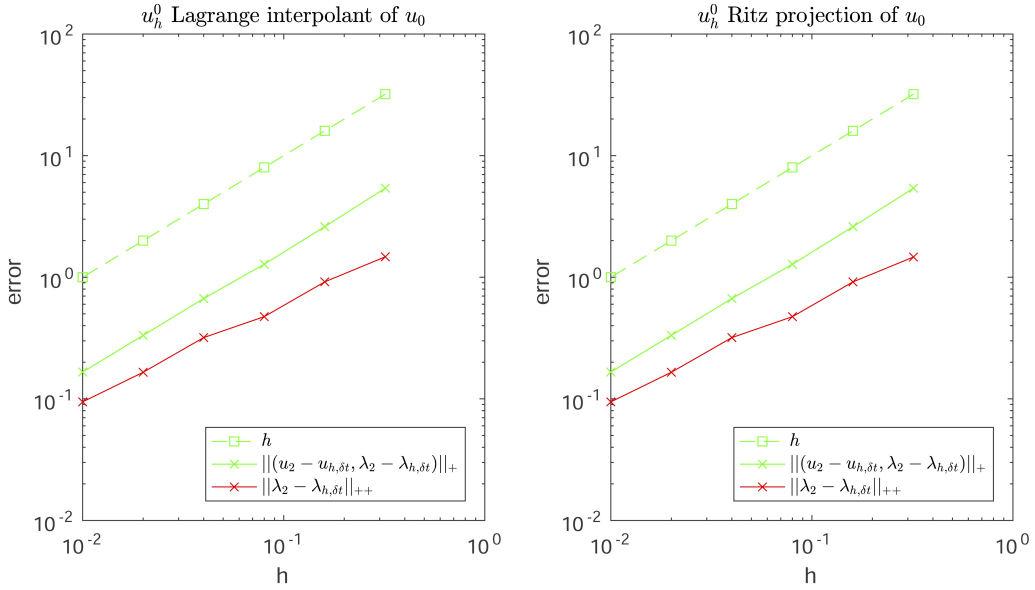

FIGURE 7. Convergence history of (2.9) for Solution (5.2).

The results depicted on the right-hand side of the above figures are aligned with the results presented in Section 4, even if some calculations have been performed using the wrong initial condition. In fact, Lemma 4.1 does present an optimal convergence estimate independent of the choice initial condition, although in a weak norm, which only measures the fluctuations of λ, multiplied by a time step. To stress this fact, we have performed a numerical experiment reminiscent of the one from [22]. That is, we have fixed one level $n = 1$ and have taken $\delta t \longrightarrow 0$.

We have then measured $\|\lambda(\delta t) - \lambda_{\tilde{h}}^1\|_{0,\gamma}$ both considering $u_h^0 = i_h(u_0)$ and $u_h^0 = P_h(u_0, 0)$ defined before. The results, depicted in Figs. 8-11 show that, unless the initial condition is chosen appropriately, the error in λ grows as $\delta t \longrightarrow 0$, i. e. the approximation of λ cannot be guaranteed. These results confirm the sharpness of the stability results presented in Section 3.

