

On two simple and sharp lower bounds for $\exp(x^2)$

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Abstract

In this note, we determine two simple and sharp lower bounds for $\exp(x^2)$, improving two well-known lower bounds. They are defined by continuous even functions on \mathbb{R} using exponential functions for the first lower bound, and polynomial-exponential functions for the second lower bound. The results are illustrated by a numerical study and some graphics.

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1. Introduction

In this note, we propose two sharp lower bounds for the function $\exp(x^2)$. This can be of interest in many area of mathematics when the function $\exp(x^2)$ naturally appears. One of the most famous example is the gaussian probability density function; a lower bound for $\exp(x^2)$ gives an upper bound for this function. The first lower bound proposes an improvement of the well-known exponential (hyperbolic) lower bound: for any $x \in \mathbb{R}$, we have $\exp(x^2) \geq \cosh(\sqrt{2}x)$, whereas the second lower bound proposes an improvement of the well-known polynomial lower bound: for any $x \in \mathbb{R}$, we have $\exp(x^2) \geq 1 + x^2 + x^4/2$. Both of them can be proved using the exponential series: $\exp(x^2) = \sum_{k=0}^{+\infty} x^{2k}/(k!)$. For other classical inequalities involving exponential functions, we refer to Mitrinović (1964) and Bullen (1998). The motivation of this note is based on the following remark: the two lower bounds presented above are sharp for small values of $|x|$, but not when $|x|$ is greater to a certain threshold. We thus propose to improve them by an analytic approach; we split \mathbb{R} into two parts, an interval A centered in 0 with a length depending on a tuning parameter $a \geq 0$, and its complementary A^c , we keep the existing lower bound for $\exp(x^2)$ on A , and we consider a particular exponentiated warped version of it on A^c . The obtained lowers bounds are simple continuous even function on \mathbb{R} using exponential functions for the first one and polynomial-exponential functions for the second one; there is no power of x^2 . We thus provide a simple and intuitive alternative to the sharp lower bound proposed by Brenner and Alzer (1991), *i.e.* for any $a > 0$ and $x \in \mathbb{R}$, $m(x; a) = (1 + x^2/a)^{(a(x+x^2))^{1/2}}$.

The rest of the note is organized as follows. Section 2 present the first lower bound, supported by a short numerical study and some graphics. Section 3 presents the second lower bound, supported by a short numerical study and some graphics. The proofs of our results are postponed in Section 4.

2. First lower bound

The following lower bound of $\exp(x^2)$ is well-known: for any $x \in \mathbb{R}$, we have $\exp(x^2) \geq \cosh(\sqrt{2}x)$. In the result below, we determine a continuous even function on \mathbb{R} denoted by $f(x; a)$ such that, for any $a \geq 0$, $\exp(x^2) \geq f(x; a) \geq \cosh(\sqrt{2}x)$, $x \in \mathbb{R}$.

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Proposition 1. For any $a \geq 0$, we defined the function $f(x; a)$ by

$$f(x; a) = \cosh(\sqrt{2}x) \mathbf{1}_{\{|x| < a/2\}} + \cosh(\sqrt{2}(|x| - a)) \exp(2a|x| - a^2) \mathbf{1}_{\{|x| \geq a/2\}}, \quad x \in \mathbb{R},$$

where $\mathbf{1}_A$ denotes the indicator function over A , i.e. $\mathbf{1}_A = 1$ if $x \in A$ and 0 elsewhere. Then, for any $a \geq 0$ and $x \in \mathbb{R}$, we have

$$\exp(x^2) \geq f(x; a) \geq \cosh(\sqrt{2}x).$$

Note that $f(x; a)$ is a continuous function on \mathbb{R} according to x ; $\lim_{|x| \rightarrow a/2} f(x; a) = \cosh(a/\sqrt{2}) = f(a/2; a) = f(-a/2; a)$. Moreover, the lower bound of the second inequality can be attained: $f(x; 0) = \cosh(\sqrt{2}x)$. Several criteria are possible for a suitable choice for a . For instance, one can consider $a_* = \arg \min_{a \geq 0} \int_{-2.5}^{2.5} [f(x; a) - \exp(x^2)]^2 dx$.

