Geodesics toward corners in first passage percolation

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GEODESICS TOWARD CORNERS IN FIRST PASSAGE PERCOLATION

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ABSTRACT. For stationary first passage percolation in two dimensions, the existence and uniqueness of semi-infinite geodesics directed in particular directions or sectors has been considered by Damron and Hanson [4], Ahlberg and Hoffman [1], and others. However the main results do not cover geodesics in the direction of corners of the limit shape $\mathcal{B}$, where two facets meet. We construct an example with the following properties: (i) the limiting shape is an octagon, (ii) semi-infinite geodesics exist only in the four axis directions, and (iii) in each axis direction there are multiple such geodesics. Consequently, the set of points of $\partial\mathcal{B}$ which are in the direction of some geodesic does not have all of $\mathcal{B}$ as its convex hull.

1. INTRODUCTION

We consider stationary first passage percolation (FPP) on a lattice $\mathbb{L}$ with site set $\mathbb{Z}^2$, and with a set of bonds which we denote $\mathcal{E}$. We are mainly interested in the usual set of nearest-neighbor bonds $\mathcal{E} = \{(x, y); \|x - y\|_1 = 1\}$, though in Section 2 we consider $\mathcal{E}$ with added diagonal bonds to construct a simpler example. Each bond e of $\mathcal{E}$ is assigned a random passage time $\tau_e \geq 0$, and the configuration τ is assumed stationary under lattice translations; the measure on the space $\Omega = [0, \infty)^{\mathcal{E}}$ of configurations τ is denoted $\mathbf{P}$, with corresponding expectation $\mathbf{E}$. For sites x, y of $\mathbb{L}$, a *path* γ from x to y in $\mathbb{L}$ is a sequence $x = x_0, \ldots, x_n = y$ with x_i, x_{i+1} adjacent in $\mathbb{L}$ for all i; we may equivalently view γ as a sequence of edges. The *passage time* $T(\gamma)$ of γ is $T(\gamma) = \sum_{e\in\gamma} \tau_e$. For sites x, y we define

$$T(x, y) = \inf\{T(\gamma) : \gamma \text{ is a path from } x \text{ to } y\}.$$

A *geodesic* from x to y is a path which achieves this infimum. A *semi-infinite geodesic* Γ from a site x is a path with (necessarily distinct) sites $x = x_0, x_1, \ldots$ for which every finite segment is a geodesic, and the *direction* of Γ, denoted $\mathrm{Dir}(\Gamma)$, is the set of limit points of $\{x_n/|x_n| : n \geq 1\}$. It is of interest to understand semi-infinite geodesics, and in particular the set of directions in which they exist.

It is standard to make the following assumptions, from [6].

Assumption A1.

(i) $\mathbf{P}$ is ergodic with respect to lattice translation;
(ii) $\mathbf{E}(\tau_e^{2+\epsilon}) < \infty$ for any $e \in \mathcal{E}$, for some $\epsilon > 0$.

Under A1, Boivin [2] showed that for each $x \in \mathbb{Z}^2$ the limit

$$\mu(x) = \lim_{n\to\infty} \frac{T(0, nx)}{n}$$

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exists. This μ extends to $\mathbb{Q}^2$ by restricting to n for which $nx \in \mathbb{Z}^2$, and then to $\mathbb{R}^2$ by continuity; the resulting function is a norm. Its unit ball is a nonempty convex symmetric set which we denote $\mathcal{B}$. The *wet region* at time t is $\mathcal{B}(t) = \{x + [-\frac{1}{2}, \frac{1}{2}]^2 : T(0, x) \le t\}$. The shape theorem of Boivin [2] says that with probability one, given $\epsilon > 0$, for all sufficiently large t we have

$$(1 - \epsilon)\mathcal{B} \subset \frac{\mathcal{B}(t)}{t} \subset (1 + \epsilon)\mathcal{B},$$

so $\mathcal{B}$ is called the *limit shape*. Haggstrom and Meester [5] showed that every compact convex B arises as the limit shape for some stationary FPP process.

We add the following assumptions, also from [6].

Assumption A2.

(iii) $\mathbf{P}$ has all the symmetries of the lattice $\mathbb{L}$;
(iv) if α, γ are finite paths, with the same endpoints, differing in at least one edge then $T(\alpha) \neq T(\gamma)$ a.s.;
(v) $\mathbf{P}$ has *upward finite energy*: for any bond e and any t such that $\mathbf{P}(\tau_e > t) > 0$, we have

$$\mathbf{P}(\tau_e > t \mid \{\tau_f : f \neq e\}) > 0;$$

(vi) the limit shape $\mathcal{B}$ is bounded (equivalently, μ is strictly positive except at the origin.)

Thanks to (iv), the union of all geodesics from a fixed site x to sites $y \in \mathbb{Z}^2$ is a tree, and we denote it $\mathcal{T}_x = \mathcal{T}_x(\tau)$. By [6], $\mathcal{T}_x$ contains at least 4 semi-infinite geodesics, and Brito and Hoffman [3] give an example in which there are only 4 geodesics, and the direction for each corresponds to an entire closed quadrant of the lattice.

To describe the directions in which semi-infinite geodesics may exist, we introduce some terminology. A *facet* of $\mathcal{B}$ is a closed line segment F contained in $\partial\mathcal{B}$; the unique linear functional equal to 1 on F is denoted ρ_F. For each angle from 0 there corresponds a unique point of $\partial\mathcal{B}$ in the ray from 0 at that angle; a facet thus corresponds to a sector of angles, or of unit vectors. We say a point $v \in \partial\mathcal{B}$ is of *type* i ($i = 0, 1, 2$) if v is an endpoint of i facets. We may divide points of $\partial\mathcal{B}$ (or equivalently, all angles) into 6 classes:

(1) *exposed points of differentiability*, that is, exposed points of $\partial\mathcal{B}$ where $\partial\mathcal{B}$ is differentiable, necessarily type 0;
(2) *facet endpoints of differentiability*, or equivalently, type-1 points where $\partial\mathcal{B}$ is differentiable;
(3) *facet interior points*, necessarily type-0;
(4) *half rounded corners*, that is, type-1 points where $\partial\mathcal{B}$ is not differentiable;
(5) *fully rounded corners*, that is, type-0 points where $\partial\mathcal{B}$ is not differentiable;
(6) *true corners*, meaning type-2 points.

Associated to any semi-infinite geodesic $\Gamma = \{x_0, x_1, \dots\}$ is its *Busemann function* $B_\Gamma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ given by

$$B_\Gamma(x, y) = \lim_{n \to \infty} (T(x, x_n) - T(y, x_n)).$$

From [1, Theorems 2.5 and 2.6], we know the following, under Assumptions A1 and A2. There exists a linear functional ρ_Γ on $\mathbb{R}^2$ with the property that B_Γ is *linear to* ρ_Γ, that is,

$$\lim_{|x|\to\infty} \frac{1}{|x|} |B_\Gamma(0,x) - \rho_\Gamma(x)| = 0.$$

The set $\{\rho_\Gamma = 1\}$ is always a supporting line of $\mathcal{B}$, so its intersection with $\partial\mathcal{B}$ is either an exposed point v or a facet F, and then $\mathrm{Dir}(\Gamma)$ is equal to $\{v\}$ or contained in F (modulo normalizing to unit vectors.) Thus $\mathrm{Dir}(\Gamma)$ determines ρ_Γ, unless $\mathrm{Dir}(\Gamma)$ consists of only a corner of some type. Furthermore, there is a closed set $\mathcal{C}_*$ of linear functionals such that the set of functionals ρ_Γ which appear for some Γ is almost surely equal to $\mathcal{C}_*$.

In [1], Ahlberg and Hoffman define a *random coalescing geodesic* (or *RC geodesic*), which is, in loose terms, a mapping which selects measurably for each τ a semi-infinite geodesic $\Gamma_0 = \Gamma_0(\tau)$ in $\mathcal{T}_0(\tau)$, in such a way that when the mapping is applied via translation to obtain $\Gamma_x \in \mathcal{T}_x$, Γ_0 and Γ_x coalesce a.s. for all x. The following statements are valid under Assumptions A1 and A2: they are part of, or immediate consequences of, results of Ahlberg and Hoffman [1, Theorems 12.1 and 12.2], strengthening earlier results from [6] and [4].

(I) For each exposed point of differentiability $v \in \partial\mathcal{B}$, there is a.s. a unique RC geodesic Γ with $\mathrm{Dir}(\Gamma) = \{v/|v|\}$.

(II) For each half rounded corner $v \in \partial\mathcal{B}$, there is a.s. at least one RC geodesic Γ with $\mathrm{Dir}(\Gamma) = \{v/|v|\}$; for one such Γ the linear functional ρ_Γ corresponds to a limit of supporting lines taken from the non-facet side of v. This uses the fact that $\mathcal{C}_*$ is closed.

(III) For each fully rounded corner $v \in \partial\mathcal{B}$, there are a.s. at least two RC geodesics Γ with $\mathrm{Dir}(\Gamma) = \{v/|v|\}$, with distinct linear functionals ρ_Γ corresponding to limits of supporting lines from each side of v.

(IV) Given a facet F with corresponding sector S_F of unit vectors, there is a.s. a unique RC geodesic Γ with $\mathrm{Dir}(\Gamma) \subset S_F$ and $\rho_\Gamma = \rho_F$. If F has one facet endpoint of differentiability, then $\mathrm{Dir}(\Gamma)$ contains it; if it has two then $\mathrm{Dir}(\Gamma) = S_F$. For any other RC geodesic Γ with $\mathrm{Dir}(\Gamma) \cap S_F \neq \emptyset$, this intersection is a single endpoint of S_F which must be a corner.

But little has been proved about geodesics in the directions of true corners. One may ask, must every true-corner direction be in $\mathrm{Dir}(\Gamma)$ for some geodesic Γ? Equivalently, must the convex hull of

$$\mathcal{V}_{\mathrm{geo}} := \{v \in \partial\mathcal{B} : v/|v| \in \mathrm{Dir}(\Gamma) \text{ for some semi-infinite geodesic } \Gamma\}$$

be all of $\mathcal{B}$? Further, we can consider the nonuniqueness set

$$\mathcal{N} := \big\{u \in S^1 : \text{ there exist multiple semi-infinite geodesics } \Gamma \text{ with } \mathrm{Dir}(\Gamma) = \{u\}\big\}.$$

For each fully rounded corner v we have $\mathbf{P}(v \in \mathcal{N}) = 1$. For each non-corner $v \in \partial\mathcal{B}$ we have $\mathbf{P}(v \in \mathcal{N}) = 0$, but in the case of $\mathcal{B}$ with no corners this does not mean $\mathcal{N}$ is empty. If every point of $\partial\mathcal{B}$ is an exposed point of differentiability then there is at least one geodesic in every direction; the union of all semi-infinite geodesics from 0 is therefore a tree with infinitely many branches, and each branching produces a point of $\mathcal{N}$, so $\mathcal{N}$ is infinite a.s. In the example of Brito and Hoffman [3], the limit shape is a diamond with true corners on the

axes, and for each of these corners v there is a.s. no geodesic Γ with $\mathrm{Dir}(\Gamma) = \{v/|v|\}$, so $\mathbf{P}(v \in \mathcal{N}) = 0$. This suggests the question, must $\mathbf{P}(v \in \mathcal{N}) = 0$ for true corners?

Our primary result is an example of FPP process, which we call *fast diagonals FPP*, in which some true corners have no geodesic, and others have multiple geodesics a.s. This means in particular that the convex hull of $\mathcal{V}_{\mathrm{geo}}$ is not all of $\mathcal{B}$.

Theorem 1. *The fast diagonals FPP process (defined in Section 3.2) satisfies Assumptions A1-A2, and has the following properties:*

(i) *The limit shape is an octagon, with corners on the axes and main diagonals.*
(ii) *Every semi-infinite geodesic Γ is directed in an axis direction (that is, Dir(Γ) consists of a single axis direction.)*
(iii) *For each axis direction there exist at least two random coalescing geodesics directed in that direction.*

We first introduce a simpler example in Section 2, which does not satisfy Assumption A2 (in particular (iv) and (v)), but encapsulates the key ideas of our construction. The remaining main part of the paper is devoted to modifying this example in order to satisfy Assumption A2, which brings many complications, see Section 3.

2. Simple example: diagonal highways only

For this section we consider the lattice $\mathbb{L}$ with site set $\mathbb{Z}^2$, with the set of bonds $\mathcal{E} = \{(x, y); \|x-y\|_\infty = 1\}$, which adds diagonal bonds to the usual square lattice. We frequently identify bonds and path steps by map directions: either SW/NE or SE/NW for diagonal bonds, and N, NE, etc. for steps. By *axis directions* we mean horizontal and vertical, or N, E, W, S, depending on the context. For a preceding b in a path γ, we write $\gamma[a, b]$ for the segment of γ from a to b.