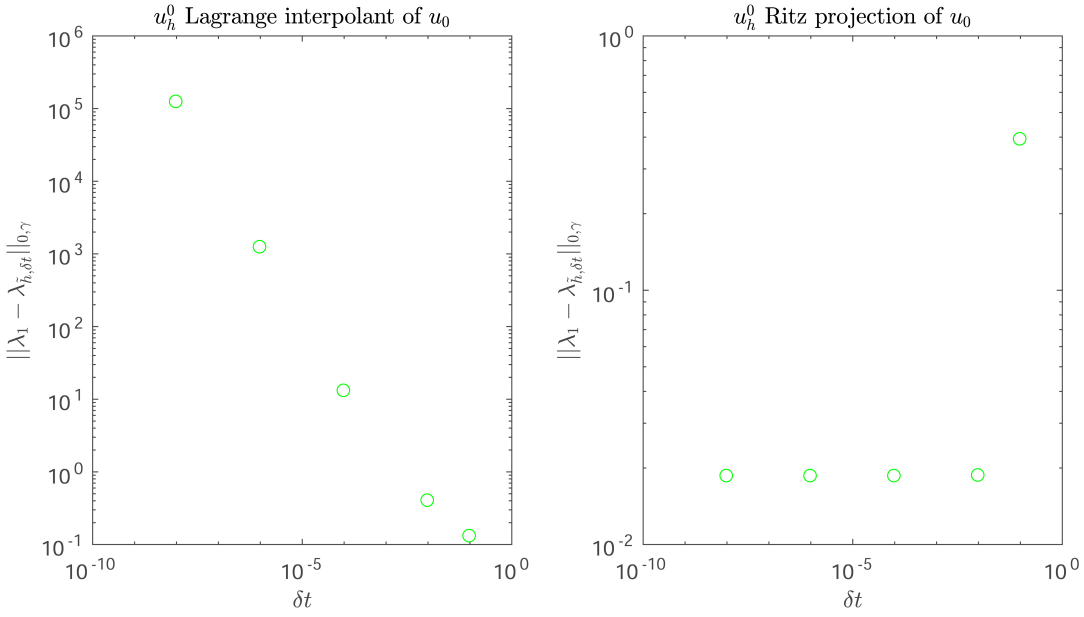

FIGURE 8. Error $\|\lambda(\delta t) - \lambda_{\tilde{h}}^1\|_{0,\gamma}$ in method (2.8) for a fixed mesh and $\delta t \to 0$ for Solution (5.1).

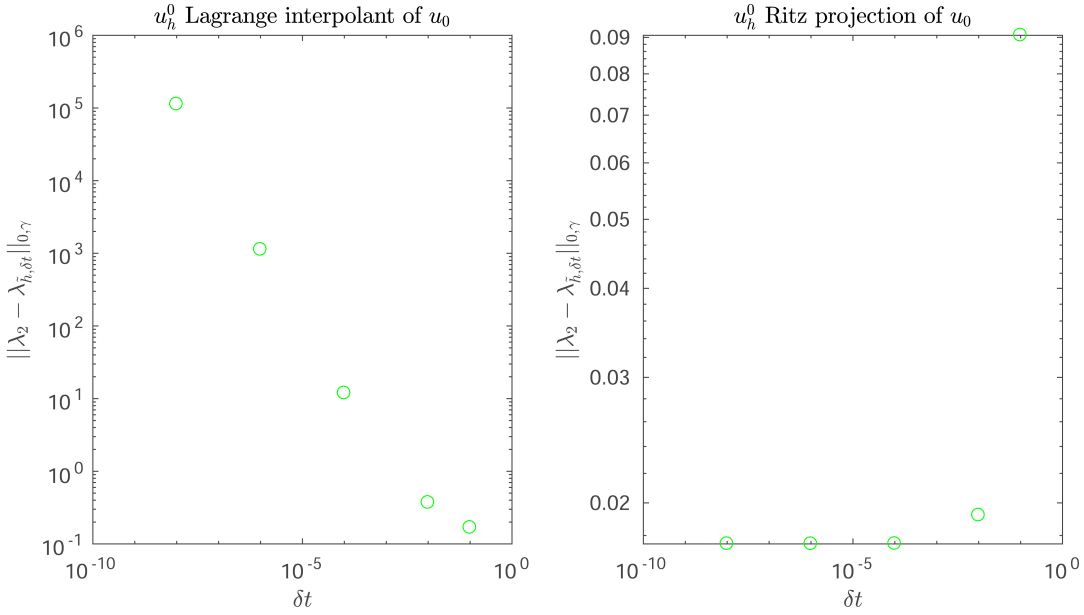

FIGURE 9. Error $\|\lambda(\delta t) - \lambda_{\tilde{h}}^1\|_{0,\gamma}$ in method (2.8) for a fixed mesh and $\delta t \to 0$ for Solution (5.2).