Table 1 shows the numerical values of the integral squared error $R(a, b)$ measuring an error between $f(x; a)$ and $\exp(x^2)$ over $[-b, b]$: $R(a, b) = \int_{-b}^b [f(x; a) - \exp(x^2)]^2 dx$ and the integral squared error $R_*(b)$ measuring an error between $\cosh(\sqrt{2}x)$ and $\exp(x^2)$: $R_*(b) = \int_{-b}^b [\cosh(\sqrt{2}x) - \exp(x^2)]^2 dx$, for several values of a and b . For the considered values, we see that $R(a, b) \leq R_*(b)$.

Then several figures are proposed. Figure 1 illustrates the well-known inequality: $\exp(x^2) \geq \cosh(\sqrt{2}x)$, Figure 2 shows graphically the first inequality in Proposition 1: $\exp(x^2) \geq f(x; a)$ with $a = 1$ and Figure 3 illustrates the second inequality in Proposition 1: $f(x; a) \geq \cosh(\sqrt{2}x)$ with $a = 1$. Let us mention that the choice of $a = 1$ is far to be optimal (as we can see in Table 1), it is arbitrary taken. The numerical study and the figures are done with Mathematica (v.11.1). See Wolfram (1999).

3. Second lower bound

Let us now focus on a second well-known lower bound of $\exp(x^2)$: for any $x \in \mathbb{R}$, we have $\exp(x^2) \geq 1 + x^2 + x^4/2$. In the result below, we determine a continuous even function on \mathbb{R} denoted by $k(x; a)$ such that, for any $a \geq 0$, $\exp(x^2) \geq k(x; a) \geq 1 + x^2 + x^4/2$, $x \in \mathbb{R}$.

Proposition 2. For any $a \geq 0$, we defined the function $k(x; a)$ by

$$k(x; a) = \left(1 + x^2 + \frac{x^4}{2}\right) \mathbf{1}_{\{|x| < a/2\}} + \left(1 + (|x| - a)^2 + \frac{(|x| - a)^4}{2}\right) \exp(2a|x| - a^2) \mathbf{1}_{\{|x| \geq a/2\}}, \quad x \in \mathbb{R}.$$

Then, for any $a \geq 0$ and $x \in \mathbb{R}$, we have

$$\exp(x^2) \geq k(x; a) \geq 1 + x^2 + \frac{x^4}{2}.$$

Note that $k(x; a)$ is a continuous function according to x ; $\lim_{|x| \rightarrow a/2} k(x; a) = 1 + a^2/4 + a^4/32 = k(a/2; a) = k(-a/2; a)$. The lower bound of the second inequality can be attained: $k(x; 0) = 1 + x^2 + x^4/2$. Among the possible criteria to chose a suitable a , one can evaluate $a_* = \arg \min_{a \geq 0} \int_{-2.5}^{2.5} [k(x; a) - \exp(x^2)]^2 dx$.

As for Table 1, Table 2 shows the numerical values of error measures: $R(a, b) = \int_{-b}^b [k(x; a) - \exp(x^2)]^2 dx$ and $R_*(b) = \int_{-b}^b [(1 + x^2 + x^4/2) - \exp(x^2)]^2 dx$, for several values of a and b . For all the considered values, we see that $R(a, b) \leq R_*(b)$.

Figure 4 illustrates the well-known lower bound: $\exp(x^2) \geq 1 + x^2 + x^4/2$, Figure 5 illustrates the first inequality in Proposition 2: $\exp(x^2) \geq k(x; a)$ with $a = 1$ and Figure 6 illustrates the second inequality in Proposition 2: $k(x; a) \geq 1 + x^2 + x^4/2$ with $a = 1$. Again, let us mention that the choice of $a = 1$ is just for the illustration; it is far to be optimal.