We assign all horizontal and vertical bonds passage time 1. Let $\frac{1}{2} < \theta < 1$ and $(2\theta)^{-1} < \eta < 1$. For diagonal bonds, for $k \geq 1$, a *highway of class k* consists of $2^k - 1$ consecutive bonds, all oriented SW/NE or all SE/NW. The collection of all highways of all classes is denoted $\mathbb{H}$, and a highway configuration, denoted ω, is an element of $\{0, 1\}^{\mathbb{H}}$. When a coordinate is 1 in ω we say the corresponding highway is *present* in ω. To obtain a random highway configuration, for each of the two orientations we let southernmost points of class-k highways occur at each $x \in \mathbb{Z}^2$ independently with probability $(\theta/2)^k$, for each $k \geq 1$. Every diagonal bond is a highway of class 0. For each diagonal bond e we have

$$\mathbf{P}(e \text{ is in a present class-}k \text{ highway}) = 1 - \left(1 - \frac{\theta^k}{2^k}\right)^{2^k - 1} \leq \theta^k, \quad k \geq 1, \tag{1}$$

so with probability one, e is in only finitely many present highways. Thus for diagonal e we can define $k(e) = \max\{k : e \text{ is in a present class-}k \text{ highway}\}$ if this set is nonempty, and $k(e) = 0$ otherwise, and then define its passage time

$$\tau_e = \begin{cases} \sqrt{2}(1 + \eta^{k(e)}) & \text{if } k(e) \geq 1; \\ 3 & \text{if } k(e) = 0. \end{cases} \tag{2}$$

For all horizontal and vertical bonds we define $\tau_e = 1$. Note that the value 3 ensures non-highway diagonal bonds never appear in geodesics.

Let A_1, A_2 denote the positive horizontal and vertical axes, respectively, each including 0. Let

$$r_k = \sum_{j=k}^{\infty} \theta^j = \frac{\theta^k}{1-\theta},$$

so that $\mathbf{P}(e \text{ is in some present highway of class} \geq k) \leq r_k$. We fix C and take k_0 large enough so

$$\eta^{k_0} < \frac{1}{32}, \quad r_{k_0} \leq \frac{1}{2}, \quad 0.1 \cdot 2^k(\eta^{k-1} - \eta^k) > \frac{4C}{r_k} \quad \text{for all } k \geq k_0, \tag{3}$$

the last being possible by our choice of η.

Proposition 1. *The stationary first passage percolation process defined as above has the following properties:*

(i) *The limit shape is an octagon, with corners on the axes and main diagonals.*
(ii) *The only infinite geodesics are vertical and horizontal lines.*

Proof. It is readily shown (see Section 3.4, for the modified example) that the asymptotic speed is 1 in each axis and diagonal direction, so the radius of B is 1 in those directions, and that B has a facet in each of the 8 sectors of angle $\pi/4$ between an axis and a diagonal. This proves item (i) and we focus on item (ii).

Step 1. Construction of a "success" event. For x in the first quadrant W, let $\Delta(x)$ denote the distance in the SW direction from x to $A_1 \cup A_2$. Let $\widehat{G}_k = \{x \in W : \Delta(x) = (2^{k-1}+1)\sqrt{2}\}$, which is a translate of $A_1 \cup A_2$. For $j > k$ define three random sets of highways:

$$\mathcal{G}_{k,j} = \{\text{all present SW/NE highways of class } j \text{ intersecting both } A_1 \cup A_2 \text{ and } \widehat{G}_k\},$$

$$\mathcal{G}'_{k,j} = \{\text{all present SW/NE highways of class } j \text{ crossing } A_1 \cup A_2 \text{ but not } \widehat{G}_k\},$$

$$\mathcal{G}''_{k,j} = \{\text{all present SW/NE highways of class } j \text{ crossing } \widehat{G}_k \text{ but not } A_1 \cup A_2\}.$$

Note that these three sets are independent of each other, and intersections with each along any given line have density at most θ^j. Intersections with $\mathcal{G}_{k,k}$ have density (over sites)

$$\mathbf{P}(0 \in \mathcal{G}_{k,k}) \geq 1 - \exp(-\theta^k/2) \geq \frac{\theta^k}{3}. \tag{4}$$

Here $0 \in \mathcal{G}_{k,k}$ is a shorthand notation for 0 being in a highway in $\mathcal{G}_{k,k}$. Let

$$\widehat{H}_{i,k} = \text{the highway intersecting } A_i \text{ closest to 0 among all in } \cup_{j \geq k} \mathcal{G}_{k,j}, \quad i = 1, 2,$$

and let $\widehat{U}_k = (\widehat{X}_{1,k}, 0)$ and $\widehat{V}_k = (0, \widehat{X}_{2,k})$ denote the corresponding intersection points in A_1 and A_2. Let $\widehat{\Omega}_k$ denote the *open* region bounded by $A_1 \cup A_2, \widehat{G}_k, \widehat{H}_{1,k}$, and $\widehat{H}_{2,k}$, see Figure 1.

We define the event $\widehat{F}_k = \widehat{I}_k \cap \widehat{M}_k$ (success at stage k) where

(i) $\widehat{I}_k : \max(\widehat{X}_{1,k}, \widehat{X}_{2,k}) \leq C/r_k$, with C from (3);
(ii) $\widehat{M}_k$: every SW/NE highway intersecting $\widehat{\Omega}_k$ is in classes $1, \ldots, k-1$.

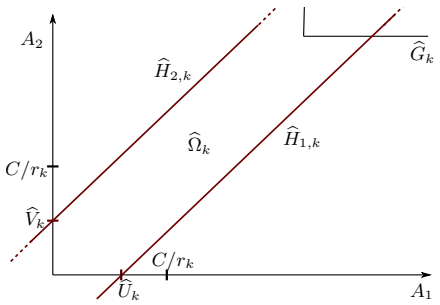

FIGURE 1. Representation of the region $\widehat{\Omega}_k$, which is enclosed by $A_1, A_2, \widehat{G}_k$ and $\widehat{H}_{1,k}, \widehat{H}_{2,k}$. Here, the event $\widehat{I}_k$ is realized.

Note that a highway intersecting $\widehat{\Omega}_k$ cannot intersect both $A_1 \cup A_2$ and $\widehat{G}_k$, by definition of $\widehat{H}_{i,k}$.

We claim there exists $\lambda > 0$ such that

$$\mathbf{P}(\widehat{F}_k) \geq \lambda \text{ for all } k \geq k_0. \tag{5}$$

Let us first prove that $\mathbf{P}(\widehat{I}_k)$ is bounded away from 0. Similarly to (4), we have that $\mathbf{P}\big(0 \in \bigcup_{j \geq k} \mathcal{G}_{k,j}\big) \geq r_k/3$, so for $i = 1, 2$, by independence,

$$\mathbf{P}(\widehat{X}_{i,k} > C/r_k) \leq \exp(-C/3) =: \zeta < 1, \quad \text{for all } k \geq k_0. \tag{6}$$

By independence, we get that $\mathbf{P}(\widehat{I}_k) \geq (1-\zeta)^2$.

We next prove that $\mathbf{P}(\widehat{M}_k \mid \widehat{I}_k)$ is bounded away from 0 for $k \geq k_0$. In fact, by the above-mentioned independence of the three sets of highways, by (1) and (3) we have

$$\mathbf{P}(\widehat{M}_k \mid \widehat{I}_k) \geq \min_{x_1, x_2 \leq C/r_k} \mathbf{P}(\widehat{M}_k \mid \widehat{X}_{1,k} = x_1, \widehat{X}_{2,k} = x_2) \geq \min_{x_1, x_2 \leq C/r_k} (1-r_k)^{x_1+x_2-1} \geq e^{-2C}.$$

This completes the proof of (5). A slight modification of this proof shows that for fixed ℓ, for k sufficiently large we have $\mathbf{P}(\widehat{F}_k \mid \sigma(\widehat{F}_1, \ldots, \widehat{F}_\ell)) > \lambda/2$, and it follows that $\mathbf{P}(\widehat{F}_k \ i.o.) = 1$.

Step 2. Properties of geodesics in case of a success. We now show that when $\widehat{F}_k$ occurs, for every $x \notin A_1 \cup A_2 \cup \widehat{\Omega}_k$ in the first quadrant, every geodesic $\widehat{\Gamma}_{0x}$ from 0 to x follows A_1 from 0 to $\widehat{U}_k$, or A_2 from 0 to $\widehat{V}_k$. Since x is in the first quadrant, it is easily seen that any geodesic from 0 to x has only N, NE and E steps. Let $\widehat{p}_x = (r, s)$ be the first site of $\widehat{\Gamma}_{0x}$ not in $A_1 \cup A_2 \cup \widehat{\Omega}_k$. Besides the geodesic $\widehat{\Gamma}_{0x}[0, \widehat{p}_x]$, we define an alternate path ψ_x from 0 to $\widehat{p}_x$ as follows; for this we assume $\widehat{p}_x$ is in the first quadrant on or below the main diagonal, and make the definition symmetric for $\widehat{p}_x$ elsewhere.

(i) If $\widehat{p}_x \in \widehat{H}_{1,k}$ we let ψ_x follow A_1 east from 0 to $\widehat{U}_k$, then NE from $\widehat{U}_k$ to $\widehat{p}_x$ on $\widehat{H}_{1,k}$.

(ii) If $\widehat{p}_x$ is in the horizontal part of $\widehat{G}_k$ we let $\widehat{U}'_k$ be the intersection of $\widehat{H}_{1,k}$ with the vertical line through $\widehat{p}_x$, and let ψ_x be the path east from 0 to $\widehat{U}_k$, then NE to $\widehat{U}'_k$, then north to $\widehat{p}_x$.

(iii) Otherwise $\widehat{p}_x$ is adjacent to A_1 and the final step of $\widehat{\Gamma}_{0x}[0, \widehat{p}_x]$ is from some $\widehat{U}''_k \in A_1$ to $\widehat{p}_x$, N or NE. We let ψ_x go east from 0 to $\widehat{U}''_k$, then take one step (N or NE) to $\widehat{p}_x$.

In case (i), the path ψ_x has no N steps, and it is easily seen that any path in $A_1 \cup A_2 \cup \Omega_k$ from 0 to $\widehat{p}_x$ containing some N steps will be strictly slower than ψ_x, and hence is not a geodesic. Thus every geodesic from 0 to $\widehat{p}_x$ has s NE and $r-s$ E steps. Since success occurs at stage k, any diagonal bonds in $\widehat{\Gamma}_{0x}[0,\widehat{p}_x]\backslash\widehat{H}_{1,k}$ have passage time strictly more than $\sqrt{2}(1+\eta^k)$, making $\widehat{\Gamma}_{0x}[0,\widehat{p}_x]$ strictly slower than ψ_x, a contradiction. It follows that we must have $\widehat{\Gamma}_{0x}[0,\widehat{p}_x] = \psi_x$, which means $\widehat{\Gamma}_{0x}$ indeed follows A_1 from 0 to $\widehat{U}_k$.

In case (ii), we have $s = 2^{k-1}+1$ and $0 \le r-s \le \widehat{X}_{1,k}$. Let q be the number of NE steps in $\widehat{\Gamma}_{0x}[0,\widehat{p}_x]$, so it must have $s-q$ N and $r-q$ E steps. Each of the diagonal bonds has passage time at least $\sqrt{2}(1+\eta^{k-1})$, so its passage time satisfies

$$T(\widehat{\Gamma}_{0x}[0,\widehat{p}_x]) \ge \sqrt{2}(1+\eta^{k-1})q + (s-q) + (r-q) \ge \sqrt{2}(1+\eta^{k-1})q + 2(s-q) \ge \sqrt{2}(1+\eta^{k-1})s.$$

By contrast, the northward segment of ψ_x has length $\widehat{X}_{1,k} - (r-s)$, so

$$\begin{aligned} T(\psi_x) &\le 2\widehat{X}_{1,k} - (r-s) + \sqrt{2}(1+\eta^k)(r-\widehat{X}_{1,k}) \\ &\le 2\widehat{X}_{1,k} + \sqrt{2}(1+\eta^k)s \\ &\le \sqrt{2}(1+\eta^k)s + \frac{2C}{r_k}\,. \end{aligned}$$

Hence by (3),

$$T(\widehat{\Gamma}_{0x}[0,\widehat{p}_x]) - T(\psi_x) \ge \sqrt{2}(\eta^{k-1}-\eta^k)s - \frac{2C}{r_k} = (2^{k-1}+1)\sqrt{2}(\eta^{k-1}-\eta^k) - \frac{2C}{r_k} > 0.$$

But this means $\widehat{\Gamma}_{0x}$ is not a geodesic, a contradiction. Thus we cannot have $\widehat{p}_x$ in the horizontal part of $\widehat{G}_k$ – and similarly not in the vertical part.

In case (iii), since the unique geodesic between any two points of A_1 is a segment of A_1, it is straightforward that we must have $\widehat{\Gamma}_{0x}[0,\widehat{p}_x] = \psi_x$. Again, $\widehat{\Gamma}_{0x}$ follows A_1 from 0 to $\widehat{U}_k$.