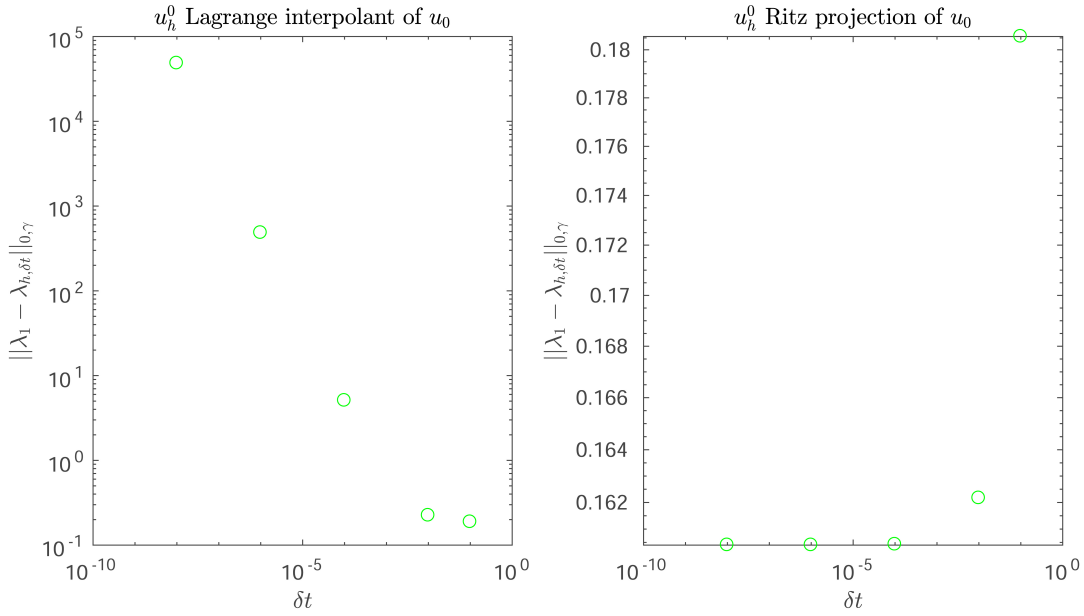

FIGURE 10. Error $\|\lambda(\delta t) - \lambda_h^1\|_{0,\gamma}$ in method (2.9) for a fixed mesh and $\delta t \to 0$ for Solution (5.1).

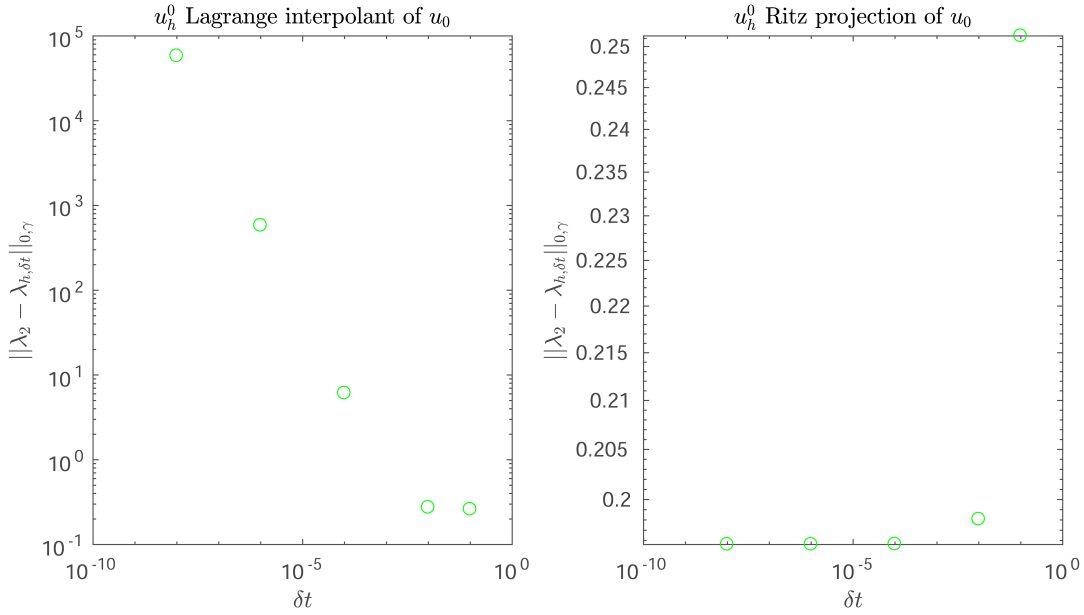

FIGURE 11. Error $\|\lambda(\delta t) - \lambda_h^1\|_{0,\gamma}$ in method (2.9) for a fixed mesh and $\delta t \to 0$ for Solution (5.2).

6. Conclusion

In this paper we have proposed and analised a stabilised finite element method for the transient heat equation solved with a fictitious domain strategy. The choice of stabilising terms allows us to use any combination of meshes to define the finite element spaces for the primal

variable and Lagrange multiplier. Concerning the time discretisation, the main result is that unconditional stability and optimal convergence can be obtained assuming the right choice of initial conditions. Interestingly, this is not exclusive to the stabilised finite element method, but is also a requirement if an inf-sup stable method is used. This has been confirmed by numerical experiments.

Acknowledgment

A part of this work was done during the visits of ChG and GRB to the Laboratoire de Mathématiques de Besançon. The authors want to thank Alexei Lozinski for many helpful discussions and useful suggestions. The work of Ch.G. has been funded by the Faculty of Science, University of Strathclyde, through a Tranche 1 Scholarship. F.Ch. thanks Région Bourgogne Franche-Comté for partial funding ("Convention Région 2015C-4991.").

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