Note that the comparison of the two proposed lower bounds is not immediate; from an analytic point of view, there exists $x_* > 0$ such that $\cosh(\sqrt{2}x) < 1 + x^2 + x^4/2$ for $x \in [-x_*, x_*]$ and the reverse for $|x| \geq x_*$. A fair numerical comparison needs to take the optimal a for each of them with a given criteria, which remains arbitrary.

Table 1: Numerical evaluations of $R(a, b)$ and $R_*(b)$ for the first lower bound.

$R(a, b)$	$a = 0.1$	$a = 0.5$	$a = 0.7$	$a = 1$	$a = 1.3$	$a = 1.7$	$a = 2$	$a = 2.5$	$a = 3$	$R_*(b)$
$b = 1.5$	4.4441	0.566336	0.113316	0.0027994	0.00173972	0.0300955	0.201991	2.34954	6.29601	6.29601
$b = 2$	419.788	170.47	82.1288	16.2676	1.21222	0.0311728	0.207063	4.96995	482.842	482.842
$b = 2.5$	48470.4	35358.7	26319.9	12962.2	4044.96	269.723	7.01922	5.16422	208.093	50481.3

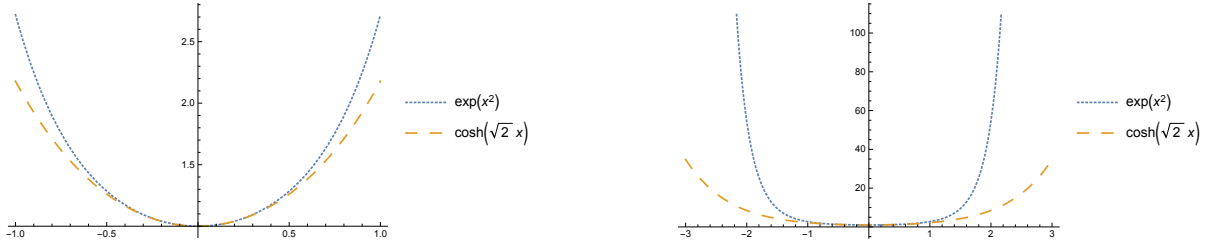


Figure 1: Superimposed curves of $\exp(x^2)$ and $\cosh(\sqrt{2}x)$ for $x \in [-b, b]$ with $b = 1$ for the first figure, and $b = 3$ for the second figure.

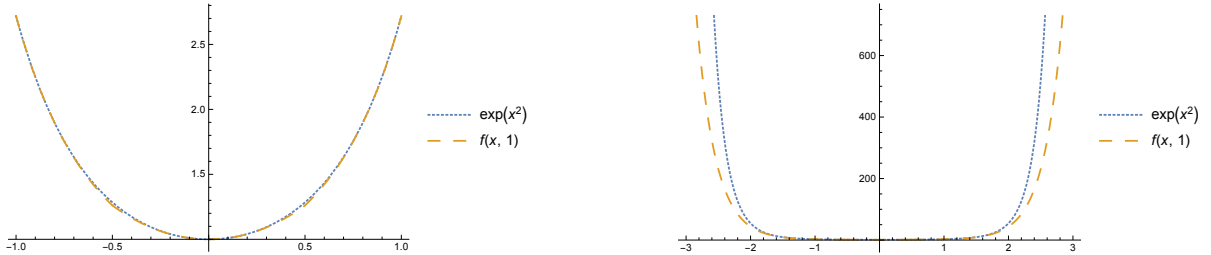


Figure 2: Superimposed curves of $\exp(x^2)$ and $f(x, 1)$ for $x \in [-b, b]$ with $b = 1$ for the first figure, and $b = 3$ for the second figure.