Step 3. Conclusion. If $\widehat{\Gamma}$ is any semi-infinite geodesic from 0 which has infinitely many points in the (closed) first quadrant, then for each of the infinitely many k for which $\widehat{F}_k$ occurs, the initial segment of $\widehat{\Gamma}$ must follow an axis from the origin to $\widehat{U}_k$ or $\widehat{V}_k$. But since $\widehat{X}_{1,k}, \widehat{X}_{2,k} \to \infty$, this means $\widehat{\Gamma}$ itself must be one of these axes. It follows that all horizontal and vertical lines are semi-infinite geodesics, and no other paths. □

3. Modification for square lattice, finite energy, and unique geodesics

The preceding simpler example does not satisfy A2 (iv) or (v), and it does not allow the use of results known only for the usual square lattice. To create an example on the square lattice, the main changes we have to make are as follows: replace diagonal highways with *zigzag highways* (alternating horizontal and vertical steps) as done in [5]; introduce horizontal/vertical highways; make highways of class k not have a fixed length; add auxiliary randomization. This adds significant complications. Primarily, since the graph is planar, there is sharing of bonds between, for example, horizontal and zigzag highways where they cross. Since the passage times are different in the two types of highways, each shared bond slows or speeds the total passage time along at least one of the highways, compared to what

it would be without the other highway. We must ensure that the number of such crossings is not a primary determinant of which paths are geodesics.

Once we have properly define a passage time configuration (the *fast diagonals* FPP), our strategy will be similar to that of the simple example of Section 2: we will define a "success" event, and show that when a success occurs geodesics approximately follow an axis for a long distance, at least until they reach a very fast zigzag highway.

3.1. **Definition of the *fast diagonals* FPP process.** We will need auxiliary randomization, so for each bond e we let ξ_e, ξ'_e be uniform in $[0,1]$, independent of each other and from bond to bond. We select parameters $\eta, \widetilde{\eta}, \theta, \widetilde{\theta}, \mu$ as follows: fix $c_\theta \in (0.4, 0.5)$. Then $c_\theta/(1-c_\theta) > 2/3$ and $2^{-c_\theta^2/(1-c_\theta)} < 7/8$ so we can choose $0 < 4\delta < c_{\widetilde{\theta}} < 1$ such that $c_\theta c_{\widetilde{\theta}}/(1 - c_\theta(c_{\widetilde{\theta}} - 4\delta)) > 2/3$ and $2^{-c_\theta^2 c_{\widetilde{\theta}}/(1-c_\theta(c_{\widetilde{\theta}}-4\delta))} < 7/8$. Then let

$$\theta = 2^{-c_\theta}, \quad \widetilde{\theta} = 2^{-c_{\widetilde{\theta}}}, \quad \theta^{c_\theta c_{\widetilde{\theta}}/(1-c_\theta(c_{\widetilde{\theta}}-4\delta))} < \eta < \min\left(\frac{7}{8}, \theta^{2/3}\right), \quad \widetilde{\eta} < \frac{1}{2\theta}, \tag{7}$$

and

$$\theta < \mu < \min(\widetilde{\theta}^{c_\theta}, \eta, (\theta\eta)^{1-c_\theta(c_{\widetilde{\theta}}-4\delta)}\theta^{-4c_\theta\delta}). \tag{8}$$

Note that since $c_\theta c_{\widetilde{\theta}} < 0.5$, the third condition in (7) guarantees $\eta > \theta$; since $c_\theta < 1/2$ this also means

$$2\eta\theta > 2\theta^2 > 1, \tag{9}$$

guaranteeing that (3) holds. Further, the first inequality in that third condition is equivalent to $\theta < (\theta\eta)^{1-c_\theta(c_{\widetilde{\theta}}-4\delta)}\theta^{-4c_\theta\delta}$. Together these show μ can be chosen to satisfy (8).

Highways and types of bonds. A *zigzag highway* is a set of (adjacent) bonds in any finite path which either (i) alternates between N and E steps, starting with either, called a SW/NE highway, or (ii) alternates between N and W steps, starting with either, called a SE/NW highway. If the first step in the path is N, we say the highway is *V-start*; if the first step is W or E we say it is *H-start*. A SW/NE highway is called *upper* if it is above the main diagonal, and *lower* if it is below, and analogously for SE/NW highways. Note a SW/NE highway does not have a direction toward SW or NE, it is only a set of bonds, and similarly for SE/NW. The *length* $|H|$ of a zigzag highway H is the number of bonds it contains, and class(H) denotes its class. To each zigzag highway H we associate a random variable W_H uniformly distributed in $[0,1]$ and independent from highway to highway.

For each $k \geq 1$ we construct a configuration $\omega^{(k)}$ of zigzag highways of class k: these highways can have any length $1, \dots, 2^{k+3}$. Specifically, for each length $j \leq 2^{k+3}$ and each $x \in \mathbb{Z}^2$, a SW-most endpoint of a present length-j H-start SW/NE highway of class k occurs at x with probability $\theta^k/2^{2k+4}$, independently over sites x, with the same for V-start, and similarly for SE/NW highways. Formally we can view $\omega^{(k)}$ as an element of $\{0,1\}^{\mathbb{H}_k}$, where $\mathbb{H}_k$ is the set of all possible class-k zigzag highways; when a coordinate is 1 in $\omega^{(k)}$ we say the corresponding highway is *present* in $\omega^{(k)}$. We write $\omega = (\omega^{(1)}, \omega^{(2)}, \dots)$ the configuration of zigzag highways of all classes.

A *horizontal highway* of length k is a collection of k consecutive horizontal bonds, and similarly for a *vertical highway*. Highways of both these types are called *HV highways*. For each $k \geq 1$ we construct a configuration $\widetilde{\omega}^{(k)}$ of HV highways of class k: these highways can

have any length $1, \dots, 2^k$, and for each length $j \le 2^k$ and each $x \in \mathbb{Z}^2$, a leftmost endpoint of a present length-j horizontal highway of class k occurs at x with probability $\widetilde{\theta}^k/2^{2k}$, independently over sites x, and similarly for vertical highways. We write $\widetilde{\omega} = (\widetilde{\omega}^{(1)}, \widetilde{\omega}^{(2)}, \dots)$ for the configuration of HV highways of all classes.

We now combine the classes of zigzag highways and "thin" them into a single configuration by deletions. We do this in two stages, first removing those which are too close to certain other zigzag highways, then those which are crossed by a sufficiently long HV highway.

Specifically, for *stage-1 deletions* we define a linear ordering (a *ranking*) of the SW/NE highways in ω, as follows. Highway H' ranks above highway H if one of the following holds: (i) $\mathrm{class}(H') > \mathrm{class}(H)$; (ii) $\mathrm{class}(H') = \mathrm{class}(H)$ and $|H'| > |H|$; (iii) $\mathrm{class}(H') = \mathrm{class}(H)$, $|H'| = |H|$, and $W_{H'} > W_H$. Let $d_1(A, B)$ denote the ℓ_1 distance between the sets A and B of sites or bonds. We then delete any SW/NE highway H from any $\omega^{(k)}$ if there exists another SW/NE highway H', with $d_1(H, H') \le 22$ which ranks higher than H. We then do the same for SE/NW highways. The configuration of highways that remain in some $\omega^{(k)}$ after stage-1 deletions is denoted $\omega^{\mathrm{zig,thin},1}$. Here the condition $d_1(H, H') \le 22$ is chosen to follow from $d_1(H, H') < 3/(1-\eta) - 1$; we have chosen $\eta < 7/8$ in (7) so the value 22 works.

For *stage-2 deletions*, we let $\zeta = \delta/(c_{\widetilde{\theta}} - \delta)$, and for each $m \ge 1$ we delete from $\omega^{\mathrm{zig,thin},1}$ each zigzag highway of class m which shares a bond with an HV highway of class $(1+\zeta)m/c_{\widetilde{\theta}}$ or more. The configuration of highways that remain in some $\omega^{(k)}$ after both stage-1 and stage-2 deletions is denoted $\omega^{\mathrm{zig,thin},2}$. For a given class-$m$ zigzag highway H in $\omega^{\mathrm{zig,thin},1}$, for each bond e of H there are at most 2^ℓ possible lengths and 2^ℓ possible endpoint locations for a class-ℓ HV highway containing e, so the probability H is deleted in stage 2 is at most

$$2^{m+3} \sum_{\ell \ge (1+\zeta)m/c_{\widetilde{\theta}}} 2^{2\ell} \cdot \frac{\widetilde{\theta}^\ell}{2^{2\ell}} = \frac{8}{1-\widetilde{\theta}}\, 2^{-\zeta m}. \tag{10}$$

Following stage-2 deletions we make one further modification, which we call *stage-3 trimming*. Suppose H, H' are zigzag highways of opposite orientation (SW/NE vs SE/NW) in $\omega^{\mathrm{zig,thin},2}$, and x is an endpoint of H. If $d_1(x, H') \le 1$, then we delete from H the 4 final bonds of H, ending at x. The resulting configuration is denoted $\omega^{\mathrm{zig,thin}}$. This ensures that for any SW/NE zigzag highway H and SE/NW zigzag highway H', either H and H' *fully cross* (meaning they intersect, and there are at least 2 bonds of each highway on either side of the intersection bond) or they satisfy $d_1(H, H') \ge 2$.

This construction create several types of bond, which will have different definitions for their passage times. A bond e which is in no highway in $\omega^{\mathrm{zig,thin}}$ but which has at least one endpoint in some highway in $\omega^{\mathrm{zig,thin}}$ is called a *boundary bond*. A bond e in any highway in $\omega^{\mathrm{zig,thin}}$ is called a *zigzag bond*. A *HV bond* is a bond in some HV highway in some $\widetilde{\omega}^{(k)}$. An *HV-only bond* is an HV bond which is not a zigzag bond. A *backroad bond* is a bond which is not a zigzag bond, HV-only bond, or boundary bond.

Moreover, there are special types of boundary and zigzag bonds that arise when a SW/NE zigzag highway crosses a SE/NW one, so we need the following definitions for bonds in $\omega^{\mathrm{zig,thin}}$. For zigzag bonds, we distinguish:

(i) The first and last bonds (or sites) of any zigzag highway are called *terminal bonds* (or *terminal sites*.) Bonds which are not the first or last bond of a specified path are called *interior bonds*.

(ii) An adjacent pair of zigzag bonds in the same direction (both N/S or both E/W, which are necessarily from different highways, one SW/NE and one SE/NW) is called a *meeting pair*, and each bond in the pair is a *meeting zigzag bond*.

(iii) A zigzag bond for which both endpoints are meeting-pair midpoints is called an *intersection zigzag bond*. Equivalently, when a SW/NE highway intersects a SE/NW one, the bond forming the intersection is an intersection zigzag bond.

(v) A zigzag bond which is not a meeting, intersection, or terminal zigzag bond is called a *normal zigzag bond*.

For boundary bounds, we distinguish the following:

(vi) A boundary bond is called a *semislow boundary bond* if either (a) it is adjacent to two meeting bonds (and is necessarily parallel to the intersection bond, separated by distance 1), called an *entry/exit bond*, or (b) it is adjacent to an intersection bond.

(vi) A boundary bond e is called a *skimming boundary bond* if one endpoint is the terminal site of a zigzag highway, and the corresponding terminal bond is perpendicular to e.

(vii) A boundary bond which is not a semislow or skimming boundary bond is called a *normal boundary bond*.

These special type of bonds are represented in Figure 2.

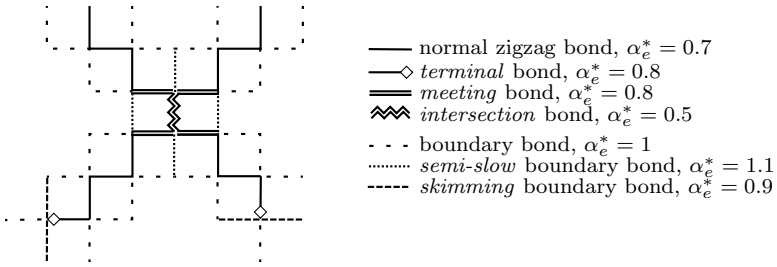

FIGURE 2. Different types of bonds near a crossing of two zigzag highways, and their compensated core passage times, see (13).

Definition of the passage times. For each edge e we associate its zigzag class

$$k(e) = \begin{cases} \max\{k : e \text{ is in a class-}k \text{ zigzag highway in } \omega^{\mathrm{zig,thin}}\} & \text{if this set is nonempty,} \\ 0, & \text{otherwise,} \end{cases}$$

and its HV class

$$\widetilde{k}(e) = \begin{cases} \max\{k : e \text{ is in a class-}k \text{ HV highway in } \widetilde{\omega}^{(k)}\} & \text{if this set is nonempty,} \\ 0, & \text{otherwise.} \end{cases}$$

We then define a bond e to be *slow* if

$$\xi'_e \le 4^{-k(e)\vee\widetilde{k}(e)}. \tag{11}$$

Slow bonds exist only to make clear that upward finite energy holds.

We next define the *raw core passage time* $\alpha_e = \alpha_e(\widetilde{\omega}, \omega^{\text{zig,thin}})$ of each bond e, mimicking ideas of the simple example of Section 2. For non-slow e we set

$$\alpha_e = \begin{cases} 0.7 & \text{if } e \text{ is a zigzag bond} \\ 0.9 & \text{if } e \text{ is an HV-only bond} \\ 1 & \text{if } e \text{ is a backroad bond or non-HV boundary bond,} \end{cases} \tag{12}$$

and for slow e, we set $\alpha_e = 1.2$. Then we define the *compensated core passage time* $\alpha_e^* = \alpha_e^*(\omega, \omega^{\text{zig,thin}})$ for non-slow e by

$$\alpha_e^* = \begin{cases} 0.5 & \text{if } e \text{ is an intersection zigzag bond} \\ 0.7 & \text{if } e \text{ is a normal zigzag bond} \\ 0.8 & \text{if } e \text{ is a meeting or terminal zigzag bond} \\ 0.9 & \text{if } e \text{ is an HV-only bond which is non-boundary or skimming boundary} \\ 1 & \text{if } e \text{ is a backroad bond or normal boundary bond} \\ 1.1 & \text{if } e \text{ is a semislow boundary bond,} \end{cases} \tag{13}$$

and for slow e, we set $\alpha_e^* = 1.2$. The term "compensated" refers mainly to the following: when an HV highway crosses a zigzag one, it typically intersects one normal zigzag bond and two normal boundary bonds. The sum of the raw core passage times for these 3 bonds is $0.9 + 0.7 + 0.9 = 2.5$, whereas in the absence of the zigzag highway the sum would be $3 \times 0.9 = 2.7$. With the compensated times, the sum is restored to 2.7, and in that sense the HV highway does not "feel" the zigzag highway. The compensation picture is more complicated when the HV highway crosses near the intersection of a SW/NE highway and a SE/NW highway (which necessarily fully cross.) It must be done so that (15) and (16) below hold, whether the HV highway contains intersection bonds, meeting bonds or entry/exit bonds, see Figure 2.

The idea in the definition of α_e^* is that we compensate for the "too fast" zigzag bonds in an HV highway (0.7 vs 0.9) by extracting a toll of 0.1 for entering or exiting a zigzag highway. If the entry/exit is through a terminal or meeting zigzag bond (as when passing through a meeting block), then the toll is paid by increasing the time of that bond from 0.7 to 0.8. In the meeting case, to avoid increasing the total time along the zigzag highway, the core passage time of the adjacent intersection bond is reduced to 0.5. If the entrance/exit for the zigzag highway is made through any other type of zigzag bond, the toll is paid by increasing the adjacent boundary bond in the path from 0.9 (if it's an HV bond) to 1. There is an exception in an entry/exit bond, which may be both entrance and exit: the toll for such a bond is 0.2.

Due to the stage-1 deletions, any two parallel zigzag highways H, H' in $\omega^{\text{zig,thin}}$ satisfy

$$d_1(H, H') \geq 23, \tag{14}$$

and as a result, no further type of compensation beyond (13) needs to be considered.

We thus have the following property: suppose H is an HV highway for which the first and last bonds are HV-only. Then

$$\sum_{e \in H} \alpha_e^* = 0.9|H|. \tag{15}$$

End effects may alter this for general HV highways, but it is easily checked that every HV highway H satisfies

$$\Big|\sum_{e\in H}\alpha_e^* - 0.9|H|\Big| \le 0.4. \tag{16}$$

We can now define the full passage times τ_e (based on the configuration $\omega^{\mathrm{zig,thin}}$) by

$$\tau_e = \alpha_e^* + \sigma_e, \tag{17}$$

where σ_e is defined to be $0.1\xi'_e$ if e is a slow bond, and for non-slow e

$$\sigma_e = 0.1 \times \begin{cases} \eta^{k(e)} + \widetilde{\eta}^{k(e)}\xi_e & \text{if } e \text{ is a zigzag bond,} \\ \widetilde{\eta}^{\widetilde{k}(e)} + \widetilde{\eta}^{\widetilde{k}(e)}\xi_e & \text{if } e \text{ is an HV-only bond,} \\ \widetilde{\eta}^{\widetilde{k}(e)}\xi_e & \text{if } e \text{ is a boundary bond which is not HV.} \end{cases} \tag{18}$$

We refer to the resulting stationary FPP process as the *fast diagonals FPP process*. We stress that the presence of the independent variables ξ_e ensures that Assumption A2(iv) is satisfied, and the presence of slow bonds ensures A2(v); the rest of A1 and A2 are straightforward.

First observations and notations. It is important that since $\eta > \widetilde{\eta}$, passage times along long zigzag highways are much more affected by the class of the highway than are times along HV highways. In fact, by increasing k_0 we may assume $(2\widetilde{\eta})^{k_0} < \eta^{k_0} < 1/16$. Then if H is an HV or zigzag highway of class k (so of length at most 2^{k+3}), we have

$$\sum_{e\in H}\sigma_e \le 0.1\sum_{e\in H}(\widetilde{\eta}^k + \widetilde{\eta}^k\xi_e) \le 1.6(2\widetilde{\eta})^k < 0.1, \tag{19}$$

so the maximum effect on $\sum_{e\in H}\tau_e$ of all the variables involving $\widetilde{\eta}$ is less than 0.1, which is less than the effect of the α_e^* value for any single bond e.

Henceforth we consider only $k \ge k_0$. Let $r_k = \sum_{j=k}^{\infty}\theta^j$ and $\widetilde{r}_k = \sum_{j=k}^{\infty}\widetilde{\theta}^j$. Define $q_k = \log_2(2C/r_k) = c_\theta k + b$ where b is a constant.

The following subsections prove Theorem 1. The strategy is similar to that of Section 2: we first construct in Section 3.2 an event F_k that a.s. occurs for a positive fraction of all k's, and then show in Section 3.3 that when F_k occurs geodesics have to stay near the axis.

3.2. **Construction of a "success" event.** Analogously to Section 2, we construct a random region Ω_k (which is an enlarged random version of $\widehat{\Omega}_k$), and a deterministic region Θ_k which may contain Ω_k, as follows.

As before we write A_1, A_2 for the positive horizontal and vertical axes, and now also A_3, A_4 for the negative horizontal and vertical axes, respectively. We write G_k^1 for the set $\{x \in W : \Delta(x) = 2^k\sqrt{2}\}$ (formerly denoted $\widehat{G}_k$) and $\widetilde{G}_k^1$ for $\{x \in W : \Delta(x) = (2^k+4)\sqrt{2}\}$; successively rotating the lattice by 90 degrees yields corresponding sets G_k^2, G_k^3, G_k^4 and $\widetilde{G}_k^2, \widetilde{G}_k^3, \widetilde{G}_k^4$ in the second, third and fourth quadrants, respectively. Let $H_{NE,L,k}$ and $H_{NE,U,k}$ be the lower and upper zigzag highways in $\omega^{\mathrm{zig,thin},2}$ of class k or more, intersecting both $\widetilde{G}_k^1$ and $\widetilde{G}_k^3$, which intersect A_1 and A_2, respectively, closest to 0. Let $U_k^{NE} = (X_{1,k}^{NE}, 0)$ be the leftmost point of $A_1 \cap H_{NE,L,k}$, and $V_k^{NE} = (0, X_{2,k}^{NE})$ the lowest point of $A_2 \cap H_{NE,U,k}$. Rotating the lattice 90 degrees yields analogous highways $H_{NW,L,k}$ and $H_{NW,U,k}$ each intersecting $\widetilde{G}_k^2$ and $\widetilde{G}_k^4$,

and intersections points $U_k^{NW} = (X_{1,k}^{NW}, 0)$ and $V_k^{NW} = (0, X_{2,k}^{NW})$ with axes A_3 and A_2, respectively. Here we have used $\widetilde{G}_k^i$ and not G_k^i so that stage-3 trimming does not prevent $H_{*,\cdot,k}$ from reaching appropriate G_k^i.

Let $\ell_{1,k}$ and $\ell_{3,k}$ be the vertical lines $\{\pm 2C/r_k\} \times \mathbb{R}$ crossing A_1 and A_3 respectively, and $\ell_{2,k}$ and $\ell_{4,k}$ the horizontal lines $\mathbb{R} \times \{\pm 2C/r_k\}$ crossing A_2 and A_4. Let $J_{N,k}$ (and $J_{S,k}$, respectively) denote the lowest (and highest) horizontal highway above (and below) the horizontal axis intersecting both $\ell_{1,k}$ and $\ell_{3,k}$. Analogously, let $J_{E,k}$ (and $J_{W,k}$) be the leftmost (rightmost) vertical highway to the right (left) of the vertical axis intersecting both $\ell_{2,k}$ and $\ell_{4,k}$. Let $(Y_{E,k}, 0)$ be the intersection of $J_{E,k}$ with A_1, and analogously for $(0, Y_{N,k}), (Y_{W,k}, 0)$ and $(0, Y_{S,k})$ in A_2, A_3, and A_4.

Let Ω_k^{NE} be the open region bounded by $H_{NE,L,k}, H_{NE,U,k}, G_k^1$, and G_k^3, and let Ω_k^{NW} be the open region bounded by $H_{NW,L,k}, H_{NW,U,k}, G_k^2$, and G_k^4. Then let $\Omega_k = \Omega_k^{NW} \cup \Omega_k^{NE}$ (an X-shaped region), see Figure 3 below.

Let $h_{NE,L,k}$ and $h_{NE,U,k}$ denote the SW/NE diagonal lines through $(C/r_k, 0)$ and $(0, C/r_k)$ respectively, and let $h_{NW,L,k}$ and let $h_{NW,U,k}$ denote the SE/NW diagonal lines through $(-C/r_k, 0)$ and $(0, C/r_k)$, respectively. Let Θ_k^{NE} denote the closed region bounded by $h_{NE,L,k}, h_{NE,U,k}, G_k^1$ and G_k^3, and Θ_k^{NW} the closed region bounded by $h_{NW,L,k}, h_{NW,U,k}, G_k^2$ and G_k^4. Then let $\Theta_k = \Theta_k^{NW} \cup \Theta_k^{NE}$. In the event of interest to us, the nonrandom region Θ_k will contain the random region Ω_k.

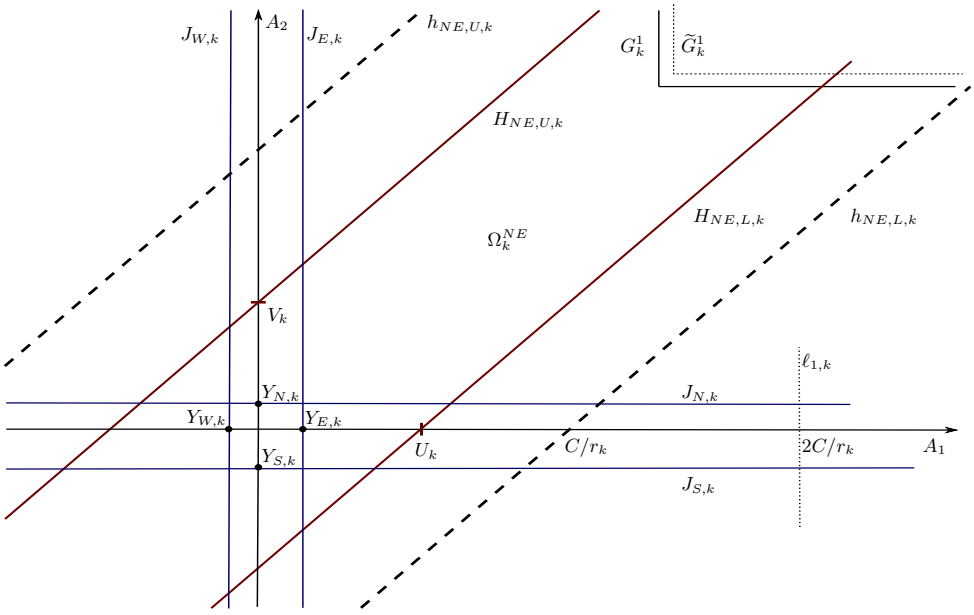

FIGURE 3. Representation of the different regions of interest (the picture focuses on the upper-right quadrant, and the scales are not respected): Ω_k^{NE} is the region enclosed by $H_{NE,U,k}, H_{NE,L,k}$ and G_k^i $(i = 1, 2)$; Θ_k^{NE} is the region enclosed by $h_{NE,U,k}, h_{NE,L,k}$ and G_k^i $(i = 1, 2)$. The construction of Ω_k^{NW} and Θ_k^{NW} is symmetric by rotation of 90°.