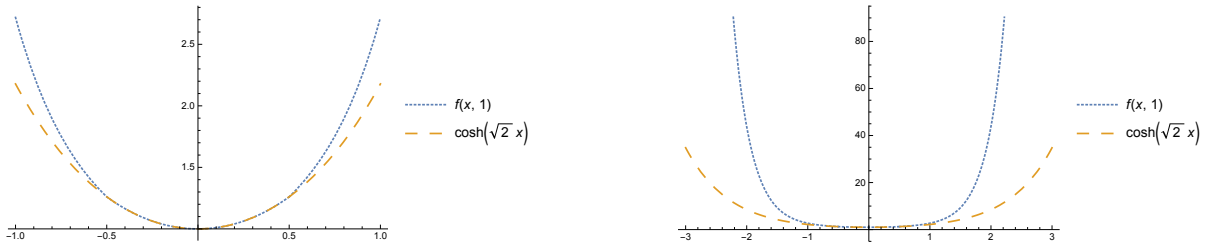


Figure 3: Superimposed curves of $f(x, 1)$ and $\cosh(\sqrt{2}x)$ for $x \in [-b, b]$ with $b = 1$ for the first figure, and $b = 3$ for the second figure.

Table 2: Numerical evaluations of $R(a, b)$ and $R_*(b)$ for the second lower bound.

$R(a, b)$	$a = 0.1$	$a = 0.5$	$a = 0.7$	$a = 1$	$a = 1.3$	$a = 1.7$	$a = 2$	$a = 2.5$	$a = 3$	$R_*(b)$
$b = 1.5$	1.55921	0.073	0.0068	0.000029	0.0000439	0.00188203	0.0206829	0.562207	2.62638	2.62638
$b = 2$	291.674	73.3948	24.3381	2.15825	0.0459892	0.00188357	0.0207268	0.780871	21.17	362.355
$b = 2.5$	44507.6	26997.6	17012.7	5707.33	973.879	16.9692	0.0944469	0.78247	30.9922	47492.8

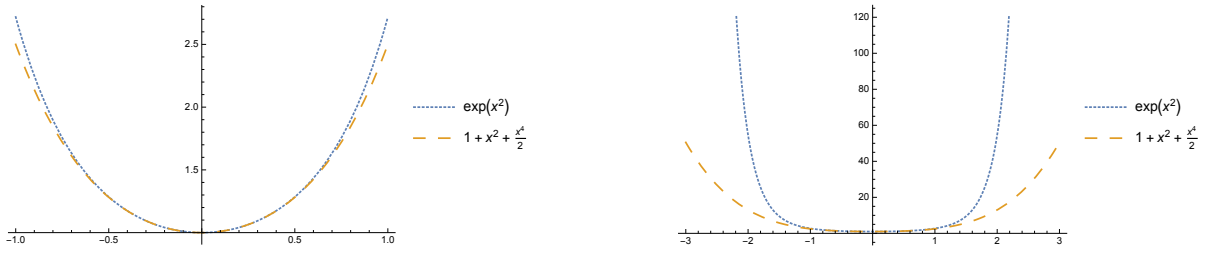


Figure 4: Superimposed curves of $\exp(x^2)$ and $1 + x^2 + x^4/2$ for $x \in [-b, b]$ with $b = 1$ for the first figure, and $b = 3$ for the second figure.

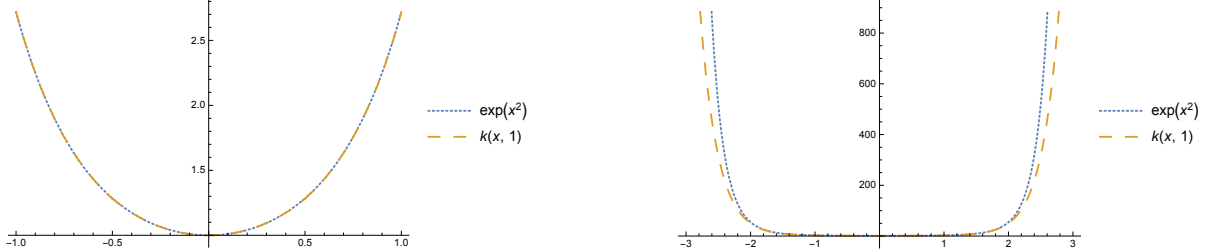


Figure 5: Superimposed curves of $\exp(x^2)$ and $k(x; 1)$ for $x \in [-b, b]$ with $b = 1$ for the first figure, and $b = 3$ for the second figure.