We now construct an event F_k, which we will show occurs for a positive fraction of all k, a.s. We show in Section 3.3 that F_k ensures that all finite geodesics from 0 to points outside

Ω_k must stay near one of the axes until leaving $\Omega_k^{NW} \cap \Omega_k^{NE}$. For $m > c_{\widetilde{\theta}} q_k/(1+\zeta)$ let $R_{k,m}^*$ be the number of class-m zigzag highways in $\omega^{(m)}$ intersecting $J_{*,k}$ in Θ_k, for $* =$ N, E, S, W. Here ζ is from the definition of stage-2 deletions. Note that since the class of $J_{*,k}$ is at least q_k, any intersecting highways of class $m \le c_{\widetilde{\theta}} q_k/(1+\zeta)$ are removed in stage-2 deletions, so $R_{k,m}^*$ counts those which might remain (depending on the class of $J_{*,k}$).

Fix $c \in (0, C)$ and define the events

$$I_k^* := \Big\{ \frac{c}{r_k} \le X_{i,k}^* \le \frac{C}{r_k} \text{ for } i = 1, 2 \Big\}, \text{ for } * = \text{NE, NW}, \qquad I_k = I_k^{NE} \cap I_k^{NW},$$

$$M_k^{NE} := \begin{Bmatrix} \text{every SW/NE highway } H \notin \{H_{NE,L,k}, H_{NE,U,k}\} \\ \text{in } \omega \text{ intersecting } \Theta_k^{NE} \text{ is in classes } 1, \dots, k-1 \end{Bmatrix}, \quad \text{and analogously for } M_k^{NW},$$

and let $M_k = M_k^{NE} \cap M_k^{NW}$. Define also

$$\widetilde{D}_k : |Y_{*,k}| \le \frac{C}{\widetilde{r}_{q_k}} \text{ for } * = E, N, W, S.$$

$$E_{1,k} : \sum_{m > c_{\widetilde{\theta}} q_k/(1+\zeta)} R_{k,m}^* \eta^m \le c_1 (\eta^3 \theta^{-2})^{c_\theta \delta k} (\mu^{-1}\eta)^k, \text{ for } * = E, N, W, S.$$

with c_1 to be specified and δ from (7), noting that by the bound on η in (7) we have $\eta^3\theta^{-2} < 1$, and

$$E_{2,k} : \text{there are no slow bonds in any } J_{*,k} \ (* = \text{N, E, S, W}) \\ \text{nor in any } H_{*,\cdot,k} \ (* = \text{NE, NW}, \cdot = \text{U, L}).$$

Finally, we set $F_k = I_k \cap M_k \cap \widetilde{D}_k \cap E_{1,k} \cap E_{2,k}$. Note that when F_k occurs we have $\Omega_k \subset \Theta_k$.

Lemma 1. *There exists some κ_1 such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{F_k} \ge \kappa_1 \quad \text{a.s.} \tag{20}$$

Moreover, letting $n_1(\omega) < n_2(\omega) < \cdots$ be the indices for which $\omega \in F_k$, we have

$$\limsup_{j\to\infty} \frac{n_{j+1}}{n_j} = 1 \quad \text{a.s.} \tag{21}$$

Proof. In the events I_k and M_k, the highways $H_{*,\cdot,k}$ and associated values $X_{i,k}^*$ are taken from the configuration $\omega^{\text{zig,thin},2}$. Using instead the configuration $\omega^{\text{zig,thin},1}$ yields different events, which we denote I_k^1 and M_k^1 respectively. Let $F_k^1 = I_k^1 \cap M_k^1 \cap \widetilde{D}_k$ (noting we do not intersect with $E_{1,k}, E_{2,k}$ here). First, we prove the following.

Claim 1. *There exists $\kappa_1 > 0$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{F_k^1} \ge \kappa_1 \quad \text{a.s.}$$

Proof of Claim 1. As a first step, we show that

$$\inf_{k\geq k_0} \mathbf{P}(I_k^1 \cap M_k^1) > 0. \tag{22}$$

For a lower SW/NE zigzag highway H, let $(x_0(H), 0)$ be the intersection point in A_1 closest to $(0,0)$ when one exists, and similarly for an upper SW/NE zigzag highway H let $(0, y_0(H))$ be the intersection point in A_2 closest to $(0,0)$. We call H *k-connecting* if H intersects both G_k^1 and G_k^3. The event $I_k^1 \cap M_k^1$ contains the event

A_k : there exists exactly one lower SW/NE highway H in ω of class k or more intersecting Θ_k^{NE}, and this H is k-connecting and satsifies $\frac{c}{r_k} \leq x_0(H) \leq \frac{C}{r_k} - 23$; further, the analogous statement holds for upper SW/NE highways with $y_0(\cdot)$ in place of $x_0(\cdot)$, and for lower and upper SE/NW highways.

The parts of A_k for the 4 types of zigzag highways (upper vs lower, SW/NE vs SE/NW) are independent, so to bound the probability of A_k we can consider just one of these parts and take the 4th power of the corresponding probability. In particular, considering lower SW/NE highways H of class $\ell \geq k$ intersecting Θ_k^{NE}, there are at most $2^{\ell+3}$ possible lengths for H, at most $2^{\ell+3}C/r_k$ possible SW-most points, and the choice of H-start or V-start, so at most $2^{2\ell+7}C/r_k$ possible highways. Also, considering lower SW/NE highways H of class $m \geq k$ which are k-connecting and satisfy $\frac{c}{r_k} \leq x_0(H) \leq \frac{C}{r_k} - 23$, there are at least $2^{m+1}(C-c)/r_k$ possible lower endpoints, and 2^{m+1} possible lengths, for H. Therefore considering only lower highways, assuming $c < C/2$ we have

$$\begin{aligned}
\mathbf{P}(I_k^1 \cap M_k^1)^{1/4} \geq \mathbf{P}(A_k)^{1/4} &\geq \sum_{m\geq k} \frac{2^{2m+2}(C-c)}{r_k} \, \frac{\theta^m}{2^{2m+4}} \prod_{\ell \geq k} \left(1 - \frac{\theta^\ell}{2^{2\ell+4}}\right)^{2^{2\ell+7}C/r_k} \\
&\geq \left(\sum_{m\geq k} \frac{C\theta^m}{8r_k}\right) \exp\left(-\sum_{\ell \geq k} \frac{16C\theta^\ell}{r_k}\right) \\
&= \frac{C}{8} e^{-16C}, \qquad (23)
\end{aligned}$$

proving (22).

A similar but simpler proof also using a count of highways yields that $\inf_{k\geq k_0} \mathbf{P}(\widetilde{D}_k) > 0$, so by independence we have

$$\inf_{k\geq k_0} \mathbf{P}(F_k^1) \geq \kappa_0 > 0 \tag{24}$$

for some κ_0. But Claim 1 is a stronger statement, and we now complete its proof, using (24).

Let $\mathcal{H}_k$ denote the set of all SW/NE highways in ω which intersect Θ_k^{NE}, together with all SE/NW highways which intersect Θ_k^{NW}. Let $\mathcal{F}_k$ denote the σ-field generated by $\mathcal{H}_k$, and let $\mathcal{C}_k$ denote the largest j such that $\mathcal{H}_k$ contains a highway of class j. For $m \geq k$, class-m SW/NE highways intersecting Θ_k^{NE} have at most $(2C/r_k)(2^{m+3} + 2^{k+3})$ possible SW-most endpoints, and 2^{m+3} possible lengths, and 2 directions for the initial step (V- or H-start), so the number of such highways is bounded by a sum of Bernoulli random variables, each with

parameter (success probability) less than $1/2$ and with total mean at most

$$2^{m+4}\frac{2C}{r_k}(2^{m+3}+2^{k+3})\frac{\theta^m}{2^{2m+4}} \le \frac{32C\theta^m}{r_k}.$$

The same is true for SE/NW highways. Hence for $n \ge 0$ the number of highways of class at least $k+n$ is bounded by a similar sum of Bernoulli variables of total mean at most $32Cr_{k+n}/r_k$. It follows that for any $n \ge 0$,

$$\begin{aligned}\mathbf{P}(\mathcal{C}_k - k \ge n) &\le \mathbf{P}(\text{some highway in } \omega \text{ of class } k+n \text{ or more intersects } \Theta_k)\\ &\le \frac{32Cr_{k+n}}{r_k} = 32C\theta^n. \qquad (25)\end{aligned}$$

Let n_0 be the least integer with $\theta^{n_0} < c/C$. Define a random sequence of indices $1 = K_1 < K_2 < \dots$ inductively as follows: having defined K_i, let $K_{i+1} := \max\{\mathcal{C}_{K_i}, (K_i + n_0)\} + 1$. Here $k \ge K_i + n_0$ ensures $C/r_{K_i} < c/r_k$, and $k > \mathcal{C}_{K_i}$ ensures that $H_{NE,L,k}, H_{NE,U,k}$ do not intersect $\Theta^{NE}_{K_i}$, and likewise for NW in place of NE. For any j, ℓ, and any $A \in \mathcal{F}_j$, the event $A \cap \{K_i = j, \mathcal{C}_j = \ell\}$ only conditions zigzag highways in $\mathcal{H}_j$, and ensures that no SW/NE (or SE/NW) highways of class $\ge k = \max(j + n_0, \ell) + 1$ intersect Θ^{NE}_j (or Θ^{NW}_j, respectively); in particular it ensures that $\widehat{X}^*_{i,k} > C/r_j$ for $* =$ NE, NW. Therefore since $C/r_j < c/r_k$ this event increases the probability of $I^1_k \cap M^1_k$: for such j, ℓ, k, A,

$$\mathbf{P}\big(I^1_k \cap M^1_k \mid A \cap \{K_i = j, \mathcal{C}_j = \ell\}\big) \ge \mathbf{P}(I^1_k \cap M^1_k).$$

Similarly the bound in (25) is still valid conditionally: for such j, ℓ, k,

$$\mathbf{P}\big(\mathcal{C}_k - k \ge n \mid A \cap \{K_i = j, \mathcal{C}_j = \ell\}\big) \le 32C\theta^n \quad \text{for all } n \ge 0. \qquad (26)$$

It follows that for some $c_2 > 0$,

$$\limsup_{i\to\infty} \frac{K_i}{i} \le c_2 \quad \text{a.s.} \qquad (27)$$

On the other hand, it is straightforward that for some $\kappa_2 > 0$, for all $j < k$ and all $B \in \sigma(\widetilde{D}_1, \dots, \widetilde{D}_j)$,

$$\mathbf{P}(\widetilde{D}_k \mid B) \ge \kappa_2 \mathbf{P}(\widetilde{D}_k) \quad \text{a.s.},$$

so by independence and (24),

$$\mathbf{P}\big(F^1_k \mid A \cap B \cap \{K_i = j, \mathcal{C}_j = \ell\}\big) \ge \kappa_2 \mathbf{P}(F^1_k) \ge \kappa_2\kappa_0. \qquad (28)$$

Since A, B are arbitrary, it follows that the variables $1_{F_{K_i}}$ dominate an independent Bernoulli sequence with parameter $\kappa_2\kappa_0$, so

$$\liminf_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} 1_{F^1_{K_i}} \ge \kappa_2\kappa_0 \quad \text{a.s.}$$

This, combined with (27), proves Claim 1 with $\kappa_1 = \kappa_2\kappa_0/c_2$. □

Note that by (26), the variables $K_{i+1} - K_i, i \ge 1$ are dominated by an i.i.d. sequence of the form "constant plus geometric random variable."

Let us go back to the proof of Lemma 1. Let B_k denote the event that none of the 4 highways $H_{*,\cdot,k}(\omega^{\text{zig,thin},1})$ are deleted in stage-2 deletions, then we have

$$I_k^1 \cap M_k^1 \cap B_k \subset I_k \cap M_k. \tag{29}$$

Claim 2. *We have*

$$\sum_k \mathbf{P}(B_k^c \cap I_k^1) < \infty \ ; \quad \sum_k \mathbf{P}(E_{2,k}^c \cap \widetilde{D}_k) < \infty \quad \text{and} \quad \sum_k \mathbf{P}(E_{1,k}^c) < \infty.$$

Proof of Claim 2. Let $Z^H_{*,\cdot,k}$ denote the class of the zigzag highway $H_{*,\cdot,k}(\omega^{\text{zig,thin},1})$, for $*$ = NE, NW and $\cdot$ = U, L, and let $Z^J_{*,k}$ denote the class of the HV highway $J_{*,k}$ for $*$ = N, E, W, S. Then, by (10), for $n \geq 0$

$$\mathbf{P}\big(B_k^c \mid I_k^1 \cap \{Z^H_{NE,L,k} = k+n\}\big) \leq \frac{8}{1-\widetilde{\theta}}\, 2^{-\zeta(k+n)} \leq \frac{8}{1-\widetilde{\theta}}\, 2^{-\zeta k}\,. \tag{30}$$

Since the upper bound is independent of n, summing over n we get that $\mathbf{P}(B_k^c \cap I_k^1) \leq 8(1-\widetilde{\theta})^{-1}2^{-\zeta k}$, and the first item of Claim 2 is proven.