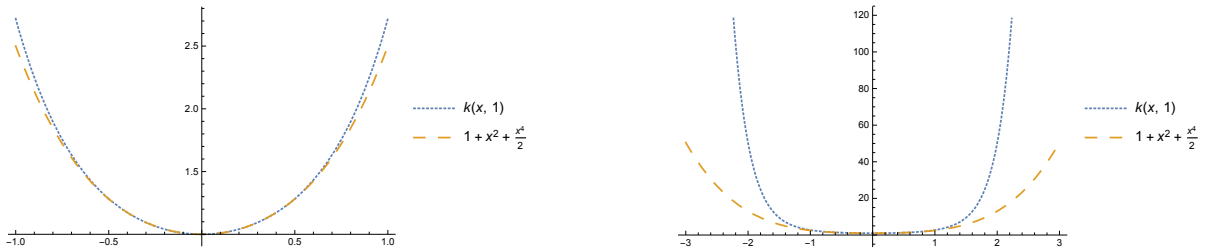


Figure 6: Superimposed curves of $k(x; 1)$ and $1 + x^2 + x^4/2$ for $x \in [-b, b]$ with $b = 1$ for the first figure, and $b = 3$ for the second figure.

4. Proofs

Proof of Proposition 1. First of all, let us recall that $\exp(x^2) \geq \cosh(\sqrt{2}x)$, $x \in \mathbb{R}$. Therefore, using $x^2 = (|x| - a)^2 + 2a|x| - a^2$, we have

$$\exp(x^2) = \exp((|x| - a)^2) \exp(2a|x| - a^2) \geq \cosh\left(\sqrt{2}(|x| - a)\right) \exp(2a|x| - a^2).$$

Let us set $g(x; a) = \cosh(\sqrt{2}(|x| - a)) \exp(2a|x| - a^2)$. We now propose to study $g(x; a)$ according to the variable a . We have

$$\frac{\partial}{\partial a} g(x; a) = \exp(2a|x| - a^2) h(|x| - a),$$

where $h(y)$ is the function defined by $h(y) = 2y \cosh(\sqrt{2}y) - \sqrt{2} \sinh(\sqrt{2}y)$, $y \in \mathbb{R}$. We have immediately $\exp(2a|x| - a^2) > 0$. Let us investigate the sign of $h(|x| - a)$. Since, for any $y \in \mathbb{R}$,

$$\frac{\partial}{\partial y} h(y) = 2\sqrt{2}y \sinh(\sqrt{2}y) \geq 0,$$

$h(y)$ is increasing. Thanks to $h(0) = 0$, if $a < |x|$, we have $h(|x| - a) > 0$, implying that $\frac{\partial}{\partial a} g(x; a) > 0$, so $g(x; a)$ is strictly increasing according to a . If $a > |x|$, we have $h(|x| - a) < 0$, implying that $\frac{\partial}{\partial a} g(x; a) < 0$, so $g(x; a)$ is strictly decreasing according to a . Moreover, $\frac{\partial}{\partial a} g(x; a) = 0$ if and only if $a = |x|$. Noticing that $g(x; 0) = g(x; 2|x|) = \cosh(\sqrt{2}x)$, for any $|x| \geq a/2$, we have

$$\exp(x^2) = g(x; |x|) \geq g(x; a) \geq \inf_{a \in [0, 2|x|]} g(x; a) = \inf [g(x; 0), g(x; 2|x|)] = \cosh(\sqrt{2}x).$$

Thus $\exp(x^2) \mathbf{1}_{\{|x| \geq a/2\}} \geq g(x; a) \mathbf{1}_{\{|x| \geq a/2\}} \geq \cosh(\sqrt{2}x) \mathbf{1}_{\{|x| \geq a/2\}}$. For $|x| < a/2$, we have the reverse of the second inequality:

$$\cosh(\sqrt{2}x) = \inf_{a \in [0, 2|x|]} g(x; a) \geq \sup_{a \notin [0, 2|x|]} g(x; a) \geq g(x; a),$$

but the well-known lower bound is still valid: $\exp(x^2) \mathbf{1}_{\{|x| < a/2\}} \geq \cosh(\sqrt{2}x) \mathbf{1}_{\{|x| < a/2\}}$. So, for any $x \in \mathbb{R}$, we have

$$\exp(x^2) \geq \cosh(\sqrt{2}x) \mathbf{1}_{\{|x| < a/2\}} + g(x; a) \mathbf{1}_{\{|x| \geq a/2\}} = f(x; a), \quad f(x; a) \geq \cosh(\sqrt{2}x).$$

Proposition 1 is proved. ■

Proof of Proposition 2. First of all, let us recall that $\exp(x^2) \geq 1 + x^2 + x^4/2$, $x \in \mathbb{R}$. It follows from $x^2 = (|x| - a)^2 + 2a|x| - a^2$ that

$$\exp(x^2) = \exp((|x| - a)^2) \exp(2a|x| - a^2) \geq \left(1 + (|x| - a)^2 + \frac{(|x| - a)^4}{2}\right) \exp(2a|x| - a^2).$$

Let us set $\ell(x; a) = (1 + (|x| - a)^2 + (|x| - a)^4/2) \exp(2a|x| - a^2)$. We now propose to study $\ell(x; a)$ according to the variable a . Algebraic manipulations and simplifications give a simple expression for $\frac{\partial}{\partial a} \ell(x; a)$:

$$\frac{\partial}{\partial a} \ell(x; a) = \exp(2a|x| - a^2) (|x| - a)^5,$$

Since $\exp(2a|x| - a^2) > 0$, we have $\frac{\partial}{\partial a} \ell(x; a) = 0$ if and only if $a = |x|$.

Moreover, if $a < |x|$, we have $\frac{\partial}{\partial a} \ell(x; a) > 0$, so $\ell(x; a)$ is strictly increasing according to a , and if $a > |x|$, we have $\frac{\partial}{\partial a} \ell(x; a) < 0$, so $\ell(x; a)$ is strictly decreasing according to a . Noticing that $\ell(x; 0) = \ell(x; 2|x|) = 1 + x^2 + x^4/2$, for any $|x| \geq a/2$, we have

$$\exp(x^2) = \ell(x; |x|) \geq \ell(x; a) \geq \inf_{a \in [0, 2|x|]} \ell(x; a) = \inf [\ell(x; 0), \ell(x; 2|x|)] = 1 + x^2 + \frac{x^4}{2}.$$

This proves that $\exp(x^2) \mathbf{1}_{\{|x| \geq a/2\}} \geq \ell(x; a) \mathbf{1}_{\{|x| \geq a/2\}} \geq (1 + x^2 + x^4/2) \mathbf{1}_{\{|x| \geq a/2\}}$. For $|x| < a/2$, we have the reverse of the second inequality:

$$1 + x^2 + \frac{x^4}{2} = \inf_{a \in [0, 2|x|]} \ell(x; a) \geq \sup_{a \notin [0, 2|x|]} \ell(x; a) \geq \ell(x; a),$$

but the well-known lower bound still holds: $\exp(x^2) \mathbf{1}_{\{|x| < a/2\}} \geq (1 + x^2 + x^4/2) \mathbf{1}_{\{|x| < a/2\}}$. Hence, for any $x \in \mathbb{R}$, we have

$$\exp(x^2) \geq \left(1 + x^2 + \frac{x^4}{2}\right) \mathbf{1}_{\{|x| < a/2\}} + \ell(x; a) \mathbf{1}_{\{|x| \geq a/2\}} = k(x; a), \quad k(x; a) \geq 1 + x^2 + \frac{x^4}{2}.$$

This completes the proof of Proposition 2. ■

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