Similarly,

$$\mathbf{P}\big(E_{2,k}^c \mid \widetilde{D}_k \cap \{Z^J_{N,k} = q_k + n\}\big) \leq 2^{q_k+n+3}4^{-(q_k+n)} \leq 8 \cdot 2^{-q_k}.$$

so we get that $\mathbf{P}\big(E_{2,k}^c \cap \widetilde{D}_k\big) \leq 2^{-q_k+3}$, and the second item of Claim 2 is proven.

For the last item, recall $c_{\widetilde{\theta}}/(1+\zeta) = c_{\widetilde{\theta}} - \delta$. We have from the upper bound for μ in (8)

$$\begin{aligned}\sum_{m > c_{\widetilde{\theta}} q_k/(1+\zeta)} \theta^{m-k-c_\theta \delta m}\eta^m &\leq c_1(\theta^{1-c_\theta\delta}\eta)^{(c_\theta(c_{\widetilde{\theta}}-\delta)-1)k}\mu^k\theta^{-c_\theta\delta k}(\mu^{-1}\eta)^k \\ &\leq c_1(\eta^3\theta^{-2})^{c_\theta\delta k}(\mu^{-1}\eta)^k, \end{aligned} \tag{31}$$

so

$$\mathbf{P}(E_{1,k}^c) \leq 4 \sum_{m > c_{\widetilde{\theta}} q_k/(1+\zeta)} \mathbf{P}\left(R^N_{k,m} > \theta^{m-k-c_\theta\delta m}\right).$$

Analogously to (25) we have that $\mathbf{E}(R^N_{k,m}) \leq c_3\theta^m/r_k$ and by Markov's inequality,

$$\mathbf{P}(E_{1,k}^c) \leq c_4 \sum_{m > c_{\widetilde{\theta}} q_k/(1+\zeta)} \theta^{c_\theta\delta m} \leq c_5\theta^{c_6 k}.$$

This proves the last item of Claim 2. □

Equation (20) in Lemma 1 now follows from (29), Claims 1 and 2, and the Borel-Cantelli lemma. Equation (21) comes additionally from the remark made at the end of the proof of Claim 1. □

3.3. **Properties of geodesics in case of a success.** For $x \notin \Omega_k$, we let Γ_{0x} be the geodesic from 0 to x (unique since the ξ_e are continuous random variables.) For $p, q \in \Gamma_{0x}$ we denote by $\Gamma_{0x}[p,q]$ the segment of Γ_{0x} from p to q. We let p_x be the first point of Γ_{0x} outside Ω_k. We then define t_x in the boundary of the "near-rectangle" $\Omega_k^{NE} \cap \Omega_k^{NW}$ as below.

Note that this boundary consists of 4 zigzag segments, one from each highway $H_{*,\cdot,k}$ ($*$ = NE, NW; $\cdot$ = U, L). Some boundary points (one bond or site at each "corner") are contained in 2 such segments; we call these *double points*. Removing all double points leaves 4 connected

components of the boundary, which we call *disjoint sides* of $\Omega_k^{NE} \cap \Omega_k^{NW}$, each contained in a unique highway $H_{*,\cdot,k}$. The set $\Omega_k \backslash (\Omega_k^{NE} \cap \Omega_k^{NW})$ has 4 connected components, which we call *arms*, extending from $\Omega_k^{NE} \cap \Omega_k^{NW}$ in the directions NW, NE, SE, SW. Each arm includes one disjoint side of $\Omega_k^{NE} \cap \Omega_k^{NW}$.

If p_x is not a double point then it is contained in the boundary of one of the arms, and we let t_x be the first point of $\Gamma_{0x}[0, p_x]$ in that arm (necessarily in a disjoint side; see Case 3 of Figure 4.) If instead p_x is a double point, then we pick arbitrarily one of the two highways $H_{*,\cdot,k}$ containing it, and let t_x be the first site of $\Gamma_{0x}[0, p_x]$ in that highway.

For μ from (8), let $L_{N,k}$ denote the horizontal line $\mathbb{R} \times \{\mu^{-k}\}$; $L_{*,k}$ is defined analogously for $* = \mathrm{E}, \mathrm{S}, \mathrm{W}$. We define $\Lambda_{H,k}$ to be the "horizontal axis corridor," meaning the closure of the portion of $\Omega_k^{NE} \cap \Omega_k^{NW}$ strictly between $L_{S,k}$ and $L_{N,k}$, and let $\Lambda_{V,k}$ denote the similar "vertical axis corridor."

Lemma 2. *For sufficiently large k, when F_k occurs, for all $x \notin \Omega_k$ we have either $\Gamma_{0x}[0, t_x] \subset \Lambda_{H,k}$ or $\Gamma_{0x}[0, t_x] \subset \Lambda_{V,k}$.*

From this lemma, we can conclude the proof of Theorem 1(ii). For a configuration ω let $n_1(\omega) < n_2(\omega) < \dots$ be the indices k for which $\omega \in F_k$. Let Γ_0 be an infinite geodesic starting from the origin, with sites $0 = x_0, x_1, \dots$. Then Lemma 2 says that for each $j \geq 1$, Γ_0 is contained in either Λ_{H,n_j} or Λ_{V,n_j} until it leaves $\Omega_{n_j}^{NE} \cap \Omega_{n_j}^{NW}$; accordingly, we say Γ_0 is *horizontal at stage j* or *vertical at stage j*.

Suppose Γ_0 is horizontal at stage j, and vertical at stage $j+1$. The horizontal coordinate of the first point of Γ_0 outside $\Omega_{n_j}^{NE} \cap \Omega_{n_j}^{NW}$ then has magnitude at least $c/r_{n_j} - \mu^{-n_j}$ but (from the definition of $\widetilde{D}_k$) at most $C/\widetilde{r}_{q_{n_j+1}}$. Since $\theta < \mu$ by (8), this means that for large j we have $c/2r_{n_j} \leq C/\widetilde{r}_{q_{n_j+1}}$, and hence $\widetilde{r}_{q_{n_j+1}}/r_{n_j} \leq 2C/c$. We fix $\epsilon > 0$ be small enough so $\widetilde{\theta}^{(1+\epsilon)c_\theta} > \theta$, which is possible by (8). Then for j large enough, thanks to (21), we get that $q_{n_{j+1}} \leq (1+\epsilon)c_\theta n_j$. Hence we have

$$\Big(\frac{\widetilde{\theta}^{(1+\epsilon)c_\theta}}{\theta}\Big)^{n_j} \leq \frac{\widetilde{r}_{q_{n_j+1}}}{r_{n_j}} \leq \frac{2C}{c}.$$

But this can only be true for finitely many j, so there exists a random J_0 such that for $j \geq J_0$, either Γ_0 is horizontal at stage j for all $j \geq J_0$, or Γ_0 is vertical at stage j for all $j \geq J_0$. (We call Γ_0 *horizontal* or *vertical*, accordingly.) Using again that $n_{j+1}/n_j \to 1$, we get that

$$\mu^{-n_{j+1}} \ll \frac{c}{r_{n_j}} \quad \text{as } j \to \infty,$$

that is, the width of $\Lambda_{*,n_{j+1}}$ is much less than the length of Λ_{*,n_j}, for $* = \mathrm{H}, \mathrm{V}$. This guarantees that for such Γ_0, the angle to x_i from an axis approaches 0, that is, Γ_0 is directed in an axis direction, proving Theorem 1(ii). □

Remark 1. The above reasoning gives that geodesics reaching horizontal distance $n = c/r_k = c'\theta^{-k}$ deviate from the horizontal axis by at most μ^{-k}. In the other direction, heuristically, in order to reach horizontal distance $n = c/r_k$, the most efficient way should involve a route going as soon as possible (through a succession of horizontal and zigzag highways) to the closest horizontal highway that reaches at least distance c/r_k, and then

following that highway. This suggests that in order to reach distance $n = c/r_k$, geodesics have a transversal fluctuation of order at least $1/\widetilde{r}_{q_k} = c\widetilde{\theta}^{-c_\theta k} = c\theta^{-c_{\widetilde{\theta}}k}$, or equivalently order $n^{c_{\widetilde{\theta}}}$, as this is the typical vertical distance to the closest horizontal highway reaching n.

Suppose $c_{\widetilde{\theta}} > 10/11$. Then choosing c_θ slightly less than 0.5, then η slightly less than $2/3$, and then δ sufficiently small, we satisfy (7), (8), and the conditions preceding them, and further, $\widetilde{\theta}^{c_\theta}$ is the smallest of the 3 quantities on the right side of (8). This means we can choose μ arbitrarily close to $\widetilde{\theta}^{c_\theta} = \theta^{c_{\widetilde{\theta}}}$. Thus our upper bound of μ^{-k} becomes $n^{c_{\widetilde{\theta}}+o(1)}$, nearly matching the heuristic lower bound. Hence in this case we expect geodesics reaching horizontal distance n to have transversal fluctuations of order $n^{c_{\widetilde{\theta}}+o(1)}$, with at least all values $c_{\widetilde{\theta}} \in (10/11, 1)$ being possible.

Proof of Lemma 2. Let us start with a claim analogous to what we proved in Section 2. Recall the definitions of Γ_{0x} and p_x.

Claim 3. *When F_k occurs, for all $x \notin \Omega_k$ we have $p_x \notin G_k^1 \cup G_k^2 \cup G_k^3 \cup G_k^4$.*

Proof of Claim 3. Suppose $p_x = (r, s)$ is in the horizontal part of G_k^1 (so $r \geq s = 2^k$) and let U_k' be the upper endpoint of the bond which is the intersection of $H_{NE,L,k}$ and the vertical line through p_x. Define the alternate path π_x from 0 east to U_k^{NE}, then NE along $H_{NE,L,k}$ to U_k', then north to p_x. We now compare the passage times of $\Gamma_{0x}[0, p_x]$ versus π_x.

We divide the bonds of the lattice into NW/SE *diagonal rows*: the jth diagonal row R_j consists of those bonds with one endpoint in $\{(x_1, x_2) : x_1 + x_2 = j - 1\}$ and the other in $\{(x_1, x_2) : x_1 + x_2 = j\}$. We call a bond $e \in \Gamma_{0x}[0, p_x]$ a *first bond* if for some $j \geq 1$, e is the first bond of $\Gamma_{0x}[0, p_x]$ in R_j, and we let N be the number of first bonds in $\Gamma_{0x}[0, p_x]$ which are in SE/NW highways. There are at least $2s$ first bonds, and any first bond e not in any SE/NW highway satisfies $\tau_e \geq 0.7 + 0.1\eta^{k-1}$, since by the definitions of M_k and F_k we have $k(e) \leq k - 1$. From (14), letting $q = |\Gamma_{0x}[0, p_x]| \geq 2s$, at most $q/24$ SE/NW highways intersect $\Gamma_{0x}[0, p_x]$. Since no two first bonds can be in the same SE/NW highway, we thus have $N \leq q/24$. Therefore

$$\begin{aligned} T(\Gamma_{0x}[0, p_x]) &\geq (2s - N)(0.7 + 0.1\eta^{k-1}) + (q - 2s + N)0.7 \\ &= 2s(0.7 + 0.1\eta^k) + 2s(0.1\eta^{k-1} - 0.1\eta^k) + (q - 2s)0.7 - 0.1N\eta^{k-1} \\ &\geq 2s(0.7 + 0.1\eta^k) + 2s(0.1\eta^{k-1} - 0.1\eta^k) + (q - 2s)0.7 - \frac{q}{240}\eta^{k-1}. \end{aligned} \tag{32}$$

If $q \geq 3s$ then $q(0.7 - \eta^{k-1}/240) \geq 1.4s$ so from (32) we get

$$T(\Gamma_{0x}[0, p_x]) \geq 2s(0.7 + 0.1\eta^k) + 2s(0.1\eta^{k-1} - 0.1\eta^k). \tag{33}$$

If instead $q < 3s$ then since $1 - \eta \geq 1/8$ we have $s(0.1\eta^{k-1} - 0.1\eta^k) \geq s\eta^{k-1}/80 \geq q\eta^{k-1}/240$, so

$$T(\Gamma_{0x}[0, p_x]) \geq 2s(0.7 + 0.1\eta^k) + s(0.1\eta^{k-1} - 0.1\eta^k). \tag{34}$$

By contrast, the NE segment of π_x has length $2(r - X^{NE}_{1,k})$ and the N segment has length $X^{NE}_{1,k} - (r-s)$, so we have for large k

$$\begin{aligned} T(\pi_x) &\le 1.2X^{NE}_{1,k} + 1.2(X^{NE}_{1,k} - (r-s)) + 2(0.7 + 0.1\eta^k + 0.1\widetilde{\eta}^k)(r - X^{NE}_{1,k}) \\ &\le 2.4X^{NE}_{1,k} + 2s(0.7 + 0.1\eta^k + 0.1\widetilde{\eta}^k) \\ &\le 2s(0.7 + 0.1\eta^k) + \frac{4C}{r_k}, \end{aligned}$$

where we used that $X^{NE}_{1,k} \le C/r_k$, and that $\widetilde{\eta}^k s = (2\widetilde{\eta})^k \le C/r_k$, using (7). Therefore by (9) we get

$$T(\Gamma_{0x}[0,p_x]) - T(\pi_x) \ge 0.1 \cdot 2^k(\eta^{k-1} - \eta^k) - \frac{4C}{r_k} > 0.$$

This contradicts $\Gamma_{0x}[0,p_x]$ being a geodesic, so p_x cannot be in the horizontal part of G^1_k. All other cases are symmetric, so Claim 3 is proved. □

We need to further restrict the location of $\Gamma_{0x}[0,p_x]$. When F_k occurs, we begin by dividing the sites of each $H_{*,\cdot,k}$ (with $* =$ NW, NE and $\cdot =$ L, U) into *accessible* and *inaccessible* sites In $H_{NE,L,k}$ we define as *inaccessible* the sites strictly between its intersection with $J_{N,k}$ and its intersection with $J_{W,k}$; the rest of the sites are *accessible*. Lattice symmetry yields the definition of accessible in the other 3 zigzag paths.

This definition enables us to define canonical paths to reach accessible points, which we need below. Given a site $a \in J_{N,k} \cap \Omega^{NE}_k$ and an accessible site $b \in H_{NE,L,k}$ in the first quadrant, there is a *canonical path* from a to b which follows $J_{N,k}$ from a to $H_{NE,L,k}$, then (changing direction 45 degrees) follows $H_{NE,L,k}$ to b. Similarly we can define canonical paths from sites $a \in J_{S,k} \cap \Omega^{NW}_k$ to accessible $b \in H_{NW,U,k}$ in the 4th quadrant, with further extension by lattice symmetries. When two points a, b lie in the same horizontal or vertical line, we define the *canonical path* from a to b to be the one which follows that line.

We say $\Gamma_{0x}[0,p_x]$ is *returning* if it contains a point of $L_{*,k}$, followed by a point of $J_{*,k}$, where both values $*$ (N, E, W or S) are the same. We recall that by definition, from (8) we have $\theta < \mu < \widetilde{\theta}^{c_\theta}$ which ensures that, when F_k occurs, the height of $\Lambda_{H,k}$ is much less than its length, but much more than the height of $J_{N,k}$. Recall the definitions of p_x, t_x from the beginning of the section.

Claim 4. *If F_k occurs and $\Gamma_{0x}[0,t_x] \not\subset \Lambda_{H,k} \cup \Lambda_{V,k}$, then either $\Gamma_{0x}[0,p_x]$ is returning, or one of t_x, p_x is accessible .*

Proof of Claim 4. Suppose F_k occurs, $\Gamma_{0x}[0,t_x] \not\subset \Lambda_{H,k} \cup \Lambda_{V,k}$, and neither t_x nor p_x is accessible. We may assume p_x lies in $H_{NE,L,k}$ on or above $H_{NW,U,k}$, as other cases are symmetric; then $t_x \in H_{NW,U,k}$ (or we may assume so, if p_x is a double point.) Since $\Gamma_{0x}[0,t_x] \not\subset \Lambda_{H,k} \cup \Lambda_{V,k}$, $\Gamma_{0x}[0,t_x]$ must intersect $L_{N,k} \cup L_{S,k}$; let q_x be the first such point of intersection. If $q_x \in L_{S,k}$, then the fact that t_x is not accessible (it must lie above $J_{S,k}$) means that $\Gamma_{0x}[0,p_x]$ is returning. If $q_x \in L_{N,k}$, then the fact that p_x is not accessible (hence lying below $J_{N,k}$) again means that $\Gamma_{0x}[0,p_x]$ is returning. □

We now complete the proof of Lemma 2, by a contradiction argument. Assume that F_k occurs, but for some $x \notin \Omega_k$ we have $\Gamma_{0x}[0,t_x] \not\subset \Lambda_{H,k} \cup \Lambda_{V,k}$. As in the proof of Claim 4, we may assume p_x lies in $H_{NE,L,k}$ on or above $H_{NW,U,k}$, and then that $t_x \in H_{NW,U,k}$. Let

$q_x \in L_{N,k} \cup L_{S,k}$ be as in the proof of Claim 4, and let a_x be the last point of $\Gamma_{0x}[0, q_x]$ in $J_{*,k}$, with subscript $* = \mathrm{N}$ or S according as $q_x \in L_{N,k}$ or $q_x \in L_{S,k}$. Thus $\Gamma_{0x}[0, p_x]$ follows a path $0 \to a_x \to q_x \to t_x \to p_x$.

By Claim 4, we now have three cases:

Case 1. $\Gamma_{0x}[0, p_x]$ is returning. In this case we define b_x to be the first point of $\Gamma_{0x}[q_x, p_x]$ in the line $J_{*,k}$ ($* = \mathrm{N}$ or S) containing a_x.

Case 2. $\Gamma_{0x}[0, p_x]$ is not returning, and p_x is accessible (hence above $J_{N,k}$, so necessarily $q_x \in L_{N,k}$). In this case we define $b_x = p_x$.

Case 3. $\Gamma_{0x}[0, p_x]$ is not returning, p_x is inaccessible (hence below $J_{N,k}$, so necessarily $q_x \in L_{S,k}$), and t_x is accessible. In this case we define $b_x = t_x$.

In all three cases we compare passage times for $\Gamma_{0x}[a_x, b_x]$ to that of the canonical path from a_x to b_x, which we denote γ_x, and we obtain our contradiction by showing that

$$T(\gamma_x) < T(\Gamma_{0x}[a_x, b_x]), \tag{35}$$

see Figure 4.

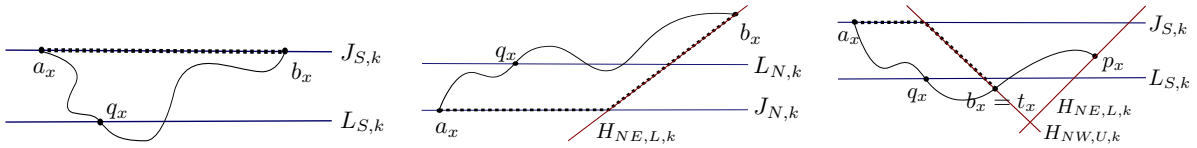

FIGURE 4. Cases 1-2-3 (from left to right). The path Γ_{0x} is the curved line going through a_x, q_x and b_x, and the canonical path γ_x is the dashed line joining a_x to b_x through highways.

Since $E_{2,k}$ occurs, there are no slow bonds in γ_x, so to prove (35) we may and do assume there are no slow bonds at all. Cases 2 and 3 are essentially symmetric, so we focus on Case 2 first, then the simpler Case 1.

We write (in Case 2)

$$a_x = (u, Y_{N,k}), \quad q_x = (y, \mu^{-k}), \quad b_x = p_x = (r, s), \tag{36}$$

and we let $(d, Y_{N,k})$ denote the left endpoint of the bond $J_{N,k} \cap H_{NE,L,k}$.

The class of the highway $J_{N,k}$ is at least q_k, so due to stage-2 deletions, the only zigzag highways intersecting $J_{N,k}$ have class greater than $c_{\tilde{\theta}} q_k/(1+\zeta)$. From (16), (19) and the definition of $E_{1,k}$ we have

$$\begin{aligned} T(\gamma_x) &\le 0.9(d-u) + 0.4 + \sum_{m > c_{\tilde{\theta}} q_k/(1+\zeta)} R_{k,m}\eta^m + 2(s - Y_{N,k})(0.7 + 0.1\eta^k) + 0.2 \\ &\le 0.9(d-u) + 2(s - Y_{N,k})(0.7 + 0.1\eta^k) + 0.6 + c_1(\eta^3\theta^{-2})^{c_\theta \delta k}(\mu^{-1}\eta)^k. \end{aligned} \tag{37}$$

Our aim is to show that for some c_7,

$$T(\Gamma_{0x}[a_x, b_x]) \ge 0.9(d-u) + 2(s - Y_{N,k})(0.7 + 0.1\eta^k) + c_7(\mu^{-1}\eta)^k, \tag{38}$$

which with (37) is sufficient to yield (35) for all large k, since $\mu^{-1}\eta > 1$ by (8) and $\eta^3\theta^{-2} < 1$ by (7).

The rest of the proof is devoted to showing (38), by analyzing the type (and number of each type) of bonds that the path Γ_{0x} uses.

A *singleton bond* in a path Γ is a zigzag bond adjacent in Γ to two boundary bonds. We divide $\Gamma_{0x}[a_x, b_x] = (a_x = z_0, z_1, \dots, z_n = b_x)$ into the following types of segments:

(i) *zigzag segments*: maximal subsegments which do not contain two consecutive non-zigzag bonds. A zigzag segment must start and end with a boundary bond, unless it starts at a_x or ends at b_x.

(ii) *intermediate segments*: the segments in between two consecutive zigzag segments, the segment up to the first zigzag segment, and the segment after the final zigzag segment (any of which may be empty.)

For maximal subsegments contained in a single zigzag highway, we can associate a unique highway orientation $\Phi(e)$ (SW/NE or SE/NW) to each zigzag bond in $e \in \Gamma_{0x}[a_x, b_x]$, with the exception of singleton intersection bonds in Γ, for which we assign $\Phi(e)$ arbitrarily.

We let $z_{n_{2j-2}}, z_{n_{2j-1}}$ be the endpoints of the jth intermediate segment, $1 \le j \le J+1$, so $\beta_j = \Gamma_{0x}[z_{n_{2j-1}}, z_{n_{2j}}]$ is the jth zigzag segment, $1 \le j \le J$. A *transition site* in a path Γ is a site in Γ which is an endpoint of a zigzag bond in Γ and a boundary bond in Γ. For any path Γ we define

$$T_{\alpha^*}(\Gamma) = \sum_{e \in \Gamma} \alpha_e^*. \tag{39}$$

For $* =$ Z, B, H, V we define

$$N_*(\Gamma) = \big|\{e \in \Gamma : e \text{ has property } *\}\big|,$$

where subscripts and corresponding properties are as follows:

$$Z: \text{ zigzag}, \qquad B: \text{ not zigzag}, \qquad H: \text{ horizontal}, \qquad V: \text{ vertical},$$
$$N: \text{ northward step in } \Gamma \text{ (E, W, S similar)}.$$

We may combine subscripts to require multiple properties, for example $N_{ZH}(\Gamma) = |\{e \in \Gamma : e \text{ is a horizontal zigzag bond}\}|$, and we use superscripts NE or NW to restrict the count to zigzag bonds belonging to SW/NE highways or SE/NW, respectively. We also let $N_{Hi}(\Gamma)$ be the number of zigzag highways intersecting Γ and $N_T(\Gamma)$ the number of transition sites in Γ. The intersection of a geodesic with any one zigzag highway is always connected, and if for example a SW/NE highway segment is traversed by Γ in the NE direction, the number of N and E steps differs by at most 1. It follows that for every geodesic Γ we have

$$\begin{aligned} N_{Hi}(\Gamma) \ge D(\Gamma) := {} & |N_{ZE}^{NE}(\Gamma) - N_{ZN}^{NE}(\Gamma)| + |N_{ZW}^{NE}(\Gamma) - N_{ZS}^{NE}(\Gamma)| \\ & + |N_{ZE}^{NW}(\Gamma) - N_{ZS}^{NW}(\Gamma)| + |N_{ZW}^{NW}(\Gamma) - N_{ZN}^{NW}(\Gamma)|. \end{aligned} \tag{40}$$

We now make some observations about geodesics and zigzag segments. We show that a zigzag segment β_j may contain multiple bonds of at most one zigzag highway (which we call *primary*)—the bonds in β_j belonging to other zigzag highways are necessarily singleton bonds, and there are at most 2 of these; if β_j intersects 3 zigzag highways then the primary highway must lie between the two singletons. Moreover, any non-zigzag interior bond of some β_j must be an entry/exit bond. Indeed, due to the stage-3 trimming, in order for Γ to switch from one zigzag highway to another within β_j – with at least two bonds on each – there would need to be one of the following succession of steps (or some lattice rotation thereof): (i) N, E, E, E, S with the middle one being an exit/entry bond; (ii) N, E, E, S

with the middle 2 being meeting zigzag bonds; (iii) N, E, S with the middle one being an intersection bond. (We refer to Figure 2 for a picture.) But none of these patterns can occur in a geodesic, because omitting the N and S steps always produces a faster path. Then (14) and the stage-3 trimming establish the remaining properties mentioned.

For $2 \le j \le J-1$, if β_j contains a primary highway then the bonds of β_j, from the initial bond through the first bond of the primary highway, must follow one of the following patterns:

(i) boundary (not skimming), zigzag (not intersection)
(ii) skimming boundary, terminal
(iii) semislow boundary, intersection (necessarily followed by a meeting bond)
(iv) boundary (not skimming), meeting, meeting
(v) boundary (not skimming), zigzag (not intersection), entry/exit, zigzag (not intersection).

The same is true in reverse order at the opposite end of β_j. If there is no primary highway then the full β_j follows one of the following patterns, or its reverse:

(vi) boundary (not skimming), zigzag (not intersection), boundary (not skimming)
(vii) skimming boundary, terminal, boundary (not skimming)
(viii) semislow boundary, intersection, semislow boundary
(ix) normal boundary, zigzag (not intersection), entry/exit, zigzag (not intersection), normal boundary.

It is readily checked from this that in all cases

$$\begin{aligned} T_{\alpha^*}(\beta_j) &\ge 0.9N_B(\beta_j) + 0.7N_Z(\beta_j) + 0.1N_T(\beta_j) \\ &= 0.9N_B(\beta_j) + 0.7N_Z(\beta_j) + 0.2N_{Hi}(\beta_j), \quad 2 \le j \le J-1. \end{aligned} \tag{41}$$

For $j = 1, J$, β_j is a truncation of a path as in (i)–(ix), omitting a (possibly empty) segment of bonds at one end, and we similarly have

$$T_{\alpha^*}(\beta_j) \ge 0.9N_B(\beta_j) + 0.7N_Z(\beta_j) + 0.2N_{Hi}(\beta_j) - 0.2, \quad j = 1, J,$$

and therefore

$$T_{\alpha^*}(\Gamma_{0x}[a_x, b_x]) \ge 0.9N_B(\Gamma_{0x}[a_x, b_x]) + 0.7N_Z(\Gamma_{0x}[a_x, b_x]) + 0.2N_{Hi}(\Gamma_{0x}[a_x, b_x]) - 0.4. \tag{42}$$

By (42) and (40) we have

$$T_{\alpha^*}(\Gamma_{0x}[a_x, b_x]) \ge 0.9N_B(\Gamma_{0x}[a_x, b_x]) + 0.7N_Z(\Gamma_{0x}[a_x, b_x]) + 0.2D(\Gamma_{0x}[a_x, b_x]) - 0.4, \tag{43}$$

and from the definition of M_k^{NE},

$$T(\Gamma_{0x}[a_x, b_x]) \ge T_{\alpha^*}(\Gamma_{0x}[a_x, b_x]) + 0.1\eta^{k-1}\big(N_{ZN}^{NE}(\Gamma_{0x}[a_x, b_x]) + N_{ZE}^{NE}(\Gamma_{0x}[a_x, b_x])\big). \tag{44}$$

In view of (36) and (43)-(44), let us consider the question of minimizing

$$\begin{aligned} &0.9(n_{BE} + n_{BW} + n_{BN} + n_{BS}) \\ &\quad + 0.7\left(n_{ZN}^{NE} + n_{ZE}^{NE} + n_{ZW}^{NE} + n_{ZS}^{NE} + n_{ZN}^{NW} + n_{ZE}^{NW} + n_{ZW}^{NW} + n_{ZS}^{NW}\right) \\ &\quad + 0.2\Big(|n_{ZE}^{NE} - n_{ZN}^{NE}| + |n_{ZW}^{NE} - n_{ZS}^{NE}| + |n_{ZE}^{NW} - n_{ZS}^{NW}| + |n_{ZW}^{NW} - n_{ZN}^{NW}|\Big) \\ &\quad + 0.1\eta^{k-1}(n_{ZN}^{NE} + n_{ZE}^{NE}) - 0.4 \end{aligned} \tag{45}$$

subject to all 8 variables being nonnegative integers satisfying

$$n_{ZE}^{NE} + n_{ZE}^{NW} - n_{ZW}^{NE} - n_{ZW}^{NW} + n_{BE} - n_{BW} = (d-u) + (s - Y_{N,k}) = N_H(\gamma_x), \tag{46}$$

$$n_{ZN}^{NE} - n_{ZS}^{NW} - n_{ZS}^{NE} + n_{ZN}^{NW} + n_{BN} - n_{BS} = s - Y_{N,k} = N_V(\gamma_x), \tag{47}$$

$$n_{ZN}^{NE} + n_{ZN}^{NW} + n_{BN} = (s - Y_{N,k}) + g, \tag{48}$$

$$n_{ZS}^{NE} + n_{ZS}^{NW} + n_{BS} = g, \tag{49}$$

$$n_{ZN}^{NW} + n_{BN} = j, \tag{50}$$

for some fixed $g \geq 0$ and $0 \leq j \leq (s - Y_{N,k}) + g$. Here (49) is redundant but we include it for ready reference, and despite (50) we formulate the problem with n_{ZN}^{NW} as a variable, to match the rest of the problem. Further, g may be viewed as an "overshoot", the number of northward steps beyond the minimum needed to reach the height s of p_x, and j is the number of northward steps taken "inefficiently," that is, not in NE/SW zigzag highways. From the definition of q_x we may restrict to

$$s - Y_{N,k} + g \geq \mu^{-k}. \tag{51}$$

To study this we use the concept of *shifting mass* from one variable $n_\bullet^*$ to a second one, by which we mean incrementing the second by 1 and the first by -1. We also use *canceling mass* between two variables in (46) or two in (47), one appearing with "$+$" and the other with "$-$", by which we mean decreasing each variable by 1.

Shifting mass from n_{BS} to n_{ZS}^{NE}, or from n_{BN} to n_{ZN}^{NW}, does not increase (45), so a minimum exists with $n_{BS} = n_{BN} = 0$, so we may eliminate those two variables. Among the variables $n_\bullet^*$ in (46), if a variable with "+" and a variable with "−" are both nonzero, then canceling (unit) mass between them decreases (45) by at least 1; this means that a minimum exists with all negative terms in (46) equal to 0. Then shifting mass in (47) from n_{ZS}^{NE} to n_{ZS}^{NW}, or in (46) from n_{BE} to n_{ZE}^{NW}, does not increase (45) (since the preceding step has set n_{ZW}^{NE} and n_{ZW}^{NW} to 0), so there is a minimum with also $n_{ZS}^{NE} = n_{BE} = 0$. With these variables set to 0, the problem becomes minimizing

$$\begin{aligned} &0.7\left(n_{ZE}^{NE} + n_{ZE}^{NW} + s - Y_{N,k} + 2g\right) + 0.2\Big(|n_{ZE}^{NE} - (s - Y_{N,k} + g - j)| + |n_{ZE}^{NW} - g| + j\Big) \\ &\qquad + 0.1\eta^{k-1}(s - Y_{N,k} + g - j + n_{ZE}^{NE}) - 0.4 \end{aligned} \tag{52}$$

subject to

$$n_{ZE}^{NE} + n_{ZE}^{NW} = (d-u) + (s - Y_{N,k}). \tag{53}$$

Setting $n_{ZE}^{NE} = z, n_{ZE}^{NW} = (d-u) + (s - Y_{N,k}) - z$ and considering the effect of incrementing z by 1, we see that (52) is minimized (not necessarily uniquely) at

$$z = \min\Big((s - Y_{N,k}) + g - j, [(s - Y_{N,k}) + (d-u) - g] \vee 0\Big). \tag{54}$$

We now consider two cases.

Case 2A. $2g-j \le d-u$. Here we have $z = (s-Y_{N,k})+g-j$ in (54), and the corresponding minimum value of (52) is

$$\begin{aligned}
&0.9(d-u) + 0.7\cdot 2(s-Y_{N,k}) + g + 0.4j + 0.1\eta^{k-1}(2(s-Y_{N,k}) + 2(g-j)) - 0.4\\
&\quad \ge 0.9(d-u) + 0.7\cdot 2(s-Y_{N,k}) + g + 0.1\eta^{k-1}\cdot 2(s-Y_{N,k}) - 0.4\\
&\quad \ge 0.9(d-u) + (0.7+0.1\eta^{k-1})2(s-Y_{N,k}) + 0.1\eta^k(\eta^{-1}-1)\cdot 2(s-Y_{N,k}+g) - 0.4.
\end{aligned}$$

For the first inequality, we used that $0.4j \ge 0.2\eta^k j$, and for the second one that $g \ge 0.2\eta^k(\eta^{-1}-1)g$, for k sufficiently large. With (44) and (51) this shows that

$$T(\Gamma_{0x}[a_x, b_x]) \ge 0.9(d-u) + (0.7+\eta^k)\cdot 2(s-Y_{N,k}) + 0.2\eta^k(\eta^{-1}-1)\mu^{-k} - 0.4.$$

Since $\eta > \mu$ by (8), this proves (38).

Case 2B. $2g - j > d - u$, so that

$$0.8g \ge 0.4(d-u) + 0.1\eta^{k-1}j. \tag{55}$$

Here we have $z = [(s-Y_{N,k})+(d-u)-g]\vee 0$ in (54), and the corresponding minimum value of (52) in the case $z = (s-Y_{N,k})+(d-u)-g > 0$ (the case $z = 0$ being treated similarly) is

$$\begin{aligned}
&0.5(d-u) + 0.7\cdot 2(s-Y_{N,k}) + 1.8g + 0.1\eta^{k-1}((d-u) + 2(s-Y_{N,k}) - j) - 0.4\\
&\quad \ge 0.9(d-u) + 0.7\cdot 2(s-Y_{N,k}) + g + 0.1\eta^{k-1}\cdot 2(s-Y_{N,k}) - 0.4
\end{aligned}$$

where the inequality follows from (55). Then (38) follows as in Case 2A.

We now briefly explain the modifications of the above argument to treat Case 1. Analogously to (36), we write

$$a_x = (u, Y_{N,k}), \quad q_x = (y, \mu^{-k}), \quad b_x = (v, Y_{N,k}),$$

and we may assume $v - u > 0$, since otherwise we can consider the path $\Gamma_{0x}[a_x, b_x]$ running backwards. Similarly to (37), we have

$$\begin{aligned}
T(\gamma_x) &\le 0.9(v-u) + 0.5 + \sum_{m > c_{\widetilde{\theta}} q_k/(1+\zeta)} R_{k,m}\eta^m\\
&\le 0.9(v-u) + 0.5 + c_1(\eta^3\theta^{-2})^{c_\theta \delta k}(\mu^{-1}\eta)^k. \qquad (56)
\end{aligned}$$

and similarly to (38) we want to show

$$T(\Gamma_{0x}[a_x, b_x]) \ge 0.9(v-u) + c_7(\mu^{-1}\eta)^k \tag{57}$$

in order to obtain (35). Then, in the proof above, we replace $d-u$ with $v-u$ and $s - Y_{N,k}$ with 0 in the constraints (46)–(50) and in (51). Otherwise the proof remains the same, establishing (57). In the end, in all the Cases 1–3 we have the contradiction (35), and Lemma 2 is proven. □

3.4. **Proof of items (i) and (iii) in Theorem 1.** Item (i). It follows readily from the fact that $X_{i,k} = o(2^k)$ (meaning there are long diagonal highways close to the origin) that the asymptotic speed in any diagonal direction is $\sqrt{2}/1.4$, and similarly along an axis it is $1/0.9$. The proof of Lemma 2 shows that for any (a, b) with $0 \le b \le a$, the passage time is

$$\tau((0,0),(a,b)) \ge (.9(a-b) + 1.4b)\,(1 + o(1)) \quad \text{as } a \to \infty.$$

From the fact that there are, with high probability, long HV and zigzag highways close to any given point (reflected in the fact that $X_{*,k} = o(2^k)$ and $Y_{*,k} = o(2^k)$), we readily obtain the reverse inequality. The linearity of the asymptotic expression $.9(a-b)+1.4b$ means that the limit shape is flat between any diagonal and an adjacent axis. It follows that the limit shape $\mathcal{B}$ is an octogon, with vertex $(1/.9, 0)$ on the horizontal axis, $(\sqrt{2}/1.4, \sqrt{2}/1.4)$ on the SW/NE diagonal, and symmetrically in other quadrants, proving Theorem 1(i).

Item (iii). Let F be the facet of B in the first quadrant between the horizontal axis and the main diagonal, and let ρ_F be the linear functional equal to 1 on F. Recalling (IV) in the introduction, we have from [1] that there is a.s. a unique RC geodesic Γ_F with $\mathrm{Dir}(\Gamma_F) \subset \{v/|v| : v \in F\}$, and Γ_F has Busemann function linear to ρ_F. But all geodesics are directed in axis directions so we must have $\mathrm{Dir}(\Gamma_F) = \{(1,0)\}$. Let $\widehat{F}$ be the facet which is the mirror image of F across the horizontal axis. From lattice symmetry, we have $\mathrm{Dir}(\Gamma_{\widehat{F}}) = \{(1,0)\}$ and its Busemann function is linear to $\rho_{\widehat{F}}$. Since the Busemann functions differ, we must have $\Gamma_F \ne \Gamma_{\widehat{F}}$. This and lattice symmetry prove Theorem 1(iii).

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